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The action of operator semigroups on the topological dual of the Beurling–Björck space

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Abstract

We investigate the action of a class of operator semigroups on generalized functions of almost exponential growth, proving that these generalized functions are admissible initial conditions for the associated heat equation.

Keywords: Beurling–Björck space; w-Tempered distributions; Representation theorem; Admissible initial values; Gauss–Weierstrass semigroup; Ornstein–Uhlenbeck semigroup

1. Introduction

We investigate the action of a class of operator semigroups $\{e^{tA}\}_{t\geqslant 0}$, on generalized functions of almost exponential growth. Specifically, we consider generalized functions in the topological dual $\mathcal{S}'_w$ of the Beurling–Björck space $\mathcal{S}_w$, proposed by A. Beurling in [3], and studied by G. Björck in [4] and by H.-J. Schmeisser and H. Triebel in [16]. The space $\mathcal{S}_w$ and its topological dual $\mathcal{S}'_w$ provide an extension, away from the context of polynomial growth and decay, for the rich theory of generalized functions created by L. Schwartz. The operator semigroups we consider are integral operators with kernels defined by functions in the space $\bigcap_w \mathcal{S}_w$. We characterize this intersection by a condition that resembles the definition of the Denjoy–Carleman classes $\mathcal{C}^{\{a_k\}}$ [4]. Relevant examples of the operator semigroups we consider are the Gauss–Weierstrass semigroup defined by $A = \Delta$, where Δ is the Laplace operator, and the Ornstein–Uhlenbeck semigroup associated with the operator $A = \frac{1}{2}\Delta - x \cdot \nabla$, where ∇ denotes the gradient.

We also prove that the functionals in $\mathcal{S}'_w$ are admissible initial values for the generalized heat operator $\partial_t - A$. More precisely, we prove that given $T \in \mathcal{S}'_w$, there is a solution $u(x,t)$ of $(\partial_t - A)u = 0$, for which $u(\cdot,t)$ converges to T in $\mathcal{S}'_w$, in the strong dual topology. In this respect, our work is inspired by a substantial body of work on the realization of various types of generalized functions as initial values of solutions for the classical heat equation. This work, pioneered by T. Matsuzawa [13,14] for hyperfunctions and inspired by the work of L. Hörmander [10], typically

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uses functional estimates that are variations of those describing real analyticity. In our case, however, the definition of the space $\mathcal{S}_w$ involves conditions on the function and on its Fourier transform. For this reason, we base our approach on a representation theorem for functionals in $\mathcal{S}'_w$, which we obtain using the topological version proved in [1] of the characterization of $\mathcal{S}_w$ proved by S.-Y. Chung, D. Kim and S. Lee in [6]. This representation theorem provides as well a new characterization for tempered distributions.

We point out that in a very interesting paper, B.P. Dhungana [7] has considered the heat operator associated to the Hermite operator $-\Delta + |x|^2$, in the context of tempered distributions.

Beurling's motivation for studying almost exponential growth was his research on quasi-analyticity and in this direction, we mention the recent work of M. Andersson and B. Berndtsson [2].

Ultra-rapidly decreasing test functions have found a renewed source of interest in the study of modulation spaces via the short time Fourier transform. In this regard we mention, as an example, the pioneering work of K. Gröchenig and G. Zimmermann [8]. In a related field, earlier work of J. Dziubański and E. Hernández [9] showed how to construct band-limited wavelets that are functions with almost exponential decay. It was known already that it is not possible to have band-limited wavelets with exponential decay.

The organization of this paper is as follows: In Section 2 we define the Beurling–Björck space $\mathcal{S}_w$ and its topological dual $\mathcal{S}'_w$, and we recall several properties of these spaces. In particular, we state a characterization of $\mathcal{S}_w$ to be used later. In Section 3, we prove a representation theorem for generalized functions in $\mathcal{S}'_w$ and discuss its implications for the study of integral operators acting on $\mathcal{S}'_w$. In Section 4 we characterize the space $\bigcap_w \mathcal{S}_w$. We use this characterization in Section 5 to define the integral kernels we consider in this paper and to prove that the functionals in $\mathcal{S}'_w$ are admissible initial values for the generalized heat operator $\partial_t - A$. Section 5 concludes with an application of these results to the study of the Gauss–Weierstrass and Ornstein–Uhlenbeck semigroups acting on $\mathcal{S}'_w$.

The notation we use is standard. The symbols C^∞, C_0^∞, L^p, D', etc., indicate the usual spaces of functions or distributions defined on $\mathbb{R}^n$, with complex values. We denote $|\cdot|$ the Euclidean norm on $\mathbb{R}^n$, while $\|\cdot\|_p$ indicates the norm in the space L^p. When we do not work on the general Euclidean space $\mathbb{R}^n$, we will write $L^p(\mathbb{R})$, etc., as appropriate. Partial derivatives will be denoted ∂^α, where α is a multi-index $(\alpha_1, \dots, \alpha_n)$. If it is necessary to indicate on which variables we are taking the derivative, we will do so by attaching sub-indexes. We will use the standard abbreviations $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$. With $\alpha \leqslant \beta$ we mean that $\alpha_j \leqslant \beta_j$ for every j. The Fourier transform of a function g will be denoted $\mathcal{F}(g)$ or $\hat{g}$ and it will be defined as $\int_{\mathbb{R}^n} e^{-2\pi i x\xi} g(x)\, dx$. The inverse Fourier transform is then $\mathcal{F}^{-1}(g) = \int_{\mathbb{R}^n} e^{2\pi i x\xi} g(\xi)\, d\xi$. The letter C will indicate a positive constant, that may be different at different occurrences. If it is important to indicate that a constant depends on certain parameters, we will do so by attaching sub-indexes to the constant. We will not indicate the dependence of constants on the dimension n or other fixed parameters. Other notation will be introduced at the appropriate time.

2. Preliminary definitions and results

The space $\mathcal{S}_w$ and its topological dual $\mathcal{S}'_w$ provide an extension, away from the context of polynomial growth and decay, of the rich theory created by L. Schwartz.

Let us recall that the Schwartz space $\mathcal{S}$ of test functions consists of those C^∞ functions $\varphi : \mathbb{R}^n \to \mathbb{C}$ for which the norm

$$p_{k,m}(\varphi) = \sup_{|\alpha| \leqslant m} \left\| \left(1 + |x|\right)^k \partial^\alpha \varphi \right\|_\infty$$

is finite, for $k, m = 0, 1, 2, \dots$. The topological dual of $\mathcal{S}$ is the space $\mathcal{S}'$ of tempered distributions.

Observing that $(1 + |x|)^k = e^{k \ln(1+|x|)}$, A. Beurling proposed in [3] to measure growth and decay using an exponential $e^{kw(x)}$. The conditions imposed in [3] and [4] on the function w imply that $\mathcal{S}_w$ enjoys most of properties that the space $\mathcal{S}$ has. For instance, $\mathcal{S}_w$ is a Fréchet algebra with respect to both pointwise multiplication and convolution, and the Fourier transform is an isomorphism of one structure to the other; moreover, the functions in $\mathcal{S}_w$ with compact support form a dense subspace, D_w, containing partitions of unity.

The function $e^{kw(x)}$ does not have in general a special connection with the derivative operator ∂^α via the Fourier transform. For this reason, the definition of $\mathcal{S}_w$ imposes conditions on both the function and the Fourier transform.

