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An Extension of the Notion of Orthogonality to Banach Spaces

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We extend the usual notion of orthogonality to Banach spaces. We show that the extension is quite rich in structure by establishing some of its main properties and consequences. Geometric characterizations and comparison results with other extensions are established. Also, we establish a characterization of compact operators on Banach spaces that admit orthonormal Schauder bases. Finally, we characterize orthogonality in the spaces $l_2^p(C)$. © 2002 Elsevier Science (USA)

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1. INTRODUCTION

Throughout this paper K is the field of real or complex numbers, E is a Banach space over K with unit ball denoted by B and norm denoted by $\|\cdot\|$, and $(x_n) := (x_n)_{n=1}^N := (x_n)_{n\in L}$ is a finite or infinite sequence in E, where either N is a positive integer and $L := \{1, 2, \ldots, N\}$, or $N = \infty$ and $L := \{1, 2, \ldots\}$. For $J(\neq \emptyset) \subset L$, the closure of the span of the set $\{x_n : n \in J\}$ is denoted by $[x_n : n \in J]$. The unit ball in $[x_n : n \in J]$ is denoted by B_J.

The notion of orthogonality goes a long way back in time. Usually this notion is associated with Hilbert spaces or, more generally, inner product spaces. Various extensions have been introduced through the decades. Thus, for instance, x is orthogonal to y in E

(a) in the sense of G. Birkhoff [1], if for every $\alpha \in K$

$$\|x + \alpha y\| \geq \|x\|;$$

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(b) in the sense of B. D. Roberts [5], if for every $\alpha \in K$

$$\|x + \alpha y\| = \|x - \alpha y\|;$$

(c) in the isosceles sense (R. C. James [4]), if

$$\|x + y\| = \|x - y\|;$$

(d) in the Pythagorean sense (R. C. James [4]), if

$$\|x - y\|^2 = \|x\|^2 + \|y\|^2;$$

(e) in the sense of I. Singer [8], if

$$\left\| \frac{x}{\|x\|} + \frac{y}{\|y\|} \right\| = \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\|.$$

More recently, the following two definitions were introduced in [6]:

DEFINITION 1. A finite or infinite sequence $(x_n)_{n\in L}$ in E is said to be semi-orthonormal if $\|x_n\| = 1$ for all $n \in L$ and if

$$\left\| \sum_{n\in L} a_n x_n \right\| \geq \sup_{n\in L} |a_n|, \qquad \text{for each } \sum_{n\in L} a_n x_n \in E. \tag{1.1}$$

Note that if $(x_n)_{n\in L}$ is semi-orthonormal then $\{x_n : n \in L\}$ is linearly independent, and, for each $i \in L$, if we set

$$\left(x_i^*, \sum_{n\in L} a_n x_n \right) := a_i, \tag{1.2}$$

then x_i^* is the unique element in $[x_n : n \in L]^*$ that satisfies, for all $j \in L$,

$$(x_i^*, x_j) = \delta_{i,j} := \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j, \end{cases}$$

where $(x_i^*, x_j) := x_i^*(x_j)$. By the Hahn Banach theorem, each x_n^* can be extended to an element of E^*, denoted also by x_n^*, without changing its norm. The sequence $(x_n^*)_{n\in L}$ in E^* is called a sequence of associated (or corresponding) coefficient functionals for the sequence $(x_n)_{n\in L}$. Note that the sequence of extensions $(x_n^*)_{n\in L}$ is not unique. Now we state the definition of orthogonality introduced in [6]:

DEFINITION 2. Let $(x_n)_{n\in L}$ be a semi-orthonormal sequence in E, and let $(x_n^*)_{n\in L}$ be a sequence of corresponding coefficient functionals. The sequence $(x_n)_{n\in L}$ is said to be orthonormal if, for any $(\lambda_n)_{n\in L} \in \ell^\infty$ and any $x \in E$, $\sum_{n\in L} \lambda_n (x_n^*, x) x_n$ converges and

$$\left\| \sum_{n\in L} \lambda_n (x_n^*, x) x_n \right\| \leq \|x\| \sup_{n\in L} \|\lambda_n\|. \tag{1.3}$$

The sequence $(x_n)_{n\in L}$ is said to be orthogonal if the sequence $(x_n/\|x_n\|)$ obtained from the sequence $(x_n)_{n\in L}$ after the trivial terms are deleted is orthonormal.

In Section 3, we show that this definition of orthogonality does indeed depend on the choice of coefficient functionals.

The object of this paper is to introduce a new but simple and natural extension of the notion of orthogonality that is quite rich in structure and that can be useful in the study of various classes of function spaces and operators.

In Section 2, we introduce the new definition together with some of its main properties and consequences. Among other things, we show that if (x_n) is orthonormal in the sense of our definition, then (x_n) is semi-orthonormal. Also we show that if (x_n) is orthogonal in the sense of Definition 2, then (x_n) is orthogonal in the sense of our definition (Definition 3). Moreover, we prove that most of the results established in [6] still hold true under the weaker condition that (x_n) is orthogonal in our sense.

In Section 3, we establish some geometric characterizations for the various notions. An example is constructed to show that it is possible for (x_n) to be orthonormal in our sense while, for any choice of corresponding coefficient functionals, (x_n) is not orthogonal in the sense of Definition 2. Moreover, we give a necessary and sufficient condition under which orthogonality in our sense implies (hence is equivalent to) orthogonality in the sense of Definition 2. We finish by constructing another example that shows that we could have x_i orthogonal to x_j for all $i \neq j$ while (x_n) is not orthogonal.

Let $L(F, E)$ denote the set of bounded linear operators from the normed space F into the Banach space E. It is known that, if F and E are Hilbert spaces then the set of compact operators in $L(F, E)$ is the closure in $L(F, E)$ of the set of finite-rank operators. This gives a convenient and practical characterization of compact operators. In Section 4, we show that this characterization still holds true when F is any normed space and E is any Banach space that admits an orthonormal Schauder basis.

Finally, in Section 5, we establish a characterization for orthogonality in the spaces $l_2^p(C)$, where C is the set of complex numbers.

