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Space Homogeneous Solutions of the Linear Semiconductor Boltzmann Equation

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The linear Boltzmann equation describing electron flow in a semiconductor is considered. The Cauchy problem for space-independent solutions is investigated, and without requiring a bounded collision frequency the existence of integrable solutions is established. Mass conservation, an H-theorem, and moment estimates also are obtained, assuming weak conditions. Finally, the uniqueness of the solution is demonstrated under a suitable hypothesis on the collision frequency.

Key Words: Boltzmann equation; Cauchy problem; semiconductors.

1. INTRODUCTION

The classical Boltzmann equation for a perfect rarefied gas has been studied extensively by many authors, and many interesting results have been obtained (see for example [1, 2, and 3]). In contrast, the Boltzmann equation describing electron flow in semiconductors has only recently become a subject of research. For a wide range of applications, it has proved a useful model for studying hot electrons or high field effects in semiconductors. Due to the particular nature of the scattering mechanisms in the semiconductor case, the collision operator assumes a distinct aspect. In common with the classical Boltzmann equation, even in relatively simple cases many mathematical difficulties arise. The existence of solutions to the initial value problem was proved in [10–12]. However, in order to obtain general results, in these papers smooth kernels for the collision operator were assumed. In fact, physical kernels contain the Dirac distribution, and so spaces of continuous functions are the most natural frame-

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work. A collision operator based on these kernels was studied in detail in [5], where space homogeneous continuous solutions to the Cauchy problem were found.

The existence and uniqueness of space homogeneous continuous solutions to the linear Boltzmann equation were proved in [6] and [7]. These results were obtained under the assumption of a bounded collision frequency. This holds, for example, if the collision kernels are based on polar optical phonon or impurity scattering rates [4]. However, other important scattering processes give unbounded collision frequencies. In applications, the choice of kernel depends on the material considered (silicon, germanium, etc.).

In this paper we relax the assumption on the collision frequency so that it can be unbounded. This allows us to prove the existence of Lebesgue integrable solutions, but not their continuity. If additional hypotheses hold, uniqueness is also demonstrated.

Recently, Banasiak [13, 14] obtained very general results concerning the existence and uniqueness of integrable solutions to linear transport equations. As a particular case, the Boltzmann equation for semiconductor devices in the parabolic band approximation was included. The main tools of those papers are interesting new theorems of the semigroup theory. Our existence theorem seems not to be a direct consequence of Banasiak's results, because in our case one of the hypotheses concerning mass conservation does not hold in general.

The plan of this paper is as follows. In Section 2 we briefly introduce the semiconductor Boltzmann equation and describe the collision operator. In Section 3 we write a truncated kinetic equation and solve a Cauchy problem. Moreover, some properties of the solutions are characterized. In Section 4 existence and uniqueness results are presented.

2. BASIC EQUATIONS

In the low density case the Boltzmann equation for an electron gas in a semiconductor is

$$\frac{\partial f}{\partial t}(t,\mathsf{k}) = \int_{\mathbb{R}^3} \left[S(\mathsf{k}',\mathsf{k}) f(t,\mathsf{k}') - S(\mathsf{k},\mathsf{k}') f(t,\mathsf{k}) \right] d\mathsf{k}'. \tag{1}$$

Here we consider the case of no electric field and we look for space homogeneous solutions. The unknown is the electron probability density f, which is a function of the time $t \in \mathbb{R}_0^+ = [0, +\infty)$ and the wave vector

$\mathsf{k} \in \mathbb{R}^3$. The kernel S is given by

$$S(\mathsf{k},\mathsf{k}') = \mathscr{G}(\mathsf{k},\mathsf{k}')\big[(\mathsf{n}_q + 1)\delta(\varepsilon(\mathsf{k}') - \varepsilon(\mathsf{k}) + \hbar\omega) + \mathsf{n}_q\delta(\varepsilon(\mathsf{k}') - \varepsilon(\mathsf{k}) - \hbar\omega)\big] + \mathscr{G}_0(\mathsf{k},\mathsf{k}')\delta(\varepsilon(\mathsf{k}') - \varepsilon(\mathsf{k})), \tag{2}$$

where the functions $\mathscr{G}(\mathsf{k},\mathsf{k}')$ and $\mathscr{G}_0(\mathsf{k}',\mathsf{k})$ are continuous on $\mathbb{R}^3 \times \mathbb{R}^3$ and satisfy the symmetry conditions $\mathscr{G}(\mathsf{k},\mathsf{k}') = \mathscr{G}(\mathsf{k}',\mathsf{k})$ and $\mathscr{G}_0(\mathsf{k},\mathsf{k}') = \mathscr{G}_0(\mathsf{k}',\mathsf{k})$. The constant positive parameter n_q is given by

$$\mathsf{n}_q = \left[\exp\left(\frac{\hbar\omega}{k_B T_L}\right) - 1\right]^{-1}, \tag{3}$$

where $\hbar$ is the Planck constant divided by 2π, ω is the positive constant phonon frequency, k_B is the Boltzmann constant, and T_L is the lattice temperature. The symbol δ denotes the Dirac distribution and $\varepsilon(\mathsf{k})$ is the electron energy. The latter is a continuous function and it will be defined later in this section. As an example, the expression $\delta(\varepsilon(\mathsf{k}') - \varepsilon(\mathsf{k}) + \hbar\omega)$ is a new distribution given by the composition of the delta distribution and the continuous function ε. The meaning of the collision operator on the right-hand side of (1) and some of its properties may be found in [5].

The collision operator describes the scattering processes between free electrons and phonons. In this model the ensemble of phonons is assumed to be in thermal equilibrium. The electrons move inside the crystal and collide with the phonons. Hence, the electron distribution function changes over time. When an electron collides, it may gain or lose a quantum of energy $\hbar\omega$, according to the probability scattering rates depending on the functions $\mathscr{G}$ and $\mathscr{G}_0$. For further details, we refer the reader to [4] and [8]. From the mathematical point of view, the main difficulties in studying this transport equation arise from these irregular integral kernels.

The form of the electron energy function ε depends on the band structure of the crystal. Many simple analytical expressions are used in applications. The most common are the parabolic band approximation and the Kane model [4]. The formula

$$\varepsilon(\mathsf{k}) = \frac{\hbar^2|\mathsf{k}|^2}{m^* + \sqrt{m^*(m^* + 2|\mathsf{k}|^2\alpha\hbar^2)}} \tag{4}$$

gives the electron energy for both models. The real number α is called the non-parabolicity parameter. It is positive for the Kane model, while $\alpha = 0$ gives the parabolic band approximation. The constant m^* is the effective electron mass, and $|\mathsf{k}|$ denotes the modulus of the vector k. It is possible to consider other expressions (see [4] or [5]) instead of Eq. (4). Our results hold in these cases also without essential differences, but for the sake of clarity we refer to Eq. (4) in the following.