Before we define the space $\mathcal{S}_w$ we need to introduce the space $\mathcal{M}_c$ of admissible functions w.

Definition 1. (See [4,16].) With $\mathcal{M}_c$ we indicate the space of functions $w : \mathbb{R}^n \to \mathbb{R}$ of the form $w(x) = \Omega(|x|)$, where

1. $\Omega : [0, \infty) \to [0, \infty)$ is increasing, continuous and concave,
2. $\Omega(0) = 0$,
3. $\int_{\mathbb{R}} \frac{\Omega(t)}{(1+t^2)}\, dt < \infty$,
4. $\Omega(t) \geqslant a + b \ln(1 + t)$ for some $a \in \mathbb{R}$ and some $b > 0$.

In the next remark we collect a few important consequences of Definition 1 to be used later.

Remark 2. As observed in [4], the first condition in Definition 1 implies that the function w is subadditive, $w(x + y) \leqslant w(x) + w(y)$. For a proof of this result, see Proposition 4.6 in [2]. Condition 4 implies that the function $e^{-Nw(x)}$ is integrable for some $N \in \{0, 1, 2, \ldots\}$. Condition 4 also implies that $w(x) = o(\frac{|x|}{\ln|x|})$ as $|x| \to \infty$, as demonstrated in [4, Corollary 1.2.8].

Definition 3. (See [4,16].) Given $w \in \mathcal{M}_c$, we denote by $\mathcal{S}_w$ the space of C^∞ functions $\varphi : \mathbb{R}^n \to \mathbb{C}$ satisfying the conditions,

$$q_{k,m}(\varphi) = \sup_{|\beta| \leqslant m} \left\| e^{kw} \partial^\beta \varphi \right\|_\infty < \infty, \tag{1}$$

$$q_{k,m} \circ \mathcal{F}(\varphi) = \sup_{|\beta| \leqslant m} \left\| e^{kw} \partial^\beta \hat{\varphi} \right\|_\infty < \infty, \tag{2}$$

for all $k, m = 0, 1, 2, \ldots$. The topological dual of $\mathcal{S}_w$ is the space $\mathcal{S}'_w$ of w-tempered functionals, also called tempered ultradistributions.

We observe that (1) implies that $\varphi \in L^1$ and $\hat{\varphi} \in C^\infty$. So, the formulation of (2) makes sense. The space $\mathcal{S}_w$ is a Fréchet space with the topology defined by the family of norms $\{q_{k,m}, q_{k,m} \circ \mathcal{F}\}_{k,m=0}^\infty$. By a Fréchet space we mean a Hausdorff locally convex topological vector space that is metrizable and complete.

It is clear from the definition that the space $\mathcal{S}_w$ is invariant under the Fourier transform. Furthermore, the conditions imposed on the function w assure that the space $\mathcal{S}_w$ satisfies additional properties that are expected from a space of testing functions intended to generalize the space $\mathcal{S}$. For instance, the operators of differentiation and of multiplication by x^α are continuous from $\mathcal{S}_w$ into itself; the space $\mathcal{S}_w$ is a Fréchet algebra under both pointwise multiplication and convolution, and the Fourier transform is an isomorphism of one structure to the other. Condition 3 is equivalent to the existence of a dense subspace $\mathcal{D}_w$ of $\mathcal{S}_w$, containing partitions of unity. As a consequence of 4, $\mathcal{S}_w \subseteq \mathcal{S}$ continuously and thus, $\mathcal{S}'_w \supseteq \mathcal{S}'$, for all $w \in \mathcal{M}_c$. We refer to [4] and [16, p. 16], for a more detailed discussion of the role played in the properties of $\mathcal{S}_w$ by each of the conditions 1–4 in Definition 1.

When $w(x) = \ln(1 + |x|)$, the space $\mathcal{S}_w$ becomes the Schwartz space $\mathcal{S}$ and the space $\mathcal{D}_w$ becomes C_0^∞. The space $\mathcal{D}_w$ plays for $\mathcal{S}_w$ the role that C_0^∞ plays for $\mathcal{S}$. Another important example of a function in $\mathcal{M}_c$ is $w(x) = |x|^d$ for $0 < d < 1$. The function $|x|$ does not satisfy condition 4 and thus, it does not belong to $\mathcal{M}_c$. As discussed in [4] and [16], this fact implies that the zero function is the only compactly supported function in $\mathcal{S}_{|x|}$, showing the profound difference between exponential decay and almost exponential decay. The results by J. Dziubański and E. Hernández [9] cited in the introduction, are a practical manifestation of this difference.

We note that the inclusions $\mathcal{S}_w \subseteq \mathcal{S}$ and $\mathcal{S}'_w \supseteq \mathcal{S}'$ are strict, in general. In fact, the function $e^{-(1+|x|^2)^\alpha}$, $0 < \alpha < \frac{1}{2}$, belongs to $\mathcal{S}$, but it does not belong to $\mathcal{S}_w$ when $w(x) = |x|^\beta$, $2\alpha < \beta < 1$. On the other hand, the function $f(x) = e^{|x|^\alpha}$, $0 < \alpha < 1$, defines by integration a functional T_f in $\mathcal{S}'_w$ for $w(x) = |x|^\alpha$. Moreover, the functional T_f is a distribution in $\mathcal{D}'$. However, there is no tempered distribution T so that $T_f(\varphi) = T(\varphi)$ for all $\varphi \in C_0^\infty$. In fact, let $\theta \in C_0^\infty$ be a usual cut-off function, $0 \leqslant \theta \leqslant 1$, $\theta(x) = 1$ for $|x| \leqslant 1$ and $\theta(x) = 0$ for $|x| \geqslant 2$. If $\theta_j(x) = \theta(\frac{x}{j})$ and $\varphi \in \mathcal{S}$, the sequence $\{\theta_j \varphi\}$ converges to φ in $\mathcal{S}$ as $j \to \infty$. If there were a tempered distribution T that coincides with T_f on C_0^∞, we would have $T_f(\theta_j\varphi) \to T(\varphi)$ as $j \to \infty$. However, if we pick a nonnegative function φ in $\mathcal{S}$ that is equal to $e^{-|x|^\alpha}$ for $|x| \geqslant 1$, we can write

$$T_f(\theta_j\varphi) = \int_{\mathbb{R}^n} e^{|x|^\alpha}\theta_j(x)\varphi(x)\,dx \geqslant \int_{1\leqslant|x|\leqslant j} dx,$$

which contradicts the convergence of $T_f(\theta_j\varphi)$ as $j \to \infty$.

Remark 4. When $w(x) = \ln(1 + |x|)$, the conditions $q_{k,m}(\varphi) < \infty$, $q_{k,m} \circ \mathcal{F}(\varphi) < \infty$ become redundant, due to the very special role that the function $(1 + |x|)$ plays with respect to the Fourier transform and its inverse. The characterizations of $\mathcal{S}$ and $\mathcal{S}_w$ proved in [11] and [6] avoid this problem, by formulating conditions that turn out to be the same for both spaces. We state now these characterizations in the form given in [1].

