

Rearrangement of conditionally convergent series on a small set<sup>☆</sup>Rafał Filipów, Piotr Szuca<sup>\*</sup>

Institute of Mathematics, University of Gdańsk, ul. Wita Stwosza 57, 80-952 Gdańsk, Poland

## ARTICLE INFO

## Article history:

Received 1 March 2008

Available online 23 July 2009

Submitted by M. Laczkovich

## Keywords:

Riemann's theorem

Bolzano–Weierstrass property

Positive Summability Property

Statistical density

Rearrangement of series

Extending ideals

Summable ideals

Analytic ideals

P-ideals

## ABSTRACT

We consider ideals  $\mathcal{I}$  of subsets of the set of natural numbers such that for every conditionally convergent series  $\sum_{n \in \omega} a_n$  and every  $r \in \mathbb{R}$  there is a permutation  $\pi_r : \omega \rightarrow \omega$  such that  $\sum_{n \in \omega} a_{\pi_r(n)} = r$  and

$$\{n \in \omega : \pi_r(n) \neq n\} \in \mathcal{I}.$$

We characterize such ideals in terms of extendability to a summable ideal (this answers a question of Wilczyński). Additionally, we consider Sierpiński-like theorems, where one can rearrange only indices with positive  $a_n$ .

© 2009 Elsevier Inc. All rights reserved.

## 1. Introduction

A well-known theorem of Riemann [21, p. 235] says that if a series of real numbers is conditionally convergent, then it can be rearranged to converge to an arbitrarily taken real number or to diverge to  $+\infty$  or  $-\infty$ . In other words, for every conditionally convergent series  $\sum_{n \in \omega} a_n$  and  $r \in \mathbb{R}$  there exists a permutation  $\pi : \omega \rightarrow \omega$  such that  $\sum_{n \in \omega} a_{\pi(n)} = r$ . (The set of natural numbers is denoted by  $\omega$ .)

In this paper, we consider a question if it is always possible to take the permutation  $\pi : \omega \rightarrow \omega$ , in Riemann's theorem, so that it changes only a small set of terms of the series.

Of course, the answer depends on the notion of smallness we will consider. In this paper we focus on the notion of smallness induced by ideals of subsets of the set of natural numbers. Namely, for an ideal  $\mathcal{I} \subset \mathcal{P}(\omega)$  we say that a permutation  $\pi : \omega \rightarrow \omega$  changes only small set of  $\omega$  if  $\{n \in \omega : \pi(n) \neq n\} \in \mathcal{I}$ .

We will say that an ideal  $\mathcal{I} \subset \mathcal{P}(\omega)$  has the (R) property if for every conditionally convergent series  $\sum_{n \in \omega} a_n$  and  $r \in \mathbb{R}$  there is a permutation  $\pi_r : \omega \rightarrow \omega$  such that  $\sum_{n \in \omega} a_{\pi_r(n)} = r$  and

$$\{n \in \omega : \pi_r(n) \neq n\} \in \mathcal{I}.$$

It was already proved by Wilczyński [28] that the ideal  $\mathcal{I}_d$  (the ideal of sets of asymptotic density zero) has the (R) property. In his paper, Wilczyński also proved the following theorem: if  $\sum_{n \in \omega} a_n$  is conditionally convergent series then there

<sup>☆</sup> The work of both authors was supported by BW grants 5100-5-0292-7 and 5100-5-0157-9. The first author was also supported by the Skirball postdoctoral fellowship of the Center of Advanced Studies in Mathematics at the Mathematics Department of Ben Gurion University.

<sup>\*</sup> Corresponding author.

E-mail addresses: rfilipow@math.univ.gda.pl (R. Filipów), pszuca@radix.com.pl (P. Szuca).

URL: <http://rfilipow.math.univ.gda.pl/> (R. Filipów).

exists a set  $A \subset \omega$  such that  $A \in \mathcal{I}_d$  and the series  $\sum_{n \in A} a_n$  is also conditionally convergent. It is not difficult to see that from this results one can easily get that the ideal  $\mathcal{I}_d$  has the (R) property.

We will say that an ideal  $\mathcal{I} \subset \mathcal{P}(\omega)$  has the (W) property if for every conditionally convergent series  $\sum_{n \in \omega} a_n$  there is  $A \in \mathcal{I}$  such that  $\sum_{n \in A} a_n$  is also conditionally convergent. Wilczyński asked [28, Question 1] to find a characterization of ideals which have the (W) property.

In Section 3, we show (Theorem 3.3) that both properties (R) and (W) are equivalent. Moreover, we characterize these properties in terms of extendability to summable ideals (Theorem 3.3)—and this answers the question of Wilczyński.