It is useful to introduce the collision frequency

$$\nu(\mathsf{k}) := \int_{\mathbb{R}^3} S(\mathsf{k}, \mathsf{k}')\, d\mathsf{k}' \tag{5}$$

and the gain operator

$$J(f)(t, \mathsf{k}) := \int_{\mathbb{R}^3} S(\mathsf{k}', \mathsf{k}) f(t, \mathsf{k}')\, d\mathsf{k}'. \tag{6}$$

Thus, Eq. (1) becomes

$$\frac{\partial f}{\partial t}(t, \mathsf{k}) + \nu(\mathsf{k}) f(t, \mathsf{k}) = J(f)(t, \mathsf{k}). \tag{7}$$

It is easy to verify [7, 9] that the function

$$M(\mathsf{k}) := \exp\left[-\frac{\varepsilon(\mathsf{k})}{k_B T_L}\right] \tag{8}$$

is a solution of (7), because $J(M) = \nu M$. The function $M(\mathsf{k})$ is called the Maxwellian distribution and describes the equilibrium state of the electron gas.

In this paper, we look for solutions of Eq. (7) satisfying the initial condition

$$f(0, \mathsf{k}) = \Phi(\mathsf{k}), \tag{9}$$

where $\Phi(\mathsf{k})$ is an assigned non-negative continuous function on $\mathbb{R}^3$.

3. TRUNCATED EQUATIONS

Let Ω be a closed subset of a finite dimensional Euclidean space. We denote by $C(\Omega)$ the set of all continuous functions $g : \Omega \to \mathbb{R}$. Moreover, the spaces $C_b(\Omega)$ and $C_c(\Omega)$ are the subsets of $C(\Omega)$ of all bounded functions or having compact support, respectively. $L^1(\Omega)$ is the set of Lebesgue integrable functions on Ω.

In order to prove the existence of a solution to (7) satisfying (9), we introduce a modified equation, where the gain term $J(f)$ is replaced by a bounded operator in $C(\mathbb{R}_0^+ \times \mathbb{R}^3)$. We define, for every $n \in \mathbb{N}$,

$$\overline{\psi}_n(u) := \begin{cases} 1 & \text{if } \dfrac{u}{k_B T_L} \leq n \\ 1 + n - \dfrac{u}{k_B T_L} & \text{if } n < \dfrac{u}{k_B T_L} < n + 1 \\ 0 & \text{if } \dfrac{u}{k_B T_L} \geq n + 1, \end{cases} \tag{10}$$

and $\psi_n(\mathsf{k}, \mathsf{k}') := \overline{\psi}_n(\varepsilon(\mathsf{k}))\overline{\psi}_n(\varepsilon(\mathsf{k}'))$, for every $\mathsf{k}, \mathsf{k}' \in \mathbb{R}^3$.

For every $f \in C(\mathbb{R}_0^+ \times \mathbb{R}^3)$, we define

$$J_n(f)(t, \mathsf{k}) := \int_{\mathbb{R}^3} S(\mathsf{k}', \mathsf{k}) f(t, \mathsf{k}') \psi_n(\mathsf{k}, \mathsf{k}')\, d\mathsf{k}'. \tag{11}$$

Using Lemma $(B.4)$ of [5], it is immediate to prove that the operator J_n maps $C(\mathbb{R}_0^+ \times \mathbb{R}^3)$ into itself and that $\nu \in C(\mathbb{R}^3)$. Moreover, $J_n(f)(t, \cdot\,) \in C_c(\mathbb{R}^3)$ for every $t \in \mathbb{R}_0^+$, and $J_n(f)$ is non-negative if f is, too. We consider the Cauchy problem

$$\begin{aligned} \frac{\partial f}{\partial t}(t, \mathsf{k}) + \nu(\mathsf{k}) f(t, \mathsf{k}) &= J_n(f)(t, \mathsf{k}) \\ f(0, \mathsf{k}) &= \Phi_n(\mathsf{k}), \end{aligned} \tag{12}$$

where

$$\Phi_n(\mathsf{k}) := \Phi(\mathsf{k}) \overline{\psi}_n(\varepsilon(\mathsf{k})). \tag{13}$$

We look for solutions of problem (12) belonging to the space $C(\mathbb{R}_0^+ \times \mathbb{R}^3)$.

We remark that only the gain term of the true equation (7) is changed. The use of truncated kernels is well known in the literature (see, for instance, Ref. [2]), but usually both terms of the collision operator are modified.

THEOREM 3.1. *If* $\Phi \in C(\mathbb{R}^3)$ *and it is non-negative, then the problem* (12) *admits a unique global non-negative continuous solution.*

Proof. Let λ be a real non-negative parameter, to be chosen later. By $C_\lambda(\mathbb{R}_0^+ \times \mathbb{R}^3)$ we denote the linear space of all continuous real-valued functions g defined in $\mathbb{R}_0^+ \times \mathbb{R}^3$, such that

$$\sup\left\{ e^{-\lambda t} |g(t, \mathsf{k})| : t \in \mathbb{R}_0^+, \mathsf{k} \in \mathbb{R}^3 \right\} < +\infty.$$

If we define

$$\|g\|_\lambda := \sup\left\{e^{-\lambda t}|g(t,\mathsf{k})| : t \in \mathbb{R}_0^+, \mathsf{k} \in \mathbb{R}^3\right\}$$

for all $g \in C_\lambda(\mathbb{R}_0^+ \times \mathbb{R}^3)$, then it is simple to see that $\|\cdot\|_\lambda$ is a norm and $C_\lambda(\mathbb{R}_0^+ \times \mathbb{R}^3)$ is a Banach space.

Of course, the Cauchy problem (12) is equivalent in $C_\lambda(\mathbb{R}_0^+ \times \mathbb{R}^3)$ to the integral equation

$$f(t,\mathsf{k}) = e^{-\nu(\mathsf{k})t}\Phi_n(\mathsf{k}) + e^{-\nu(\mathsf{k})t}\int_0^t e^{\nu(\mathsf{k})r} J_n(f)(r,\mathsf{k})\,dr. \qquad (14)$$

We denote by $T_n(f)$ the right-hand side of Eq. (14). We prove that $T_n(f)$ admits a unique fixed point by using the well-known Banach–Caccioppoli Theorem.

It is a simple matter to show that $T_n(f)$ maps $C_\lambda(\mathbb{R}_0^+ \times \mathbb{R}^3)$ into itself. Moreover, $f(t,\mathsf{k}) \geq 0$ (for every $(t,\mathsf{k}) \in \mathbb{R}_0^+ \times \mathbb{R}^3$) implies that $T_n(f)(t,\mathsf{k}) \geq 0$ for every $t \in \mathbb{R}_0^+$ and $\mathsf{k} \in \mathbb{R}^3$. Setting

$$j_n(\mathsf{k}) := J_n(1)(t,\mathsf{k}) = \int_{\mathbb{R}^3} S(\mathsf{k}',\mathsf{k})\,\psi_n(\mathsf{k},\mathsf{k}')\,d\mathsf{k}',$$