Proposition 5. *(See [6,11].) Given $w \in \mathcal{M}_c$, the space $\mathcal{S}_w$ can be described as a set as well as topologically, as*

$$\mathcal{S}_w = \{\varphi : \mathbb{R}^n \to \mathbb{C} : \varphi \text{ is continuous and } q_{k,0}(\varphi) < \infty,\ q_{k,0} \circ \mathcal{F}(\varphi) < \infty \text{ for } k = 0, 1, 2, \ldots\}. \tag{3}$$

Remark 6. The condition $q_{k,0}(\varphi) < \infty$ for $k = 0, 1, 2, \ldots$ implies that $\varphi \in L^1$, so the formulation of the condition $q_{k,0} \circ \mathcal{F}(\varphi) < \infty$ makes sense for $k = 0, 1, 2, \ldots$. Moreover, (3) implies that φ and $\hat{\varphi}$ are C^∞ functions. The two families of norms $\{q_{k,0}\}_k$ and $\{q_{k,0} \circ \mathcal{F}\}_k$ are not equivalent, as can be seen by considering the function $(1 + |x|^2)^{-\frac{\alpha}{2}}$ for $\alpha > 0$. We refer to [17, p. 132] for details.

3. A representation theorem for functionals in the space $\mathcal{S}'_w$

According to Proposition 5, the Fréchet space structure defined in $\mathcal{S}_w$ can be described by the family $\{r_k\}_k$ of norms given by

$$r_k(\varphi) = \|e^{kw}\varphi\|_\infty + \|e^{kw}\hat{\varphi}\|_\infty \tag{4}$$

for $k = 0, 1, 2, \ldots$. We observe that

$$\lim_{|x|\to\infty} (e^{kw}\varphi)(x) = 0,$$
$$\lim_{|x|\to\infty} (e^{kw}\hat{\varphi})(x) = 0,$$

for each $k = 0, 1, 2, \ldots$. Thus, $e^{kw}\varphi$ and $e^{kw}\hat{\varphi}$ both belong to $\mathcal{C}_0$, the Banach space of continuous functions vanishing at infinity, equipped with the supremum norm.

Theorem 7. *Given a functional L in $\mathcal{S}'_w$ there exist two regular complex Borel measures μ_1 and μ_2 of finite total variation and $k \in \{0, 1, 2, \ldots\}$ so that*

$$L = e^{kw}\mu_1 + \mathcal{F}[e^{kw}\mu_2], \tag{5}$$

in the sense of $\mathcal{S}'_w$. Conversely, any pair of such measures and $k \in \{0, 1, 2, \ldots\}$ define as in (5) *a functional in $\mathcal{S}'_w$.*

Proof. Given $L \in \mathcal{S}'_w$, according to (4) there exist k, C so that

$$L(\varphi) \leqslant C(\|e^{kw}\varphi\|_\infty + \|e^{kw}\hat{\varphi}\|_\infty),$$

for all $\varphi \in \mathcal{S}_w$. Moreover, the map

$$\mathcal{S}_w \to \mathcal{C}_0 \times \mathcal{C}_0$$
$$\varphi \to (e^{kw}\varphi, e^{kw}\hat{\varphi})$$

is well defined, linear, continuous and injective. Let $\mathcal{R}$ be the range of this map. We define on $\mathcal{R}$ the map

$$l_1(f, g) = L(\varphi),$$

where $f = e^{kw}\varphi$, $g = e^{kw}\hat{\varphi}$ for a unique $\varphi \in \mathcal{S}_w$. The map $l_1 : \mathcal{R} \to \mathbb{C}$ is linear and continuous. By the Hahn–Banach theorem there exists a functional L_1 in the topological dual $(\mathcal{C}_0 \times \mathcal{C}_0)'$ of $\mathcal{C}_0 \times \mathcal{C}_0$ such that $\|L_1\| = \|l_1\|$ and the restriction of L_1 to $\mathcal{R}$ is l_1.

The spaces $(\mathcal{C}_0 \times \mathcal{C}_0)'$ and $\mathcal{C}_0' \times \mathcal{C}_0'$ are isomorphic, as Banach spaces, since we can write $L_1(f,g) = L_1(f,0) + L_1(0,g)$. Using the classical F. Riesz representation theorem (see for instance [15, p. 130]), there exist regular complex Borel measures μ_1 and μ_2 of finite total variation so that

$$L_1(f,g) = \int_{\mathbb{R}^n} f\,d\mu_1 + \int_{\mathbb{R}^n} g\,d\mu_2,$$

for all $(f,g) \in \mathcal{C}_0 \times \mathcal{C}_0$. If $(f,g) \in \mathcal{R}$, we conclude

$$L(\varphi) = \int_{\mathbb{R}^n} e^{kw}\varphi\,d\mu_1 + \int_{\mathbb{R}^n} e^{kw}\hat{\varphi}\,d\mu_2,$$

for all $\varphi \in \mathcal{S}_w$. Or, in the sense of the pairing $(\mathcal{S}_w', \mathcal{S}_w)$,

$$L = e^{kw}\mu_1 + \mathcal{F}\big[e^{kw}\mu_2\big].$$

This completes the proof of Theorem 7. □

Remark 8. When $w(x) = (1+|x|)$, (5) becomes

$$L = \big(1+|x|\big)^k \mu_1 + \mathcal{F}\big[\big(1+|\xi|\big)^k \mu_2\big],$$

which gives a new way of representing tempered distributions.

G. Björck defines [4, p. 373] the convolution $T * \varphi$ of a functional T and a test function φ as the function given by $(T, \varphi(z - \cdot))$. Using this definition, he proves in Theorem 1.8.12 of [4] that the convolution $T * \varphi$ belongs to $\mathcal{S}_w'$ when $T \in \mathcal{S}_w'$ and $\varphi \in \mathcal{S}_w$. As a first application of Theorem 7, we will show now that the definition of convolution used by G. Björck coincides with the classical definition $(U * V, \psi) = (U_x, (V_z, \psi(x+z)))$. Here we indicate with a subscript the variable on which the functional acts.

Lemma 9. *Given $L \in \mathcal{S}_w'$ and $\varphi \in \mathcal{S}_w$, we can use the classical definition of $L * \varphi$,*

$$(L * \varphi, \psi)_{\mathcal{S}_w', \mathcal{S}_w} = \big(L_x, \big(\varphi_z, \psi(x+y)\big)\big)_{\mathcal{S}_w', \mathcal{S}_w}$$

for all $\psi \in \mathcal{S}_w$. Furthermore, the functional $L * \varphi$ coincides with the functional given by integration against the function $f(y) = (L, \varphi(y - \cdot))_{\mathcal{S}_w', \mathcal{S}_w}$.

