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Lipschitz stability in determining density and two Lamé coefficients

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Abstract

We consider an inverse problem of determining spatially varying density and two Lamé coefficients in a non-stationary isotropic elastic equation by a single measurement of data on the whole lateral boundary. We prove the Lipschitz stability provided that initial data are suitably chosen. The proof is based on a Carleman estimate which can be obtained by the decomposition of the Lamé system into the rotation and the divergence components.

Keywords: Carleman estimate; Lamé system; Inverse problem

1. Introduction and the main result

We consider the three-dimensional isotropic non-stationary Lamé system:

$$\rho(x)\partial_t^2\mathbf{u}(x,t) - (L_{\lambda,\mu}\mathbf{u})(x,t) = \mathbf{f}(x,t), \quad (x,t) \in Q \equiv \Omega \times (-T,T), \tag{1.1}$$

where

$$\begin{aligned}(L_{\lambda,\mu}\mathbf{v})(x) \equiv{}& \mu(x)\Delta\mathbf{v}(x) + \big(\mu(x)+\lambda(x)\big)\nabla \operatorname{div}\mathbf{v}(x)\\ &+ \big(\operatorname{div}\mathbf{v}(x)\big)\nabla\lambda(x) + \big(\nabla\mathbf{v} + (\nabla\mathbf{v})^T\big)\nabla\mu(x), \quad x \in \Omega\end{aligned} \tag{1.2}$$

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(e.g., Gurtin [12]). Throughout this paper, $\Omega \subset \mathbb{R}^3$ is a bounded domain whose boundary $\partial\Omega$ is sufficiently smooth, t and $x = (x_1, x_2, x_3)$ denote the time variable and the spatial variable respectively, and $\mathbf{u} = (u_1, u_2, u_3)^T$, where $\cdot^T$ denotes the transpose of matrices,

$$\partial_j \phi = \frac{\partial \phi}{\partial x_j}, \quad j = 1, 2, 3, \qquad \partial_t \phi = \frac{\partial \phi}{\partial t}.$$

For $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \{\mathbb{N} \cup \{0\}\}^3$, we set $\partial_x^\alpha = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \partial_3^{\alpha_3}$ and $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$, and $\partial_{x,t}^\alpha$ is similarly defined. We set $\nabla \mathbf{v} = (\partial_k v_j)_{1 \leqslant j,k \leqslant 3}$, $\nabla_{x,t} \mathbf{v} = (\nabla \mathbf{v}, \partial_t \mathbf{v})$ for a vector function $\mathbf{v} = (v_1, v_2, v_3)^T$. Moreover, the coefficients ρ, λ, μ under consideration, satisfy

$$\rho, \lambda, \mu \in C^2(\overline{\Omega}), \quad \rho(x) > 0,\ \mu(x) > 0,\ \lambda(x) + \mu(x) > 0 \text{ for } x \in \overline{\Omega}. \tag{1.3}$$

Let $\mathbf{u} = \mathbf{u}(\lambda, \mu, \rho; \mathbf{p}, \mathbf{q})(x, t)$ be sufficiently smooth and satisfy

$$\rho(x)\big(\partial_t^2 \mathbf{u}\big)(x, t) = (L_{\lambda,\mu} \mathbf{u})(x, t), \quad (x, t) \in Q, \tag{1.4}$$

$$\mathbf{u}(x, 0) = \mathbf{p}(x), \quad (\partial_t \mathbf{u})(x, 0) = \mathbf{q}(x), \quad x \in \Omega. \tag{1.5}$$

We consider

Inverse problem with a finite number of measurements

Let $\omega \subset \Omega$ be a suitable subdomain and let $\mathbf{p}_j$, $\mathbf{q}_j$, $1 \leqslant j \leqslant \mathcal{N}$, be appropriately given. Then determine $\lambda(x)$, $\mu(x)$, $\rho(x)$, $x \in \Omega$, by

$$\mathbf{u}(\lambda, \mu, \rho; \mathbf{p}_j, \mathbf{q}_j)|_{\omega \times (-T,T)}. \tag{1.6}$$

As for the inverse problem of determining some (or all) of λ, μ and ρ with a finite number of measurements, we can first refer to:

Isakov [26] where the author proved the uniqueness in determining a single coefficient $\rho(x)$, using four measurements (i.e., $\mathcal{N} = 4$).

Ikehata, Nakamura and Yamamoto [14] which reduced the number $\mathcal{N}$ of measurements to three for determining ρ.

Imanuvilov, Isakov and Yamamoto [16] which proved conditional stability and the uniqueness in the determination of the three functions $\lambda(x)$, $\mu(x)$, $\rho(x)$, $x \in \Omega$, with two measurements (i.e., $\mathcal{N} = 2$). See also Isakov [30].

Imanuvilov and Yamamoto [23–25] which reduced $\mathcal{N} = 2$ to $\mathcal{N} = 1$ (i.e., a single measurement) in determining all of λ, μ, ρ by a single measurement $\mathbf{u}|_{\omega \times (-T,T)}$, and established conditional stability of Hölder type by means of an H^{-1}-Carleman estimate. See also [21].

As for similar inverse problems for the Lamé system with residual stress, see Isakov, Wang and Yamamoto [31], Lin and Wang [44].

Our method is based on a method by Carleman estimates, which was originally introduced in the field of coefficient inverse problems by Bukhgeim and Klibanov [8] simultaneously and independently on each other for the proofs of global uniqueness and stability theorems for these problems. Also see Klibanov [36]. In particular, for the Lamé system, we use a modification of the method in [8] by Imanuvilov and Yamamoto [23]. In [23], only a Hölder stability estimate is proved, but by the ideas in Klibanov and Timonov [40], Klibanov and Yamamoto [41], we can prove the Lipschitz stability for our inverse problem with $\mathcal{N} = 1$. For a related technique, see Chapter 3.5 in Klibanov and Timonov [39]. In [16,23], an H^{-1}-Carleman estimate is a key but requires more technical details. Here we will use a Carleman estimate for the Lamé system which is derived from a usual L^2-Carleman estimate for a scalar hyperbolic equation.

Thus the advantages of this paper are:

(1) the Lipschitz stability in our inverse problem with $\mathcal{N} = 1$,
(2) use of a conventional Carleman estimate.

On the other hand, for (2) we have to choose a neighborhood ω of $\partial\Omega$, although it is sufficient that ω is a neighborhood of a sufficiently large subboundary [23–25]. Then, $\mathbf{u}(\lambda, \mu, \rho; \mathbf{p}, \mathbf{q})(\cdot, t)$, $t \in (-T, T)$, is given in a neighborhood of $\partial\Omega$, so that we do not directly assign boundary values but the observation data in $\omega \times (-T, T)$ include information of boundary values.

For the statement of the main result, we introduce notations and an admissible set of unknown coefficients λ, μ, ρ. Set

$$d = \Big(\sup_{x\in\Omega} |x - x_0|^2 - \inf_{x\in\Omega} |x - x_0|^2\Big)^{\frac{1}{2}}, \tag{1.7}$$

where $x_0 \notin \overline{\Omega}$ is arbitrarily fixed. Let $M_0 \geqslant 0$, $0 < \theta_0 \leqslant 1$ and $\theta_1 > 0$ be arbitrarily fixed and let us introduce the conditions on a scalar function β:

$$\begin{cases} \beta(x) \geqslant \theta_1 > 0, & x \in \overline{\Omega}, \\ \|\beta\|_{C^3(\overline{\Omega})} \leqslant M_0, & \dfrac{(\nabla\beta(x) \cdot (x - x_0))}{2\beta(x)} \leqslant 1 - \theta_0, \quad x \in \overline{\Omega \setminus \omega}. \end{cases} \tag{1.8}$$

For a fixed subdomain $\partial\omega_0 \supset \partial\Omega$ and fixed sufficiently smooth functions λ_0, μ_0, ρ_0 on $\overline{\Omega}$, we set

$$\begin{aligned} \mathcal{W} = \mathcal{W}_{M_0,\theta_0,\theta_1} = \bigg\{&(\lambda, \mu, \rho) \in \big\{C^3(\overline{\Omega})\big\}^3;\ \rho = \rho_0,\ \lambda = \lambda_0,\ \mu = \mu_0 \text{ in } \omega_0, \\ &\frac{\lambda + 2\mu}{\rho},\ \frac{\mu}{\rho} \text{ satisfy (1.8)}\bigg\}. \end{aligned} \tag{1.9}$$

We choose $\theta > 0$ such that

$$\theta + \frac{M_0 d}{\sqrt{\theta_1}}\sqrt{\theta} < \theta_0\theta_1, \qquad \theta_1 \inf_{x\in\Omega} |x - x_0|^2 - \theta \sup_{x\in\Omega} |x - x_0|^2 > 0. \tag{1.10}$$

Here we note that since $x_0 \notin \overline{\Omega}$, such $\theta > 0$ exists.

Let E_3 the 3×3 identity matrix. We note that $(L_{\lambda,\mu}\mathbf{p})(x)$ is a 3-column vector for 3-column vector $\mathbf{p}$. Moreover, by $\{\mathbf{a}\}_j$ we denote the matrix (or vector) obtained from $\mathbf{a}$ after deleting the jth row and $\det_j A$ means $\det\{A\}_j$ for a square matrix A. Let (λ, μ, ρ) be an arbitrary element of $\mathcal{W}$.