we have

$$\begin{aligned}
&|J_n(f)(t,\mathsf{k}) - J_n(g)(t,\mathsf{k})| \\
&\qquad \leq \int_{\mathbb{R}^3} S(\mathsf{k}',\mathsf{k})\,\psi_n(\mathsf{k},\mathsf{k}')|f(t,\mathsf{k}') - g(t,\mathsf{k}')|\,d\mathsf{k}' \\
&\qquad \leq \|f-g\|_\lambda \int_{\mathbb{R}^3} S(\mathsf{k}',\mathsf{k})\,\psi_n(\mathsf{k},\mathsf{k}')e^{\lambda t}\,d\mathsf{k}' \\
&\qquad = j_n(\mathsf{k})e^{\lambda t}\|f-g\|_\lambda
\end{aligned}$$

for every f and g belonging to $C_\lambda(\mathbb{R}_0^+ \times \mathbb{R}^3)$. Hence,

$$\begin{aligned}
|T_n(f)(t,\mathsf{k}) - T_n(g)(t,\mathsf{k})| &\leq e^{-\nu(\mathsf{k})t}\int_0^t e^{\nu(\mathsf{k})r} j_n(\mathsf{k})e^{\lambda r}\|f-g\|_\lambda\,dr \\
&= e^{-\nu(\mathsf{k})t} j_n(\mathsf{k})\|f-g\|_\lambda \frac{e^{[\nu(\mathsf{k})+\lambda]t} - 1}{\lambda + \nu(\mathsf{k})} \\
&\leq \frac{j_n(\mathsf{k})}{\lambda + \nu(\mathsf{k})}\|f-g\|_\lambda e^{\lambda t}.
\end{aligned}$$

Since $j_n \in C_c(\mathbb{R}^3)$, it is possible to choose λ such that

$$\max\left\{ \frac{j_n(\mathsf{k})}{\lambda + \nu(\mathsf{k})} : \mathsf{k} \in \mathbb{R}^3 \right\} < 1.$$

Therefore T_n is a contraction in $C_\lambda(\mathbb{R}_0^+ \times \mathbb{R}^3)$. It is evident that the unique solution of (14) can be obtained by iteration starting from $f = 0$. Hence the solution of Eq. (14) is non-negative and the proof is accomplished. ∎

We denote by $f_n(t, \mathsf{k})$ the solution of Eq. (14). Taking into account (10), we note that, for every (t, k) such that $\varepsilon(\mathsf{k}) \geq (n+1)k_B T_L$, Eq. (14) is simply $f_n(t, \mathsf{k}) = 0$.

3.1. *Mass Inequality*

Since in the first equation of (12) only the gain term is truncated, mass conservation does not hold in general. Setting

$$\nu_n(\mathsf{k}) := \int_{\mathbb{R}^3} S(\mathsf{k}, \mathsf{k}') \psi_n(\mathsf{k}, \mathsf{k}') \, d\mathsf{k}',$$

it is clear that $\nu_n(\mathsf{k}) \leq \nu(\mathsf{k})$ for every $\mathsf{k} \in \mathbb{R}^3$.

THEOREM 3.2. *If* $\Phi \in C(\mathbb{R}^3)$ *and it is non-negative, then*

$$\int_{\mathbb{R}^3} f_n(t, \mathsf{k}) \, d\mathsf{k} \leq \int_{\mathbb{R}^3} \Phi_n(\mathsf{k}) \, d\mathsf{k} \quad \textit{for all } t \in \mathbb{R}_0^+.$$

Proof. Since

$$f_n(t, \mathsf{k}) - \Phi_n(\mathsf{k}) = \int_0^t \left[J_n(f_n)(r, \mathsf{k}) - \nu(\mathsf{k}) f_n(r, \mathsf{k}) \right] dr$$

and for each $t \in \mathbb{R}_0^+$ the function $f_n(t, \cdot) \in C_c(\mathbb{R}^3)$, it follows that

$$\begin{aligned}
&\int_{\mathbb{R}^3} f_n(t, \mathsf{k}) \, d\mathsf{k} - \int_{\mathbb{R}^3} \Phi_n(\mathsf{k}) \, d\mathsf{k} \\
&\quad = \int_{\mathbb{R}^3} d\mathsf{k} \int_0^t \left[J_n(f_n)(r, \mathsf{k}) - \nu(\mathsf{k}) f_n(r, \mathsf{k}) \right] dr \\
&\quad = \int_0^t dr \int_{\mathbb{R}^3} \left[J_n(f_n)(r, \mathsf{k}) - \nu(\mathsf{k}) f_n(r, \mathsf{k}) \right] d\mathsf{k} \\
&\quad \leq \int_0^t dr \int_{\mathbb{R}^3} \left[J_n(f_n)(r, \mathsf{k}) - \nu_n(\mathsf{k}) f_n(r, \mathsf{k}) \right] d\mathsf{k} = 0.
\end{aligned}$$

Then

$$0 \leq \int_{\mathbb{R}^3} f_n(t,\mathsf{k})\, d\mathsf{k} \leq \int_{\mathbb{R}^3} \Phi_n(\mathsf{k})\, d\mathsf{k},$$

which is the desired conclusion. ∎

3.2. *Monotonic Property*

Since n is an arbitrary integer, we have obtained a sequence $\{f_n\}$ of solutions of problem (12) belonging to $C(\mathbb{R}_0^+ \times \mathbb{R}^3)$. We prove that the sequence $\{f_n\}$ is monotonically increasing.

We first establish the following Gronwall-type lemma.

LEMMA 3.1. *If $h \in C(\mathbb{R}_0^+ \times \mathbb{R}^3)$ verifies the inequality*

$$h(t,\mathsf{k}) \leq e^{-\nu(\mathsf{k})t} \int_0^t e^{\nu(\mathsf{k})r} J_n(h)(r,\mathsf{k})\, dr \quad \textit{for every } (t,\mathsf{k}) \in \mathbb{R}_0^+ \times \mathbb{R}^3,$$

and $h(0,\mathsf{k}) \leq 0$ for all $k \in \mathbb{R}^3$, then $h(t,\mathsf{k}) \leq 0$ for all $(t,\mathsf{k}) \in \mathbb{R}_0^+ \times \mathbb{R}^3$.

Proof. Let ϱ be a positive parameter. For every $t \in \mathbb{R}_0^+$ and $\mathsf{k} \in \mathbb{R}^3$, we have

$$\begin{aligned} \varrho M(\mathsf{k}) &= e^{-\nu(\mathsf{k})t} \varrho M(\mathsf{k}) + e^{-\nu(\mathsf{k})t} \int_0^t e^{\nu(\mathsf{k})r} J(\varrho M)(\mathsf{k})\, dr \\ &\geq e^{-\nu(\mathsf{k})t} \varrho M(\mathsf{k}) + e^{-\nu(\mathsf{k})t} \int_0^t e^{\nu(\mathsf{k})r} J_n(\varrho M)(\mathsf{k})\, dr. \end{aligned}$$

Let

$$t_* = \sup\left\{t \in \mathbb{R}_0^+ : h(t,\mathsf{k}) \leq \varrho M(\mathsf{k}) \text{ for all } \mathsf{k} \in \mathbb{R}^3\right\}.$$

We note that the above set is not empty because

$$h(0,\mathsf{k}) \leq \varrho M(\mathsf{k}) \quad \text{for all } \mathsf{k} \in \mathbb{R}^3.$$