Proof. First we observe that the function $f(y) = (L_x, \varphi(y-x))_{\mathcal{S}_w', \mathcal{S}_w}$ is continuous. Due to the inequality

$$\big|f(y)\big| \leqslant C\Big(\sup_x \big|e^{kw(x)}\varphi(y-x)\big| + \sup_x \big|e^{kw(x)}\hat{\varphi}(x)\big|\Big)$$

and the subadditivity of the function w (see Remark 2), the pairing (f, ψ) is well defined by means of integration. Then, using (5), we can write for each y,

$$\big(L_x, \varphi(y-x)\big)_{\mathcal{S}_w', \mathcal{S}_w} = \int_{\mathbb{R}^n} e^{kw(x)}\varphi(y-x)\,d\mu_1(x) + \big(\big(e^{kw}\mu_2\big)(\xi), e^{-2\pi i y\xi}\,\overline{\mathcal{F}}(\varphi)(\xi)\big)_{\mathcal{S}_w', \mathcal{S}_w}.$$

So,

$$\big(\big(L_x, \varphi(y-x)\big)_{\mathcal{S}_w', \mathcal{S}_w}, \psi\big) = \int_{\mathbb{R}^n} e^{kw(x)}\bigg(\int_{\mathbb{R}^n} \varphi(y-x)\psi(y)\,dy\bigg)\,d\mu_1(x) + \int_{\mathbb{R}^n} e^{kw(\xi)}\,\overline{\mathcal{F}}(\varphi)(\xi)\hat{\psi}(\xi)\,d\mu_2(\xi). \tag{6}$$

We note that $\overline{\mathcal{F}}(\varphi)(\xi)\hat{\psi}(\xi) = \mathcal{F}[\check{\varphi} * \psi]$, where $\check{\varphi}(x) = \varphi(-x)$. Thus, we can write the second term in (6) as

$$\big(\mathcal{F}\big[e^{kw}\mu_2\big], \check{\varphi} * \psi\big)_{\mathcal{S}_w', \mathcal{S}_w}.$$

This implies that

$$\begin{aligned}\big((L_x, \varphi(y-x))_{\mathcal{S}'_w,\mathcal{S}_w}, \psi\big) &= \big(e^{kw(x)}\mu_1(x), (\varphi(y-x), \psi(y))\big)_{\mathcal{S}'_w,\mathcal{S}_w} \\ &\quad + \big(\mathcal{F}\big[e^{kw}\mu_2\big](x), (\varphi(y-x), \psi(y))\big)_{\mathcal{S}'_w,\mathcal{S}_w} \\ &= \big(L_x, (\varphi(y-x), \psi(y))\big)_{\mathcal{S}'_w,\mathcal{S}_w},\end{aligned}$$

for all $\psi \in \mathcal{S}_w$. This completes the proof of Lemma 9. □

We now mention a further application of Theorem 7 and Lemma 9 to the study of operator semigroups acting on w-tempered distributions. To begin, we recall the definition of a semigroup [12] as a collection of linear operators $\{E_t\}_{t\geqslant 0}$ on a Banach space $\mathcal{X}$ satisfying the following conditions:

$$E_0 = \mathrm{id}_{\mathcal{X}},$$
$$E_t E_s = E_{t+s}$$

and the map

$$t \to E_t\phi$$

is continuous for each $\phi \in \mathcal{X}$. Semigroups that are of interest to us are those defined by convolution operators acting on the Fréchet space $\mathcal{X} = \mathcal{S}_w$. In the sequel, our goal will be to extend their action to the dual space $\mathcal{S}'_w$. From Lemma 9, we make the following observation:

Corollary 10. *Given $G \in S_w$, let $\{E_t\}_{t\geqslant 0}$ be the semigroup defined, for $t > 0$, by the convolution kernel $t^{-n}G(\frac{x-y}{t})$. Then, the action of E_t on $\mathcal{S}'_w$ is given by integration against the function*

$$\rho(x) = \left(L_y, t^{-n}G\left(\frac{x-y}{t}\right)\right)_{\mathcal{S}'_w,\mathcal{S}_w}. \tag{7}$$

Proof. It suffices to show that $G(\frac{x}{t})$ is in $\mathcal{S}_w$ for each $t > 0$. We fix $t > 0$, and choose an integer N such that $N \geqslant t$. Then, by the properties of w (see Definition 1 and Remark 2), it follows that $w(x) \leqslant Nw(\frac{x}{N}) \leqslant Nw(\frac{x}{t})$. Thus, for every x, we have

$$\left|e^{kw(x)}G\left(\frac{x}{t}\right)\right| \leqslant \left|e^{Nkw(\frac{x}{t})}G\left(\frac{x}{t}\right)\right| \leqslant \left\|e^{Nkw}G\right\|_\infty.$$

Now, we consider $\mathcal{F}[G(\frac{x}{t})](\xi) = t^n\widehat{G}(t\xi)$. If $t \geqslant 1$, then $w(x) \leqslant w(tx)$, and hence

$$\left|e^{kw(x)}\widehat{G}(t\xi)\right| \leqslant \left|e^{kw(tx)}\widehat{G}(t\xi)\right| \leqslant \left\|e^{kw}\widehat{G}\right\|_\infty.$$

As for $0 < t < 1$, we choose an integer N such that $N \geqslant \frac{1}{t}$, so, we have $w(x) \leqslant Nw(\frac{x}{N}) \leqslant Nw(tx)$ and thus

$$\left|e^{kw(x)}\widehat{G}(t\xi)\right| \leqslant \left|e^{Nkw(tx)}\widehat{G}(t\xi)\right| \leqslant \left\|e^{Nkw}\widehat{G}\right\|_\infty.$$

Using Proposition 5, the result follows. □

While Corollary 10 gives sufficient conditions for an operator semigroup $\{E_t\}_{t\geqslant 0}$ to be well defined on $\mathcal{S}'_w$ for a fixed $w \in \mathcal{M}$, we are interested in studying semigroups that may be defined on $\mathcal{S}'_w$ for any such w. To this end, we consider the intersection $\bigcap_{w\in\mathcal{M}_c}\mathcal{S}_w$ as a class of functions from which to obtain convolution kernels.

4. On the intersection of the spaces S_w

Our goal in this section is to characterize the space $\bigcap_{w\in\mathcal{M}_c}\mathcal{S}_w$, as this will yield sufficient conditions for the convolution operators given by (7) to be well defined on $\mathcal{S}'_w$ for all $w \in \mathcal{M}_c$. To begin, we let $a = \{a_k\}_{k=0}^\infty$ be a sequence of positive real numbers such that $\sum_{k\geqslant 0}\frac{1}{a_k} < \infty$ and $\frac{a_k}{k!^{1/k}}$ is increasing for $k \geqslant 2$. We define the function $\Omega_a : \mathbb{R}_+ \to \mathbb{R}$ as

$$\Omega_a(t) = \sum_{k \geqslant 0} \frac{t^k}{a_k^k}. \tag{8}$$

From [4, Theorem 1.5.11], we have that the function $w_a(x) = \ln \Omega_a(|x|)$ is in $\mathcal{M}_c$. Next, we define the following subspace of $\mathcal{C}_0$:

Definition 11. Let $\mathcal{S}_0$ be the set of all continuous functions f for which there exists a constant $C > 0$ such that

$$\left\| \frac{|x|^k f}{(k \ln k)^k} \right\|_\infty + \left\| \frac{|\xi|^k \hat{f}}{(k \ln k)^k} \right\|_\infty \leqslant C^{k+1},$$

for all integers $k \geqslant 2$.

We seek to prove the following assertion concerning the intersection over w of the spaces $\mathcal{S}_w$:

Theorem 12. $\bigcap_{w \in \mathcal{M}_c} \mathcal{S}_w = \mathcal{S}_0$.