2. ORTHOGONALITY IN BANACH SPACES

One of the natural and simple properties of orthogonality in a Hilbert space H that one would like to hold true in a Banach space is that x is orthogonal to y in H if and only if

$$\|x + \lambda_1 y\| = \|x + \lambda_2 y\|, \qquad \text{for all } \lambda_1, \lambda_2 \in K, |\lambda_1| = |\lambda_2|. \tag{2.1}$$

Clearly, in any Banach space, Eq. (2.1) is equivalent to

$$\|\lambda x + \mu y\| = \||\lambda|x + |\mu|y\|, \qquad \text{for all } \lambda, \mu \in K. \tag{2.2}$$

Hence, we introduce the following definition:

DEFINITION 3. A finite or infinite sequence $(x_n)_{n\in L}$ in a Banach space E is said to be orthogonal if

$$\left\|\sum_{n\in L} a_n x_n\right\| = \left\|\sum_{n\in L} |a_n| x_n\right\|, \qquad \text{for each } \sum_{n\in L} a_n x_n \in E. \tag{2.3}$$

If, in addition, $\|x_n\| = 1$ for all $n \in L$, then $(x_n)_{n\in L}$ is said to be orthonormal. We write $x \perp y$ if x is orthogonal to y.

It is clear from the definition that $(x_n)_{n\in L}$ is orthogonal in E if and only if $(x_n)_{n\in L}$ is orthogonal in $[x_n : n \in L]$.

For the remainder of this paper, to avoid confusion, we reserve the word "orthogonal" for a sequence $(x_n)_{n\in L}$ that is orthogonal in our sense (Definition 3). If $(x_n)_{n\in L}$ is orthogonal in the sense of Definition 2, then $(x_n)_{n\in L}$ is said to be (*)-orthogonal.

Note that Definition 3 is an extension of the usual notion of orthogonality since in a Hilbert space H, $x \perp y$ in our sense if and only if $\langle x, y\rangle = 0$, where $\langle\cdot,\cdot\rangle$ denotes the inner product in H. Indeed, simple calculations after both sides of Eq. (2.2) are squared give that $x \perp y$ if and only if $\mathrm{Re}(\lambda\bar{\mu}\langle x, y\rangle) = \mathrm{Re}(|\lambda||\mu|\langle x, y\rangle)$ for all scalars λ, μ. But this is true if and only if $\langle x, y\rangle = 0$. Here, $\mathrm{Re}(z)$ is the real part of z and $\bar{z}$ is the conjugate of z.

Remark 1. If $(x_n)_{n\in L}$ is orthogonal then, for each pair of sequences $(b_n)_{n\in L}$ and $(c_n)_{n\in L}$ in K satisfying $|b_n| = |c_n|$ for all $n \in L$, we have

$$\sum_{n\in L} b_n x_n \text{ converges} \qquad \text{if and only if } \sum_{n\in L} c_n x_n \text{ converges}, \tag{2.4}$$

and, if both summations converge, then

$$\left\|\sum_{n\in L} b_n x_n\right\| = \left\|\sum_{n\in L} c_n x_n\right\|.$$

Proof. It follows directly from Eq. (2.3) that, if $(x_n)_{n\in L}$ is orthogonal, then, for every nonempty and finite set $I \subset L$, we have

$$\left\|\sum_{n\in I} b_n x_n\right\| = \left\|\sum_{n\in I} |b_n| x_n\right\| = \left\|\sum_{n\in I} |c_n| x_n\right\| = \left\|\sum_{n\in I} c_n x_n\right\|. \tag{2.5}$$

The remark follows directly from Eq. (2.5). ∎

Equation (2.2) is clearly equivalent to

$$\|x + \lambda y\| = \|x + |\lambda| y\|, \qquad \text{for all } \lambda \in K, \lambda \neq 0. \tag{2.6}$$

Two nonempty subsets F and G of E are said to be orthogonal, and we write $F \perp G$, if $x \perp y$ for all $x \in F$ and all $y \in G$. In particular, if $F = \{x\}$, then we write $x \perp G$.

When extending the notion of orthogonality, one of the main properties that one would like to have is that $(\sum_{n\in I} a_n x_n) \perp (\sum_{n\in J} a_n x_n)$ whenever I and J are disjoint subsets of L and $(x_n)_{n\in L}$ is orthogonal in E. In that respect we have

THEOREM 1. *Given a sequence $(x_n)_{n\in L}$ in E, the following are equivalent:*

(i) *The sequence $(x_n)_{n\in L}$ is orthogonal in E.*

(ii) *For each pair of nonempty and disjoint sets $I, J \subset L$,*

$$[x_n : n \in I] \perp [x_n : n \in J].$$

(iii) *For each $i \in L$,*

$$x_i \perp [x_n : n \in L, n \neq i]. \tag{2.7}$$

Proof. (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) are trivial.

(iii) $\Rightarrow$ (i): Let l and m be any two positive integers satisfying $1 \le l \le m \le N$. Using (2.7) successively for $i = l, l+1, \ldots, m$, we get

$$\left\|\sum_{n=l}^{m} a_n x_n\right\| = \left\||a_l|x_l + \sum_{n=l+1}^{m} a_n x_n\right\| = \cdots = \left\|\sum_{n=l}^{m-1} |a_n| x_n + a_m x_m\right\| \tag{2.8}$$
$$= \left\|\sum_{n=l}^{m} |a_n| x_n\right\|.$$

This implies, since $\sum_{n\in L} a_n x_n$ converges, that the sequence $(\sum_{n=1}^{m} |a_n| \times x_n)_{m\in L}$ is Cauchy. Hence, $\sum_{n\in L} |a_n| x_n$ converges and, again by Eq. (2.8),

$$\left\|\sum_{n=1}^{N} a_n x_n\right\| = \left\|\sum_{n=1}^{N} |a_n| x_n\right\|,$$

which implies that the sequence $(x_n)_{n\in L}$ is orthogonal. ∎

A function g defined on the field K is said to be radial if

$$g(z) = g(|z|), \qquad \text{for all } z \in K.$$

The following lemma is interesting in itself:

LEMMA 1. *If g is a convex real-valued function defined on the field K, then g is radial if and only if $g(z)$ is nondecreasing as $|z|$ increases in $[0, \infty)$.*

Proof. First suppose that g is radial. Since g is convex, we have

$$g(0) \leq \tfrac{1}{2}g(z) + \tfrac{1}{2}g(-z) \qquad \text{for all } z \in K.$$