If $t_* < +\infty$, since $J_n(h)(t,\cdot) \in C_c(\mathbb{R}^3)$, then there exists $\delta > 0$ such that

$$\int_{t_*}^{t_*+\delta} e^{\nu(\mathsf{k})r} J_n(h)(r,\mathsf{k})\, dr \leq \varrho M(\mathsf{k}).$$

Now,

$$
\begin{aligned}
h(t_* + \delta, \mathsf{k}) &\le e^{-\nu(\mathsf{k})(t_* + \delta)} \int_0^{t_* + \delta} e^{\nu(\mathsf{k})r} J_n(h)(r, \mathsf{k})\, dr \\
&\le e^{-\nu(\mathsf{k})t_*} \int_0^{t_* + \delta} e^{\nu(\mathsf{k})r} J_n(h)(r, \mathsf{k})\, dr \\
&= e^{-\nu(\mathsf{k})t_*} \int_0^{t_*} e^{\nu(\mathsf{k})r} J_n(h)(r, \mathsf{k})\, dr \\
&\qquad + e^{-\nu(\mathsf{k})t_*} \int_{t_*}^{t_* + \delta} e^{\nu(\mathsf{k})r} J_n(h)(r, \mathsf{k})\, dr \\
&\le e^{-\nu(\mathsf{k})t_*} \int_0^{t_*} e^{\nu(\mathsf{k})r} J_n(\varrho M)(\mathsf{k})\, dr + e^{-\nu(\mathsf{k})t_*} \varrho M(\mathsf{k}) \\
&\le \varrho M(\mathsf{k}).
\end{aligned}
$$

This contradicts the assumption that t_* is finite. Thus $t_* = +\infty$, i.e.,

$$
h(t, \mathsf{k}) \le \varrho M(\mathsf{k}) \quad \text{for every } (t, \mathsf{k}) \in \mathbb{R}_0^+ \times \mathbb{R}^3.
$$

Since ϱ is an arbitrary positive real number, the assertion is achieved. ∎

THEOREM 3.3. *The sequence* $\{f_n\}$ *is monotonically increasing.*

Proof. We have

$$
\begin{aligned}
&f_n(t, \mathsf{k}) - f_{n+1}(t, \mathsf{k}) \\
&\quad = e^{-\nu(\mathsf{k})t} \Phi_n(\mathsf{k}) - e^{-\nu(\mathsf{k})t} \Phi_{n+1}(\mathsf{k}) \\
&\qquad + e^{-\nu(\mathsf{k})t} \int_0^t e^{\nu(\mathsf{k})r} \left[J_n(f_n)(r, \mathsf{k}) - J_n(f_{n+1})(r, \mathsf{k}) \right] dr \\
&\qquad - e^{-\nu(\mathsf{k})t} \int_0^t e^{\nu(\mathsf{k})r} \left[J_{n+1}(f_{n+1})(r, \mathsf{k}) - J_n(f_{n+1})(r, \mathsf{k}) \right] dr \\
&\quad \le e^{-\nu(\mathsf{k})t} \int_0^t e^{\nu(\mathsf{k})r} \left[J_n(f_n)(r, \mathsf{k}) - J_n(f_{n+1})(r, \mathsf{k}) \right] dr.
\end{aligned}
$$

Therefore,

$$
f_n(t, \mathsf{k}) - f_{n+1}(t, \mathsf{k}) \le e^{-\nu(\mathsf{k})t} \int_0^t e^{\nu(\mathsf{k})r} J_n(f_n - f_{n+1})(r, \mathsf{k})\, dr \qquad \text{for all } n \in \mathbb{N}.
$$

Putting

$$h(t,\mathsf{k}) = f_n(t,\mathsf{k}) - f_{n+1}(t,\mathsf{k}) \quad \text{for every } (t,\mathsf{k}) \in \mathbb{R}_0^+ \times \mathbb{R}^3$$

and taking into account Lemma 3.1, the proof is complete. ∎

3.3. *H-Theorem and Moment Estimates*

In Ref. [6] it was shown that the H-Theorem can be used to obtain an estimate of the hydrodynamic energy. The H-Theorem was generalized in a simple way by Markowich and Schmeiser [9]. We use this to derive upper bounds for the moments.

LEMMA 3.2. *Let $G \in C(\mathbb{R})$ be a monotonically increasing function. If $g \in C(\mathbb{R}^3)$ then*

$$\int_{\mathbb{R}^3} G(g(\mathsf{k})/M(\mathsf{k}))[J_n(g)(\mathsf{k}) - \nu_n(\mathsf{k})g(\mathsf{k})]\, d\mathsf{k} \le 0. \tag{15}$$

Proof. It is evident that the integral in (15) exists because $J_n(g) - \nu_n g \in C_c(\mathbb{R}^3)$. In order to make the formulas of this proof compact, we will often omit the argument k of the functions and only the prime symbol will indicate that a function depends on k'. So, we have

$$\begin{aligned}
&\int_{\mathbb{R}^3} G(g(\mathsf{k})/M(\mathsf{k}))[J_n(g)(\mathsf{k}) - \nu_n(\mathsf{k})g(\mathsf{k})]\, d\mathsf{k} \\
&\quad = \int_{\mathbb{R}^6} S(\mathsf{k},\mathsf{k}')\psi_n(\mathsf{k},\mathsf{k}')g(\mathsf{k})\left[G\left(\frac{g(\mathsf{k}')}{M(\mathsf{k}')}\right) - G\left(\frac{g(\mathsf{k})}{M(\mathsf{k})}\right)\right] d\mathsf{k}\, d\mathsf{k}' \\
&\quad = \int_{\mathbb{R}^6} \mathscr{G}(n_q + 1)\psi_n \delta(\varepsilon' - \varepsilon + \hbar\omega) g\left[G\left(\frac{g'}{M'}\right) - G\left(\frac{g}{M}\right)\right] d\mathsf{k}\, d\mathsf{k}' \\
&\qquad + \int_{\mathbb{R}^6} \mathscr{G} n_q \psi_n \delta(\varepsilon' - \varepsilon - \hbar\omega) g\left[G\left(\frac{g'}{M'}\right) - G\left(\frac{g}{M}\right)\right] d\mathsf{k}\, d\mathsf{k}' \\
&\qquad + \int_{\mathbb{R}^6} \mathscr{G}_0 \psi_n \delta(\varepsilon' - \varepsilon) g\left[G\left(\frac{g'}{M'}\right) - G\left(\frac{g}{M}\right)\right] d\mathsf{k}\, d\mathsf{k}' \\
&\quad = \int_{\mathbb{R}^6} \mathscr{G}\psi_n \delta(\varepsilon' - \varepsilon - \hbar\omega) M n_q \left[G\left(\frac{g}{M}\right) - G\left(\frac{g'}{M'}\right)\right] \\
&\qquad \times \left[\frac{g'}{M'} - \frac{g}{M}\right] d\mathsf{k}\, d\mathsf{k}' + \frac{1}{2}\int_{\mathbb{R}^6} \mathscr{G}_0 \psi_n \delta(\varepsilon' - \varepsilon) M \\
&\qquad \times \left[G\left(\frac{g'}{M'}\right) - G\left(\frac{g}{M}\right)\right]\left[\frac{g}{M} - \frac{g'}{M'}\right] d\mathsf{k}\, d\mathsf{k}' \\
&\le 0,
\end{aligned}$$

where we used Eq. (3) and Eq. (8). ∎

THEOREM 3.4. *Let $G \in C(\mathbb{R})$ be non-negative and monotonically increasing. Then*

$$t \to \int_{\mathbb{R}^3} \left[M(\mathsf{k}) \int_0^{f_n(t,\mathsf{k})/M(\mathsf{k})} G(x)\, dx \right] d\mathsf{k} \tag{16}$$

is a monotonic decreasing function on $\mathbb{R}_0^+$.

