



Autonomous self-similar ordinary differential equations and the Painlevé connection

K. Andriopoulos*, P.G.L. Leach

School of Mathematical Sciences, Howard College, University of KwaZulu-Natal, Durban 4041, South Africa

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Abstract

We demonstrate an intimate connection between nonlinear higher-order ordinary differential equations possessing the two symmetries of autonomy and self-similarity and the leading-order behaviour and resonances determined in the application of the Painlevé Test. Similar behaviour is seen for systems of first-order differential equations. Several examples illustrate the theory. In an integrable case of the *ABC* system the singularity analysis reveals a positive and a negative resonance and the method of leading-order behaviour leads naturally to a Laurent expansion containing both.

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1. Introduction

The equation

$$\ddot{x} + 3x\dot{x} + x^3 = 0, \tag{1.1}$$

already noted many years ago [14,22], to a lesser extent its generalisation

$$\ddot{x} + ax\dot{x} + bx^3 = 0, \tag{1.2}$$

* Corresponding author.

E-mail addresses: kand@aegean.gr (K. Andriopoulos), leachp@ukzn.ac.za (P.G.L. Leach).

known as the modified Painlevé–Ince equation [1], has many attractive properties in that it is linearisable, consequently has eight Lie point symmetries [20], and possesses the Painlevé Property. It occurs in the most diverse places such as the theory of univalent functions [13], modelling the fusion of pellets [7], the study of stellar structure [16,21], the reduction of Yang–Baxter equations [18] and motion in a space of constant curvature [6]. It has also been the object of study from a dynamical point of view [17]. Abraham-Shrauner showed that the modified Painlevé–Ince equation was linearisable for all values of the parameters, α and β , by means of a nonlocal transformation. The essence of the transformation is to increase the order of the equation to one of the third order by means of a Riccati transformation and then to reduce the third-order equation to a second-order equation by means of a different transformation based upon the invariance of the third-order equation under time translation. The modified Painlevé–Ince equation possesses the two symmetries of invariance under time translation and self-similarity for general values of the parameters. There has been a number of papers devoted to more general equations possessing these two symmetries [4,9,10,12]. In most studies these equations were of the second order. Recently Andriopoulos and Leach [3] extended the class of equations linearisable by the method of Abraham-Shrauner and showed that the process of linearisation was intimately connected with two parameters which arise in the application of singularity analysis to these second-order equations. The two parameters are the coefficient of the leading-order term and the value of the nongeneric resonance. We should emphasise that this connection exists independently of an acceptable value for the nongeneric resonance. A value of the resonance which is compatible with an analytic solution simply implies that the reversal of the linearisation procedure leads to an analytic solution of the original equation.

In this paper we extend our considerations to equations of higher order which are invariant under the same symmetries as the modified Painlevé–Ince equation. We note that the self-similar symmetry is selected to give the exponent of the leading-order term as -1 . An equation invariant under a self-similar symmetry with different coefficients is readily converted to the same by a transformation of the dependent variable. Consequently we have not lost generality. Naturally we are aware that the transformation just mentioned can affect the analytic nature of the solution of the original equation, but our primary concern is with the relationship between the parameters of the singularity analysis and the transformations which we make. The major result is that the coefficient of the leading-order term and the resonances occur naturally in the final equation obtained after a process of increase of order followed by a decrease of order of the very specific nature. We illustrate the results with examples drawn from the Riccati Hierarchy [8] and the *ABC* system of Lotka–Volterra equations [5,15,19,23].

To give a flavour of the procedure we illustrate the process with the modified Painlevé–Ince equation before commencing a more general analysis of the higher-order equations.

Firstly we calculate the parameters of the singularity analysis. The exponent of the leading-order term is -1 by construction. The coefficient of the leading-order term is determined by the solutions of $2 - a\alpha + b\alpha^2 = 0$, where in the usual notation we have written the leading-order term as $\alpha\tau^{-1}$. The nongeneric resonance is given by $r = 4 - a\alpha$.

We apply the Riccati transformation, $x = \alpha\dot{w}/w$, to (1.2) and obtain the third-order equation

$$w^2\ddot{w} + (a\alpha - 3)w\dot{w}\ddot{w} + (b\alpha^2 - a\alpha + 2)\dot{w}^3 = 0. \quad (1.3)$$

It is evident that a simplifying choice of the value of the parameter in the Riccati transformation is the coefficient of the leading-order term as found by the singularity analysis. It then follows that (1.3) can be written as

$$w\ddot{w} + (1 - r)\dot{w}\dot{\dot{w}} = 0. \tag{1.4}$$

Since (1.4) is autonomous and homogeneous in w , we may reduce the order by the variation of the standard transformation, videlicet $u = \log w$ and $v = \dot{w}^2$, to obtain

$$v'' - rv' = 0. \tag{1.5}$$

Thus we see that the linearised form of the modified Painlevé–Ince equation is obtained by a transformation to the third order using the coefficient of the leading-order term as a parameter in the Riccati transformation and the linearised form of the equation contains the value of the nongeneric resonance corresponding to that coefficient. The linearised equation may take two forms since there are two possible values of α . Each value has its own peculiar value of the nongeneric resonance.