Now we are ready to state

Theorem. *Let $\omega \subset \Omega$ be a subdomain such that $\partial\omega \supset \partial\Omega$. For $\mathbf{p} = (p_1, p_2, p_3)^T$ and $\mathbf{q} = (q_1, q_2, q_3)^T$, we assume that there exist $j_1, j_2 \in \{1, 2, 3, 4, 5, 6\}$ such that*

$$\det_{j_1}\begin{pmatrix} (L_{\lambda,\mu}\mathbf{p})(x) & (\operatorname{div}\mathbf{p}(x))E_3 & (\nabla\mathbf{p}(x) + (\nabla\mathbf{p}(x))^T)(x - x_0) \\ (L_{\lambda,\mu}\mathbf{q})(x) & (\operatorname{div}\mathbf{q}(x))E_3 & (\nabla\mathbf{q}(x) + (\nabla\mathbf{q}(x))^T)(x - x_0) \end{pmatrix} \neq 0, \quad \forall x \in \overline{\Omega \setminus \omega_0}, \tag{1.11}$$

$$\det_{j_2}\begin{pmatrix} (L_{\lambda,\mu}\mathbf{p})(x) & \nabla\mathbf{p}(x) + (\nabla\mathbf{p}(x))^T & (\operatorname{div}\mathbf{p})(x - x_0) \\ (L_{\lambda,\mu}\mathbf{q})(x) & \nabla\mathbf{q}(x) + (\nabla\mathbf{q}(x))^T & (\operatorname{div}\mathbf{q})(x - x_0) \end{pmatrix} \neq 0, \quad \forall x \in \overline{\Omega \setminus \omega_0}, \tag{1.12}$$

and that

$$T > \frac{1}{\sqrt{\theta}} d. \tag{1.13}$$

Then, for any $M_1 > 0$, there exists a constant $C_1 = C_1(\mathcal{W}, M_1, \omega, \Omega, T, \lambda, \mu, \rho) > 0$ such that

$$\begin{aligned}
&\|\tilde{\lambda} - \lambda\|_{H^2(\Omega)} + \|\tilde{\mu} - \mu\|_{H^2(\Omega)} + \|\tilde{\rho} - \rho\|_{H^1(\Omega)} \\
&\quad \leqslant C_1 \big(\big\| \mathbf{u}(\lambda, \mu, \rho; \mathbf{p}, \mathbf{q}) - \mathbf{u}(\tilde{\lambda}, \tilde{\mu}, \tilde{\rho}; \mathbf{p}, \mathbf{q}) \big\|_{H^5(-T,T;H^2(\omega))} \\
&\qquad + \big\| \mathbf{u}(\lambda, \mu, \rho; \mathbf{p}, \mathbf{q}) - \mathbf{u}(\tilde{\lambda}, \tilde{\mu}, \tilde{\rho}; \mathbf{p}, \mathbf{q}) \big\|_{H^4(-T,T;H^{\frac{5}{2}}(\omega))} \big),
\end{aligned} \tag{1.14}$$

provided that $(\tilde{\lambda}, \tilde{\mu}, \tilde{\rho}) \in \mathcal{W}$ and

$$\big\| \mathbf{u}(\lambda, \mu, \rho; \mathbf{p}, \mathbf{q}) \big\|_{W^{7,\infty}(Q)}, \big\| \mathbf{u}(\tilde{\lambda}, \tilde{\mu}, \tilde{\rho}; \mathbf{p}, \mathbf{q}) \big\|_{W^{7,\infty}(Q)} \leqslant M_1. \tag{1.15}$$

Inequality (1.14) gives the Lipschitz stability by a single measurement in a neighborhood of the whole boundary, and after artificial choice (1.11) and (1.12) of initial values, a single measurement yields such stability. Moreover, conditions (1.11) and (1.12) depend on a fixed (λ, μ, ρ), so that in our conclusion (1.14), we cannot change both (λ, μ, ρ), $(\tilde{\lambda}, \tilde{\mu}, \tilde{\rho}) \in \mathcal{W}$.

As the following example shows, we can take such $\mathbf{p}$ and $\mathbf{q}$.

Example of p, q satisfying (1.11) and (1.12). For simplicity, we assume that λ, μ are positive constants. Noting that the fifth columns of the matrices in (1.11) and (1.12) have $x - x_0$ as factors, we will take quadratic functions in x. For example, we take

$$\mathbf{p}(x) = \begin{pmatrix} 0 \\ x_1 x_2 \\ 0 \end{pmatrix}, \qquad \mathbf{q}(x) = \begin{pmatrix} x_2^2 \\ 0 \\ x_2^2 \end{pmatrix}.$$

Then, by choosing $j_1 = 6$ and $j_2 = 5$, we can satisfy (1.11) and (1.12).

Remark. The regularity and the bounds in (1.15) can be derived by the compatibility conditions of the known λ_0, μ_0, ρ_0 and $\mathbf{p}$, $\mathbf{q}$ and the regularity of λ, μ, ρ, $\tilde{\lambda}$, $\tilde{\mu}$, $\tilde{\rho}$. For example, as $\mathbf{u}(\lambda, \mu, \rho; \mathbf{p}, \mathbf{q})$ and $\mathbf{u}(\tilde{\lambda}, \tilde{\mu}, \tilde{\rho}; \mathbf{p}, \mathbf{q})$, we can consider the solutions to the initial value/boundary value problem (1.4) and (1.5) with the homogeneous Dirichlet boundary condition $\mathbf{u}(x, t) = 0$, $x \in \partial\Omega$, $-T < t < T$. We choose a subdomain $\omega_1 \subset \omega_0$ with $\partial\omega_1 \supset \partial\Omega$ and assume

$$\mathbf{p} \in C^9(\overline{\Omega}), \quad \mathbf{p}|_{\omega_1} = 0, \quad \mathbf{q} \in C^8(\overline{\Omega}), \quad \mathbf{q}|_{\omega_1} = 0, \qquad \lambda, \mu \in C^8(\overline{\Omega}), \quad \rho \in C^7(\overline{\Omega}).$$

Then, applying the general theory for the hyperbolic system (e.g., Lions and Magenes [45]) to $\partial_t^8 \mathbf{u}(\lambda, \mu, \rho; \mathbf{p}, \mathbf{q})$, the elliptic regularity of the operator $L_{\lambda,\mu}$ and the Sobolev embedding theorem, we have

$$\partial_t^j \mathbf{u}(\lambda, \mu, \rho; \mathbf{p}, \mathbf{q}) \in C\big([-T, T]; H^{9-j}(\Omega)\big) \subset C\big([-T, T]; C^{7-j}(\overline{\Omega})\big), \quad 0 \leqslant j \leqslant 7.$$

Moreover, the constant M_1 in (1.15) is bounded by

$$\begin{aligned}
\widetilde{M_1} = \max\big\{ &\|\mathbf{p}\|_{C^9(\overline{\Omega})}, \|\mathbf{q}\|_{C^8(\overline{\Omega})}, \|\lambda\|_{C^8(\overline{\Omega})}, \|\mu\|_{C^8(\overline{\Omega})}, \\
&\|\tilde{\lambda}\|_{C^8(\overline{\Omega})}, \|\tilde{\mu}\|_{C^8(\overline{\Omega})}, \|\rho\|_{C^7(\overline{\Omega})}, \|\tilde{\rho}\|_{C^7(\overline{\Omega})} \big\}.
\end{aligned}$$

In this case, thanks to $\mathbf{p}|_{\omega_1} = \mathbf{q}|_{\omega_1} = 0$ and $\mathbf{u}(\lambda, \mu, \rho; \mathbf{p}, \mathbf{q}) = 0$ on $\partial\Omega \times (-T, T)$, we see that the compatibility conditions are automatically satisfied. As is seen from the proof of the main result in Section 2, the constant C_1 in (1.14) depends on $\widetilde{M}_1$, ω, ω_1, Ω, T, λ, μ, ρ, λ_0, μ_0, ρ_0, M_0, θ_0, θ_1.

We can give more comprehensive conditions for (1.15) by means of the theory of the forward problem such as the initial value/boundary value problem, but for concentrating on the inverse problem, throughout this paper, we assume the boundedness condition (1.15).

We conclude this section with the references to other publications concerning inverse problems by Carleman estimates after the originating paper Bukhgeim and Klibanov [8].

(1) Baudouin and Puel [2], Bukhgeim [6] for an inverse problem of determining potentials in Schrödinger equations.
(2) Imanuvilov and Yamamoto [17,20], Isakov [27,28], Klibanov [37] for the corresponding inverse problems for parabolic equations.
(3) Amirov and Yamamoto [1], Bellassoued [3,4], Bellassoued and Yamamoto [5], Bukhgeim et al. [7], Imanuvilov and Yamamoto [18,19,22] (especially for conditional stability), Isakov [27–29], Isakov and Yamamoto [32], Khaĭdarov [34,35], Klibanov [36,37], Klibanov and Timonov [39], [40], Klibanov and Yamamoto [41], Puel and Yamamoto [46,47], Yamamoto [49] for inverse problems of determining potentials, damping coefficients or the principal terms in scalar hyperbolic equations.
(4) Li [42], Li and Yamamoto [43] for Maxwell's equations.
(5) Yuan and Yamamoto [50] for plate equations.

2. Proof of theorem

We set

$$\psi(x,t) = |x - x_0|^2 - \theta t^2, \quad \varphi(x,t) = e^{\tau\psi(x,t)}, \quad (x,t) \in Q,$$

and

$$Q_\omega = \omega \times (-T, T).$$

First, in terms of (1.9) and (1.10), we can deduce the following lemma in the same way as in [23]. Henceforth C, C_j denote constants which are independent of s but dependent on Ω, ω, T and the choice of fixed λ, μ, ρ.

Lemma 2.1. *Let* $(\lambda, \mu, \rho) \in \mathcal{W}$ *and let* (1.10) *and* (1.13) *hold. There exists* $\hat{\tau} > 0$ *such that for any* $\tau > \hat{\tau}$, *we can choose* $s_0 = s_0(\tau) > 0$ *and* $C_1 = C_1(s_0, \tau_0, \Omega, \omega, T) > 0$ *such that*

$$\begin{aligned}
&\int_Q \Bigg(s^4|\mathbf{y}|^2 + s^2|\nabla_{x,t}\mathbf{y}|^2 + \sum_{|\alpha|=2} \big|\partial_x^\alpha \mathbf{y}\big|^2 + s^2\big|\nabla_{x,t}(\operatorname{rot}\mathbf{y})\big|^2 \\
&\qquad + s^4|\operatorname{rot}\mathbf{y}|^2 + s^2\big|\nabla_{x,t}(\operatorname{div}\mathbf{y})\big|^2 + s^4|\operatorname{div}\mathbf{y}|^2 \Bigg) e^{2s\varphi}\,dx\,dt \\
&\leqslant C_1 \int_Q \big(s|\operatorname{div}\mathbf{f}|^2 + s|\operatorname{rot}\mathbf{f}|^2 + s|\mathbf{f}|^2\big) e^{2s\varphi}\,dx\,dt + Ce^{Cs}\|\mathbf{y}\|^2_{H^1(-T,T;H^2(\omega))}, \\
&s \geqslant s_0, \qquad (2.1)
\end{aligned}$$

for any $\mathbf{y} \in H^3(Q)$ *such that*

$$\rho \partial_t^2 \mathbf{y} - L_{\lambda,\mu} \mathbf{y} = \mathbf{f}, \qquad \partial_t^j \mathbf{y}(\cdot, \pm T) = 0, \quad j = 0, 1.$$

The constants in (2.1) *can be taken uniformly as long as* $(\lambda, \mu, \rho) \in \mathcal{W}$.