But $g(z) = g(-z)$, since g is radial. Therefore $g(0) \leq g(z)$ for all $z \in K$. This implies that the restriction of g to the set of real numbers is a convex function attaining its minimum at zero. Therefore g: $[0, \infty) \to (-\infty, \infty)$ is nondecreasing. Hence, for all $z_1, z_2 \in K$, $|z_1| \leq |z_2|$, we have

$$g(z_1) = g(|z_1|) \leq g(|z_2|) = g(z_2).$$

Conversely, suppose that $g(z)$ is nondecreasing as $|z|$ increases in $[0, \infty)$, and let $z_o \in K$ be fixed. We need to prove that $g(z_o) = g(|z_o|)$. Suppose that $g(z_o) \neq g(|z_o|)$, say $g(z_o) < g(|z_o|)$ (the case $g(z_o) > g(|z_o|)$ is similar). Then, since g is convex (hence continuous), there exists z, $|z| > |z_o|$, such that

$$|g(z) - g(z_0)| < |g(|z_o|) - g(z_0)|.$$

Hence, since $g(z_o) < g(|z_o|)$, we get that $g(z) < g(|z_o|)$ while $|z| > |z_o|$. This contradicts the assumption. ∎

Using the previous lemma, we can now establish some useful necessary and sufficient conditions for $(x_n)_{n\in L}$ to be orthogonal. We have

THEOREM 2. *Given a sequence $(x_n)_{n\in L}$ in E, the following are equivalent*:

(i) *The sequence $(x_n)_{n\in L}$ is orthogonal in E.*

(ii) *For each pair of sequences $(b_n)_{n\in L}$ and $(c_n)_{n\in L}$ in K satisfying $|b_n| \leq |c_n|$ for all $n \in L$, if $\sum_{n\in L} c_n x_n$ converges then $\sum_{n\in L} b_n x_n$ converges, and*

$$\left\| \sum_{n\in L} b_n x_n \right\| \leq \left\| \sum_{n\in L} c_n x_n \right\|. \tag{2.9}$$

(iii) *For each pair of sequences $(b_n)_{n\in L}$ and $(c_n)_{n\in L}$ in K satisfying $|b_n| = |c_n|$ for all $n \in L$, $\sum_{n\in L} c_n x_n$ converges, if and only if $\sum_{n\in L} b_n x_n$ converges, and, if both converge,*

$$\left\| \sum_{n\in L} b_n x_n \right\| = \left\| \sum_{n\in L} c_n x_n \right\|.$$

Proof. (i) $\Rightarrow$ (ii): Suppose that $(x_n)_{n\in L}$ is orthogonal and let $(b_n)_{n\in L}$ and $(c_n)_{n\in L}$ be two sequences in K satisfying $|b_n| \leq |c_n|$ for all $n \in L$ and such that $\sum_{n\in L} c_n x_n$ converges. For each $i \in L$ and each $v_i \in [x_n : n \in L, n \neq i]$, the function $g(\lambda) := \|\lambda x_i + v_i\|$ is convex and radial, since by Theorem 1 (iii), $x_i \perp v_i$. Hence, by Lemma 1 and since $|b_i| \leq |c_i|$, we obtain that $g(b_i) \leq g(c_i)$. In other words, for each $i \in L$ and each $v_i \in [x_n : n \in L, n \neq i]$, we have

$$\|b_i x_i + v_i\| \leq \|c_i x_i + v_i\|. \tag{2.10}$$

Applying Eq. (2.10) successively, we obtain, for any finite set $\{l, l+1, \ldots, m\} \subset L$, that

$$\begin{aligned} \left\|\sum_{n=l}^{m} b_n x_n\right\| &\leq \left\|c_l x_l + \sum_{n=l+1}^{m} b_n x_n\right\| \leq \cdots \leq \left\|\sum_{n=l}^{m-1} c_n x_n + b_m x_m\right\| \\ &\leq \left\|\sum_{n=l}^{m} c_n x_n\right\|. \end{aligned} \tag{2.11}$$

Part (ii) follows directly from Eq. (2.11).

(ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (i) are trivial. ∎

The following lemma will enable us to state a uniqueness property.

Lemma 2. *Let F and G be two subspaces of E. If $F \perp G$ and $u \in F + G$, then*

(i) $F \cap G = \{0\}$.

(ii) *$u \perp F$ if and only if $u \in G$.*

Proof. (i) Let $v \in F \cap G$. Then $v \perp v$, and consequently we have

$$2\|v\| = \|v + v\| = \|v - v\| = 0.$$

(ii) Clearly, if $u \in G$ then $u \perp F$, since $G \perp F$. Conversely, suppose that $u \perp F$. By (i), u can be written uniquely as $u = u_1 + u_2$, where $u_1 \in F$ and $u_2 \in G$. We must show that $u_1 = 0$. Note that $u_2 \perp u_1$ and $u \perp u_1$. It follows that, for every $r \in [0, \infty)$,

$$\begin{aligned} \|u_2 + ru_1\| &= \|u_2 - ru_1\| = \|u - (r+1)u_1\| \\ &= \|u + (r+1)u_1\| = \|u_2 + (r+2)u_1\|. \end{aligned}$$

Considering the cases where $r = 0, 2, 4, \ldots,$ we obtain that

$$\|u_2\| = \|u_2 + 2ku_1\|$$

for all positive integers k, which is possible only if $\|u_1\| = 0$. ∎

As an immediate consequence of Lemma 2 we obtain that every nonzero element x of a two-dimensional subspace F of E admits at most one orthogonal direction in F. More precisely, we have

COROLLARY 1. *Let x and y be two nonzero elements in E satisfying $x \perp y$. Then we have*

$$\{z \in \text{ span } \{x, y\} : z \perp x\} = \text{span } \{y\}.$$

Now we give some comparison results. First we start with the simple observation that

LEMMA 3. *If $(x_n)_{n\in L}$ is an orthonormal sequence in E, then $(x_n)_{n\in L}$ is semi-orthonormal.*

Proof. Let $\sum_{n\in L} a_n x_n \in E$. Then for each $i \in L$ we have, by Theorem 2,

$$|a_i| = \|a_i x_i\| \leq \left\| \sum_{n\in L} a_n x_n \right\|.$$

Therefore $\sup_{n\in L} |a_n| \leq \| \sum_{n\in L} a_n x_n\|$, which ends the proof. ∎

In Section 3, Example 1, we will show that the reverse of Lemma 3 is not true.

Recall that if a sequence $(x_n)_{n\in L}$ is orthogonal in the sense of Definition 2, then $(x_n)_{n\in L}$ is said to be $(^*)$-orthogonal.