This paper is structured as follows. In Section 2 we present the theoretical development connecting the parameters of the singularity analysis and the increase and decrease of order as in the example of the modified Painlevé–Ince equation presented above. We do this for an equation of order $(n + 1)$ and for a system of first-order equations. In Section 3 we provide two examples to illustrate the former and one system to illustrate the latter. One of the examples for the higher-order equations is a member of the Riccati hierarchy which is characterised by being linearisable and hence possessing considerable symmetry. The second example is a variant of the first with general coefficients and hence possessing only the two symmetries of the class. For the system we consider the well-known *ABC* Lotka–Volterra system. This system is known to be integrable [5] in the sense of having two first integrals under a constraint on the values of two of the parameters. For a specific value of the third parameter the system is integrable in the sense of Painlevé even though one of the nongeneric resonances is positive and the other is negative. For this case we indicate how one obtains the Laurent series. We conclude with some observations in Section 4.

2. Theoretical development

2.1. An $(n + 1)$ th-order equation

An $(n + 1)$ th-order ordinary differential equation invariant under the two symmetries

$$\Gamma_1 = \partial_t \quad \text{and} \quad \Gamma_2 = -t\partial_t + x\partial_x$$

has the form of

$$\frac{x^{(n+1)}}{x^{n+2}} = f\left(\frac{\dot{x}}{x^2}, \frac{\ddot{x}}{x^3}, \frac{\dddot{x}}{x^4}, \dots, \frac{x^{(n)}}{x^{n+1}}\right) = f\left(\frac{x^{(j)}}{x^{j+1}}\right) \tag{2.1}$$

in a standard notation.

From the structure of Γ_2 it is evident that the exponent of the leading-order term, $x = \alpha\tau^p$, is $p = -1$. The coefficient, α , is a solution of the equation

$$(-1)^{n+1} \frac{(n+1)!}{\alpha^{n+1}} = f\left(\frac{(-1)^j j!}{\alpha^j}\right). \tag{2.2}$$

To determine the resonances we firstly expand the general term in (2.1) to the first order in μ , where we set $x = \alpha\tau^{-1} + \mu\tau^{r-1}$. We have

$$\begin{aligned} \frac{x^{(j)}}{x^{j+1}} &= \frac{(\alpha\tau^{-1} + \mu\tau^{r-1})^{(j)}}{(\alpha\tau^{-1} + \mu\tau^{r-1})^{j+1}} \\ &= \frac{(-1)^j j!}{\alpha^j} + \mu \left\{ \frac{(r-1)!}{\alpha^{j+1}(r-1-j)!} + \frac{(-1)^{j+1}(j+1)!}{\alpha^{j+1}} \right\} \tau^r. \end{aligned} \tag{2.3}$$

We substitute (2.3) into (2.1), make a formal expansion of the Taylor series for each side and use (2.2) to remove the leading-order terms. The resonances are the solutions of

$$\begin{aligned} &\frac{1}{\alpha^{n+2}} \left[\frac{(r-1)!}{(r-2-n)!} + (-1)^{n+2}(n+2)! \right] \\ &= \sum_{j=1}^n \frac{1}{\alpha^{j+1}} \frac{\partial f}{\partial \xi_j} \left\{ \frac{(r-1)!}{(r-1-j)!} + (-1)^{j+1}(j+1)! \right\} \tau^r, \end{aligned} \tag{2.4}$$

where

$$\xi_j = \frac{x^{(j)}}{x^{j+1}}$$

and the derivatives are evaluated at $(-1)^j j!/\alpha^j$. That one of the resonances is -1 is evident from (2.4) independently of the functional form of f and its derivatives. The values of the other resonances do depend on the value of α .

To see the connection between the coefficient of the leading-order behaviour and the values of the resonances on the one hand and the symmetries on the other hand we make use of a combination of increase of order and decrease of order. As we demonstrated in the Introduction with Eq. (1.2), we increase the order of the equation by means of the Riccati transformation

$$x = \alpha \frac{\dot{w}}{w}, \tag{2.5}$$

where the α is the coefficient of the leading-order behaviour determined by the singularity analysis above, and decrease the order with the use of the symmetry of time translation. The resulting equation is of the same order as the original equation and is the type of an Euler equation. Consequently we use the standard Euler transformation to render this equation autonomous. This is not a necessary part of the demonstration of the relationship between the symmetries and the parameters of the singularity analysis, but we make the additional transformation to obtain an equation of the same status, i.e., autonomous, as the original equation.