Proof. Let us set $v = \operatorname{div} \mathbf{y}$ and $\mathbf{w} = \operatorname{rot} \mathbf{y}$. Then we have (e.g., Eller et al. [11], Imanuvilov and Yamamoto [23]):

$$\rho \partial_t^2 \mathbf{y} - \mu \Delta \mathbf{y} + Q_1(\mathbf{y}, v) = \mathbf{f} \quad \text{in } Q,$$
$$\rho \partial_t^2 v - (\lambda + 2\mu) \Delta v + Q_2(\mathbf{y}, v, \mathbf{w}) = \operatorname{div} \mathbf{f} \quad \text{in } Q$$

and

$$\rho \partial_t^2 \mathbf{w} - \mu \Delta \mathbf{w} + Q_3(\mathbf{y}, v, \mathbf{w}) = \operatorname{rot} \mathbf{f} \quad \text{in } Q,$$

where $Q_1(\mathbf{y}, v) = \sum_{|\alpha|=1} a_\alpha^{(1)}(x) \partial_x^\alpha \mathbf{y} + \sum_{|\alpha| \leqslant 1} b_\alpha^{(1)}(x) \partial_x^\alpha v$, $Q_j(\mathbf{y}, v, \mathbf{w}) = \sum_{|\alpha|=1} a_\alpha^{(j)}(x) \partial_x^\alpha \mathbf{y} + \sum_{|\alpha|=1} b_\alpha^{(j)}(x) \partial_x^\alpha v + \sum_{|\alpha|=1} c_\alpha^{(j)}(x) \partial_x^\alpha \mathbf{w}$, $j = 2, 3$, and $a_\alpha^{(j)}, b_\alpha^{(j)}, c_\alpha^{(j)} \in L^\infty(Q)$. Therefore we apply a Carleman estimate by Imanuvilov [15] to the system, so that

$$\int_Q \bigl\{ s^3 \bigl(|\operatorname{rot} \mathbf{y}|^2 + |\operatorname{div} \mathbf{y}|^2 + |\mathbf{y}|^2 \bigr) + s \bigl(\bigl| \nabla_{x,t} (\operatorname{rot} \mathbf{y}) \bigr|^2 + \bigl| \nabla_{x,t} (\operatorname{div} \mathbf{y}) \bigr|^2 + |\nabla_{x,t} \mathbf{y}|^2 \bigr) \bigr\} e^{2s\varphi} \, dx \, dt$$
$$\leqslant C \int_Q \bigl(|\operatorname{div} \mathbf{f}|^2 + |\operatorname{rot} \mathbf{f}|^2 + |\mathbf{f}|^2 \bigr) e^{2s\varphi} \, dx \, dt + C e^{Cs} \|\mathbf{y}\|_{H^1(-T,T;H^1(\omega))} \tag{2.2}$$

and

$$\int_Q \bigl(s^4 |\mathbf{y}|^2 + s^2 |\nabla_{x,t} \mathbf{y}|^2 \bigr) e^{2s\varphi} \, dx \, dt$$
$$\leqslant C \int_Q \bigl(s |\operatorname{div} \mathbf{f}|^2 + s |\operatorname{rot} \mathbf{f}|^2 + s |\mathbf{f}|^2 \bigr) e^{2s\varphi} \, dx \, dt + C e^{Cs} \|\mathbf{y}\|_{H^1(-T,T;H^1(\omega))}^2. \tag{2.3}$$

Next for all large $s > 0$, we have

$$\begin{aligned}
\Delta \bigl(\mathbf{y} e^{s\varphi} \bigr) &= \nabla \bigl(\operatorname{div} \bigl(\mathbf{y} e^{s\varphi} \bigr) \bigr) - \operatorname{rot} \bigl(\operatorname{rot} \bigl(\mathbf{y} e^{s\varphi} \bigr) \bigr) \\
&= \sum_{j=1}^3 \bigl\{ \bigl\{ \nabla \bigl(\partial_j e^{s\varphi} \bigr) \bigr\} y_j + \bigl(\partial_j e^{s\varphi} \bigr) \nabla y_j \bigr\} + s (\nabla \varphi) e^{s\varphi} \operatorname{div} \mathbf{y} + e^{s\varphi} \nabla (\operatorname{div} \mathbf{y}) \\
&\quad + (\mathbf{y} \cdot \nabla) \bigl(\nabla e^{s\varphi} \bigr) - \bigl(\bigl(\nabla e^{s\varphi} \bigr) \cdot \nabla \bigr) \mathbf{y} \\
&\quad + \bigl(\nabla e^{s\varphi} \bigr) \operatorname{div} \mathbf{y} - \mathbf{y} \operatorname{div} \bigl(\nabla e^{s\varphi} \bigr) - \bigl(\nabla e^{s\varphi} \bigr) \times \operatorname{rot} \mathbf{y} - e^{s\varphi} \operatorname{rot} (\operatorname{rot} \mathbf{y}) \\
&= e^{s\varphi} \nabla (\operatorname{div} \mathbf{y}) + O\bigl(s^2 \bigr) K_1(\mathbf{y}) e^{s\varphi} + O(s) K_2(\nabla \mathbf{y}) e^{s\varphi} - \bigl(\operatorname{rot} (\operatorname{rot} \mathbf{y}) \bigr) e^{s\varphi},
\end{aligned}$$

where K_1, K_2 are linear operators. Therefore

$$\bigl| \Delta \bigl(\mathbf{y} e^{s\varphi} \bigr) \bigr| \leqslant C e^{s\varphi} \bigl\{ s^2 |\mathbf{y}| + s |\nabla \mathbf{y}| + \bigl| \nabla (\operatorname{div} \mathbf{y}) \bigr| + \bigl| \nabla (\operatorname{rot} \mathbf{y}) \bigr| \bigr\},$$

so that

$$\int_\Omega \left|\Delta\big(\mathbf{y}(x,t)e^{s\varphi(x,t)}\big)\right|^2 dx$$
$$\leqslant C\int_\Omega \big(s^4|\mathbf{y}|^2 + s^2|\nabla\mathbf{y}|^2 + \left|\nabla(\operatorname{div}\mathbf{y})\right|^2 + \left|\nabla(\operatorname{rot}\mathbf{y})\right|^2\big)e^{2s\varphi}\,dx \tag{2.4}$$

for any $t\in[-T,T]$. The elliptic regularity and (2.4) yield

$$\sum_{|\alpha|=2}\int_\Omega \left|\partial_x^\alpha\big(\mathbf{y}(x,t)e^{s\varphi(x,t)}\big)\right|^2 dx$$
$$\leqslant C\int_\Omega \big(\left|\Delta\big(\mathbf{y}e^{s\varphi}\big)\right|^2 + \left|\mathbf{y}e^{s\varphi}\right|^2\big)\,dx + C\left\|\mathbf{y}e^{s\varphi}\right\|^2_{H^{\frac{3}{2}}(\partial\Omega)}$$
$$\leqslant C\int_\Omega \big(s^4|\mathbf{y}|^2 + s^2|\nabla\mathbf{y}|^2 + \left|\nabla(\operatorname{div}\mathbf{y})\right|^2 + \left|\nabla(\operatorname{rot}\mathbf{y})\right|^2\big)e^{2s\varphi}\,dx + C\left\|\mathbf{y}e^{s\varphi}\right\|^2_{H^2(\omega)}.$$

Therefore

$$\sum_{|\alpha|=2}\int_Q \left|\big(\partial_x^\alpha\mathbf{y}\big)(x,t)e^{s\varphi(x,t)}\right|^2 dx\,dt - Cs^2\sum_{j=1}^3\int_Q |\partial_j\mathbf{y}|^2e^{2s\varphi}\,dx\,dt - Cs^4\int_Q |\mathbf{y}|^2e^{2s\varphi}\,dx\,dt$$
$$\leqslant C\int_Q \big(s^4|\mathbf{y}|^2 + s^2|\nabla\mathbf{y}|^2 + \left|\nabla(\operatorname{div}\mathbf{y})\right|^2 + \left|\nabla(\operatorname{rot}\mathbf{y})\right|^2\big)e^{2s\varphi}\,dx\,dt$$
$$+ Ce^{Cs}\|\mathbf{y}\|^2_{L^2(-T,T;H^2(\omega))}. \tag{2.5}$$

Thus, in terms of (2.2), (2.3) and (2.5), we have

$$\int_Q \Bigg(s^4|\mathbf{y}|^2 + s^2|\nabla_{x,t}\mathbf{y}|^2 + \sum_{|\alpha|=2}\left|\partial_x^\alpha\mathbf{y}\right|^2 + s^2\left|\nabla_{x,t}(\operatorname{rot}\mathbf{y})\right|^2$$
$$+ s^4|\operatorname{rot}\mathbf{y}|^2 + s^2\left|\nabla_{x,t}(\operatorname{div}\mathbf{y})\right|^2 + s^4|\operatorname{div}\mathbf{y}|^2\Bigg)e^{2s\varphi}\,dx\,dt$$
$$\leqslant C\int_Q \big(s|\operatorname{div}\mathbf{f}|^2 + s|\operatorname{rot}\mathbf{f}|^2 + s|\mathbf{f}|^2\big)e^{2s\varphi}\,dx\,dt + Ce^{Cs}\|\mathbf{u}\|^2_{H^1(-T,T;H^2(\omega))}.$$

Thus the proof of Lemma 2.1 is complete. □

As for Carleman estimates, see also Hörmander [13], Triggiani and Yao [48].

Next we consider a first-order partial differential operator

$$(P_0g)(x) = B(x)\cdot\nabla g(x) + B_0(x)g(x), \quad x\in\Omega, \tag{2.6}$$

where $B=(b_1,b_2,b_3)\in\{W^{2,\infty}(\Omega)\}^3$ and $B_0\in W^{2,\infty}(\Omega)$. Then

Lemma 2.2. *We assume*

$$\left|\big(B(x)\cdot(x-x_0)\big)\right| > 0, \quad x\in\overline{\Omega}. \tag{2.7}$$

Then there exists a constant $\tau_0 > 0$ *such that for all* $\tau > \tau_0$, *there exist* $s_0 = s_0(\tau) > 0$ *and* $C_2 = C_2(s_0, \tau_0, \Omega, \omega) > 0$ *such that*

$$s^2 \int_\Omega \Bigl(\sum_{|\alpha|\leqslant 2} \bigl|\partial_x^\alpha g(x)\bigr|^2 \Bigr) e^{2s\varphi(x,0)}\,dx \leqslant C_2 \int_\Omega \Bigl(\sum_{|\alpha|\leqslant 2} \bigl|\partial_x^\alpha (P_0 g)(x)\bigr|^2 \Bigr) e^{2s\varphi(x,0)}\,dx \tag{2.8}$$

for all $s > s_0$ *and* $g \in H_0^3(\Omega)$.