Another comparison result is

LEMMA 4. *If the sequence $(x_n)_{n\in L}$ is $(^*)$-orthonormal with respect to some sequence of corresponding coefficient functionals $(x_n^*)_{n\in L}$, then $(x_n)_{n\in L}$ is orthonormal.*

Proof. Let $(x_n)_{n\in L}$ be $(^*)$-orthonormal with respect to a sequence $(x_n^*)_{n\in L}$ of corresponding coefficient functionals. Also, let $\sum_{n\in L} a_n x_n$, $\sum_{n\in L} b_n x_n \in E$ satisfy $|a_n| \leq |b_n|$ for all $n \in L$. For each n there exists $\lambda_n \in K$ such that

$$a_n = \lambda_n b_n \quad \text{and} \quad |\lambda_n| \leq 1.$$

If $x := \sum_{n\in L} b_n x_n$ then, for all $n \in L$, we have $(x_n^*, x) = b_n$. Therefore, by assumption and since $|\lambda_n| \leq 1$ for all $n \in L$, we get

$$\begin{aligned} \left\| \sum_{n\in L} a_n x_n \right\| &= \left\| \sum_{n\in L} \lambda_n b_n x_n \right\| = \left\| \sum_{n\in L} \lambda_n (x_n^*, x) x_n \right\| \\ &\leq \|x\| \sup_{n\in L} |\lambda_n| \leq \|x\| = \left\| \sum_{n\in L} b_n x_n \right\|. \end{aligned}$$

Therefore, by Theorem 2, $(x_n)_{n\in L}$ is orthonormal. ∎

Before we continue, we mention that in Section 3, Example 2, we will show that the reverse of Lemma 4 is not true.

Recall that, given a semi-orthonormal sequence (x_n) in E, there exists a unique sequence $(x_n^*)_{n\in L}$ in $[x_n : n \in L]^*$ satisfying

$$(x_i^*, x_j) = \delta_{i,j}.$$

It follows from [6, Lemma 1.9] that if $(x_n)_{n\in L}$ is $(^*)$-orthonormal then the sequence $(x_n^*)_{n\in L}$ in $[x_n : n \in L]^*$ is $(^*)$-orthonormal. We prove the stronger result:

THEOREM 3. *Let $(x_n)_{n\in L}$ be a finite or infinite sequence in E. The following are equivalent:*

(i) *$(x_n)_{n\in L}$ is orthonormal.*

(ii) *$(x_n)_{n\in L}$ is semi-orthonormal and $(x_n^*)_{n\in L}$ is orthonormal in $[x_n : n \in L]^*$.*

Proof. (i) $\Rightarrow$ (ii): First, by Lemma 3 we get that $(x_n)_{n\in L}$ is semi-orthonormal. Also, by the definition of $(x_n^*)_{n\in L}$, $\|x_n^*\| = 1$ for all $n \in L$. Now, let $x^* := \sum_{n\in L} a_n x_n^*$ and $y^* := \sum_{n\in L} b_n x_n^*$ be two elements in $[x_n^* : n \in L]$ satisfying $|a_n| \le |b_n|$ for all $n \in L$. For each n there exists $\lambda_n \in K$ such that

$$a_n = \lambda_n b_n \quad \text{and} \quad |\lambda_n| \le 1.$$

We need to show that $\|x^*\| \le \|y^*\|$. Let $x \in [x_n : n \in L]$, $x := \sum_{n\in L} \mu_n x_n$. Then, since $|\lambda_n| \le 1$, $\sum_{k\in L} \lambda_k \mu_k x_k$ converges and

$$\left\| \sum_{k\in L} \lambda_k \mu_k x_k \right\| \le \left\| \sum_{k\in L} \mu_k x_k \right\|.$$

Therefore, since $\sum_{n\in L} \lambda_n b_n x_n^*$ is a continuous functional and since $(x_i^*, x_j) = \delta_{i,j}$, we get

$$\begin{aligned} |(x^*, x)| &= \left| \left(\sum_{n\in L} \lambda_n b_n x_n^*, \sum_{k\in L} \mu_k x_k \right) \right| = \left| \sum_{n\in L} \lambda_n b_n \mu_n (x_n^*, x_n) \right| \\ &= \left| \left(\sum_{n\in L} b_n x_n^*, \sum_{k\in L} \lambda_k \mu_k x_k \right) \right| = \left| \left(y^*, \sum_{k\in L} \lambda_k \mu_k x_k \right) \right| \\ &\le \|y^*\| \left\| \sum_{k\in L} \lambda_k \mu_k x_k \right\| \le \|y^*\| \left\| \sum_{k\in L} \mu_k x_k \right\| = \|y^*\| \|x\|. \end{aligned}$$

Therefore $\|x^*\| \le \|y^*\|$, and, consequently, $(x_n^*)_{n\in L}$ is orthonormal in $[x_n : n \in L]^*$.

(ii) $\Rightarrow$ (i): Let $(x_n)_{n\in L}$ be semi-orthonormal and suppose that $(x_n^*)_{n\in L}$ is orthonormal in $[x_n : n \in L]^*$. Now let $x := \sum_{n\in L} a_n x_n$ and $y := \sum_{n\in L} b_n x_n$ be two elements in E satisfying $|a_n| \le |b_n|$ for all $n \in L$ and let λ_n be as above. We need to show that $\|x\| \le \|y\|$. This follows immediately from the proof of $\|x^*\| \le \|y^*\|$ in (i) $\Rightarrow$ (ii) by interchanging y^* with y, x^* with x, and x_n^* with x_n. ∎

3. GEOMETRIC CHARACTERIZATIONS

We start this section with a geometric characterization for the notion of semi-orthonormality.

Recall that B is the unit ball in E and that, if J is a nonempty subset of L, then B_J is the unit ball in $[x_n : n \in J]$.

LEMMA 5. *Let $(x_n)_{n\in L}$ be a sequence in E satisfying $\|x_n\| = 1$ for all $n \in L$. The following are equivalent*:

(i) *$(x_n)_{n\in L}$ is semi-orthonormal.*

(ii) *For each $\sum_{n\in L} a_n x_n \in B$, we have $\sup_{n\in L} |a_n| \le 1$.*

Proof. (i) $\Rightarrow$ (ii): This is direct since, by (i), $\sup_{n\in L} |a_n| \le \| \sum_{n\in L} a_n x_n\|$.