Proof. From the first part of Eq. (12), we have

$$G\left(\frac{f_n(t,\mathsf{k})}{M(\mathsf{k})} \right) \frac{\partial f_n(t,\mathsf{k})}{\partial t} \le G\left(\frac{f_n(t,\mathsf{k})}{M(\mathsf{k})} \right) [J_n(f_n)(t,\mathsf{k}) - \nu_n(\mathsf{k}) f_n(t,\mathsf{k})],$$

which is equivalent to

$$\frac{\partial}{\partial t} \left\{ \left[M(\mathsf{k}) \int_0^{f_n(t,\mathsf{k})/M(\mathsf{k})} G(x)\, dx \right] \right\} \le G\left(\frac{f_n(t,\mathsf{k})}{M(\mathsf{k})} \right) [J_n(f_n)(t,\mathsf{k}) - \nu_n(\mathsf{k}) f_n(t,\mathsf{k})].$$

By integrating with respect to k and taking into account Lemma 3.2, this yields

$$\frac{\partial}{\partial t} \left\{ \int_{\mathbb{R}^3} \left[M(\mathsf{k}) \int_0^{f_n(t,\mathsf{k})/M(\mathsf{k})} G(x)\, dx \right] d\mathsf{k} \right\} \le 0,$$

which completes the proof. ∎

COROLLARY 3.1. *Putting*

$$S_n(t) = \int_{\mathbb{R}^3} [f_n(t,\mathsf{k}) + M(\mathsf{k})] \left\{ \log\left[\frac{f_n(t,\mathsf{k})}{M(\mathsf{k})} + 1 \right] \right\} d\mathsf{k},$$

then

$$0 \le S_n(t) \le S_n(0) \quad \textit{for all } t \in \mathbb{R}_0^+.$$

Proof. If we consider the function

$$G(x) := \begin{cases} \log(x+1) & \text{if } x \ge 0 \\ 0 & \text{if } x \le 0 \end{cases},$$

then the assumptions of Theorem 3.4 are verified. Now the integral of G

in Eq. (16) is elementary. Hence the function

$$t \to \int_{\mathbb{R}^3} M(\mathsf{k}) \left\{ \left[\frac{f_n(t,\mathsf{k})}{M(\mathsf{k})} + 1 \right] \log\left[\frac{f_n(t,\mathsf{k})}{M(\mathsf{k})} + 1 \right] - \frac{f_n(t,\mathsf{k})}{M(\mathsf{k})} \right\} d\mathsf{k}$$

is decreasing in time. This implies that

$$\int_{\mathbb{R}^3} \left\{ [f_n(t,\mathsf{k}) + M(\mathsf{k})] \log\left[\frac{f_n(t,\mathsf{k})}{M(\mathsf{k})} + 1 \right] - f_n(t,\mathsf{k}) \right\} d\mathsf{k}$$
$$\le \int_{\mathbb{R}^3} \left\{ [\Phi_n(\mathsf{k}) + M(\mathsf{k})] \log\left[\frac{\Phi_n(\mathsf{k})}{M(\mathsf{k})} + 1 \right] - \Phi_n(\mathsf{k}) \right\} d\mathsf{k}.$$

Now it is immediate to see that

$$0 \le S_n(t) \le S_n(0) + \int_{\mathbb{R}^3} f_n(t,\mathsf{k})\, d\mathsf{k} - \int_{\mathbb{R}^3} \Phi_n(\mathsf{k})\, d\mathsf{k} \le S_n(0).$$

This proves the assertion. ∎

Note that we have actually proved that the function

$$t \to S_n(t) - \int_{\mathbb{R}^3} f_n(t,\mathsf{k})\, d\mathsf{k}$$

is decreasing in $\mathbb{R}_0^+$. This differs from the H-theorem of Ref. [7]. In fact, it is not true in general that mass conservation holds. The next result shows that upper bounds to the thermodynamical energy and higher order moments can be derived by means of Theorem 3.4.

LEMMA 3.3. *If μ is a real number greater than or equal to* 1, *then the function*

$$t \to \int_{\mathbb{R}^3} M(\mathsf{k}) \left[\frac{f_n(t,\mathsf{k})}{M(\mathsf{k})} + 1 \right] \left\{ \log\left[\frac{f_n(t,\mathsf{k})}{M(\mathsf{k})} + 1 \right] \right\}^{\mu} d\mathsf{k}$$

belongs to $C_b(\mathbb{R}_0^+)$.

Proof. Let us consider the function

$$G(x) := \begin{cases} [\log(x+1)]^{\mu}, & \text{if } x \ge 0, \\ 0, & \text{if } x \le 0. \end{cases}$$

It verifies the assumptions of Theorem 3.4, so that the function

$$\int_{\mathbb{R}^3} \left[M(\mathsf{k}) \int_0^{f_n(t,\mathsf{k})/M(\mathsf{k})} [\log(x+1)]^{\mu}\, dx \right] d\mathsf{k} \tag{17}$$

is decreasing on $\mathbb{R}_0^+$.

Letting

$$I_\mu(z) = \int_0^z [\log(x+1)]^\mu \, dx \quad \text{and} \quad c_\mu = \int_{\mathbb{R}^3} \left[M(\mathsf{k}) I_\mu\left(\frac{\Phi_n(\mathsf{k})}{M(\mathsf{k})} \right) \right] d\mathsf{k}, \tag{18}$$

then

$$\int_{\mathbb{R}^3} \left[M(\mathsf{k}) I_\mu\left(\frac{f_n(t,\mathsf{k})}{M(\mathsf{k})} \right) \right] d\mathsf{k} \le c_\mu \quad \forall t \ge 0.$$

By simple calculus, it is easy to see that

$$I_\mu(z) + \mu I_{\mu-1}(z) = (z+1)[\log(z+1)]^\mu,$$

and so, for all $t \ge 0$,

$$\begin{aligned} 0 &\le \int_{\mathbb{R}^3} M(\mathsf{k}) \left[\frac{f_n(t,\mathsf{k})}{M(\mathsf{k})} + 1 \right] \left\{ \log\left[\frac{f_n(t,\mathsf{k})}{M(\mathsf{k})} + 1 \right] \right\}^\mu d\mathsf{k} \\ &= \int_{\mathbb{R}^3} M(\mathsf{k}) \left[I_\mu\left(\frac{f_n(t,\mathsf{k})}{M(\mathsf{k})} \right) + \mu I_{\mu-1}\left(\frac{f_n(t,\mathsf{k})}{M(\mathsf{k})} \right) \right] d\mathsf{k} \le c_\mu + \mu c_{\mu-1}. \end{aligned}$$

This concludes the proof. ∎

THEOREM 3.5. *If $p \in]0,1[$ then for all $t \in \mathbb{R}_0^+$*

$$\begin{aligned} &\int_{\mathbb{R}^3} f_n(t,\mathsf{k}) [\varepsilon(\mathsf{k})]^\mu \, d\mathsf{k} \\ &\qquad \le (c_\mu + \mu c_{\mu-1}) \left(\frac{k_B T_L}{1-p} \right)^\mu + \int_{\mathbb{R}^3} [M(\mathsf{k})]^p [\varepsilon(\mathsf{k})]^\mu \, d\mathsf{k}, \end{aligned} \tag{19}$$

where c_μ is given by (18).