Proof. We set $F = P_0 g$ and $\varphi_0(x) = \varphi(x,t)$. By integration by parts, we can prove

$$s^2 \int_\Omega |g|^2 e^{2s\varphi_0}\,dx \leqslant C_2 \int_\Omega |F|^2 e^{2s\varphi_0}\,dx \tag{2.9}$$

(e.g., [23]). Since $P_0(\partial_j g) = \partial_j F - (\partial_j P_0) g$ and $\partial_j g|_{\partial\Omega} = 0$, we apply (2.9) to $\partial_j g$, so that

$$\begin{aligned} s^2 \int_\Omega |\partial_j g|^2 e^{2s\varphi_0}\,dx &\leqslant C_2 \int_\Omega \bigl(|g|^2 + |\nabla g|^2\bigr) e^{2s\varphi_0}\,dx + C_2 \int_\Omega |\partial_j F|^2 e^{2s\varphi_0}\,dx \\ &\leqslant C_2 \int_\Omega \bigl(|F|^2 + |\partial_j F|^2\bigr) e^{2s\varphi_0}\,dx + C_2 \int_\Omega |\nabla g|^2 e^{2s\varphi_0}\,dx. \end{aligned}$$

Therefore

$$s^2 \int_\Omega |\nabla g|^2 e^{2s\varphi_0}\,dx \leqslant C_2 \int_\Omega \bigl(|F|^2 + |\nabla F|^2\bigr) e^{2s\varphi_0}\,dx + C_2 \int_\Omega |\nabla g|^2 e^{2s\varphi_0}\,dx.$$

Taking $s_0 > 0$ sufficiently large, we have

$$s^2 \int_\Omega |\nabla g|^2 e^{2s\varphi_0}\,dx \leqslant C_2 \int_\Omega \bigl(|F|^2 + |\nabla F|^2\bigr) e^{2s\varphi_0}\,dx. \tag{2.10}$$

Next we have

$$P_0(\partial_k \partial_\ell g) = \partial_k \partial_\ell F - \sum_{j=1}^3 (\partial_k b_j)(\partial_\ell \partial_j g) + (\partial_\ell b_j)(\partial_k \partial_j g) + K(g, \nabla g),$$

where K is a linear operator of g and ∇g. Noting that $\partial_k \partial_\ell g = 0$ on $\partial\Omega$, we apply (2.9) to $\partial_k \partial_\ell g$, and, similarly to (2.10), we can complete the proof of Lemma 2.2. □

Finally, we show an observability inequality, which may be an independent interest.

Lemma 2.3. *Let* $(\lambda, \mu, \rho) \in \mathcal{W}$ *and let us assume* (1.13). *Let* $\mathbf{u} \in H^3(Q)$ *satisfy* $(\rho\partial_t^2 - L_{\lambda,\mu})\mathbf{u} = \mathbf{f}$. *Then there exists a constant* $C_3 > 0$ *such that*

$$\begin{aligned} &\int_\Omega \Bigl(\sum_{|\alpha|\leqslant 2} \bigl|\partial_x^\alpha \mathbf{u}(x,t)\bigr|^2 + \bigl|\partial_t \mathbf{u}(x,t)\bigr|^2 \Bigr)\,dx + \int_Q \bigl|\nabla \partial_t \mathbf{u}(x,t)\bigr|^2\,dx\,dt \\ &\quad \leqslant C_3 \int_Q \bigl(|\operatorname{div} \mathbf{f}|^2 + |\operatorname{rot} \mathbf{f}|^2 + |\mathbf{f}|^2\bigr)\,dx\,dt + C_3 \bigl(\|\mathbf{u}\|^2_{H^1(-T,T;H^2(\omega))} + \|\mathbf{u}\|^2_{L^2(-T,T;H^{\frac{5}{2}}(\omega))} \bigr) \end{aligned}$$

for all $t \in [-T, T]$.

Starting from works of Klibanov and Malinsky [38] and Kazemi and Klibanov [33], this kind of inequality is usually proved by Carleman estimate. See, e.g., Cheng et al. [9], and we will prove it in Appendix A for completeness.

Now we proceed to

Proof of theorem. The proof is similar to Imanuvilov and Yamamoto [23]. Henceforth, for simplicity, we set

$$\mathbf{u} = \mathbf{u}(\lambda, \mu, \rho; \mathbf{p}, \mathbf{q}), \qquad \mathbf{v} = \mathbf{u}(\tilde{\lambda}, \tilde{\mu}, \tilde{\rho}; \mathbf{p}, \mathbf{q}) \tag{2.11}$$

and

$$\mathbf{y} = \mathbf{u} - \mathbf{v}, \qquad f = \rho - \tilde{\rho}, \qquad g = \lambda - \tilde{\lambda}, \qquad h = \mu - \tilde{\mu}. \tag{2.12}$$

Then

$$\tilde{\rho}\partial_t^2 \mathbf{y} = L_{\tilde{\lambda},\tilde{\mu}} \mathbf{y} + G\mathbf{u} \quad \text{in } Q \tag{2.13}$$

and

$$\mathbf{y}(x,0) = \partial_t \mathbf{y}(x,0) = 0, \quad x \in \Omega. \tag{2.14}$$

Here we set

$$\begin{aligned} G\mathbf{u}(x,t) = {} & -f(x)\partial_t^2 \mathbf{u}(x,t) + (g+h)(x)\nabla(\operatorname{div}\mathbf{u})(x,t) + h(x)\Delta\mathbf{u}(x,t) \\ & + (\operatorname{div}\mathbf{u})(x,t)\nabla g(x) + \big(\nabla\mathbf{u}(x,t) + \big(\nabla\mathbf{u}(x,t)\big)^T\big)\nabla h(x). \end{aligned} \tag{2.15}$$

By (1.13), we have the inequality $\theta T^2 > d^2$. Therefore, by the definition of d and the definition of the function φ, we have

$$\varphi(x,0) \geqslant d_1, \quad \varphi(x,T) = \varphi(x,-T) < d_1, \quad x \in \overline{\Omega},$$

with $d_1 = \exp(\tau \inf_{x\in\Omega} |x - x_0|^2)$. Thus, for given $\varepsilon > 0$, we can choose a sufficiently small $\delta = \delta(\varepsilon) > 0$ such that

$$\varphi(x,t) \geqslant d_1 - \varepsilon, \quad (x,t) \in \overline{\Omega} \times [-\delta, \delta], \tag{2.16}$$

and

$$\varphi(x,t) \leqslant d_1 - 2\varepsilon, \quad x \in \overline{\Omega},\ t \in [-T, -T+2\delta] \cup [T-2\delta, T]. \tag{2.17}$$

In order to apply Lemma 2.1, it is necessary to introduce a cut-off function χ satisfying $0 \leqslant \chi \leqslant 1$, $\chi \in C^\infty(\mathbb{R})$ and

$$\chi = \begin{cases} 0 & \text{on } [-T, -T+\delta] \cup [T-\delta, T], \\ 1 & \text{on } [-T+2\delta, T-2\delta]. \end{cases} \tag{2.18}$$

In the sequel, $C_j > 0$ denote generic constants depending on s_0, τ, M_0, M_1, θ_0, θ_1, Ω, T, x_0, ω, χ and $\mathbf{p}$, $\mathbf{q}$, ε, δ, but independent of $s > s_0$.

Setting $\mathbf{z}_1 = \chi\partial_t^2\mathbf{y}$, $\mathbf{z}_2 = \chi\partial_t^3\mathbf{y}$ and $\mathbf{z}_3 = \chi\partial_t^4\mathbf{y}$, we have

$$\begin{cases} \tilde{\rho}\partial_t^2\mathbf{z}_1 = L_{\tilde{\lambda},\tilde{\mu}}\mathbf{z}_1 + \chi G\big(\partial_t^2\mathbf{u}\big) + 2\tilde{\rho}(\partial_t\chi)\partial_t^3\mathbf{y} + \tilde{\rho}\big(\partial_t^2\chi\big)\partial_t^2\mathbf{y}, \\ \tilde{\rho}\partial_t^2\mathbf{z}_2 = L_{\tilde{\lambda},\tilde{\mu}}\mathbf{z}_2 + \chi G\big(\partial_t^3\mathbf{u}\big) + 2\tilde{\rho}(\partial_t\chi)\partial_t^4\mathbf{y} + \tilde{\rho}\big(\partial_t^2\chi\big)\partial_t^3\mathbf{y}, \\ \tilde{\rho}\partial_t^2\mathbf{z}_3 = L_{\tilde{\lambda},\tilde{\mu}}\mathbf{z}_3 + \chi G\big(\partial_t^4\mathbf{u}\big) + 2\tilde{\rho}(\partial_t\chi)\partial_t^5\mathbf{y} + \tilde{\rho}\big(\partial_t^2\chi\big)\partial_t^4\mathbf{y} \quad \text{in } Q. \end{cases} \tag{2.19}$$

We set

$$\mathcal{D}=\|\mathbf{y}\|^2_{H^5(-T,T;H^2(\omega))}+\|\mathbf{y}\|^2_{H^4(-T,T;H^{\frac{5}{2}}(\omega))}.$$

Noting that $\mathbf{u}\in W^{7,\infty}(Q)$, in view of (2.18) and Lemma 2.1, we can apply Carleman estimate (2.1) to (2.19), so that

$$\begin{aligned}
&\sum_{j=2}^{4}\int_Q\Big(s^4\big|\partial_t^j\mathbf{y}\big|^2\chi^2+s^2\big|\nabla\partial_t^j\mathbf{y}\big|^2\chi^2+\sum_{|\alpha|=2}\big|\partial_x^\alpha\partial_t^j\mathbf{y}\big|^2\chi^2\Big)e^{2s\varphi}\,dx\,dt\\
&\quad\leqslant Cs\int_Q\sum_{j=2}^{4}\big(\chi^2\big|\nabla\big(G\big(\partial_t^j\mathbf{u}\big)\big)\big|^2+\chi^2\big|G(\partial_t\mathbf{u})\big|^2\big)e^{2s\varphi}\,dx\,dt\\
&\qquad+Cs\int_Q\big(|\partial_t\chi|^2+\big|\partial_t^2\chi\big|^2\big)\Big\{\sum_{j=2}^{5}\big(\big|\operatorname{div}\big(\partial_t^j\mathbf{y}\big)\big|^2+\big|\operatorname{rot}\big(\partial_t^j\mathbf{y}\big)\big|^2+\big|\partial_t^j\mathbf{y}\big|^2\big)\Big\}e^{2s\varphi}\,dx\,dt\\
&\qquad+Ce^{Cs}\mathcal{D}.
\end{aligned}\tag{2.20}$$

Here we used $\operatorname{div}(\tilde\rho(\partial_t\chi)\partial_t^j\mathbf{y})=\nabla(\tilde\rho\partial_t\chi)\cdot\partial_t^j\mathbf{y}+\tilde\rho(\partial_t\chi)\operatorname{div}(\partial_t^j\mathbf{y})$ and $\operatorname{rot}(\tilde\rho(\partial_t\chi)\partial_t^j\mathbf{y})=\nabla(\tilde\rho\partial_t\chi)\times\partial_t^j\mathbf{y}+\tilde\rho(\partial_t\chi)\operatorname{rot}(\partial_t^j\mathbf{y})$ for $j=2,3,4,5$.