(ii) $\Rightarrow$ (i): Let $x := \sum_{n\in L} a_n x_n \in E$. The case where $x = 0$ is trivial since (ii) implies that the set $\{x_n : n \in L\}$ is linearly independent. If $x \ne 0$ then $\sum_{n\in L} a_n/\|x\|\ x_n \in B$, and, consequently, by (ii), $\sup_{n\in L} |a_n/\|x\|| \le 1$. Hence $\sup_{n\in L} |a_n| \le \|x\|$. ∎

Note that we may replace B by B_L in Lemma 5. Also, note that if $(x_n)_{n\in L}$ is a finite sequence, then Lemma 5 says that $(x_n)_{n\in L}$ is semi-orthonormal if and only if the unit ball B_L of $[x_n : n \in L]$ is contained in the l^∞-unit ball B_∞ defined by the sequence $(x_n)_{n\in L}$,

$$B_\infty := \left\{ \sum_{n\in L} a_n x_n : \sup_{n\in L} |a_n| \le 1 \right\}.$$

Another geometric observation that follows directly from Theorem 1 is

LEMMA 6. *If $(x_n)_{n\in L}$ is orthogonal in E then, for each $I \subsetneq L$, $I \ne \emptyset$, the projection P_I: $[x_n : n \in L] \to [x_n : n \in I]$ defined by*

$$P_I\left(\sum_{n\in L} a_n x_n\right) = \sum_{n\in I} a_n x_n$$

has norm one.

We note that a similar result was shown in [6, Lemma 1.5] under the assumption that $(x_n)_{n\in L}$ is $(^*)$-orthogonal.

Also, we note that the reverse of Lemma 6 is not true in general. Indeed, consider the following example in the real plane $\Re^2$:

EXAMPLE 1. Let $\{x_1, x_2\}$ be the standard basis of $\Re^2$. We can easily check that the following defines a norm on $\Re^2$:

$$\|a_1x_1 + a_2x_2\| = \max\{|a_1|, |a_2|, |a_1 + a_2|\}.$$

Simple calculations give

$$\|P_{\{1\}}\| = \|P_{\{2\}}\| = 1,$$

while

$$\|x_1 - x_2\| = 1 < 2 = \|x_1 + x_2\|.$$

Therefore the set $\{x_1, x_2\}$ is not orthogonal. Note that the set $\{x_1, x_2\}$ is semi-orthonormal, and hence semi-orthonormality is weaker than orthonormality.

In Lemma 4 we proved that, if $(x_n)_{n\in L}$ is $(^*)$-orthonormal with respect to some sequence of associated coefficient functionals, then $(x_n)_{n\in L}$ is orthonormal. Before showing that the reverse is not true in general, we give here a necessary and sufficient condition under which the reverse of Lemma 4 is true. We have

THEOREM 4. *Let $(x_n)_{n\in L}$ be an orthonormal sequence in E satisfying $[x_n : n \in L] \subsetneqq E$. Then we have*

(i) *For each sequence of corresponding coefficient functionals $(x_n^*)_{n\in L}$ in E^*, $(x_n)_{n\in L}$ is $(^*)$-orthonormal with respect to $(x_n^*)_{n\in L}$ if and only if the projection $P\colon E \to [x_n : n \in L]$, defined by $P(x) := \sum_{n\in L}(x_n^*, x)x_n$, is well defined and has norm* 1.

(ii) *There exists a sequence of corresponding coefficient functionals $(x_n^*)_{n\in L}$ in E^* such that $(x_n)_{n\in L}$ is $(^*)$-orthonormal with respect to $(x_n^*)_{n\in L}$ if and only if there exists a projection $P\colon E \to [x_n : n \in L]$ of norm* 1.

Proof. It follows by Lemma 3 that the sequence $(x_n)_{n\in L}$ is semi-orthonormal, since it is orthonormal .

(i) Let $P(x) := \sum_{n\in L}(x_n^*, x)x_n$. If $(x_n)_{n\in L}$ is $(^*)$-orthonormal with respect to $(x_n^*)_{n\in L}$, then it follows directly from Definition 2, by taking $\lambda_n := 1$ for all $n \in L$, that $P(x)$ is well defined for each $x \in E$, i.e., $\sum_{n\in L}(x_n^*, x)x_n$ converges for each $x \in E$, and that $\|P\| = 1$.

For the converse, suppose that P is well defined and that $\|P\| = 1$. Then $\sum_{n\in L}(x_n^*, x)x_n$ is convergent for each $x \in E$. Therefore it follows, by Theorem 2 and since $(x_n)_{n\in L}$ is orthonormal, that, for any $(\lambda_n)_{n\in L} \in \ell^\infty$ and any $x \in E$, $\sum_{n\in L} \lambda_n(x_n^*, x)x_n$ converges, since $|\lambda_n(x_n^*, x)| \le |(x_n^*, x)|$, and that

$$\left\|\sum_{n\in L} \lambda_n(x_n^*, x)x_n\right\| \le \left(\sup_{n\in L}\left|\lambda_n\right|\right)\left\|\sum_{n\in L}(x_n^*, x)x_n\right\|$$
$$= \left(\sup_{n\in L}\left|\lambda_n\right|\right)\|P(x)\| \le \|x\|\left(\sup_{n\in L}\left|\lambda_n\right|\right).$$

Therefore $(x_n)_{n\in L}$ is $(^*)$-orthonormal with respect to $(x_n^*)_{n\in L}$.

(ii) If there exists a sequence of corresponding coefficient functionals $(x_n^*)_{n\in L}$ in E^* such that $(x_n)_{n\in L}$ is $(^*)$-orthonormal with respect to $(x_n^*)_{n\in L}$, then, by part (i), $P(x) := \sum_{n\in L}(x_n^*, x)x_n$ is a well-defined projection of norm 1 from E onto $[x_n : n \in L]$.

Conversely, suppose that there exists a projection P: $E \to [x_n : n \in L]$ satisfying $\|P\| = 1$. Then every $x \in E$ can be written uniquely in the form

$$x = u_x + v_x,$$

where $u_x \in [x_n : n \in L]$ and $v_x \in \ker P$. Since $(x_n)_{n\in L}$ is semi-orthonormal, there exists a unique sequence $(x_n^*)_{n\in L}$ in $[x_n : n \in L]^*$ satisfying $(x_i^*, x_j) = \delta_{i,j}$. Extend each $x_n^* \in [x_n : n \in L]^*$ to an element of E^* by

$$(x_n^*, x) := (x_n^*, u_x), \tag{3.1}$$

for every $x \in E$. Then $(x_n)_{n\in L}$ is orthonormal with respect to the sequence of coefficient functionals defined by Eq. (3.1). Indeed, let $(\lambda_n)_{n\in L} \in \ell^\infty$ and let $x \in E$. Then