Proof. For every $t \in \mathbb{R}_0^+$, let us define

$$D_n(t) = \left\{ \mathsf{k} \in \mathbb{R}^3 : f_n(t,\mathsf{k}) \ge [M(\mathsf{k})]^p \right\} \quad \text{and} \quad D_n^c(t) = \mathbb{R}^3 \setminus D_n(t).$$

From Lemma 3.3, if $\mu \geq 1$ then we have

$$
\begin{aligned}
c_\mu + \mu c_{\mu-1} &\geq \int_{\mathbb{R}^3} [M(\mathsf{k}) + f_n(t,\mathsf{k})] \left[\log \frac{f_n(t,\mathsf{k}) + M(\mathsf{k})}{M(\mathsf{k})} \right]^\mu d\mathsf{k} \\
&\geq \int_{D_n(t)} f_n(t,\mathsf{k}) \left\{ \log [M(\mathsf{k})]^{p-1} \right\}^\mu d\mathsf{k} \\
&= \left(\frac{1-p}{k_B T_L} \right)^\mu \int_{D_n(t)} f_n(t,\mathsf{k}) [\varepsilon(\mathsf{k})]^\mu \, d\mathsf{k}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\int_{\mathbb{R}^3} f_n(t,\mathsf{k}) [\varepsilon(\mathsf{k})]^\mu \, d\mathsf{k} \\
&\quad = \int_{D_n(t)} f_n(t,\mathsf{k}) [\varepsilon(\mathsf{k})]^\mu \, d\mathsf{k} + \int_{D_n^c(t)} f_n(t,\mathsf{k}) [\varepsilon(\mathsf{k})]^\mu \, d\mathsf{k} \\
&\quad \leq (c_\mu + \mu c_{\mu-1}) \left(\frac{k_B T_L}{1-p} \right)^\mu + \int_{\mathbb{R}^3} [M(\mathsf{k})]^p [\varepsilon(\mathsf{k})]^\mu \, d\mathsf{k},
\end{aligned}
$$

which gives immediately the claim. ∎

4. EXISTENCE AND UNIQUENESS

We have determined an increasing sequence of functions $\{f_n\} \subseteq C(\mathbb{R}_0^+ \times \mathbb{R}^3)$ which satisfy Eq. (14). If $\Phi \in L^1(\mathbb{R}^3)$ then Theorem 3.2 and the Monotone Convergence Theorem tell us that there exists a function $f(t,\cdot) \in L^1(\mathbb{R}^3)$ such that, for each $t \in \mathbb{R}_0^+$,

$$
f(t,\mathsf{k}) = \lim_{n \to +\infty} f_n(t,\mathsf{k}) \quad \text{for almost every } \mathsf{k} \in \mathbb{R}^3 \tag{20}
$$

and

$$
\lim_{n \to +\infty} \int_{\mathbb{R}^3} f_n(t,\mathsf{k}) \, d\mathsf{k} = \int_{\mathbb{R}^3} f(t,\mathsf{k}) \, d\mathsf{k} \leq \int_{\mathbb{R}^3} \Phi(\mathsf{k}) \, d\mathsf{k}.
$$

We prove that the function f satisfies the equation

$$
f(t,\mathsf{k}) = e^{-\nu(\mathsf{k})t} \Phi(\mathsf{k}) + e^{-\nu(\mathsf{k})t} \int_0^t e^{\nu(\mathsf{k})r} J(f)(r,\mathsf{k}) \, dr. \tag{21}
$$

Since the continuity of the function f is not guaranteed, we must define the gain term J also for Lebesgue integrable functions. This is possible by the same arguments as those used for the classical Boltzmann equation (see Appendix A).

THEOREM 4.1. *If* $\Phi \in C(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$ *and it is non-negative, then the function* f *of Eq.* (20) *is a non-negative solution of Eq.* (21) *for almost every* $(t, \mathsf{k}) \in \mathbb{R}_0^+ \times \mathbb{R}^3$.

Proof. We have

$$f_n(t,\mathsf{k}) = e^{-\nu(\mathsf{k})t}\Phi_n(\mathsf{k}) + e^{-\nu(\mathsf{k})t}\int_0^t e^{\nu(\mathsf{k})r} J_n(f_n)(r,\mathsf{k})\,dr.$$

Since

$$\begin{aligned}
&|J(f)(t,\mathsf{k}) - J_n(f_n(t,\mathsf{k}))| \\
&\quad \le |J(f)(t,\mathsf{k}) - J(f_n)(t,\mathsf{k})| + |J(f_n)(t,\mathsf{k}) - J_n(f_n)(t,\mathsf{k})| \\
&\quad = \int_{\mathbb{R}^3} S(\mathsf{k}',\mathsf{k})\left[f(t,\mathsf{k}') - f_n(t,\mathsf{k}')\right] d\mathsf{k}' \\
&\qquad + \int_{\mathbb{R}^3} S(\mathsf{k}',\mathsf{k})\left[1 - \psi_n(\mathsf{k},\mathsf{k}')\right] f_n(t,\mathsf{k}')\,d\mathsf{k}',
\end{aligned}$$

it is a simple matter to prove that

$$\lim_{n\to+\infty} J_n(f_n) = J(f) \quad \text{a.e. on } \mathbb{R}_0^+ \times \mathbb{R}^3.$$

So, using again the Monotone Convergence Theorem, we obtain the assertion. ∎

Of course the function f satisfies almost everywhere the problem

$$\begin{aligned}
\frac{\partial f}{\partial t}(t,\mathsf{k}) + \nu(\mathsf{k})f(t,\mathsf{k}) &= J(f)(t,\mathsf{k}), \\
f(0,\mathsf{k}) &= \Phi(\mathsf{k}).
\end{aligned} \tag{22}$$

Moreover, the upper bounds for the moments of f_n (see Theorem 3.5) can give analogous estimates for the solution f to the Cauchy problem (22). As an example, the thermodynamical energy

$$\int_{\mathbb{R}^3} f(t,\mathsf{k})\,\varepsilon(\mathsf{k})\,d\mathsf{k}$$

is bounded in time whenever

$$\Phi(\mathsf{k}) \text{ and } M(\mathsf{k})I_1(\Phi(\mathsf{k})/M(\mathsf{k})) \text{ belong to } L^1(\mathbb{R}^3).$$

This follows immediately from Eq. (19) and the definition of c_μ.