Proof. Here, we will use the characterization of $\mathcal{S}_w$ given by Proposition 5. We first establish that $\mathcal{S}_0 \subseteq \bigcap_{w \in \mathcal{M}_c} \mathcal{S}_w$. If $f \in \mathcal{S}_0$, there exists $C > 0$ such that for every $k \geqslant 2$ we have

$$\left| f(x) \right| \leqslant C \left(\frac{Ck \ln k}{|x|} \right)^k \tag{9}$$

and similarly for $|\hat{f}(\xi)|$. We claim that this is sufficient to ensure that $f \in \mathcal{S}_w$ for every $w \in \mathcal{M}_c$ [4, Lemma 1.5.13]. Indeed, let $b > 2$, $C > 0$, and for $t > e$ we consider the function

$$h_b(t) = \left(\frac{Ct \ln t}{b} \right)^t.$$

We define $t_b = \frac{b}{Ce \ln b}$, and note that $t_b \to \infty$ as $b \to \infty$. It follows that $t_b > e$ for sufficiently large b and we have

$$h_b(t_b) = \left(\frac{1 - \frac{\ln(Ce \ln b)}{\ln b}}{e} \right)^{\frac{b}{Ce \ln b}}.$$

Furthermore, since $\frac{\ln(Ce \ln b)}{\ln b} \to 0^+$ as $b \to \infty$, it follows that for sufficiently large b we have the inequality

$$h_b(t_b) \leqslant \left(\frac{1}{e} \right)^{\frac{b}{Ce \ln b}} = e^{-\frac{b}{Ce \ln b}}. \tag{10}$$

Now, from an explicit calculation we find that $h_b'' > 0$ for $t > e$. Thus, for sufficiently large b, it follows that the supremum of $h_b(t)$ on the interval $[t_b - 1, t_b + 1]$ is given by the maximum of $h_b(t_b - 1)$ and $h_b(t_b + 1)$. Let us first consider $h_b(t_b + 1)$. Again, if b is sufficiently large, we have the bound

$$h_b(t_b + 1) = \left(\frac{C(t_b + 1) \ln(t_b + 1)}{b} \right)^{t_b + 1} \leqslant \left(e^{1/2} \frac{Ct_b \ln t_b}{b} \right)^{t_b + 1},$$

since $(x + 1) \ln(x + 1) \leqslant e^{1/2} x \ln x$ for sufficiently large x. Coupling this with (10), we obtain

$$h_b(t_b + 1) \leqslant e^{-\frac{b}{2Ce \ln b}}.$$

A similar argument reveals that for sufficiently large b, we have

$$\begin{aligned} h_b(t_b - 1) &= \left(\frac{C(t_b - 1) \ln(t_b - 1)}{b} \right)^{t_b - 1} \\ &\leqslant \left(\frac{Ct_b \ln t_b}{b} \right)^{t_b - 1} \leqslant e^{-\frac{b}{2Ce \ln b}}. \end{aligned}$$

Hence, $h_b(t) \leqslant e^{-\frac{b}{2Ce\ln b}}$ on $[t_b-1, t_b+1]$. Applying this to (9), it follows that for sufficiently large $|x|$ there is an integer $k \in [t_{|x|}-1, t_{|x|}+1]$ with $t_{|x|}-1 > e$, and thus, we obtain the estimate

$$\left|f(x)\right| \leqslant Ce^{-\frac{|x|}{2Ce\ln|x|}}.$$

Now, let us fix $w \in \mathcal{M}_c$. Since $w(x) = o(\frac{|x|}{\ln|x|})$ as $|x| \to \infty$, it follows that for any $\lambda \in \mathbb{N}$, there is $C_\lambda > 0$ for which

$$\left|f(x)\right| \leqslant C_\lambda e^{-\lambda w(x)},$$

for all $x \in \mathbb{R}^n$. An analogous result holds for $|\hat{f}(\xi)|$, and the inclusion $\mathcal{S}_0 \subseteq \bigcap_{w\in\mathcal{M}_c} \mathcal{S}_w$ is established.

We now prove the opposite inclusion via the contrapositive. Let f be a continuous function not in $\mathcal{S}_0$. Our goal is to find $w \in \mathcal{M}_c$ for which $f \notin \mathcal{S}_w$. We assume that $f \in \mathcal{S}$, the Schwartz space, and we define $G_k = \||x|^k f\|_\infty^{1/k}$, for $k \geqslant 2$. Without loss of generality, we may assume that the sequence $\{\frac{G_k}{k\ln k}\}_{k\geqslant 2}$ is unbounded. Otherwise, we can define G_k using $\hat{f}$ as opposed to f. From [5, Theorems 2 and 7], there exists a positive sequence $\{a_k\}$ with $\sum_{k\geqslant 0}\frac{1}{a_k} < \infty$ and $\frac{a_k}{k}$ increasing for $k \geqslant 1$, such that the sequence $\{\frac{G_k}{a_k}\}$ is also unbounded. Thus, for any $M \geqslant 1$, there is $k \geqslant 1$ and $x \in \mathbb{R}^n$ such that $|\frac{|x|^k f(x)}{a_k^k}| > M^k \geqslant M$. In other words, the function

$$\sum_{k\geqslant 0}\frac{|x|^k f(x)}{a_k^k} = \Omega_a\big(|x|\big) f(x),$$

where Ω_a is defined as in (8), is unbounded. Hence, it suffices to show that $w_a(x) = \ln\Omega_a(x)$ is in $\mathcal{M}_c$, i.e., that $\frac{a_k}{k!^{1/k}}$ is increasing for $k \geqslant 2$. Since $\frac{a_k}{k}$ is increasing for $k \geqslant 1$, we find that for $k \geqslant 2$ the inequality

$$\frac{a_k}{a_{k-1}} \geqslant \frac{k}{k-1}$$

holds. We observe that

$$\frac{k}{k-1} \geqslant \frac{k!^{1/k}}{(k-1)!^{1/(k-1)}}$$

holds for $k \geqslant 2$ and therefore $w \in \mathcal{M}_c$. We conclude that $f \notin \bigcap_{w\in\mathcal{M}_c} \mathcal{S}_w$, and the theorem is proved. $\square$

Remark 13. Using the fact that $\xi^\alpha \hat{f} = (\frac{-1}{2\pi i})^{|\alpha|}\mathcal{F}[\partial^\alpha f]$, we have the following sufficient condition to ensure that a smooth function f belongs to $\mathcal{S}_0$: There exists $C \geqslant 0$ such that

$$\left\|\frac{|x|^k f}{(k\ln k)^k}\right\|_\infty + \sup_{|\alpha|=k}\left\|\frac{\partial^\alpha f}{(k\ln k)^k}\right\|_1 \leqslant C^{k+1},$$

for $k \geqslant 2$.

Remark 14. Let $\{a_k\}$ be an increasing sequence of positive numbers. We recall the definition [4, Definition 1.5.3] of the Denjoy–Carleman class $\mathcal{C}^{\{a_k\}}$ as those smooth functions $f : \mathbb{R}^n \to \mathbb{C}$ for which there exists $C > 0$ such that

$$\sup_{|\alpha|=k}\left\|\frac{\partial^\alpha f}{a_k^k}\right\|_\infty \leqslant C^{k+1}.$$

The class $\mathcal{C}^{\{a_k\}}$ is said to be nonquasi-analytic if it contains nontrivial functions of compact support. In turn, this is equivalent to the condition $\sum_{k\geqslant 0}\frac{1}{a_k} < \infty$ [4, Theorem 1.5.5]. A result in [4, Theorem 1.5.12], proves that $\mathcal{S}_0 \subseteq C^{\{k\ln k\}}$. Since $\sum_{k\geqslant 2}\frac{1}{k\ln k}$ diverges to ∞, it follows that $\mathcal{S}_0$ is quasi-analytic and thus, does not contain any nontrivial function of compact support. This implies that $\bigcap_{w\in\mathcal{M}_c}\mathcal{D}_w = \{0\}$. That the intersection $\bigcap_{w\in\mathcal{M}_c}\mathcal{D}_w$ is trivial was already observed in [4].