$$P(x) = u_x := \sum_{k\in L} a_k x_k,$$

and, for all $n \in L$, we have

$$(x_n^*, x) := (x_n^*, u_x) = a_n.$$

It follows that, for all $n \in L$,

$$|\lambda_n(x_n^*, x)| \le \left(\sup_{k\in L}|\lambda_k|\right)|a_n|.$$

Therefore, since $(x_n)_{n\in L}$ is orthogonal, we obtain, by Theorem 2 and since $\sum_{n\in L} a_n x_n$ converges, that $\sum_{n\in L} \lambda_n(x_n^*, x)x_n$ converges and that

$$\left\|\sum_{n\in L} \lambda_n(x_n^*, x)x_n\right\| \le \left\|\sum_{n\in L}\left(\sup_{k\in L}|\lambda_k|\right)a_n x_n\right\| = \left\|\sum_{n\in L} a_n x_n\right\| \sup_{k\in L}|\lambda_k|$$
$$= \|P(x)\| \sup_{k\in L}|\lambda_k| \le \|x\| \sup_{k\in L}|\lambda_k|.$$

This completes the proof. ∎

Now, we construct a finite-dimensional example that shows that it is possible for $(x_n)_{n\in L}$ to be orthonormal, while, for any choice of corresponding coefficient functionals, $(x_n)_{n\in L}$ is not $(^*)$-orthonormal.

EXAMPLE 2. In $\Re^3$ as a real vector space, let $\{x_1, x_2, x_3\}$ be the standard basis. It easy to check that the following defines a norm on $\Re^3$:

$$\|a_1x_1 + a_2x_2 + a_3x_3\| := \max\left\{|a_1| + |a_2|, |a_3|, \tfrac{1}{3}|4a_2 - a_3|\right\}. \tag{3.2}$$

Setting $a_3 = 0$ in Eq. (3.2), We obtain that

$$\|a_1x_1 + a_2x_2\| = \max\left\{|a_1| + |a_2|, \tfrac{4}{3}|a_2|\right\},$$

which clearly implies that the set $\{x_1, \frac{3}{4}x_2\}$ is orthonormal, since

$$\|x_1\| = \left\|\tfrac{3}{4}x_2\right\| = \|x_3\| = 1.$$

We claim that for any choice of coefficient functionals $\{x_1^*, x_2^*\}$ associated with $\{x_1, \frac{3}{4}x_2\}$, the set $\{x_1, \frac{3}{4}x_2\}$ is not $(^*)$-orthogonal. By Theorem 4, it is enough to show that there are no projections of norm 1 on span$\{x_1, x_2\}$. Indeed, let P: $\Re^3 \to$ span$\{x_1, x_2\}$ be any projection onto span$\{x_1, x_2\}$, and let $u := u_1x_1 + u_2x_2 + u_3x_3$ be a nonzero element of ker P. Then

$$P(a_1x_1 + a_2x_2 + a_3x_3) = (a_1 - u_1a_3)x_1 + (a_2 - u_2a_3)x_2.$$

Therefore we have, if $w := a_1x_1 + a_2x_2 + a_3x_3$,

$$\|P(w)\| = \max\left\{|a_1 - u_1a_3| + |a_2 - u_2a_3|, \tfrac{4}{3}|a_2 - u_2a_3|\right\}. \tag{3.3}$$

Case 1. If $u_1 \neq 0$, say $u_1 > 0$ (the case $u_1 < 0$ is similar). Then setting $a_1 = -1$, $a_2 = 0$, and $a_3 = 1$ in Eqs. (3.2) and (3.3), we get

$$\|-x_1 + x_3\| = 1$$

and

$$\begin{aligned}\|P(-x_1 + x_3)\| &= \|(-1 - u_1)x_1 - u_2x_3\| \\ &= \max\left\{|-1 - u_1| + |-u_2|, \tfrac{4}{3}|-u_2|\right\} \geq |1 + u_1| > 1.\end{aligned}$$

Hence $\|P\| > 1$.

Case 2. If $u_1 = 0$ and $u_2 \neq 0$, say $u_2 > 0$ (the case $u_2 < 0$ is similar). Then setting $a_1 = 1$, $a_2 = 0$, and $a_3 = 1$ in Eqs. (3.2) and (3.3), we get

$$\|x_1 + x_3\| = 1$$

and

$$\begin{aligned}\|P(x_1 + x_3)\| &= \|x_1 - u_2x_2\| \\ &= \max\left\{|1| + |-u_2|, \tfrac{4}{3}|-u_2|\right\} \geq |1| + |-u_2| > 1.\end{aligned}$$

Hence $\|P\| > 1$.

Case 3. If $u_1 = u_2 = 0$, then, setting $a_1 = 0$ and $a_2 = a_3 = 1$ in Eqs. (3.2) and (3.3), we get

$$\|x_2 + x_3\| = 1$$

and

$$\|P(x_2 + x_3)\| = \|x_2\| = \tfrac{4}{3} > 1.$$

Hence $\|P\| > 1$.

Therefore, in all cases, we obtain that $\|P\| > 1$. This implies that there are no projections of norm 1 onto $\operatorname{span}\{x_1, x_2\}$.

We finish by presenting another example that shows that we could have $x_i \perp x_j$ for all $i \neq j$, while $(x_n)_{n\in L}$ is not orthogonal.

EXAMPLE 3. Again in $\Re^3$ as a real vector space, let $\{x_1, x_2, x_3\}$ be the standard basis. It easy to check that the following defines a norm on $\Re^3$:

$$\|a_1x_1 + a_2x_2 + a_3x_3\| := \max\left\{|a_1|, |a_2|, |a_3|, \tfrac{1}{2}|a_1 + a_2 - a_3|\right\}.$$

Then we have, for all $1 \leq i \neq j \leq 3$,

$$\|a_ix_i + a_jx_j\| = \max\{|a_i|, |a_j|\},$$

and, consequently,

$$x_i \perp x_j \qquad \text{for all } 1 \leq i \neq j \leq 3.$$

On the other hand, we have

$$\|x_1 + x_2 + x_3\| = 1 \quad \text{and} \quad \|x_1 + x_2 - x_3\| = 3/2.$$

Therefore $\{x_1, x_2, x_3\}$ is not orthogonal.