We are not able to prove the uniqueness of the solution of the Cauchy problem (22) without assuming further hypotheses.

THEOREM 4.2. *Assume that ν satisfies the inequality*

$$\nu(\mathsf{k}) \leq \nu_0 \left\{ 1 + \left[\frac{\varepsilon(\mathsf{k})}{k_B T_L} \right]^{\kappa} \right\},$$

where $\nu_0 > 0$ and $\kappa \geq 1$ are constants. If the initial datum Φ and $MI_{\kappa}(\Phi/M)$ and $MI_{\kappa-1}(\Phi/M)$ belong to $L^1(\mathbb{R}^3)$, then there exists a unique solution to the Cauchy problem (22) *such that f and $f\varepsilon^{\kappa}$ belong to $L^1([0,T] \times \mathbb{R}^3)$ for every $T \in \mathbb{R}^+$.*

Proof. The assumptions on Φ generate the existence of a solution f such that $f \in L^1([0,T] \times \mathbb{R}^3)$ and $f\varepsilon^{\kappa} \in L^1([0,T] \times \mathbb{R}^3)$ for every $T \in \mathbb{R}^+$.

Since the equation is linear, it is sufficient to prove that the only solution of the problem (22) with initial condition $\Phi \equiv 0$ is the function $f(t, \mathsf{k}) = 0$ almost everywhere on $\mathbb{R}_0^+ \times \mathbb{R}^3$. It is easy to prove (see, also Appendix B) that

$$\int_{\mathbb{R}^3} \operatorname{sgn}\left(\frac{f(t,\mathsf{k})}{M(\mathsf{k})} \right) [J(f)(t,\mathsf{k}) - \nu(\mathsf{k}) f(t,\mathsf{k})] \, d\mathsf{k} \leq 0, \tag{23}$$

where $\operatorname{sgn}(z)$ is the sign of the real number z ($\operatorname{sgn}(0) = 0$). In fact, (23) follows as in Lemma 3.2 and by observing that $\lim_{n \to +\infty} \tanh(nz) = \operatorname{sgn}(z)$. Now, (23) implies

$$\int_{\mathbb{R}^3} \frac{\partial f(t,\mathsf{k})}{\partial t} \operatorname{sgn}\left(\frac{f(t,\mathsf{k})}{M(\mathsf{k})} \right) d\mathsf{k} \leq 0,$$

i.e.,

$$\int_{\mathbb{R}^3} \frac{\partial |f(t,\mathsf{k})|}{\partial t} \, d\mathsf{k} \leq 0.$$

Fix $T > 0$. Since

$$f(t,\mathsf{k}) = \int_0^t [J(f)(r,\mathsf{k}) - \nu(\mathsf{k}) f(r,\mathsf{k})] \, dr,$$

then $f(\cdot, \mathsf{k})$ is absolutely continuous in $[0, T]$ for almost all $\mathsf{k} \in \mathbb{R}^3$. Hence

$$0 \geq \int_0^t dr \int_{\mathbb{R}^3} \frac{\partial |f(t,\mathsf{k})|}{\partial t} \, d\mathsf{k} = \int_{\mathbb{R}^3} |f(t,\mathsf{k})| \, d\mathsf{k},$$

and the conclusion is achieved. ∎

COROLLARY 4.1. *Under the same assumptions as those in Theorem* 4.2, *mass conservation holds*.

Proof. We have

$$\int_{\mathbb{R}^3} f_n(t,\mathsf{k})\,d\mathsf{k} - \int_{\mathbb{R}^3} \Phi_n(\mathsf{k})\,d\mathsf{k} = \int_0^t dr \int_{\mathbb{R}^3} [\nu_n(\mathsf{k}) - \nu(\mathsf{k})] f_n(r,\mathsf{k})\,d\mathsf{k}.$$

Now

$$\lim_{n\to+\infty} [\nu_n(\mathsf{k}) - \nu(\mathsf{k})] f_n(r,\mathsf{k}) = 0 \text{ pointwise}$$

$$\text{for almost all } (r,\mathsf{k}) \in \mathbb{R}_0^+ \times \mathbb{R}^3.$$

Moreover,

$$|\nu_n(\mathsf{k}) - \nu(\mathsf{k})| f_n(r,\mathsf{k}) \le 2\nu(\mathsf{k}) f_n(r,\mathsf{k}) \le 2\nu_0 \left[1 + \left(\frac{\varepsilon(\mathsf{k})}{\hbar\omega}\right)^{\kappa}\right] f(r,\mathsf{k}),$$

and the Lebesgue Dominated Convergence Theorem gives

$$\int_{\mathbb{R}^3} f(t,\mathsf{k})\,d\mathsf{k} - \int_{\mathbb{R}^3} \Phi(\mathsf{k})\,d\mathsf{k} = 0.$$

This is precisely the assertion of the corollary. ∎

The last results show that the same assumptions guarantee uniqueness and mass conservation. Then, it is reasonable to think that both properties are related. This recalls to mind the uniqueness and energy conservation relation proved by Wennberg in Ref. [15] in the case of the classical Boltzmann equation. It is not clear if in the present case it is possible to have more than one solution having different masses for some $t > 0$. In Appendix C we show a simple example where $\Phi \in L^1(\mathbb{R}^3)$ but $Q(f)$ is not integrable in $\mathbb{R}^3$.

APPENDIX A

Here we give the meaning of $J(f)$ for f Lebesgue integrable. In order to do this, first we consider the integral

$$\int_{\mathbb{R}^3} p(\mathsf{k})\,\delta(\varepsilon(\mathsf{k}) - k_B T_L q)\,d\mathsf{k}, \tag{A.1}$$

where $p \in C(\mathbb{R}_0^+ \times \mathbb{R}^3)$ and q is a real number. We consider the Kane model, and we introduce the new dimensionless variables w, μ, and ϑ

instead of k by means of the formula

$$\mathsf{k} = \frac{\sqrt{2m^* k_B T_L}}{\hbar} \sqrt{w(1+\alpha_k w)} \left(\sqrt{1-\mu^2} \cos\vartheta, \sqrt{1-\mu^2} \sin\vartheta, \mu \right),$$

where $\alpha_k = k_B T_L \alpha$. We denote by $\hat{p}$ the function p in terms of the new variables. We obtain

$$\int_{\mathbb{R}^3} p(\mathsf{k})\, \delta(\varepsilon(\mathsf{k}) - q)\, d\mathsf{k} = \int_0^{+\infty} dw \int_{-1}^{1} d\mu \int_0^{2\pi} d\vartheta\, \hat{p}(w,\mu,\vartheta)\, \delta(w-q)\, D(w), \quad \text{(A.2)}$$

where

$$D(w) = \left(\frac{\sqrt{m^*}}{\hbar} \right)^3 \sqrt{2k_B T_L}\, \sqrt{w}\, (2\alpha_k w + 1) \sqrt{\alpha_k w + 1}$$

is the Jacobian of the transformation.