As we do not seem to be able to perform the combination of increase and decrease of order on the general n th-order equation without being reduced to an absolute mess, we turn to specific equations in Section 3.

2.2. *A system of n first-order ordinary differential equations*

We now examine a system of n first-order ordinary differential equations

$$\dot{x}_i = f_i(t, x_1, x_2, \dots, x_n), \quad i = 1, n. \tag{2.6}$$

System (2.6) is invariant under the two symmetries

$$\Gamma_1 = \partial_t, \quad \Gamma_2 = -t\partial_t + \sum_{i=1}^n x_i \partial_{x_i}$$

when it has the form

$$\dot{x}_i = x_i^2 f\left(\frac{x_j}{x_i}\right), \quad i = 1, n, \quad j = 1, n, \tag{2.7}$$

where x_j/x_i , $j = 1, i - 1, i + 1, n$, represent the $n - 1$ zeroth-order invariants common to both Γ_1 and Γ_2 .

To determine the leading-order behaviour of (2.7) we substitute $x_i = \alpha_i \tau^{p_i}$ into (2.7) to obtain

$$\alpha_i p_i \tau^{p_i-1} = \alpha_i^2 \tau^{2p_i} f_i\left(\frac{\alpha_j \tau^{p_j}}{\alpha_i \tau^{p_i}}\right).$$

The terms balance when $p_i = -1$, $i = 1, n$, and the coefficients α_i are solutions of the system of equations

$$-1 = \alpha_i f_i\left(\frac{\alpha_j}{\alpha_i}\right), \quad i = 1, n, \quad j = 1, n. \tag{2.8}$$

To determine the resonances we set $x_i = \alpha_i \tau^{-1} + \mu_i \tau^{r-1}$ in (2.7). Then

$$\begin{aligned} -\alpha_i + (r - 1)\mu_i \tau^r &= (\alpha_i^2 + 2\alpha_i \mu_i \tau^r + \mu_i^2 \tau^{2r}) \\ &+ f_i\left(\frac{\alpha_j \tau^{-1} + \mu_j \tau^{r-1}}{\alpha_i \tau^{-1} + \mu_i \tau^{r-1}}\right), \quad i = 1, n. \end{aligned} \tag{2.9}$$

We write the argument of f_i as

$$\frac{\alpha_j}{\alpha_i} \left(\frac{1 + \mu_j/\alpha_j \tau^r}{1 + \mu_i/\alpha_i \tau^r} \right) = \frac{\alpha_j}{\alpha_i} + \frac{1}{\alpha_i^2} (\alpha_i \mu_j - \alpha_j \mu_i) \tau^r$$

to the terms linear in the μ_i . We make a formal Taylor expansion of f_i about α_j/α_i and, when we take the leading-order relationships into consideration, the resulting system is

$$(r + 1)\mu_i = \sum_{j=1}^n (\alpha_i \mu_j - \alpha_j \mu_i) f'_i\left(\frac{\alpha_j}{\alpha_i}\right). \tag{2.10}$$

For each j the prime on f_i denotes the differentiation of f_i with respect to the zeroth-order invariant x_j/x_i , $j = 1, i - 1, i + 1, n$. The requirement that -1 be a resonance implies that the matrix which is the coefficient of the vector μ on the right-hand side of (2.10) has zero determinant.

3. Examples

3.1. Third-order member of the Riccati Hierarchy

The third-order member of the Riccati Hierarchy is given by Euler et al. [8] as

$$y''' + 4yy'' + 3y'^2 + 6y^2y' + y^4 = 0 \tag{3.1}$$

and is rendered as an obvious member of the class of equations under consideration when it is written as

$$\frac{y'''}{y^4} + 4\frac{y''}{y^3} + 3\left(\frac{y'}{y^2}\right)^2 + 6\frac{y'}{y^2} + 1 = 0. \tag{3.2}$$

Under the Riccati transformation $y = \alpha w' / w$ (3.1) becomes

$$w^3w'''' + 4(\alpha - 1)w^2w'w'' + 3(\alpha - 1)w^2w''^2 + 6(\alpha - 1)(\alpha - 2)ww'^2w'' + (\alpha - 1)(\alpha - 2)(\alpha - 3)w'^4 = 0. \tag{3.3}$$