5. Convolution operators on $\mathcal{S}'_w$ with kernels in $\mathcal{S}_0$

In this section, we study the action of a specific class of operator semigroups on the space $\mathcal{S}'_w$of w-tempered distributions. We consider the semigroup $\{E_t\}_{t\geqslant 0}$ defined on $\mathcal{S}_w$ as the convolution with the kernel $t^{-n}G(\frac{\cdot}{t})$,

$$E_t(\varphi)(x) = \left(t^{-n}G\left(\frac{x-y}{t}\right), \varphi(y)\right), \tag{11}$$

for $t > 0$.

As a first example, we mention the Gauss–Weierstrass semigroup $\{T_t\}_{t\geqslant 0}$ defined by integration with respect to the heat kernel,

$$T_{\sqrt{t}}(\varphi)(x) = \left(\frac{1}{(4\pi t)^{n/2}}e^{-\frac{|x-y|^2}{4t}}, \varphi(y)\right) = \left(t^{-n/2}H\left(\frac{x-y}{t^{1/2}}\right), \varphi(y)\right),$$

for $t > 0$. It is well known [12] that this is the semigroup $T_{\sqrt{t}} = e^{t\Delta}$ generated by the Laplacian Δ on $\mathbb{R}^n$: Given an appropriate $\varphi : \mathbb{R}^n \to \mathbb{R}$, the function $u : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}$ defined as $u(x,t) = T_{\sqrt{t}}(\varphi)(x)$ is a solution to the heat equation $u_t - \Delta u = 0$ with boundary $u(x, 0) = \varphi(x)$, which means

$$\varphi(x) = u(x, 0) = \lim_{t\to 0^+} T_{\sqrt{t}}(\varphi)(x),$$

where the convergence is uniform on bounded subsets of $\mathbb{R}^n$. Another semigroup that we consider is the Ornstein–Uhlenbeck semigroup $K_{e^{-t}} = e^{tA}$, generated by the Ornstein–Uhlenbeck operator

$$A = \frac{1}{2}\Delta - x\cdot\nabla$$

acting on the Hilbert space $L^2(\mathbb{R}^n, e^{-|x|^2}dx)$. The action of this semigroup is given by integration with respect to the Mehler kernel [12]

$$e^{tA}\varphi(x) = K_r(\varphi)(x) = \left(\frac{e^{-\frac{|y-rx|^2}{1-r^2}}}{\pi^{n/2}(1-r)^{n/2}}, \varphi(y)\right) = \big(M_r(x,y), \varphi(y)\big), \tag{12}$$

where we have set $r = e^{-t} \in (0, 1)$. Here, the function $u(x,t) = K_r(\varphi)(x)$ solves the boundary value problem $u_t - Au = 0$, $u(x, 0) = \varphi(x)$. The boundary condition is interpreted as

$$\varphi(x) = u(x, 0) = \lim_{r\to 1^-} K_r(\varphi)(x),$$

with uniform convergence on bounded subsets of $\mathbb{R}^n$. We note that both the Gauss–Weierstrass and the Mehler kernels are nonnegative and satisfy

$$\left\|t^{-n/2}H\left(\frac{x-\cdot}{t^{1/2}}\right)\right\|_1 = 1,$$
$$\big\|M_r(x,\cdot)\big\|_1 = 1,$$

for every $t > 0$, $0 < r < 1$ and $x \in \mathbb{R}^n$. This is a general property of the class of symmetric diffusion semigroups to which they belong (cf. [12,18]), and will be part of the hypotheses we place on our general convolution kernels G in the sequel. As stated in Remark 2, $w(x) = o(\frac{|x|}{\ln|x|})$ when $|x| \to \infty$ and as a consequence, the Gauss–Weierstrass kernel and the Mehler kernel both belong to the space $\mathcal{S}_0$. Thus, we may apply Corollary 10 to conclude that the operators $T_{\sqrt{t}}$ and K_r are well defined on $\mathcal{S}'_w$ for every $w \in \mathcal{M}_c$. What interests us is whether we can prove the existence of the limits

$$\lim_{t\to 0^+} T_{\sqrt{t}}(L) = L,$$
$$\lim_{r\to 1^-} K_r(L) = L \tag{13}$$

in the sense of the strong dual topology of S'_w. Doing so would complete our assertion that the w-tempered distributions can be realized as boundary values to solutions of the generalized heat equation.

Let $\{E_t\}_{t\geqslant 0}$ be the semigroup defined by (11) for $t > 0$. We assume that the kernel $t^{-n}G(\frac{\cdot}{t})$ belongs to S_0, it is nonnegative and satisfies $\|t^{-n}G(\frac{\cdot}{t})\|_1 = 1$, for each $t > 0$. Using Corollary 10 and Theorem 12, the action of E_t on $L \in \mathcal{S}'_w$, for any $w \in \mathcal{M}_c$, is found by defining $E_t(L)$ to be the functional given by integration against the function $\rho(x)$ of (7). Using Lemma 9, this is equivalent to setting

$$\big(E_t(L), \varphi\big)_{\mathcal{S}'_w,\mathcal{S}_w} = \left(L_y, \left(t^{-n}G\left(\frac{x-y}{t}\right), \varphi(x)\right)\right)_{\mathcal{S}'_w,\mathcal{S}_w}.$$

Then, the existence of the limit

$$\lim_{t\to 0^+} E_t(L) = L,$$

in the sense of the strong dual topology of S'_w, is equivalent to

$$\left(t^{-n}G\left(\frac{x-\cdot}{t}\right), \varphi(x)\right)_{\mathcal{S}'_w,\mathcal{S}_w} \to \varphi$$

in $\mathcal{S}_w$, as $t \to 0^+$, uniformly on bounded subsets of $\mathcal{S}_w$. This is the content of our next result.

Theorem 15. *Let $t^{-n}G(\frac{\cdot}{t})$ be as above and let B be a fixed bounded subset of $\mathcal{S}_w$. Then*

$$\left(t^{-n}G\left(\frac{x-\cdot}{t}\right), \varphi(x)\right)_{\mathcal{S}'_w,\mathcal{S}_w} \to \varphi$$

in $\mathcal{S}_w$, as $t \to 0^+$, uniformly for $\varphi \in B$.