Moreover, by (1.15) we see that

$$\begin{aligned}
&\big|\nabla\big(G\big(\partial_t^j\mathbf{u}\big)\big)\big|\leqslant C\Big\{|\nabla f|+|f|+\sum_{|\alpha|\leqslant 2}\big|\partial_x^\alpha g(x)\big|+\sum_{|\alpha|\leqslant 2}\big|\partial_x^\alpha h(x)\big|\Big\}\quad\text{in }Q,\\
&\big|G\big(\partial_t^j\mathbf{u}\big)\big|\leqslant C\big(|f|+|\nabla g|+|\nabla h|+|g|+|h|\big)\quad\text{in }Q
\end{aligned}\tag{2.21}$$

and

$$|\partial_t\chi|,\big|\partial_t^2\chi\big|\neq 0\quad\text{only for }t\in(T-2\delta,T-\delta)\cup(-T+\delta,-T+2\delta).\tag{2.22}$$

On the other hand, (2.13) implies

$$\tilde\rho\partial_t^2\big(\partial_t^j\mathbf{y}\big)=L_{\tilde\lambda,\tilde\mu}\partial_t^j\mathbf{y}+G\big(\partial_t^j\mathbf{u}\big)\quad\text{in }Q,\ \ j=0,1,2,3,4.$$

Therefore Lemma 2.3 and (2.21) yield

$$\begin{aligned}
&\int_Q\Big\{\sum_{k=2}^{5}\big(\big|\nabla\partial_t^k\mathbf{y}\big|^2+\big|\partial_t^k\mathbf{y}\big|^2\big)+\sum_{j=2}^{4}\sum_{|\alpha|=2}\big|\partial_t^j\partial_x^\alpha\mathbf{y}\big|^2\Big\}\,dx\,dt\\
&\quad\leqslant C\sum_{j=0}^{4}\big(\big\|G\big(\partial_t^j\mathbf{u}\big)\big\|^2_{L^2(Q)}+\big\|\nabla G\big(\partial_t^j\mathbf{u}\big)\big\|^2_{L^2(Q)}\big)+C\mathcal{D}\\
&\quad\leqslant C\int_Q\Big(\sum_{|\alpha|\leqslant 2}\big|\partial_x^\alpha g(x)\big|^2+\sum_{|\alpha|\leqslant 2}\big|\partial_x^\alpha h(x)\big|^2+\big|f(x)\big|^2+\big|\nabla f(x)\big|^2\Big)\,dx\,dt+C\mathcal{D}\\
&\quad\leqslant C\big(\|f\|^2_{H^1(\Omega)}+\|g\|^2_{H^2(\Omega)}+\|h\|^2_{H^2(\Omega)}\big)+C\mathcal{D}.
\end{aligned}\tag{2.23}$$

Hence inequalities (2.20)–(2.23) yield

$$\sum_{j=2}^{4}\int_Q\Big(s^4\big|\partial_t^j\mathbf{y}\big|^2\chi^2+s^2\big|\nabla\partial_t^j\mathbf{y}\big|^2\chi^2+\sum_{|\alpha|=2}\big|\partial_x^\alpha\partial_t^j\mathbf{y}\big|^2\chi^2\Big)e^{2s\varphi}\,dx\,dt$$
$$\leqslant Cs\int_Q\Big(|f|^2+\big|\nabla f(x)\big|^2+\sum_{|\alpha|\leqslant 2}\big|\partial_x^\alpha g(x)\big|^2+\sum_{|\alpha|\leqslant 2}\big|\partial_x^\alpha h(x)\big|^2\Big)e^{2s\varphi}\,dx\,dt$$
$$+Cse^{2s(d_1-2\varepsilon)}\big(\|f\|^2_{H^1(\Omega)}+\|g\|^2_{H^2(\Omega)}+\|h\|^2_{H^2(\Omega)}\big)+Ce^{Cs}\mathcal{D}$$
$$\equiv Cs\mathcal{E}+Cs\big(\|f\|^2_{H^1(\Omega)}+\|g\|^2_{H^2(\Omega)}+\|h\|^2_{H^2(\Omega)}\big)e^{2s(d_1-2\varepsilon)}+Ce^{Cs}\mathcal{D}. \tag{2.24}$$

On the other hand, for $|\alpha|=2$, we use (2.23) and

$$\int_\Omega\big|\big(\partial_t^2\partial_x^\alpha\mathbf{y}\big)(x,0)\big|^2e^{2s\varphi(x,0)}\,dx$$
$$=\int_{-T}^{0}\frac{\partial}{\partial t}\Big(\int_\Omega\big|\big(\partial_t^2\partial_x^\alpha\mathbf{y}\big)(x,t)\big|^2\chi(t)^2e^{2s\varphi}\,dx\Big)\,dt$$
$$=\int_{-T}^{0}\int_\Omega 2\big(\big(\partial_t^3\partial_x^\alpha\mathbf{y}\big)\cdot\big(\partial_t^2\partial_x^\alpha\mathbf{y}\big)\big)\chi^2e^{2s\varphi}\,dx\,dt+2s\int_{-T}^{0}\int_\Omega\big|\partial_t^2\partial_x^\alpha\mathbf{y}\big|^2\chi^2(\partial_t\varphi)e^{2s\varphi}\,dx\,dt$$
$$+\int_{-T}^{0}\int_\Omega\big|\partial_t^2\partial_x^\alpha\mathbf{y}\big|^2\big(\partial_t\big(\chi^2\big)\big)e^{2s\varphi}\,dx\,dt$$
$$\leqslant C\int_Q s\chi^2\big(\big|\partial_t^3\partial_x^\alpha\mathbf{y}\big|^2+\big|\partial_t^2\partial_x^\alpha\mathbf{y}\big|^2\big)e^{2s\varphi}\,dx$$
$$+Ce^{2s(d_1-2\varepsilon)}\big(\|f\|^2_{H^1(\Omega)}+\|g\|^2_{H^2(\Omega)}+\|h\|^2_{H^2(\Omega)}\big)+C\mathcal{D}e^{Cs}.$$

Therefore (2.24) yields

$$\sum_{|\alpha|=2}\int_\Omega\big|\big(\partial_t^2\partial_x^\alpha\mathbf{y}\big)(x,0)\big|^2e^{2s\varphi(x,0)}\,dx$$
$$\leqslant Cs^2\big(\|f\|^2_{H^1(\Omega)}+\|g\|^2_{H^2(\Omega)}+\|h\|^2_{H^2(\Omega)}\big)e^{2s(d_1-2\varepsilon)}+Cs^2\mathcal{E}+Ce^{Cs}\mathcal{D}$$

for all large $s>0$. Similarly, we can estimate $\sum_{|\alpha|=2}\int_\Omega|(\partial_t^3\partial_x^\alpha\mathbf{y})(x,0)|^2e^{2s\varphi(x,0)}\,dx$ to obtain

$$\sum_{j=2}^{3}\sum_{|\alpha|=2}\int_\Omega\big|\partial_x^\alpha\partial_t^j\mathbf{y}(x,0)\big|^2e^{2s\varphi(x,0)}\,dx$$
$$\leqslant Cs^2e^{2s(d_1-2\varepsilon)}\big(\|f\|^2_{H^1(\Omega)}+\|g\|^2_{H^2(\Omega)}+\|h\|^2_{H^2(\Omega)}\big)+Cs^2\mathcal{E}+Ce^{Cs}\mathcal{D} \tag{2.25}$$

for all large $s>0$.

Now we will consider first-order partial differential equations satisfied by h, g and f. That is, by (2.13), (2.14) and (1.15), we have

$$\tilde\rho\partial_t^2\mathbf{y}(x,0)=G\mathbf{u}(x,0),\qquad \tilde\rho\partial_t^3\mathbf{y}(x,0)=G\partial_t\mathbf{u}(x,0). \tag{2.26}$$

For simplicity, we set

$$\begin{cases} \mathbf{a} = \begin{pmatrix} -\frac{1}{\rho} L_{\lambda,\mu}\mathbf{p} \\ -\frac{1}{\rho} L_{\lambda,\mu}\mathbf{q} \end{pmatrix}, \\ \mathbf{b}_1 = \begin{pmatrix} \operatorname{div}\mathbf{p} \\ 0 \\ 0 \\ \operatorname{div}\mathbf{q} \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{b}_2 = \begin{pmatrix} 0 \\ \operatorname{div}\mathbf{p} \\ 0 \\ 0 \\ \operatorname{div}\mathbf{q} \\ 0 \end{pmatrix}, \quad \mathbf{b}_3 = \begin{pmatrix} 0 \\ 0 \\ \operatorname{div}\mathbf{p} \\ 0 \\ 0 \\ \operatorname{div}\mathbf{q} \end{pmatrix}, \\ (\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3) = \begin{pmatrix} \nabla\mathbf{p} + (\nabla\mathbf{p})^T \\ \nabla\mathbf{q} + (\nabla\mathbf{q})^T \end{pmatrix}, \\ \mathbf{G} = \begin{pmatrix} \tilde{\rho}\partial_t^2\mathbf{y}(x,0) - (g+h)\nabla(\operatorname{div}\mathbf{p}) - h\Delta\mathbf{p} \\ \tilde{\rho}\partial_t^3\mathbf{y}(x,0) - (g+h)\nabla(\operatorname{div}\mathbf{q}) - h\Delta\mathbf{q} \end{pmatrix} \quad \text{on } \overline{\Omega}. \end{cases} \tag{2.27}$$

Then we can rewrite (2.26) as

$$\mathbf{a}f + \mathbf{b}_1\partial_1 g + \mathbf{b}_2\partial_2 g + \mathbf{b}_3\partial_3 g = \mathbf{G} - \mathbf{d}_1\partial_1 h - \mathbf{d}_2\partial_2 h - \mathbf{d}_3\partial_3 h.$$

Therefore for $j_1 \in \{1,2,3,4,5,6\}$, we have

$$\begin{aligned} &\{\mathbf{a}\}_{j_1} f + \{\mathbf{b}_1\}_{j_1}\partial_1 g + \{\mathbf{b}_2\}_{j_1}\partial_2 g + \{\mathbf{b}_3\}_{j_1}\partial_3 g \\ &\quad = \{\mathbf{G}\}_{j_1} - \{\mathbf{d}_1\}_{j_1}\partial_1 h - \{\mathbf{d}_2\}_{j_1}\partial_2 h - \{\mathbf{d}_3\}_{j_1}\partial_3 h \quad \text{on } \overline{\Omega}. \end{aligned} \tag{2.28}$$