Moreover, we give a sufficient condition for an ideal to have the (R) property in terms of the Bolzano–Weierstrass property (Corollary 3.5). We also show that for density ideals this condition is also necessary (Proposition 3.7).

In Section 4 we examine ideal version of some theorems of Sierpiński which strengthen Riemann's theorem.

For another characterizations of the set of permutations for which a series stays convergent (divergent, convergent to the same limit) see e.g. [1,22,18,10,15,11,19,5,20]. Smith in [26] considered very similar question if for every non-absolutely convergent series and  $r \in \mathbb{R}$  there is a permutation of a given type which makes a series convergent to  $r$ . The rearrangements of series convergent with regard to the ideal  $\mathcal{I}_d$  was considered in paper [4].

It is known that the properties (R) and (W) are equivalent to the Positive Summability Property, which was considered in the literature, see e.g. [6,2].

## 2. Preliminaries

The cardinality of a set  $X$  is denoted by  $|X|$ . We do not distinguish between natural number  $n$  and the set  $\{0, 1, \dots, n-1\}$ .

For a given sequence  $(a_n)_{n \in \omega}$  we define  $a_n^+ = \max\{a_n, 0\}$  and  $a_n^- = \min\{a_n, 0\}$ . For a series  $\sum_{n \in \omega} a_n$  and  $A \subset \omega$ , by  $\sum_{n \in A} a_n$  we denote the series  $\sum_{n \in \omega} \chi_A(n) \cdot a_n$ . By  $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$  we mean the extended real line.

An ideal on  $\omega$  is a family  $\mathcal{I} \subset \mathcal{P}(\omega)$  (where  $\mathcal{P}(\omega)$  denotes the power set of  $\omega$ ) which is closed under taking subsets and finite unions. By  $\text{Fin}$  we denote the ideal of all finite subsets of  $\omega$ . If not explicitly said we assume that all considered ideals are proper ( $\neq \mathcal{P}(\omega)$ ) and contain all finite sets. We can talk about ideals on any countable set by identifying this set with  $\omega$  via a fixed bijection.

An ideal  $\mathcal{I}$  is a *P-ideal* if for every sequence  $(A_n)_{n \in \omega}$  of sets from  $\mathcal{I}$  there is  $A \in \mathcal{I}$  such that  $A_n \setminus A \in \text{Fin}$  for all  $n$ , i.e.  $A_n$  is almost contained in  $A$  for each  $n$ .

An ideal  $\mathcal{I}$  is called *dense* if every  $A \notin \mathcal{I}$  contains an infinite subset that belongs to the ideal.

### 2.1. Analytic ideals

By identifying sets of naturals with their characteristic functions, we equip  $\mathcal{P}(\omega)$  with the Cantor-space topology and therefore we can assign the topological complexity to the ideals of sets of integers. In particular, an ideal  $\mathcal{I}$  is  $F_\sigma$  (*analytic*) if it is an  $F_\sigma$  subset of the Cantor space (if it is a continuous image of a  $G_\delta$  subset of the Cantor space, respectively).

A map  $\phi: \mathcal{P}(\omega) \rightarrow [0, \infty]$  is a *submeasure* on  $\omega$  if

$$\phi(\emptyset) = 0,$$

$$\phi(A) \leq \phi(A \cup B) \leq \phi(A) + \phi(B),$$

for all  $A, B \subset \omega$ . It is *lower semicontinuous* if for all  $A \subset \omega$  we have

$$\phi(A) = \lim_{n \rightarrow \infty} \phi(A \cap n).$$

For any lower semicontinuous submeasure on  $\omega$ , let  $\|\cdot\|_\phi: \mathcal{P}(\omega) \rightarrow [0, \infty]$  be the submeasure defined by

$$\|A\|_\phi = \limsup_{n \rightarrow \infty} \phi(A \setminus n) = \lim_{n \rightarrow \infty} \phi(A \setminus n),$$

where the second equality follows by the monotonicity of  $\phi$ . Let

$$\text{Exh}(\phi) = \{A \subset \omega: \|A\|_\phi = 0\},$$

$$\text{Fin}(\phi) = \{A \subset \omega: \phi(A) < \infty\}.$$

It is clear that  $\text{Exh}(\phi)$  and  $\text{Fin}(\phi)$  are ideals (not necessarily proper) for an arbitrary submeasure  $\phi$ .

All analytic P-ideals are characterized by the following theorem of Solecki.