Since $\hat{p}$ is a continuous function, the integral in Eq. (A.2) is simply equal to

$$H(q)\, D(q) \int_{-1}^{1} d\mu \int_0^{2\pi} d\vartheta\, \hat{p}(q,\mu,\vartheta),$$

with $H(q)$ being the Heaviside function. Now, if $p \in L^1(\mathbb{R}^3)$ then

$$\int_{\mathbb{R}^3} p(\mathsf{k})\, d\mathsf{k} = k_B T_L \int_0^{+\infty} dw \int_{-1}^{1} d\mu \int_0^{2\pi} d\vartheta\, \hat{p}(w,\mu,\vartheta)\, D(w).$$

Therefore, by using Fubini's Theorem, it follows that

$$\int_{-1}^{1} d\mu \int_0^{2\pi} d\vartheta\, \hat{p}(q,\mu,\vartheta)$$

exists for almost every $w \geq 0$. Hence, for every $p \in L^1(\mathbb{R}^3)$, we define

$$\int_{\mathbb{R}^3} p(\mathsf{k})\, \delta(\varepsilon(\mathsf{k}) - q k_B T_L)\, d\mathsf{k} = H(q)\, D(q) \int_{-1}^{1} d\mu \int_0^{2\pi} d\vartheta\, \hat{p}(q,\mu,\vartheta).$$

APPENDIX B

We prove that if $g \geq 0$, $g \in L^1(\mathbb{R}^3)$, and $\nu g \in L^1(\mathbb{R}^3)$, then $J(g) \in L^1(\mathbb{R}^3)$. In fact, using Fubini's Theorem, we have

$$\begin{aligned}\int_{\mathbb{R}^3} \overline{\psi}_n(\varepsilon(\mathsf{k})) J(g)(\mathsf{k})\, d\mathsf{k} &= \int_{\mathbb{R}^3} \overline{\psi}_n(\varepsilon(\mathsf{k})) \left[\int_{\mathbb{R}^3} S(\mathsf{k}', \mathsf{k}) g(\mathsf{k}')\, d\mathsf{k}' \right] d\mathsf{k} \\ &= \int_{\mathbb{R}^3} d\mathsf{k}' g(\mathsf{k}') \left[\int_{\mathbb{R}^3} S(\mathsf{k}', \mathsf{k}) \overline{\psi}_n(\varepsilon(\mathsf{k}))\, d\mathsf{k} \right] \\ &\leq \int_{\mathbb{R}^3} g(\mathsf{k}') \nu(\mathsf{k}')\, d\mathsf{k}'\end{aligned}$$

for all $n \in \mathbb{N}$.

Then

$$\int_{\mathbb{R}^3} J(g)(\mathsf{k})\, d\mathsf{k} \leq \int_{\mathbb{R}^3} \nu(\mathsf{k}) g(\mathsf{k})\, d\mathsf{k}.$$

APPENDIX C

Let $\psi(\varepsilon(\mathsf{k}))$ be a bounded measurable function having compact support. If

$$\mathscr{G}_0(\mathsf{k}, \mathsf{k}') \equiv 0, \mathscr{G}(\mathsf{k}, \mathsf{k}') = \Gamma(\varepsilon(\mathsf{k}), \varepsilon'(\mathsf{k}')),$$

and $\varphi(\varepsilon(\mathsf{k}))$ belongs to $L^1(\mathbb{R}^3)$,

then

$$\begin{aligned}&\int_{\mathbb{R}^3} Q(\varphi)(\mathsf{k}) \psi(\varepsilon(\mathsf{k}))\, d\mathsf{k} \\ &\quad= \int_{\mathbb{R}^6} \Gamma(\varepsilon', \varepsilon) \left[(n_q + 1) \delta(\varepsilon' - \varepsilon - \hbar\omega) + n_q \delta(\varepsilon' - \varepsilon + \hbar\omega) \right] \\ &\qquad \times \varphi'(\psi(\varepsilon) - \psi(\varepsilon'))\, d\mathsf{k}\, d\mathsf{k}' \\ &\quad= \int_{\mathbb{R}^6} \Gamma(\varepsilon', \varepsilon) \delta(\varepsilon' - \varepsilon - \hbar\omega)(\psi(\varepsilon) - \psi(\varepsilon')) \\ &\qquad \times \left[(n_q + 1)\varphi' - n_q \varphi \right] d\mathsf{k}\, d\mathsf{k}' \\ &\quad= \left(\int_{\mathbb{R}^3} \Gamma(\varepsilon, \varepsilon + \hbar\omega) [\psi(\varepsilon) - \psi(\varepsilon + \hbar\omega)] \right) \\ &\qquad \times \left(\left[(n_q + 1)\varphi(\varepsilon + \hbar\omega) - n_q \varphi(\varepsilon) \right] d\mathsf{k} \int_{\mathbb{R}^3} \delta(\varepsilon' - \varepsilon - \hbar\omega)\, d\mathsf{k} \right).\end{aligned}$$

Let

$$v(\mathsf{k}) = \Gamma(\varepsilon, \varepsilon + \hbar\omega) \int_{\mathbb{R}^3} \delta(\varepsilon' - \varepsilon - \hbar\omega)\, d\mathsf{k}'.$$

Then we have

$$\int_{\mathbb{R}^3} Q(\varphi)\psi(\varepsilon)\, d\mathsf{k} = \int_{\mathbb{R}^3} v(\mathsf{k})\left[\psi(\varepsilon) - \psi(\varepsilon + \hbar\omega)\right] \times \left[(n_q + 1)\varphi(\varepsilon + \hbar\omega) - n_q\varphi(\varepsilon)\right] d\mathsf{k}.$$

Choosing $\psi(\varepsilon(\mathsf{k})) = \chi_{[0, n\hbar]}(\varepsilon(\mathsf{k}))$ (where $\chi_{[0, n\hbar]}$ is the characteristic function of the interval $[0, n\hbar]$), we obtain

$$\int_{\mathbb{R}^3} Q(\varphi)\chi_{[0, n\hbar]}(\varepsilon)\, d\mathsf{k} = \int_{\mathbb{R}^3} v(\mathsf{k})\chi_{[n\hbar\omega, (n+1)\hbar\omega]}(\varepsilon) \times \left[n_q\varphi(\varepsilon) - (n_q + 1)\varphi(\varepsilon + \hbar\omega)\right] d\mathsf{k}. \quad \text{(C.1)}$$

Since v is positive, the signs of the above integrals depend on φ only. It is easy to see that there exist functions φ such that $n_q\varphi(\varepsilon(\mathsf{k})) - (n_q + 1)\varphi(\varepsilon(\mathsf{k}) + \hbar\omega)$ is always positive or negative for each $\mathsf{k} \in \mathbb{R}^3$ and $n \in \mathbb{N}$. Moreover, suitable choices of φ and Γ make (C.1) greater than a positive constant for every n or less than a negative constant for every n. Hence, it is evident that $\varphi \in L^1(\mathbb{R}^3)$ does not imply $Q(\varphi) \in L^1(\mathbb{R}^3)$.

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