Equality (2.28) is a system of five linear equations with respect to four unknowns f, $\partial_1 g$, $\partial_2 g$, $\partial_3 g$, and so for the existence of solutions, we need the consistency of the coefficients, that is,

$$\det{}_{j_1}(\mathbf{a}, \mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{G} - \mathbf{d}_1\partial_1 h - \mathbf{d}_2\partial_2 h - \mathbf{d}_3\partial_3 h) = 0 \quad \text{on } \overline{\Omega},$$

that is,

$$\sum_{k=1}^{3} \det{}_{j_1}(\mathbf{a}, \mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{d}_k)\partial_k h = \det{}_{j_1}(\mathbf{a}, \mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{G}) \quad \text{on } \overline{\Omega \setminus \omega_0} \tag{2.29}$$

by the linearity of the determinant. Here by (1.15) we note that $\mathbf{p}, \mathbf{q} \in W^{5,\infty}(\Omega)$ and

$$\sum_{|\alpha|\leqslant 2} \left|\partial_x^\alpha \mathbf{G}(x)\right| \leqslant C\sum_{j=2}^{3}\sum_{|\alpha|\leqslant 2}\left|\partial_x^\alpha\partial_t^j\mathbf{y}(x,0)\right| + C\sum_{|\alpha|\leqslant 2}\left|\partial_x^\alpha g(x)\right| + C\sum_{|\alpha|\leqslant 2}\left|\partial_x^\alpha h(x)\right|.$$

In terms of condition (1.11) and $h \equiv \mu - \tilde{\mu} \in H_0^3(\Omega \setminus \overline{\omega_0})$, considering (2.29) as a first order partial differential operator in h, we can apply Lemma 2.2 in $\Omega \setminus \overline{\omega_0}$ to obtain

$$\begin{aligned} &s^2 \int_{\Omega\setminus\overline{\omega_0}} \sum_{|\alpha|\leqslant 2} \left|\partial_x^\alpha h\right|^2 e^{2s\varphi(x,0)}\,dx \\ &\quad \leqslant C\int_\Omega \sum_{j=2}^{3}\sum_{|\alpha|\leqslant 2}\left|\partial_x^\alpha\partial_t^j\mathbf{y}(x,0)\right|^2 e^{2s\varphi(x,0)}\,dx \\ &\qquad + C\int_\Omega\left(\sum_{|\alpha|\leqslant 2}\left|\partial_x^\alpha g(x)\right|^2 + \sum_{|\alpha|\leqslant 2}\left|\partial_x^\alpha h(x)\right|^2\right)e^{2s\varphi(x,0)}\,dx \end{aligned}$$

$$\leqslant Cs^2 e^{2s(d_1-2\varepsilon)}\big(\|f\|^2_{H^1(\Omega)} + \|g\|^2_{H^2(\Omega)} + \|h\|^2_{H^2(\Omega)}\big) + Cs^2\mathcal{E} + Ce^{Cs}\mathcal{D}$$
$$+ C\int_\Omega \bigg(\sum_{|\alpha|\leqslant 2}\big|\partial_x^\alpha g(x)\big|^2 + \sum_{|\alpha|\leqslant 2}\big|\partial_x^\alpha h(x)\big|^2\bigg)e^{2s\varphi(x,0)}\,dx \tag{2.30}$$

for all large $s > 0$. Here we used (2.25). Similarly to (2.30), in terms of (1.12), we can argue for g. Hence with (2.30), we have

$$\int_{\Omega\setminus\overline{\omega_0}} \bigg(\sum_{|\alpha|\leqslant 2}\big|\partial_x^\alpha g(x)\big|^2 + \sum_{|\alpha|\leqslant 2}\big|\partial_x^\alpha h(x)\big|^2\bigg)e^{2s\varphi(x,0)}\,dx$$
$$\leqslant Ce^{2s(d_1-2\varepsilon)}\big(\|f\|^2_{H^1(\Omega)} + \|g\|^2_{H^2(\Omega)} + \|h\|^2_{H^2(\Omega)}\big)$$
$$+ C\int_Q \bigg(\big|f(x)\big|^2 + \big|\nabla f(x)\big|^2 + \sum_{|\alpha|\leqslant 2}\big|\partial_x^\alpha g(x)\big|^2 + \sum_{|\alpha|\leqslant 2}\big|\partial_x^\alpha h(x)\big|^2\bigg)e^{2s\varphi}\,dx\,dt$$
$$+ Ce^{Cs}\mathcal{D} + \frac{C}{s^2}\int_\Omega \bigg(\sum_{|\alpha|\leqslant 2}\big|\partial_x^\alpha g(x)\big|^2 + \sum_{|\alpha|\leqslant 2}\big|\partial_x^\alpha h(x)\big|^2\bigg)e^{2s\varphi(x,0)}\,dx$$

for all large $s > 0$. Here we recall the definition of $\mathcal{E}$ in (2.24). Taking $s > 0$ sufficiently large and noting that $g = h = 0$ on $\overline{\omega_0}$ by (1.9), we can absorb the last term into the left-hand side:

$$\int_\Omega \bigg(\sum_{|\alpha|\leqslant 2}\big|\partial_x^\alpha g(x)\big|^2 + \sum_{|\alpha|\leqslant 2}\big|\partial_x^\alpha h(x)\big|^2\bigg)e^{2s\varphi(x,0)}\,dx$$
$$\leqslant Ce^{2s(d_1-2\varepsilon)}\big(\|f\|^2_{H^1(\Omega)} + \|g\|^2_{H^2(\Omega)} + \|h\|^2_{H^2(\Omega)}\big)$$
$$+ C\int_Q \bigg(\big|f(x)\big|^2 + \big|\nabla f(x)\big|^2 + \sum_{|\alpha|\leqslant 2}\big|\partial_x^\alpha g(x)\big|^2 + \sum_{|\alpha|\leqslant 2}\big|\partial_x^\alpha h(x)\big|^2\bigg)e^{2s\varphi}\,dx\,dt$$
$$+ Ce^{Cs}\mathcal{D}. \tag{2.31}$$

Finally, by (2.28), we have

$$\mathbf{a}f = -\mathbf{b}_1\partial_1 g - \mathbf{b}_2\partial_2 g - \mathbf{b}_3\partial_3 g + \mathbf{G} - \mathbf{d}_1\partial_1 h - \mathbf{d}_2\partial_2 h - \mathbf{d}_3\partial_3 h \quad \text{in } \Omega.$$

Moreover, by (1.11) or (1.12), we see that $|\mathbf{a}(x)| > 0$ for $x \in \overline{\Omega\setminus\omega_0}$, so that

$$f(x) = \widetilde{K_1}\mathbf{G} + \widetilde{K_2}(\nabla g, \nabla h) \quad \text{on } \overline{\Omega\setminus\omega_0},$$

where $\widetilde{K_1}$, $\widetilde{K_2}$ are linear operators with $W^{1,\infty}$-coefficients. Thus

$$\big|\nabla f(x)\big| \leqslant C\bigg(\big|\nabla\mathbf{G}(x)\big| + \sum_{|\alpha|\leqslant 2}\big|\partial_x^\alpha g(x)\big| + \sum_{|\alpha|\leqslant 2}\big|\partial_x^\alpha h(x)\big|\bigg)$$
$$\leqslant C\bigg\{\sum_{j=2}^{3}\big(\big|\nabla\big(\partial_t^j\mathbf{y}\big)(x,0)\big| + \big|\partial_t^j\mathbf{y}(x,0)\big|\big) + \sum_{|\alpha|\leqslant 2}\big|\partial_x^\alpha g(x)\big| + \sum_{|\alpha|\leqslant 2}\big|\partial_x^\alpha h(x)\big|\bigg\}$$

and

$$\big|f(x)\big| \leqslant C\bigg\{\sum_{j=2}^{3}\big|\partial_t^j\mathbf{y}(x,0)\big| + \sum_{|\alpha|\leqslant 2}\big|\partial_x^\alpha g(x)\big| + \sum_{|\alpha|\leqslant 2}\big|\partial_x^\alpha h(x)\big|\bigg\}$$

for $x \in \overline{\Omega \setminus \omega_0}$. Hence, since $f = 0$ on $\overline{\omega_0}$, we have

$$\int_\Omega \left(\left|\nabla f(x)\right|^2 + \left|f(x)\right|^2\right) e^{2s\varphi(x,0)}\,dx$$
$$\leqslant C \int_\Omega \left\{ \sum_{j=2}^{3} \sum_{|\alpha|\leqslant 1} \left|\partial_x^\alpha \partial_t^j \mathbf{y}(x,0)\right|^2 + \sum_{|\alpha|\leqslant 2} \left|\partial_x^\alpha g(x)\right|^2 + \sum_{|\alpha|\leqslant 2} \left|\partial_x^\alpha h(x)\right|^2 \right\} e^{2s\varphi(x,0)}\,dx. \tag{2.32}$$

On the other hand, for $j = 2, 3$, we have by (2.23) and (2.24),

$$\int_\Omega \left|\nabla\left(\partial_t^j \mathbf{y}\right)(x,0)\right|^2 e^{2s\varphi(x,0)}\,dx$$
$$= \int_{-T}^{0} \frac{\partial}{\partial t} \int_\Omega \chi^2 \left|\nabla\left(\partial_t^j \mathbf{y}\right)\right|^2 e^{2s\varphi}\,dx\,dt$$
$$\leqslant C \int_Q \left(s\left|\nabla\left(\partial_t^j \mathbf{y}\right)\right|^2 \chi^2 + \left|\nabla\left(\partial_t^{j+1} \mathbf{y}\right)\right|^2 \chi^2\right) e^{2s\varphi}\,dx\,dt$$
$$+ Ce^{2s(d_1-2\varepsilon)}\left(\|f\|^2_{H^1(\Omega)} + \|g\|^2_{H^2(\Omega)} + \|h\|^2_{H^2(\Omega)}\right) + Ce^{Cs}\mathcal{D}$$
$$\leqslant Ce^{2s(d_1-2\varepsilon)}\left(\|f\|^2_{H^1(\Omega)} + \|g\|^2_{H^2(\Omega)} + \|h\|^2_{H^2(\Omega)}\right) + Ce^{Cs}\mathcal{D} + C\mathcal{E} \tag{2.33}$$

and

$$\int_\Omega \left|\left(\partial_t^j \mathbf{y}\right)(x,0)\right|^2 e^{2s\varphi(x,0)}\,dx$$
$$= \int_{-T}^{0} \frac{\partial}{\partial t} \int_\Omega \chi^2 \left|\left(\partial_t^j \mathbf{y}\right)\right|^2 e^{2s\varphi}\,dx\,dt$$
$$\leqslant C \int_Q \left(s\left|\partial_t^j \mathbf{y}\right|^2 \chi^2 + \left|\partial_t^{j+1} \mathbf{y}\right|^2 \chi^2\right) e^{2s\varphi}\,dx\,dt$$
$$+ Ce^{2s(d_1-2\varepsilon)}\left(\|f\|^2_{H^1(\Omega)} + \|g\|^2_{H^2(\Omega)} + \|h\|^2_{H^2(\Omega)}\right) + Ce^{Cs}\mathcal{D}$$
$$\leqslant Ce^{2s(d_1-2\varepsilon)}\left(\|f\|^2_{H^1(\Omega)} + \|g\|^2_{H^2(\Omega)} + \|h\|^2_{H^2(\Omega)}\right) + Ce^{Cs}\mathcal{D} + C\mathcal{E} \tag{2.34}$$

for all large $s > 0$.