**Theorem 2.1.** (See [27].) *The following conditions are equivalent for an ideal  $\mathcal{I}$  on  $\omega$ .*

- (1)  $\mathcal{I}$  is an analytic P-ideal;
- (2)  $\mathcal{I} = \text{Exh}(\phi)$  for some lower semicontinuous submeasure  $\phi$  on  $\omega$ .

Moreover, for  $F_\sigma$  ideals the following characterization holds.

**Theorem 2.2.** (See [17].) The following conditions are equivalent for an ideal  $\mathcal{I}$  on  $\omega$ .

- (1)  $\mathcal{I}$  is an  $F_\sigma$  ideal;
- (2)  $\mathcal{I} = \text{Fin}(\phi)$  for some lower semicontinuous submeasure  $\phi$  on  $\omega$ .

Below we present a few examples of analytic ideals. More examples can be found in Farah's book [7].

**Example 2.3.** The ideal of sets of asymptotic density 0

$$\mathcal{I}_d = \left\{ A \subset \omega : \limsup_{n \rightarrow \infty} \frac{|A \cap n|}{n} = 0 \right\},$$

is an analytic P-ideal. If we denote

$$\phi_d(A) = \sup \left\{ \frac{|A \cap n|}{n} : n \in \omega \right\},$$

then  $\bar{d}(A) = \|A\|_{\phi_d}$  and  $\mathcal{I}_d = \text{Exh}(\phi_d)$ .

**Example 2.4.** (See Just and Krawczyk [13].) For a function  $f : \omega \rightarrow \mathbb{R}^+$  we define the *Erdős–Ulam ideal* by

$$\mathcal{EU}_f = \left\{ A \subset \omega : \lim_{n \rightarrow \infty} \frac{\sum_{i \in A \cap n} f(i)}{\sum_{i \in n} f(i)} = 0 \right\}.$$

This is an analytic P-ideal. And for  $f(n) = 1$  we get the ideal  $\mathcal{I}_d$ .

**Example 2.5.** (See Farah [7].) Assume that  $I_n$  are pairwise disjoint intervals on  $\omega$ , and  $\mu_n$  is a measure that concentrates on  $I_n$ . Then  $\phi = \sup_n \mu_n$  is a lower semicontinuous submeasure and  $\mathcal{Z}(\mu) = \text{Exh}(\phi)$  is called a *density ideal*. Every Erdős–Ulam ideal is a density ideal.

**Example 2.6.** (See Louveau and Veličković [16].) Let  $\{n_i\}_{i \in \omega}$  be an increasing sequence of natural numbers. Let  $I_i$  be pairwise disjoint intervals on  $\omega$  such that  $|I_i| = 2^{n_i}$ . Let  $\phi_i$  be a submeasure on  $I_i$  given by

$$\phi_i(A) = \frac{\log_2(|A \cap I_i| + 1)}{n_i}.$$

Then  $\phi = \sup_i \phi_i$  is a lower semicontinuous submeasure and  $\mathcal{LV}_{\{n_i\}} = \text{Exh}(\phi)$  is called a *Louveau–Veličković ideal*.

**Example 2.7.** The ideal

$$\mathcal{I}_{\frac{1}{n}} = \left\{ A \subset \omega : \sum_{n \in A} \frac{1}{n} < \infty \right\}$$

is an  $F_\sigma$  P-ideal. If  $\phi$  is a submeasure defined by the formula

$$\phi(A) = \sum_{n \in A} \frac{1}{n}$$

then  $\mathcal{I}_{\frac{1}{n}} = \text{Fin}(\phi) = \text{Exh}(\phi)$ .

**Example 2.8.** (See Mazur [17].) For  $f : \omega \rightarrow \mathbb{R}^+$  such that  $\sum_{n \in \omega} f(n) = +\infty$  we define the *summable ideal* by

$$\mathcal{I}_f = \left\{ A \subset \omega : \sum_{n \in A} f(n) < \infty \right\}.$$

Every summable ideal is an  $F_\sigma$  ideal. For  $f(n) = \frac{1}{n}$  we get the ideal  $\mathcal{I}_{\frac{1}{n}}$ .

**Example 2.9.** The ideal of arithmetic progressions free sets

$$\mathcal{W} = \{W \subset \omega : W \text{ does not contain arithmetic progressions of all lengths}\}$$

is an  $F_\sigma$  ideal which is not a P-ideal. The fact that  $\mathcal{W}$  is an ideal follows from the non-trivial theorem of van der Waerden. This ideal was first considered by Kojman in [14].