The autonomy of (3.1) is clearly preserved by the Riccati transformation to the fourth-order equation, (3.3). We use the standard transformation $\exp[u] = w$ and $v = w'$ to combine the reduction and the Euler transformation. This gives

$$v^3v'' + (4\alpha - 7)v^3v'' + (6\alpha^2 - 22\alpha + 18)v^3v' + 4v^2v'v'' + (7\alpha - 11)v^2v'^2 + vv'^3 + (\alpha - 1)(\alpha - 2)(\alpha - 3)v^4 = 0. \tag{3.4}$$

From (3.2) it is evident that

$$f(\xi) = -4\xi_2 - 3\xi_1^2 - 6\xi_1 - 1. \tag{3.5}$$

Hence the coefficient of the leading-order term is given by the solution of

$$(-1)^{2+1} \frac{(2+1)!}{\alpha^{2+1}} = -4\left(\frac{(-1)^2 2!}{\alpha^2}\right) - 3\left(\frac{(-1)^1 1!}{\alpha}\right)^2 - 6\left(\frac{(-1)^1 1!}{\alpha}\right) - 1$$

which reduces to

$$\alpha^3 - 6\alpha^2 + 11\alpha - 6 = 0, \quad \alpha = 1, 2, 3.$$

The equation for the resonances, (2.4), becomes

$$\frac{1}{\alpha^4} [(r - 1)(r - 2)(r - 3) + 24] = \frac{1}{\alpha^2} (-6\xi_1 - 6)(r - 1 + 2) + \frac{1}{\alpha^3} (-4)[(r - 1)(r - 2) - 6],$$

where $\xi_1 = -1/\alpha$. We rearrange this as

$$\frac{1}{\alpha^4} (r^3 - 6r^2 + 11r + 18) = -\frac{6}{\alpha^3} = (1 + \alpha)(r + 1) - \frac{4}{\alpha^3} (r^2 - 3r - 4)$$

so that a common factor of $r + 1$ may be removed to give

$$r^2 - 7r + 18 = -6\alpha(\alpha + 1) - 4\alpha(r - 4)$$

as the equation for the nongeneric resonances. The solutions for the various values of α are

$$\begin{aligned} \alpha = 1: & \quad r = 1, 2, \\ \alpha = 2: & \quad r = -2, 1, \\ \alpha = 3: & \quad r = -3, -2. \end{aligned} \tag{3.6}$$

For these values of α (3.4) becomes

$$\begin{aligned} v^3 v''' - 3v^3 v'' + 2v^3 v' + 4v^2 v' v'' - 4v^2 v'^2 + v v'^3 &= 0, \\ v^3 v''' + v^3 v'' - 2v^3 v' + 4v^2 v' v'' - 3v^2 v'^2 + v v'^3 &= 0, \\ v^3 v''' + 5v^3 v'' + 6v^3 v' + 4v^2 v' v'' + 10v^2 v'^2 + v v'^3 &= 0 \end{aligned}$$

for $\alpha = 1, 2$ and 3 , respectively.

Remark 1. One observes that the coefficient of $v^3 v''$ is the negative of the sum of the two nongeneric resonances and that of $v^3 v'$ is their product.

Remark 2. If one goes to the fourth-order member of the Riccati hierarchy, a pattern similar to the above is observed for the coefficients of terms linear in the derivatives of v , i.e., the coefficient of v'''' is $-\sum r_i$, that of v''' is $\sum \sum_{i \neq j} r_i r_j$ and of v' is $\prod r_i$. The feature persists, *mutatis mutandis*, for the higher-order members of the hierarchy.

3.2. The second nonlinear higher-order equation

As a second example we consider the equation

$$\ddot{x} + ax\dot{x} + b\dot{x}^2 + cx^2\dot{x} + dx^4 = 0 \tag{3.7}$$

which has the same structure as (3.2) but without the coefficients peculiar to the third-order representative of the Riccati hierarchy.

The leading-order term is $\alpha\tau^{-1}$, where

$$d\alpha^3 - c\alpha^2 + (2a + b)\alpha - 6 = 0, \tag{3.8}$$

and the nongeneric resonances are the solutions of

$$r^2 - (7 - a\alpha)r + 18 - 2(2a + b)\alpha + c\alpha^2 = 0, \tag{3.9}$$

i.e.,

$$r_1 + r_2 = 7 - a\alpha \quad \text{and} \quad r_1 r_2 = 18 - 2(2a + b)\alpha + c\alpha^2. \tag{3.10}$$

Under the Riccati transformation $x = \alpha\dot{w}/w$ (3.7) becomes

$$w^3 \ddot{w} + (a\alpha - 4)w^2 \dot{w}\ddot{w} + (b\alpha - 3)w^2 \dot{w}^2 + [12 - (3a + 2b)\alpha + c\alpha^2]w\dot{w}^2 \ddot{w} = 0 \tag{3.11}$$

when we take as α one of the solutions of (3.8). Since (3.11) is autonomous and homogeneous, we reduce the order to an autonomous third-order differential equation by the standard transformation $u = \log w$ and $v = \dot{w}$. We obtain

$$v^3 \{v''' - (7 - a\alpha)v'' + [18 - 2(2a + b)\alpha + c\alpha^2]v'\} + 4v^2v'v'' - [11 - (a + b)\alpha]v^2v'^2 + vv'^3 = 0. \tag{3.12}$$

In the case of this more general equation we see that again the coefficient of v'' is $-(r_1 + r_2)$ and that of v' is r_1r_2 . We recall that this connection is independent of the possession by (3.7) of the Painlevé Property.