Substituting (2.31), (2.33) and (2.34) into (2.32), we obtain

$$\int_\Omega \left(\left|\nabla f(x)\right|^2 + \left|f(x)\right|^2\right) e^{2s\varphi(x,0)}\,dx$$
$$\leqslant Ce^{2s(d_1-2\varepsilon)}\left(\|f\|^2_{H^1(\Omega)} + \|g\|^2_{H^2(\Omega)} + \|h\|^2_{H^2(\Omega)}\right) + C\mathcal{E} + Ce^{Cs}\mathcal{D}.$$

Here with (2.31), we have

$$\int_\Omega \Bigl(\sum_{|\alpha|\leqslant 2} \bigl|\partial_x^\alpha g\bigr|^2 + \sum_{|\alpha|\leqslant 2} \bigl|\partial_x^\alpha h\bigr|^2 + |\nabla f|^2 + |f|^2 \Bigr) e^{2s\varphi(x,0)}\, dx$$
$$\leqslant Ce^{2s(d_1-2\varepsilon)} \bigl(\|f\|^2_{H^1(\Omega)} + \|g\|^2_{H^2(\Omega)} + \|h\|^2_{H^2(\Omega)} \bigr) + Ce^{Cs}\mathcal{D}$$
$$+ C\int_Q \Bigl(\sum_{|\alpha|\leqslant 2} \bigl|\partial_x^\alpha g\bigr|^2 + \sum_{|\alpha|\leqslant 2} \bigl|\partial_x^\alpha h\bigr|^2 + |\nabla f|^2 + |f|^2 \Bigr) e^{2s\varphi}\, dx\, dt.$$

Since

$$\int_Q \Bigl(\sum_{|\alpha|\leqslant 2} \bigl|\partial_x^\alpha g\bigr|^2 + \sum_{|\alpha|\leqslant 2} \bigl|\partial_x^\alpha h\bigr|^2 + |\nabla f|^2 + |f|^2 \Bigr) e^{2s\varphi}\, dx\, dt$$
$$= \int_\Omega \Bigl(\sum_{|\alpha|\leqslant 2} \bigl|\partial_x^\alpha g\bigr|^2 + \sum_{|\alpha|\leqslant 2} \bigl|\partial_x^\alpha h\bigr|^2 + |\nabla f|^2 + |f|^2 \Bigr) e^{2s\varphi(x,0)} \Biggl(\int_{-T}^{T} e^{2s(\varphi(x,t)-\varphi(x,0))}\, dt \Biggr) dx$$
$$= o(1) \int_\Omega \Bigl(\sum_{|\alpha|\leqslant 2} \bigl|\partial_x^\alpha g\bigr|^2 + \sum_{|\alpha|\leqslant 2} \bigl|\partial_x^\alpha h\bigr|^2 + |\nabla f|^2 + |f|^2 \Bigr) e^{2s\varphi(x,0)}\, dx,$$

as $s \to \infty$ by the Lebesgue theorem and $\varphi(x,t) < \varphi(x,0)$ for $t \neq 0$, we can absorb the last term on the right-hand side into the left-hand side, and

$$\int_\Omega \Bigl(\sum_{|\alpha|\leqslant 2} \bigl|\partial_x^\alpha g\bigr|^2 + \sum_{|\alpha|\leqslant 2} \bigl|\partial_x^\alpha h\bigr|^2 + |\nabla f|^2 + |f|^2 \Bigr) e^{2s\varphi(x,0)}\, dx$$
$$\leqslant Ce^{2s(d_1-2\varepsilon)} \bigl(\|f\|^2_{H^1(\Omega)} + \|g\|^2_{H^2(\Omega)} + \|h\|^2_{H^2(\Omega)} \bigr) + Ce^{Cs}\mathcal{D}$$

for all large $s > 0$. By $\varphi(x,0) \geqslant d_1$, we divide the both sides by e^{2sd_1}, we have

$$\int_\Omega \Bigl(\sum_{|\alpha|\leqslant 2} \bigl|\partial_x^\alpha g\bigr|^2 + \sum_{|\alpha|\leqslant 2} \bigl|\partial_x^\alpha h\bigr|^2 + |\nabla f|^2 + |f|^2 \Bigr) dx$$
$$\leqslant Ce^{-4s\varepsilon} \bigl(\|f\|^2_{H^1(\Omega)} + \|g\|^2_{H^2(\Omega)} + \|h\|^2_{H^2(\Omega)} \bigr) + Ce^{Cs}\mathcal{D}$$

for all large $s > 0$. Choosing $s > 0$ sufficiently large, we can absorb the first term on the right-hand side into the left-hand side, so that we have conclusion (1.14). □

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Appendix A. Proof of Lemma 2.3

Let us set $v = \operatorname{div} \mathbf{u}$ and $\mathbf{w} = \operatorname{rot} \mathbf{u}$. Then, as in the proof of Lemma 2.1, we have

$$\rho\partial_t^2 \mathbf{u} - \mu\Delta\mathbf{u} + Q_1(\mathbf{u}, v) = \mathbf{f} \quad \text{in } Q, \tag{A.1}$$
$$\rho\partial_t^2 v - (\lambda + 2\mu)\Delta v + Q_2(\mathbf{u}, v, \mathbf{w}) = \operatorname{div}\mathbf{f} \quad \text{in } Q \tag{A.2}$$

and

$$\rho\partial_t^2\mathbf{w}-\mu\Delta\mathbf{w}+Q_3(\mathbf{u},v,\mathbf{w})=\operatorname{rot}\mathbf{f}\quad\text{in } Q,\tag{A.3}$$

where $Q_1(\mathbf{u},v)=\sum_{|\alpha|=1}a_\alpha^{(1)}(x)\partial_x^\alpha\mathbf{u}+\sum_{|\alpha|\leqslant 1}b_\alpha^{(1)}(x)\partial_x^\alpha v$, $Q_j(\mathbf{u},v,\mathbf{w})=\sum_{|\alpha|=1}a_\alpha^{(j)}(x)\partial_x^\alpha\mathbf{u}+\sum_{|\alpha|=1}b_\alpha^{(j)}(x)\partial_x^\alpha v+\sum_{|\alpha|=1}c_\alpha^{(j)}(x)\partial_x^\alpha\mathbf{w}$, $j=2,3$, and $a_\alpha^{(j)},b_\alpha^{(j)},c_\alpha^{(j)}\in L^\infty(Q)$.

Let $t\geqslant 0$. We set

$$E_1(t)\equiv\int_\Omega\left(\left|\nabla_{x,t}\mathbf{u}(x,t)\right|^2+\left|\nabla_{x,t}v(x,t)\right|^2+\left|\nabla_{x,t}\mathbf{w}(x,t)\right|^2\right)dx.$$

Taking the scalar products of (A.1) and (A.3) with $\partial_t\mathbf{u}$ and $\partial_t\mathbf{w}$, respectively, and multiplying (A.2) with $\partial_t v$, we integrate by parts to have

$$\begin{aligned}E_1(t)\leqslant{}& CE_1(0)+C\left(\int_0^t E_1(\xi)\,d\xi+\|\mathbf{f}\|_{L^2(Q)}^2+\|\operatorname{div}\mathbf{f}\|_{L^2(Q)}^2+\|\operatorname{rot}\mathbf{f}\|_{L^2(Q)}^2\right)\\&+C\big(\|\partial_t\mathbf{u}\|_{L^2(\partial\Omega\times(-T,T))}^2+\|\partial_\nu\mathbf{u}\|_{L^2(\partial\Omega\times(-T,T))}^2+\|\partial_t v\|_{L^2(\partial\Omega\times(-T,T))}^2\\&+\|\partial_\nu v\|_{L^2(\partial\Omega\times(-T,T))}^2+\|\partial_t\mathbf{w}\|_{L^2(\partial\Omega\times(-T,T))}^2+\|\partial_\nu\mathbf{w}\|_{L^2(\partial\Omega\times(-T,T))}^2\big).\end{aligned}$$

Applying the trace theorem, we have

$$E_1(t)\leqslant CE_1(0)+CF+C\int_0^t E_1(\xi)\,d\xi,\quad 0\leqslant t\leqslant T.$$

Here we set

$$F=\|\mathbf{u}\|_{H^1(-T,T;H^2(\omega))}^2+\|\mathbf{u}\|_{L^2(-T,T;H^{\frac{5}{2}}(\omega))}^2+\|\operatorname{div}\mathbf{f}\|_{L^2(Q)}^2+\|\operatorname{rot}\mathbf{f}\|_{L^2(Q)}^2+\|\mathbf{f}\|_{L^2(Q)}^2.$$

The Gronwall inequality implies $E_1(t)\leqslant C(F+E_1(0))$, $0\leqslant t\leqslant T$. Similarly we can prove $C^{-1}E_1(0)\leqslant E_1(t)+CF$, $0\leqslant t\leqslant T$. For $-T\leqslant t\leqslant 0$, we can similarly argue to obtain

$$E_1(t_1)\leqslant CE_1(t_2)+CF,\quad -T\leqslant t_1,t_2\leqslant T.\tag{A.4}$$

Next we will include $\|\mathbf{u}(\cdot,t)\|_{L^2(\Omega)}^2$ into $E_1(t)$. By the Sobolev extension theorem and the trace theorem, we can find $\mathbf{u}^*(\cdot,t)\in H^1(\Omega)$ such that $\mathbf{u}^*(\cdot,t)=\mathbf{u}(\cdot,t)$ on $\partial\Omega$ and $\|\mathbf{u}^*(\cdot,t)\|_{H^1(\Omega)}\leqslant C\|\mathbf{u}(\cdot,t)\|_{H^{1/2}(\partial\Omega)}\leqslant C\|\mathbf{u}(\cdot,t)\|_{H^1(\omega)}$ for $-T\leqslant t\leqslant T$. Then $(\mathbf{u}-\mathbf{u}^*)(\cdot,t)\in H_0^1(\Omega)$ and the Poincaré inequality yield

$$\begin{aligned}\left\|(\mathbf{u}-\mathbf{u}^*)(\cdot,t)\right\|_{L^2(\Omega)}&\leqslant C\left\|(\nabla\mathbf{u}-\nabla\mathbf{u}^*)(\cdot,t)\right\|_{L^2(\Omega)}\\&\leqslant C\left\|\nabla\mathbf{u}(\cdot,t)\right\|_{L^2(\Omega)}+C\left\|\mathbf{u}(\cdot,t)\right\|_{H^1(\omega)}.\end{aligned}$$