Proof. We recall Definition 3 and Proposition 5 concerning the structure of $\mathcal{S}_w$. From this, it suffices to estimate the norms

$$\left\| e^{kw}\left[\left(t^{-n}G\left(\frac{x-\cdot}{t}\right), \varphi(x)\right) - \varphi\right]\right\|_\infty$$

and

$$\left\| e^{kw}\mathcal{F}\left[t^{-n}\left(G\left(\frac{x-\cdot}{t}\right), \varphi(x)\right) - \varphi\right]\right\|_\infty$$

given by Proposition 5 in terms of the norms $\|e^{kw}\partial^\alpha\varphi\|_\infty$ and $\|e^{kw}\partial^\alpha\hat{\varphi}\|_\infty$ given by Definition 3. We assume in all that follows that $t < 1$. We can write

$$\begin{aligned} e^{kw(y)}\left|\left(t^{-n}G\left(\frac{x-y}{t}\right), \varphi(x)\right) - \varphi(y)\right| &= \int_{\mathbb{R}^n} e^{kw(y)}G(z)\big|\varphi(y+tz) - \varphi(y)\big|\,dz \\ &\leqslant I_1 + I_2 + I_3, \end{aligned}$$

where we have defined

$$\begin{aligned} I_1 &= \int_{|z|\leqslant M} e^{kw(y)}G(z)\big|\varphi(y+tz) - \varphi(y)\big|\,dz, \\ I_2 &= \int_{|z|\geqslant M} e^{kw(y)}G(z)\big|\varphi(y+tz)\big|\,dz, \\ I_3 &= \int_{|z|\geqslant M} e^{kw(y)}G(z)\big|\varphi(y)\big|\,dz, \end{aligned}$$

for $M > 0$ to be determined later. We begin by estimating I_1. Using the Mean Value Theorem and the subadditivity of w, we may estimate I_1 as

$$I_1 \leqslant \int\limits_{|z|\leqslant M} e^{kw(y-y')} G(z) t M \left| e^{kw(y')} \nabla \varphi(y') \right| dz, \tag{14}$$

for some y' in the line segment between y and $y+tz$. Now, we note that $w(y-y') = \Omega(|y-y'|) \leqslant \Omega(|tz|) \leqslant \Omega(|z|) = w(z)$, since $t<1$. We also remark that $\|e^{kw}G\|_1 < \infty$. Thus, we obtain the estimate

$$I_1 \leqslant tM \left\| e^{kw} G \right\|_1 \left\| e^{kw} \nabla \varphi \right\|_\infty.$$

We observe that there is a constant $C>0$, depending only on the dimension n, for which $\|e^{kw} \nabla \varphi\|_\infty \leqslant C \sup_{|\alpha|=1} \|e^{kw}\partial^\alpha \varphi\|_\infty$. This leads to the inequality

$$I_1 \leqslant tMC \left\| e^{kw} G \right\|_1 \sup_{|\alpha|=1} \left\| e^{kw} \partial^\alpha \varphi \right\|_\infty. \tag{15}$$

Next, we consider I_2. Again, using the subadditivity of w, we obtain the bound

$$I_2 \leqslant \int\limits_{|z|\geqslant M} e^{kw(-tz)} G(z) \left| e^{kw(y+tz)} \varphi(y+tz) \right| dz.$$

Since $w(-tz) = w(tz) \leqslant w(z)$, this reads

$$I_2 \leqslant \left\| e^{kw} \varphi \right\|_\infty \int\limits_{|z|\geqslant M} e^{kw(z)} G(z)\, dz. \tag{16}$$

Finally, for I_3 we immediately obtain

$$I_3 \leqslant \left\| e^{kw} \varphi \right\|_\infty \int\limits_{|z|\geqslant M} e^{kw(y)} G(z)\, dz. \tag{17}$$

Thus, with k fixed and given any $\varepsilon > 0$, we may choose M sufficiently large so that both $\int_{|z|\geqslant M} e^{kw(z)} G(z)\, dz$ in (16) and $\int_{|z|\geqslant M} e^{kw(y)} G(z)\, dz$ in (17) are $< \varepsilon$. Then, choosing t sufficiently small in (15), we can ensure that $tMC\|e^{kw}G\|_1 < \varepsilon$ as well.

Now, for each integer $j \geqslant 0$ and multi-index β, there exists $C_{j,\beta} > 0$ such that $\|e^{jw}\partial^\beta \varphi\|_\infty \leqslant C_{j,\beta}$ for all $\varphi \in B$. From this estimate, we conclude that the expression

$$\left\| e^{kw} \left[\left(t^{-n} G\left(\frac{x - \cdot}{t} \right), \varphi(x) \right) - \varphi \right] \right\|_\infty$$

approaches 0 in $\mathcal{S}_w$ as $t \to 0^+$, uniformly on B.

Next, we consider

$$e^{kw(\xi)} \left| \mathcal{F}\left[t^{-n} \left(G\left(\frac{x - \cdot}{t} \right), \varphi(x) \right) - \varphi \right](\xi) \right| = \left| e^{kw(\xi)} \hat{\varphi}(\xi) \left[\widehat{G}(t\xi) - 1 \right] \right|.$$

We observe that

$$\left| \widehat{G}(t\xi) - 1 \right| \to 0,$$

as $t \to 0^+$, uniformly on compact subsets of $\mathbb{R}^n$. We note also that for any integer $j \geqslant 0$ and multi-index β, there is $C_{j,\beta} > 0$ such that $\|e^{jw}\partial^\beta \hat{\varphi}\|_\infty \leqslant C_{j,\beta}$ uniformly on B. In particular,

$$\left| e^{kw(\xi)} \hat{\varphi}(\xi) \right| \leqslant C_{k+1,0}\, e^{-w(\xi)},$$

uniformly on B. Thus, we may choose M sufficiently large so that $|\xi| \geqslant M$ implies $|e^{kw(\xi)}\hat{\varphi}(\xi)| < \varepsilon$, uniformly on B. So, for such ξ the estimate

$$\left| e^{kw(\xi)} \hat{\varphi}(\xi) \left[\widehat{G}(t\xi) - 1 \right] \right| \leqslant \varepsilon \|\widehat{G} - 1\|_\infty$$

holds. We may then choose t small enough so that $|\widehat{G}(t\xi) - 1| < \varepsilon$, whenever $|\xi| \leqslant M$.

Thus, the convergence of

$$\left\| e^{kw}\mathcal{F}\left[t^{-n}\left(G\left(\frac{x-\cdot}{t}\right),\varphi(x)\right)-\varphi\right]\right\|_{\infty}$$

to 0, uniformly on B, is established. This completes the proof of Theorem 15. □

Remark 16. Returning to the discussion of the Gauss–Weierstrass semigroup, we find, using Theorem 15, that the operator $T_{\sqrt{t}}(L)$ converges to L in the strong dual topology of $\mathcal{S}'_w$ as $t \to 0^+$.

In attempting to apply the result of Theorem 15 to the Ornstein–Uhlenbeck operator, we need to be careful of the fact that we cannot directly express the Mehler kernel $M_r(x,y)$ in the form $t^{-n}G(\frac{x-y}{t})$ for some function $G \in \mathcal{S}_0$. Instead, we observe that since $M_r(\cdot,y)$ and $M_r(x,\cdot)$ are both in $\mathcal{S}_0$, we can write the action of K_r on $\mathcal{S}'_w$ explicitly using the classical definition ([4, p. 373] and Lemma 9) as

$$(L * K_r,\varphi)_{\mathcal{S}'_w,\mathcal{S}_w} = \bigl(L_y,\bigl(M_r(x,y),\varphi(x)\bigr)\bigr)_{\mathcal{S}'_w,\mathcal{S}_w}.$$

Thus, to obtain the limit $L * K_r \to L$ as $r \to 1^-$ in the strong dual topology, it suffices to prove the following result:

Proposition 17. *Let B be a bounded subset of $\mathcal{S}_w$. Then,*

$$\bigl(M_r(x,\cdot),\varphi(x)\bigr)_{\mathcal{S}'_w,\mathcal{S}_w} \to \varphi$$

in $\mathcal{S}_w$, as $r \to 1^-$, uniformly on B.