Note that, in general,

$$(x \perp v_1 \text{ and } x \perp v_2) \not\Rightarrow (x \perp \operatorname{span}\{v_1, v_2\}),$$

since, in the previous example, we have $x_3 \perp x_1$ and $x_3 \perp x_2$, but x_3 is not orthogonal to $(x_1 + x_2)$, since

$$\|(x_1 + x_2) + x_3\| = 1 \quad \text{and} \quad \|(x_1 + x_2) - x_3\| = 3/2.$$

4. CHARACTERIZATION OF COMPACT OPERATORS

Let $L(F, E)$ denote the set of bounded linear operators from the normed space F into the Banach space E. It is known that if F and E are Hilbert spaces, then $T \in L(F, E)$ is compact if and only if T is the limit in $L(F, E)$ of a sequence of finite-rank operators [2, p. 42]. This gives a convenient and practical characterization of compact operators in Hilbert spaces. We show here that the same characterization still holds true for any Banach space E that admits an orthonormal Schauder basis and any normed space F. More precisely, we have

THEOREM 5. *Suppose that $\{e_n\}_{n=1}^{\infty}$ is an orthonormal Schauder basis of the Banach space E and that F is a normed space. For each positive integer k, let P_k be the projection on $[e_n : 1 \le n \le k]$ defined by*

$$P_k\left(\sum_{n=1}^{\infty} \alpha_n e_n\right) = \sum_{n=1}^{k} \alpha_n e_n, \qquad \sum_{n=1}^{\infty} \alpha_n e_n \in E.$$

Then, an operator $T \in L(F, E)$ is compact if and only if $P_k \circ T$ converges to T in $L(F, E)$.

Proof. The sufficiency part follows from the fact that for every Banach space E and every normed space F, the limit in $L(F, E)$ of a sequence of finite-rank operators is a compact operator [3, p. 215].

Now, suppose that $T \in L(F, E)$ is compact. For each positive integer k, let $T_k := P_k \circ T$. Note that since $\{e_n\}_{n=1}^{\infty}$ is orthonormal, it follows by Theorem 2 that $P_k \in L(E)$ and $\|P_k\| = 1$ for all k. Clearly we have, since $\{e_n\}_{n=1}^{\infty}$ is a Schauder basis of E,

$$\lim_k P_k(y) = y, \qquad \text{for each } y \in E.$$

Let B be the closed unit ball in F. Since T is compact, it follows that $K := \mathrm{cl}(T(B))$ is a compact subset of E. We need to show that

$$\lim_k \sup_{x \in B} \|T_k(x) - T(x)\| = 0.$$

Suppose this is not true. Then there exist $\varepsilon > 0$, a subsequence $\{T_{k_j}\}$, and a sequence $\{x_{k_j}\}$ in B such that

$$\|T_{k_j}(x_{k_j}) - T(x_{k_j})\| > \varepsilon, \qquad \text{for all } j. \tag{4.1}$$

Since K is compact, there exists a subsequence of $\{x_{k_j}\}$, say $\{x_{k_j}\}$, such that the sequence $\{T(x_{k_j})\}$ converges in K to some $y \in K$. Then we have, since $\|P_{k_j}\| = 1$ for all j,

$$\begin{aligned}\|T_{k_j}(x_{k_j}) - T(x_{k_j})\| &\le \|P_{k_j}(T(x_{k_j})) - P_{k_j}(y)\| + \|T(x_{k_j}) - P_{k_j}(y)\| \\ &\le \|T(x_{k_j}) - y\| + \|T(x_{k_j}) - P_{k_j}(y)\|.\end{aligned}$$

Letting $j \to \infty$, we obtain, since $\{T(x_{k_j})\}$ and $\{P_{k_j}(y)\}$ both converge to y, that

$$\lim_j \|T_{k_j}(x_{k_j}) - T(x_{k_j})\| = 0,$$

which contradicts Eq. (4.1). ∎

As a corollary we have

COROLLARY 2. *If E is a Banach space that admits an orthonormal Schauder basis and F is a normed space, then an operator $T \in L(F, E)$ is compact if and only if it is the limit in $L(F, E)$ of a sequence of finite-rank operators.*

Finally, if $\{e_n\}_{n=1}^{\infty}$ is an orthonormal sequence in a Hilbert space E and if T is the operator on E defined by

$$T(x) := \sum_{n=1}^{\infty} \lambda_n (e_n^*, x) e_n \qquad \text{for all } x \in E, \tag{4.2}$$

where e_n^* is the coefficient functional in $[e_k : k \geq 1]^*$ associated with e_n, then it is known that T is compact if and only if $\lim_{n\to\infty} \lambda_n = 0$. In [6], this result was extended to the cases where E is a reflexive Banach space and $\{e_n\}_{n=1}^{\infty}$ is a (*)-orthonormal sequence in E. We show here, as a corollary of Theorem 5, that E need not be reflexive and that it is sufficient for $\{e_n\}_{n=1}^{\infty}$ to be orthonormal in our sense. Indeed, we have

COROLLARY 3. *If $\{e_n\}_{n=1}^{\infty}$ is an orthonormal sequence in a Banach space E, then the operator T defined by Eq.* (4.2) *is compact if and only if* $\lim_{n\to\infty} \lambda_n = 0$.

Proof. Let T_k be as in the proof of Theorem 5. Then we have, for all $x \in E$,

$$\begin{aligned} \|T_k(x) - T(x)\| &= \left\| \sum_{n=k+1}^{\infty} \lambda_n e_n^*(x) e_n \right\| \leq \left(\sup_{n\geq k+1} |\lambda_n| \right) \left\| \sum_{n=k+1}^{\infty} e_n^*(x) e_n \right\| \\ &\leq \left(\sup_{n\geq k+1} |\lambda_n| \right) \|x\|, \end{aligned}$$

where the inequalities follow from Theorem 2. This implies that $\|T_k - T\| \leq \sup_{n\geq k+1} |\lambda_n|$ and, consequently, since $T(e_k) = \lambda_k$ for all k, that

$$\|T_k - T\| = \sup_{n\geq k+1} |\lambda_n|.$$

Therefore T_k converges to T in $L(E)$ if and only if $\lim_{n\to\infty} \lambda_n = 0$. The corollary now follows from Theorem 5. ∎

5. ORTHOGONALITY IN ℓ_2^p

For $p \in (2, \infty)$, we let

$$\ell_2^p(K) := \{(c_1, c_2) \in K \times K : \|(c_1, c_2)\|_p < \infty\},$$

where $\|(c_1, c_2)\|_p := (|c_1|^p + |c_2|^p)^{1/p}$. The support of $x := (c_1, c_2) \in \ell_2^p(K)$ is given by

$$\operatorname{supp}(x) := \{n : c_n \neq 0\}.$$

Let C be the field of complex numbers. We have the following characterization of orthogonality in $\ell_2^p(C)$:

THEOREM 6. *Two elements x_1 and x_2 in $\ell_2^p(C)$, $2 < p < \infty$, are orthogonal if and only if they have disjoint supports.*

Proof. Let $x_1 := (a_1, a_2)$ and $x_2 := (b_1, b_2)$ be two elements in $\ell_2^p(C)$.