The equation

$$\ddot{x} + x\ddot{x} + \dot{x}^2 = 0 \tag{3.13}$$

is a member of the class of generalised Chazy equations and is obtained from (3.7) by the choice of the parameters $a = b = 1$ and $c = d = 0$. Equation (3.8) reduces to a linear equation with the single root $\alpha = 2$. The nongeneric resonances are 2 and 3. The equation corresponding to (3.12) is

$$v^3(v''' - 5v'' + 6v') + 4v^2v'v'' - 7v^2v'^2 + vv'^3 = 0.$$

We note that the coefficient of v^3v'' is $-5 = -(2 + 3)$ and that of v^3v' is $6 = 2 \times 3$ as indicated in the general equation (3.13). Equation (3.13) has the attraction of being easily solved in explicit form as it is twice integrable to the Riccati equation

$$\dot{x} + \frac{1}{2}x^2 = 2K(t - t_0)$$

the solution of which is

$$x = \frac{2K^{1/3}[AAi'(z) + BBi'(z)]}{AAi(z) + BBi(z)},$$

where $z = K^{1/3}(t - t_0)$ and Ai and Bi are Airy's functions. Thus the solution of (3.13) is analytic apart from the simple poles of its denominator. There is no indication of an essential singularity as has been reported for the Chazy equation itself [11].

3.3. The ABC system

The ABC system is [5]

$$\begin{aligned} \dot{x} &= x(Cy + z), \\ \dot{y} &= y(Az + x), \\ \dot{z} &= z(Bx + y). \end{aligned} \tag{3.14}$$

In our formalism system (3.14) is

$$\begin{aligned} \dot{x} &= x^2 \left(C \frac{y}{x} + \frac{z}{x} \right), \\ \dot{y} &= y^2 \left(A \frac{z}{y} + \frac{x}{y} \right), \\ \dot{z} &= z^2 \left(B \frac{x}{z} + \frac{y}{z} \right), \end{aligned} \tag{3.15}$$

and the functions $f_i, i = 1, 3$, are given by the terms in parenthesis.

The coefficients of the leading-order terms are solutions of

$$\begin{pmatrix} 1 & 0 & A \\ B & 1 & 0 \\ 0 & C & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} \tag{3.16}$$

where we have replaced $\alpha_i, i = 1, n$, with α, β and γ .

For $ABC \neq -1$

$$\begin{aligned} \alpha &= [A(1 - C) - 1]/[ABC + 1], \\ \beta &= [B(1 - A) - 1]/[ABC + 1], \\ \gamma &= [C(1 - B) - 1]/[ABC + 1]. \end{aligned} \tag{3.17}$$

If $ABC = -1$, we have a one-parameter solution

$$\begin{aligned} \alpha &= -1 - As, \\ \beta &= B - 1 + ABs, \\ \gamma &= s, \end{aligned} \tag{3.18}$$

subject to the constraint $C(1 - B) = 1$.

From (2.10) the resonances are the solutions of

$$(r + 1) \begin{pmatrix} \mu \\ v \\ \lambda \end{pmatrix} = \begin{pmatrix} -\beta C - \gamma & \alpha C & \alpha \\ \beta & -\gamma A - \alpha & \beta A \\ \gamma B & \gamma & -\alpha B - \beta \end{pmatrix} \begin{pmatrix} \mu \\ v \\ \lambda \end{pmatrix}$$

and, when we use (3.16), this becomes

$$\begin{pmatrix} 1 & \alpha C & \alpha \\ \beta & 1 & \beta A \\ \gamma B & \gamma & 1 \end{pmatrix} \begin{pmatrix} \mu \\ v \\ \lambda \end{pmatrix} = (r + 1) \begin{pmatrix} \mu \\ v \\ \lambda \end{pmatrix}. \tag{3.19}$$

The determinant of the coefficient matrix on the left-hand side is zero.