Moreover, the Sobolev embedding theorem implies

$$\left\|\mathbf{u}(\cdot,t)\right\|_{H^1(\omega)}^2\leqslant C\|\mathbf{u}\|_{H^1(-T,T;H^1(\omega))}^2\leqslant CF.$$

That is,

$$\left\|\mathbf{u}(\cdot,t)\right\|_{L^2(\Omega)}^2\leqslant C\left\|\nabla\mathbf{u}(\cdot,t)\right\|_{L^2(\Omega)}^2+CF,\quad -T\leqslant t\leqslant T.$$

Therefore

$$E_1(t) \leqslant E(t) \equiv \int_\Omega \left(\left|\mathbf{u}(x,t)\right|^2 + \left|\nabla_{x,t}\mathbf{u}(x,t)\right|^2 + \left|\nabla_{x,t}\operatorname{div}\mathbf{u}(x,t)\right|^2 + \left|\nabla_{x,t}\operatorname{rot}\mathbf{u}(x,t)\right|^2\right) dx$$
$$\leqslant CE_1(t) + CF, \quad -T \leqslant t \leqslant T,$$

so that (A.4) implies

$$E(t_1) \leqslant CE(t_2) + CF, \quad -T \leqslant t_1, t_2 \leqslant T. \tag{A.5}$$

Let $\chi \in C_0^\infty(\mathbb{R})$ satisfy $0 \leqslant \chi \leqslant 1$ and (2.18). We set $\mathbf{v} = \chi\mathbf{u}$. Then $\partial_t^j \mathbf{v}(\cdot, \pm T) = 0$, $j = 0, 1$, and

$$\rho\partial_t^2\mathbf{v} - L_{\lambda,\mu}\mathbf{v} = \chi\mathbf{f} - \rho\big(2(\partial_t\chi)\partial_t\mathbf{u} + \big(\partial_t^2\chi\big)\mathbf{u}\big).$$

We can apply (2.1) to $\mathbf{v}$:

$$\int_Q \left(|\mathbf{v}|^2 + |\nabla_{x,t}\mathbf{v}|^2 + |\nabla_{x,t}\operatorname{div}\mathbf{v}|^2 + |\nabla_{x,t}\operatorname{rot}\mathbf{v}|^2\right)e^{2s\varphi}\,dx\,dt$$
$$\leqslant Ce^{Cs}F + C\int_Q \left(|\partial_t\chi|^2 + \left|\partial_t^2\chi\right|^2\right)\sum_{j=0}^{1}\left(\left|\partial_t^j\mathbf{u}\right|^2 + \left|\operatorname{div}\partial_t^j\mathbf{u}\right|^2 + \left|\operatorname{rot}\partial_t^j\mathbf{u}\right|^2\right)e^{2s\varphi}\,dx\,dt.$$

Taking $\delta > 0$ small and shrinking the domain Q into $\Omega \times (-\delta, \delta)$ on the left-hand side and using (2.16) and (2.18), we have

$$e^{2s(d_1-\varepsilon)}\int_{-\delta}^{\delta}\int_\Omega \left(|\mathbf{u}|^2 + |\nabla_{x,t}\mathbf{u}|^2 + |\nabla_{x,t}\operatorname{div}\mathbf{u}|^2 + |\nabla_{x,t}\operatorname{rot}\mathbf{u}|^2\right)dx\,dt$$
$$\leqslant Ce^{Cs}F + Ce^{2s(d_1-2\varepsilon)}\int_Q \sum_{j=0}^{1}\left(\left|\partial_t^j\mathbf{u}\right|^2 + \left|\operatorname{div}\partial_t^j\mathbf{u}\right|^2 + \left|\operatorname{rot}\partial_t^j\mathbf{u}\right|^2\right)dx\,dt.$$

Therefore by (A.5), we have

$$2\delta e^{2s(d_1-\varepsilon)}\big(E(0) - CF\big) \leqslant Ce^{Cs}F + 2TCe^{2s(d_1-2\varepsilon)}\big(E(0) + F\big),$$

that is,

$$E(0)\big(2\delta - 2CTe^{-2s\varepsilon}\big) \leqslant Ce^{Cs}F + CF.$$

Taking $s > 0$ sufficiently large, we obtain $E(0) \leqslant CF$. By (A.5), we have

$$E(t) \leqslant CF, \quad -T \leqslant t \leqslant T. \tag{A.6}$$

By the Sobolev extension theorem, we can find $\mathbf{u}^*(\cdot,t) \in H^2(\Omega)$ such that $\|\mathbf{u}^*(\cdot,t)\|_{H^2(\Omega)} \leqslant C\|u(\cdot,t)\|_{H^2(\omega)}$ and $\mathbf{u}^*(\cdot,t) = \mathbf{u}(\cdot,t)$ on $\partial\Omega$. Set $\mathbf{v} = \mathbf{u} - \mathbf{u}^*$. Then $\Delta\mathbf{v} = \Delta\mathbf{u} - \Delta\mathbf{u}^* = \nabla(\operatorname{div}\mathbf{u}) - \operatorname{rot}(\operatorname{rot}\mathbf{u}) - \Delta\mathbf{u}^*$ and $\mathbf{v}|_{\partial\Omega} = 0$. Hence the a priori estimate for the boundary value problem for Δ implies

$$\left\|\mathbf{v}(\cdot,t)\right\|_{H^2(\Omega)} \leqslant C\big(\left\|\nabla\operatorname{div}\mathbf{u}(\cdot,t)\right\|_{L^2(\Omega)} + \left\|\operatorname{rot}(\operatorname{rot}\mathbf{u})(\cdot,t)\right\|_{L^2(\Omega)} + \left\|\Delta\mathbf{u}^*(\cdot,t)\right\|_{L^2(\Omega)}\big).$$

Since $\mathbf{u} = \mathbf{v} + \mathbf{u}^*$, we have

$$\left\|\mathbf{u}(\cdot,t)\right\|_{H^2(\Omega)} \leqslant C\big(\left\|\nabla\operatorname{div}\mathbf{u}(\cdot,t)\right\|_{L^2(\Omega)} + \left\|\operatorname{rot}(\operatorname{rot}\mathbf{u})(\cdot,t)\right\|_{L^2(\Omega)} + \left\|\mathbf{u}^*(\cdot,t)\right\|_{H^2(\Omega)}\big)$$
$$\leqslant C\big(\left\|\nabla\operatorname{div}\mathbf{u}(\cdot,t)\right\|_{L^2(\Omega)} + \left\|\operatorname{rot}(\operatorname{rot}\mathbf{u})(\cdot,t)\right\|_{L^2(\Omega)} + \left\|\mathbf{u}(\cdot,t)\right\|_{H^2(\omega)}\big).$$

Since $\|\mathbf{u}(\cdot,t)\|_{H^2(\omega)} \leqslant C\|\mathbf{u}\|_{H^1(-T,T;H^2(\omega))}$, we have

$$\big\|\mathbf{u}(\cdot,t)\big\|^2_{H^2(\Omega)} \leqslant C\big(\big\|\nabla\operatorname{div}\mathbf{u}(\cdot,t)\big\|^2_{L^2(\Omega)} + \big\|\operatorname{rot}(\operatorname{rot}\mathbf{u})(\cdot,t)\big\|^2_{L^2(\Omega)} + F\big),$$

with which (A.6) yields

$$\int_\Omega \Big(\sum_{|\alpha|\leqslant 2}\big|\partial_x^\alpha \mathbf{u}(x,t)\big|^2 + \big|\partial_t\mathbf{u}(x,t)\big|^2 + \big|\partial_t\operatorname{div}\mathbf{u}(x,t)\big|^2 + \big|\partial_t\operatorname{rot}\mathbf{u}(x,t)\big|^2\Big)\,dx \leqslant CF. \tag{A.7}$$

Finally, we will estimate $\nabla(\partial_t\mathbf{u})$ on the right-hand side of the conclusion. By the Sobolev extension theorem and the trace theorem, for $-T\leqslant t\leqslant T$ we can find $\mathbf{u}_1^*$ such that $\mathbf{u}_1^*(\cdot,t) = \partial_t\mathbf{u}(\cdot,t)$ on $\partial\Omega$ and $\|\mathbf{u}_1^*(\cdot,t)\|_{H^1(\Omega)} \leqslant C\|\partial_t\mathbf{u}(\cdot,t)\|_{H^{\frac{1}{2}}(\partial\Omega)} \leqslant C\|\partial_t\mathbf{u}(\cdot,t)\|_{H^1(\omega)}$. Applying Theorem 6.1 (pp. 358–359) in Duvaut and Lions [10], we have

$$\begin{aligned}\big\|\partial_t\mathbf{u}(\cdot,t) - \mathbf{u}_1^*(\cdot,t)\big\|_{H^1(\Omega)} \leqslant{}& C\big\|\partial_t\mathbf{u}(\cdot,t) - \mathbf{u}_1^*(\cdot,t)\big\|_{L^2(\Omega)} \\ &+ C\big\|\operatorname{div}\big(\partial_t\mathbf{u}(\cdot,t) - \mathbf{u}_1^*(\cdot,t)\big)\big\|_{L^2(\Omega)} \\ &+ C\big\|\operatorname{rot}\big(\partial_t\mathbf{u}(\cdot,t) - \mathbf{u}_1^*(\cdot,t)\big)\big\|_{L^2(\Omega)},\end{aligned}$$

that is,

$$\begin{aligned}&\int_\Omega \big|\partial_t\nabla\mathbf{u}(x,t)\big|^2\,dx \\ &\quad\leqslant C\int_\Omega \big(\big|\partial_t\mathbf{u}(x,t)\big|^2 + \big|\partial_t\operatorname{div}\mathbf{u}(x,t)\big|^2 + \big|\partial_t\operatorname{rot}\mathbf{u}(x,t)\big|^2\big)\,dx + C\big\|\partial_t\mathbf{u}(\cdot,t)\big\|^2_{H^1(\omega)}.\end{aligned}$$

Hence by (A.7), we obtain

$$\begin{aligned}\int_Q |\partial_t\nabla\mathbf{u}|^2\,dx\,dt &\leqslant C\int_Q \big(\big|\partial_t\mathbf{u}(x,t)\big|^2 + \big|\partial_t\operatorname{div}\mathbf{u}(x,t)\big|^2 + \big|\partial_t\operatorname{rot}\mathbf{u}(x,t)\big|^2\big)\,dx\,dt + CF \\ &\leqslant CF.\end{aligned} \tag{A.8}$$

Inequalities (A.7) and (A.8) complete the proof of Lemma 2.3. □

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