It follows directly that, if x_1 and x_2 have disjoint supports, then they are orthogonal.

For the converse, suppose that x_1 and x_2 are orthogonal. Then, by Remark 1, we have, for all $r,\theta \in \Re$,

$$\|x_1 + re^{i\theta}x_2\|_p^p = \|x_1 - re^{i\theta}x_2\|_p^p = \|x_1 + |r|x_2\|_p^p. \tag{5.1}$$

Define

$$f(r, \theta) := \|x_1 + re^{i\theta}x_2\|_p^p = |a_1 + b_1re^{i\theta}|^p + |a_2 + b_2re^{i\theta}|^p.$$

Then, by Eq. (5.1), f is independent of θ. Therefore, we must have, for all r, $\theta \in \Re$,

$$\frac{\partial f}{\partial \theta}(r, \theta) = 0. \tag{5.2}$$

Also, for each fixed θ in $\Re$, we obtain from Eq. (5.1) that f must be an even function of r. Therefore, since f is a convex function of r, $f(., \theta)$ must attain its minimum at $r = 0$. Hence we have, for all $\theta \in \Re$,

$$\frac{\partial f}{\partial r}(0, \theta) = 0. \tag{5.3}$$

Now, given a differentiable function g: $\Re \to C$, we have, since $|g(t)|^p = |g(t)\bar{g}(t)|^{p/2}$,

$$\frac{d}{dt}(|g(t)|^p) = \frac{p}{2}|g(t)|^{p-2}\left(g(t)\frac{d\bar{g}}{dt}(t) + \bar{g}(t)\frac{dg}{dt}(t)\right).$$

This, together with Eqs. (5.2) and (5.3), implies, after simple calculations, that, for all r, $\theta \in \Re$,

$$|a_1 + b_1re^{i\theta}|^{p-2}\ \operatorname{Im}(a_1\overline{b_1}e^{i\theta}) + |a_2 + b_2re^{i\theta}|^{p-2}\ \operatorname{Im}(a_2\overline{b_2}e^{i\theta}) = 0 \tag{5.4}$$

and

$$(|a_1|^{p-2}a_1b_1 + |a_2|^{p-2}a_2b_2)\cos\theta = 0. \tag{5.5}$$

We need to prove that $a_1b_1 = a_2b_2 = 0$. Clearly, by Eq. (5.5), we must have either $|a_1\overline{b_1}| \neq 0$ and $|a_2\overline{b_2}| \neq 0$ or $a_1b_1 = a_2b_2 = 0$. If $|a_1\overline{b_1}| \neq 0$ and $|a_2\overline{b_2}| \neq 0$, then either there exists θ_o such that

$$\operatorname{Im}(a_1\overline{b_1}e^{-i\theta_o}) > 0 \quad \text{and} \quad \operatorname{Im}(a_2\overline{b_2}e^{-i\theta_o}) > 0, \tag{5.6}$$

or there exists $k \in (0, \infty)$ such that

$$a_2\overline{b_2} = -ka_1\overline{b_1}, \tag{5.7}$$

since this is possible for any pair of complex numbers.

If Eq. (5.6) holds, then we get, by Eq. (5.4), that, for all $r \in \Re$,

$$a_1 + b_1re^{i\theta_o} = a_2 + b_2re^{i\theta_o} = 0.$$

Hence, substituting in $f(r, \theta)$, we get

$$f(r, \theta) = (|b_1|^p + |b_2|^p)r^p|e^{i\theta} - e^{i\theta_o}|^p.$$

This implies, by Eq. (5.2) and since f is independent of θ, that $|b_1|^p + |b_2|^p = 0$. Hence $b_1 = b_2 = 0$, which contradicts the assumption that $|a_1\overline{b_1}| \neq 0$. Therefore Eq. (5.6) is not possible.

If Eq. (5.7) holds, then substituting in Eq. (5.4), we get, for all $r, \theta \in \Re$,

$$(|a_1 + b_1re^{i\theta}|^{p-2} - k|a_2 + b_2re^{i\theta}|^{p-2})\operatorname{Im}(a_1\overline{b_1}re^{-i\theta}) = 0.$$

But $|a_1\overline{b_1}| \neq 0$. Therefore

$$\operatorname{Im}(a_1\overline{b_1}re^{-i\theta}) \neq 0,$$

for all $r \neq 0$ and all $\theta \in \Re\backslash A$, where $A := \{\theta \in \Re : \operatorname{Im}(a_1\overline{b_1})\cos\theta - \operatorname{Re}(a_1\overline{b_1})\sin\theta = 0\}$. Therefore, we have, for all $\theta \in \Re\backslash A$,

$$h_r(\theta) := |a_1 + b_1re^{i\theta}|^{p-2} - k|a_2 + b_2re^{i\theta}|^{p-2} = 0.$$

But h is continuous on $\Re$ and $\Re\backslash A$ is dense in $\Re$. Therefore, for all $r \neq 0$ and all $\theta \in \Re$,

$$|a_1 + b_1re^{i\theta}|^{p-2} - k|a_2 + b_2re^{i\theta}|^{p-2} = 0.$$

Substituting in $f(r, \theta)$, we get, for all $r \neq 0$ and all $\theta \in \Re$,

$$f(r, \theta) = (1 + k^{p/(p-2)})|a_2 + b_2re^{i\theta}|^p.$$

This implies, since f is independent of θ, by Eq. (5.2), that

$$a_2 = 0 \quad \text{or} \quad b_2 = 0,$$

which contradicts the assumption that $|a_2\overline{b_2}| \neq 0$. Therefore Eq. (5.7) is not possible, and, consequently, we must have $a_1b_1 = a_2b_2 = 0$. ∎

Observe that Theorem 6 does not hold for $\ell_2^p(\Re)$, $2 < p < \infty$. Indeed, one can easily check that $(a, a) \perp (b, -b)$ in $\ell_2^p(\Re)$ for all $a, b \in \Re$.

We finish by noting that the situation is different in $\ell_n^p(C)$ when $n > 2$. A complete study of orthogonality in $\ell_S^p(C)$, where S is any subset of the set of positive integers, will appear in [7].

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