Proof. The proof is similar to that of Theorem 15, but with a few adjustments. We begin again with

$$\left| e^{kw(y)}\left[\int_{\mathbb{R}^n} M_r(x,y)\varphi(x)\,dx - \varphi(y)\right]\right|.$$

Noting that $\int_{\mathbb{R}^n} M_r(x,y)\,dx = r^{-n}$, we may estimate this as

$$\leqslant \int_{\mathbb{R}^n} e^{kw(y)} M_r(x,y)\bigl|\varphi(x) - r^n\varphi(y)\bigr|\,dx,$$

which we then estimate using the two terms

$$\leqslant \int_{\mathbb{R}^n} e^{kw(y)} M_r(x,y)\bigl|\varphi(x)-\varphi(y)\bigr|\,dx + \bigl(1-r^n\bigr)\int_{\mathbb{R}^n} e^{kw(y)} M_r(x,y)\bigl|\varphi(y)\bigr|\,dx.$$

Introducing the formula given in (12) for the Mehler kernel M_r and making the substitution $z = \frac{y-rx}{\sqrt{1-r^2}}$, this reads

$$\leqslant \pi^{-n/2}r^{-n}\int_{\mathbb{R}^n} e^{kw(y)}e^{-|z|^2}\left|\varphi\left(\frac{y-z\sqrt{1-r^2}}{r}\right)-\varphi(y)\right|dz$$
$$+\pi^{-n/2}r^{-n}\bigl(1-r^n\bigr)\int_{\mathbb{R}^n} e^{kw(y)}e^{-|z|^2}\bigl|\varphi(y)\bigr|\,dz. \tag{18}$$

We observe that the second term in (18) is bounded by $\pi^{-n/2}r^{-n}(1-r^n)\|e^{-|z|^2}\|_1\|e^{kw}\varphi\|_\infty$, and hence approaches 0 as $r \to 1^-$, uniformly on B. As for the first term, we estimate it as

$$\leqslant \pi^{-n/2}r^{-n}\int_{\mathbb{R}^n} e^{kw(y)}e^{-|z|^2}\left|\varphi\left(\frac{y-z\sqrt{1-r^2}}{r}\right)-\varphi\left(\frac{y}{r}\right)\right|dz$$
$$+\pi^{-n/2}r^{-n}\int_{\mathbb{R}^n} e^{kw(y)}e^{-|z|^2}\left|\varphi\left(\frac{y}{r}\right)-\varphi(y)\right|dz. \tag{19}$$

To prove that the first term in (19) approaches 0 as $r \to 1^-$, uniformly on B, we can proceed essentially as in the proof of Theorem 15. We will omit the details.

As for the second term, we note that applying the Mean Value Theorem, there is z' in the line segment between $\frac{y}{r}$ and y such that $|\varphi(\frac{y}{r}) - \varphi(y)| = \frac{1-r}{r}|y||\nabla\varphi(z')|$. Since $|y| \leqslant |z'|$, we estimate the second term in (19) as

$$\leqslant \pi^{-n/2} r^{-n} \frac{1-r}{r} \int_{\mathbb{R}^n} e^{kw(z')} e^{-|z|^2} \left|\nabla\varphi(z')\right| |y| \, dz. \tag{20}$$

Next, we recall from condition 4 in Definition 1 that there is a positive integer N and a constant $C > 0$ for which $|y| \leqslant Ce^{Nw(y)} \leqslant Ce^{Nw(z')}$. Substituting this into (20), we obtain the estimate

$$\leqslant \pi^{-n/2} r^{-n} \frac{1-r}{r} \left\|e^{-|z|^2}\right\|_1 C \sup_{|\alpha|=1} \left\|e^{(N+k)w} \partial^\alpha \varphi\right\|_\infty, \tag{21}$$

where, as in (15), the constant $C > 0$ only depends on the dimension n. This term clearly approaches 0 as $r \to 1^-$, uniformly on B. So, we conclude that the term

$$\left\| e^{kw} \left[\int_{\mathbb{R}^n} M_r(x, \cdot)\varphi(x)\, dx - \varphi \right] \right\|_\infty$$

does as well.

As for

$$\left| e^{kw(\xi)} \mathcal{F} \left[\int_{\mathbb{R}^n} M_r(x, y)\varphi(x)\, dx - \varphi(y) \right](\xi) \right|, \tag{22}$$

we use again the expression given in (12) for M_r and the Fubini–Tonelli theorem to rewrite (22) as

$$e^{kw(\xi)} \left| r^{-n} e^{-\frac{(1-r^2)|\xi|^2}{4}} \hat{\varphi}(r\xi) - \hat{\varphi}(\xi) \right|.$$

This expression can be estimated as

$$\leqslant e^{kw(\xi)} \left| \left(r^{-n} e^{-\frac{(1-r^2)|\xi|^2}{4}} - 1 \right) \hat{\varphi}(r\xi) \right| + e^{kw(\xi)} \left| \hat{\varphi}(r\xi) - \varphi(\xi) \right|.$$

If we choose an integer $M > 2$, then $M > \frac{1}{r}$ for all $r \in (\frac{1}{2}, 1)$. From the subadditivity of w, we find that $w(\xi) \leqslant Mw(r\xi)$ for all such r and this establishes the inequality

$$\leqslant e^{Mkw(r\xi)} \left| \left(r^{-n} e^{-\frac{(1-r^2)|\xi|^2}{4}} - 1 \right) \hat{\varphi}(r\xi) \right| + e^{Mkw(r\xi)} \left| \hat{\varphi}(r\xi) - \varphi(\xi) \right|,$$

for all $r \in (\frac{1}{2}, 1)$. From the discussion at the end of Theorem 15, this implies that the term

$$\sup_{\xi \in \mathbb{R}^n} \left[e^{Mkw(r\xi)} \left| \left(r^{-n} e^{-\frac{(1-r^2)|\xi|^2}{4}} - 1 \right) \hat{\varphi}(r\xi) \right| \right]$$

converges to 0 as $r \to 1^-$, uniformly on B.

As for the term

$$e^{Mkw(r\xi)} \left| \hat{\varphi}(r\xi) - \varphi(\xi) \right|,$$

the Mean Value Theorem gives z'' along the line segment between $r\xi$ and ξ for which $|\hat{\varphi}(r\xi) - \varphi(\xi)| = \frac{1-r}{r}|r\xi||\nabla\varphi(z'')|$. Thus, we obtain the estimate

$$\leqslant C \frac{1-r}{r} \sup_{|\alpha|=1} \left\| e^{(N+k)w} \partial^\alpha \hat{\varphi} \right\|_\infty,$$

using an argument similar to the one leading to (21). This completes the proof of Proposition 17. □

Remark 18. We conclude that, just as with the Gauss–Weierstrass semigroup, the functionals in the space $\mathcal{S}'_w$ can be realized as boundary values of solutions to the differential equation $u_t - Au = 0$.

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