More conventionally we can write (3.19) as the eigenvalue equation

$$\begin{pmatrix} 0 & \alpha C & \alpha \\ \beta & 0 & \beta A \\ \gamma B & \gamma & 0 \end{pmatrix} \begin{pmatrix} \mu \\ v \\ \lambda \end{pmatrix} = r \begin{pmatrix} \mu \\ v \\ \lambda \end{pmatrix}$$

in which we recognise that the coefficient matrix is the product

$$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} 0 & C & 1 \\ 1 & 0 & A \\ B & 1 & 0 \end{pmatrix},$$

where the second matrix is the matrix of the coefficients in (3.14). The nongeneric resonances are given by the solutions of $r^2 - r - \alpha\beta\gamma(1 + ABC) = 0$ so that

$$r_1 + r_2 = 1, \quad r_1 r_2 = -\alpha\beta\gamma(1 + ABC). \tag{3.20}$$

We note that the degenerate case $ABC = -1$ gives $r_1 = 0$ and $r_2 = 1$ and so then the ABC system passes the Painlevé Test [15]. In general the system does not pass the Painlevé Test.

As a particular instance we consider the system $A = 1, B = 2, C = -2/5$, namely

$$\begin{aligned} \dot{x} &= -\frac{2}{5}xy + xz, \\ \dot{y} &= yz + yx, \\ \dot{z} &= 2zx + zy, \end{aligned} \tag{3.21}$$

which is known [5] to possess the two first integrals

$$\begin{aligned} I_1 &= \frac{x^5 z^2}{y^6} \left[xz + \frac{3}{5}y(y - z) \right], \\ I_2 &= \frac{x + \frac{2}{5}y}{y^2} \left[xz + \frac{3}{5}y(y - z) \right] \end{aligned}$$

and to be integrable.

The coefficients of the leading-order terms are $\alpha = 2, \beta = -5$ and $\gamma = -3$. The nongeneric resonances are the solutions of (3.20), i.e., $r_1 = -2$ and $r_2 = 3$.

Conventional wisdom has it that one cannot pass the Painlevé Test with a negative and a positive nongeneric resonance. However, recently [2] we demonstrated by means of explicit construction that this be possible. The series represented the Laurent expansion of the solution on an annulus defined by two singularities when the expansion was made about a third singularity, here t_0 , where $\tau = t - t_0$. Consequently we are encouraged to seek a similar solution in this case. The determination of the coefficients of the series by means of direct substitution into (3.21) is impossible. We revert to a method developed by Feix et al. [9] to find the next to leading-order behaviour and subsequent terms. Normally this would be in the context of ascending or descending terms depending whether the expansion is in the neighborhood of the singularity or far from the singularity respectively. In an annulus the exponent determined by the leading-order analysis dominates both increasing and decreasing exponents. We illustrate the procedure.

We substitute

$$\begin{aligned} x &= 2\tau^{-1} + f_1(\tau), \\ y &= -5\tau^{-1} + g_1(\tau), \\ z &= -3\tau^{-1} + h_1(\tau), \end{aligned} \tag{3.22}$$

where $|\tau f_1(\tau)| < 1, |\tau g_1(\tau)| < 1$ and $|\tau h_1(\tau)| < 1$ in the annulus, into (3.21) and keep only the linear terms in f, g and h to obtain

$$\begin{aligned} \tau \dot{f}_1 &= -f_1 - \frac{4}{5}g_1 + 2h_1, \\ \tau \dot{g}_1 &= -5f_1 - g_1 - 5h_1, \\ \tau \dot{h}_1 &= -6f_1 - 3g_1 - h_1. \end{aligned}$$

This is a system of an Euler type and we make the standard substitution $\mathbf{F}_1 = \tau^s \mathbf{u}$, where \mathbf{u} is a constant vector and $\mathbf{F}_1 = (f_1, g_1, h_1)^T$.

The eigenvalues are found from

$$\begin{vmatrix} s + 1 & 4/5 & -2 \\ 5 & s + 1 & 5 \\ 6 & 3 & s + 1 \end{vmatrix} = 0 \Rightarrow s = -2, -3, 2.$$

The eigenvector for $s = -2$ corresponds to the location of the singularity at t_0 and is ignored. We obtain the solution

$$\mathbf{F}_1 = c_1 \begin{pmatrix} -11 \\ 40 \\ 27 \end{pmatrix} \tau^{-3} + c_2 \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix} \tau^2.$$

We may continue the process by writing

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -5 \\ -3 \end{pmatrix} \tau^{-1} + \mathbf{F}_1 + \mathbf{F}_2,$$

where $\mathbf{F}_2 = (f_2, g_2, h_2)^T$.

The equation for \mathbf{F}_2 is

$$\begin{aligned} \tau \dot{\mathbf{F}}_2 &= \begin{pmatrix} -1 & -4/5 & 2 \\ -5 & -1 & -5 \\ -6 & -3 & -1 \end{pmatrix} \mathbf{F}_2 + \begin{pmatrix} -121 \\ 640 \\ 486 \end{pmatrix} c_1^2 \tau^{-5} + \begin{pmatrix} -11 \\ 40 \\ -9 \end{pmatrix} c_1 c_2 \\ &+ \begin{pmatrix} 6 \\ -15 \\ -1 \end{pmatrix} c_2^2 \tau^5. \end{aligned} \tag{3.23}$$

The complementary function for \mathbf{F}_2 is the same as for \mathbf{F}_1 and is not repeated. We need only use the particular solution corresponding to the three vectors in (3.23). After not a little calculation we obtain

$$\mathbf{F}_2 = -\frac{1}{6} \begin{pmatrix} -39 \\ 2390 \\ 1695 \end{pmatrix} c_1^2 \tau^{-5} + \frac{1}{3} \begin{pmatrix} 43 \\ 95 \\ 0 \end{pmatrix} c_1 c_2 + \frac{1}{21} \begin{pmatrix} 34 \\ -85 \\ 5 \end{pmatrix} c_2^2 \tau^5.$$

The process of successive solution may be continued indefinitely. At each succession a new highest and a new lowest exponent are added. However, part of the particular solution of Eq. (3.23) changes the intermediate terms so that their values are not really established until the doubly infinite series is complete. As a practical procedure this could be regarded as unsatisfactory, but the alternative, which is the direct substitution of an infinite series into a system of nonlinear differential equations, is even less viable. The important feature of the procedure is that it demonstrates the existence of a Laurent series when we have both positive and negative resonances and that the two additional required arbitrary constants enter at the resonances.

We now consider the system (3.21) in terms of its potential for linearisation. We perform the Riccati transformation

$$x = 2 \frac{\dot{u}}{u}, \quad y = -5 \frac{\dot{v}}{v}, \quad z = -3 \frac{\dot{w}}{w}$$

to obtain

$$\begin{aligned}
 \frac{\ddot{u}}{\dot{u}} &= \frac{\dot{u}}{u} + 2\frac{\dot{v}}{v} - 3\frac{\dot{w}}{w}, \\
 \frac{\ddot{v}}{\dot{v}} &= 2\frac{\dot{u}}{u} + \frac{\dot{v}}{v} - 3\frac{\dot{w}}{w}, \\
 \frac{\ddot{w}}{\dot{w}} &= 4\frac{\dot{u}}{u} - 5\frac{\dot{v}}{v} + \frac{\dot{w}}{w}.
 \end{aligned}
 \tag{3.24}$$

We solve the first of (3.24) for \dot{w}/w to get

$$\frac{\dot{w}}{w} = -\frac{1}{3}\left(\frac{\ddot{u}}{\dot{u}} - \frac{\dot{u}}{u} - 2\frac{\dot{v}}{v}\right)
 \tag{3.25}$$

and substitute to the second of (3.24) to obtain

$$\frac{\ddot{u}}{\dot{u}} + \frac{\dot{u}}{u} = \frac{\ddot{v}}{\dot{v}} + \frac{\dot{v}}{v} \Rightarrow v^2 = M^2u^2 + N^2.
 \tag{3.26}$$

It then follows that, when (3.25), (3.26) are used appropriately and we set $Mu = Nf$, the third of (3.24) becomes

$$\frac{\ddot{f}}{\dot{f}} - \frac{\dot{f}^2}{f^2} - f\ddot{f}\left(\frac{5}{f^2} - \frac{3}{f^2+1}\right) + \dot{f}^2\left(\frac{5}{f^2} - \frac{5}{f^2+1} + \frac{6}{(f^2+1)^2}\right) = 0.$$

Reducing the order by making a suitable variant of the standard transformation, i.e., $f^2 = x$, $\dot{f} = g(x)$, we finally obtain the linear equation

$$4x^2(x+1)^2g'' - 2x(x+1)(x+4)g' + (11x+5)g = 0.
 \tag{3.27}$$

4. Conclusion

In this paper we developed general formulae for the determination of the coefficient(s) of the leading-order term(s) and the resonances of the singularity analysis for scalar higher-order equations and systems of first-order equations possessing the two symmetries of invariance under time-translation and self-similarity. We applied these results for some specific examples. In the case of scalar higher-order equations we showed that the parameters of the singularity analysis have an intimate connection with the coefficients of the equation resulting after a process of increase of order and decrease of order which has been demonstrated in the past to be very advantageous for the successful analysis of nonlinear ordinary differential equations. We concentrated on this class of equations because of the close connection of the presence of these two symmetries and the dominant terms of the singularity analysis.

The first example of a higher-order equation was chosen from the Riccati hierarchy and was particularly simple because that hierarchy has exceptionally attractive properties. The second higher-order example does not in general possess the Painlevé Property and is generically nonintegrable. Nevertheless the parameters obtained in applying the singularity analysis continued to appear in the transformed equation in the same way as if it were integrable.

In a previous paper [2] we demonstrated the explicit existence of a solution of a class of nonlinear equations, the Riccati hierarchy introduced in [8], for which we could write

Laurent expansions of three types. The first, the Right Painlevé Series corresponding to positive resonances, is the expansion valid in a punctured disc centered on the singularity, the second is valid in an annulus also centered on the singularity and the third, the Left Painlevé Series corresponding to negative resonances, is valid on the exterior of the largest disc containing singularities of the solutions. Unlike the Right Painlevé Series and the Left Painlevé Series, which are half series, the series valid on the annulus¹ is a normal Laurent series.

We know from other considerations that the *ABC* system considered in this paper with the special values of the parameters is integrable. Unlike the case of the members of the Riccati hierarchy we find just the one possible leading-order behaviour and set of resonances. As the two integral nongeneric resonances are of opposite sign, we proposed the possibility of the existence of a Laurent series valid over an annulus surrounding a singularity. By means of repeated application of the so-called method of the determination of the leading-order behaviour we have demonstrated a method to construct the series as a succession of approximations with each approximation adding a higher and lower power to the solution. This is not a practical way to construct a series expansion for the solution valid in an annulus. That was not our intention. What we have is a process whereby the series can be seen to exist when there are positive and negative nongeneric resonances even though we do not have the solution and other possible patterns for the resonances are not found. There may be some merit in reexamining other equations and systems for which nongeneric resonances of mixed sign are found.

In our treatment of the *ABC* system we demonstrated the linearisation of the system in the integrable case for which $A = 1$, $B = 2$ and $C = -2/5$. For the general *ABC* system (3.14) we may use a similar Riccati transformation

$$x = \alpha \frac{\dot{u}}{u}, \quad y = \beta \frac{\dot{v}}{v}, \quad z = \gamma \frac{\dot{w}}{w}$$

to arrive at the system of second-order differential equations

$$\begin{pmatrix} \ddot{u}/\dot{u} \\ \ddot{v}/\dot{v} \\ \ddot{w}/\dot{w} \end{pmatrix} = \begin{pmatrix} 1 & C\beta & \gamma \\ \alpha & 1 & A\gamma \\ B\alpha & \beta & 1 \end{pmatrix} \begin{pmatrix} \dot{u}/u \\ \dot{v}/v \\ \dot{w}/w \end{pmatrix}. \tag{4.1}$$

We note that the coefficient matrix in (4.1) has zero determinant when α , β and γ are the leading-order coefficients of the *ABC* system and the eigenvalue equation for the resonances is just

$$\begin{pmatrix} 1-r & C\beta & \gamma \\ \alpha & 1-r & A\gamma \\ B\alpha & \beta & 1-r \end{pmatrix} \begin{pmatrix} \mu/\alpha \\ \nu/\beta \\ \lambda/\gamma \end{pmatrix} = 0,$$

i.e., $r = 0$ is one resonance which reflects the homogeneity of system (4.1) in the dependent variables.

¹ There may be more than one annulus. Each annulus has its own peculiar Laurent series. As one moves outwards through a sequence of annuli, the number of negative resonances increases and the number of positive resonances decreases.

We recall that our technique to linearise the system as a single third-order differential equation relied upon the elimination of \dot{w}/w from (4.1a) and (4.1b) and then the performance of a double integration. When we eliminate \dot{w}/w from the general system (4.1), we obtain

$$\frac{\ddot{v}}{\dot{v}} + (CA\beta - 1)\frac{\dot{v}}{v} = A\left(\frac{\ddot{u}}{\dot{u}} + (\alpha - 1)\frac{\dot{u}}{u}\right)$$

which can be integrated immediately to give

$$\dot{v}^{CA\beta-1} = M(\dot{u}u^{\alpha-1})^A.$$

However, the second quadrature is possible only in the case that $A = 1$.² The ABC system does make simplification tough!

When we impose the single restriction that $A = 1$, we may again obtain a linearisation of the system by using the same procedure as that at the end of Section 3.3. The linearised equation is

$$\alpha^2 x^2 (x+1)^2 g'' - \alpha x^2 (x+1)(x+1-\alpha-\beta)g' + [(\beta\gamma - \alpha^2)x - \beta]g = 0, \quad (4.2)$$

where now $x = Mu^\alpha$. Thus we see that the linearising procedure applies to situations in which the original system does not possess the Painlevé Property.

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² Obviously a similar procedure is possible by cyclic permutation if one of B or C be unity.

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