## 2.2. Bolzano–Weierstrass property

Let  $\mathcal{I}$  be an ideal on  $\omega$ ,  $A \subset \omega$  and  $(x_n)_{n \in \omega}$  be a sequence of reals. By  $(x_n) \upharpoonright A$  we mean a subsequence  $(x_n)_{n \in A}$ . We say that  $(x_n) \upharpoonright A$  is  $\mathcal{I}$ -convergent to  $x \in \mathbb{R}$  if  $\{n \in A : |x_n - x| \geq \varepsilon\} \in \mathcal{I}$  for every  $\varepsilon > 0$ .

An ideal  $\mathcal{I}$  on  $\omega$  is called:

- (1) *Fin-BW* if for any bounded sequence  $(x_n)_{n \in \omega}$  of reals there is  $A \notin \mathcal{I}$  such that  $(x_n) \upharpoonright A$  is convergent;
- (2) *BW* if for any bounded sequence  $(x_n)_{n \in \omega}$  of reals there is  $A \notin \mathcal{I}$  such that  $(x_n) \upharpoonright A$  is  $\mathcal{I}$ -convergent.

By the well-known Bolzano–Weierstrass theorem, the ideal *Fin* is *Fin-BW*. For the discussion and applications of these properties see [9]. In particular, it is known that the ideal  $\mathcal{I}_d$  of sets of asymptotic density 0 is not *BW*, and every  $F_\sigma$  ideal is *Fin-BW*. Moreover, in the paper [8] the authors show how these properties are connected with Ramsey's theorem.

## 3. Riemann's theorem

**Lemma 3.1.** *If an ideal  $\mathcal{I}$  has the (R) property then it is dense.*

**Proof.** Suppose that the ideal  $\mathcal{I}$  is not dense. Let  $A \subset \omega$  be such that  $|A| = \omega$  and for every  $B \subset A$ ,  $B \in \mathcal{I} \iff |B| < \omega$ . Let  $A = \{n_k : k \in \omega\}$  and  $n_0 < n_1 < \dots$ . Let  $a_{n_k} = (-1)^k/k$  for every  $k \in \omega$  and  $a_n = 0$  for  $n \in \omega \setminus A$ . Then the series  $\sum_{n \in A} a_n$  is non-absolutely convergent. On the other hand, if  $B \subset \omega$ ,  $B \in \mathcal{I}$  then

$$\sum_{n \in B} |a_n| = \sum_{n \in B \cap A} |a_n| < \infty$$

since  $|B \cap A| < \infty$ . Thus  $\mathcal{I}$  does not have the (R) property.  $\square$

**Lemma 3.2.** *No summable ideal has the (R) property.*

**Proof.** Let  $f : \omega \rightarrow [0, \infty)$  and  $\mathcal{I} = \{A \subset \omega : \sum_{a \in A} f(a) < \infty\}$ . We have two cases:

- (1) There is  $\varepsilon > 0$  such that  $A_\varepsilon = \{n \in \omega : f(n) \geq \varepsilon\}$  is infinite.
- (2)  $\lim_{n \rightarrow \infty} f(n) = 0$ .

In the first case the ideal  $\mathcal{I}$  is not dense (see e.g. [12, Lemma 1.4]), so by Lemma 3.1 it does not have the property (R).

Now assume the second case. Since  $\lim_{n \rightarrow \infty} f(n) = 0$  and  $\sum_{n \in \omega} f(n) = \infty$ , there are sequences  $(M_n)_{n \in \omega}$  and  $(N_n)_{n \in \omega}$  such that:

- (1)  $M_0 < N_0 < M_1 < N_1 < \dots$ ,
- (2)  $\frac{1}{n+2} < \sum_{i=M_n}^{N_n} f(i) \leq \frac{1}{n+1}$ .

Let

$$A = \bigcup_{n \in \omega} [M_n, N_n] \cap \omega$$

and  $a_i = (-1)^n \cdot f(i)$  if  $i \in [M_n, N_n]$  for some  $n \in \omega$  and  $a_i = 0$  otherwise.

It is not difficult to see that, by well-known Leibnitz's criterion,  $\sum_{i \in \omega} a_i = s$  is conditionally convergent.

Note that if  $B \in \mathcal{I}$  then  $\sum_{i \in B} a_i$  is absolutely convergent. Indeed, since  $\sum_{i \in B} f(i) < \infty$ , then

$$\sum_{i \in B} |a_i| = \sum_{i \in B \cap A} |a_i| = \sum_{i \in B \cap A} f(i) < \sum_{i \in B} f(i) < \infty.$$

Let  $\pi : \omega \rightarrow \omega$  be a permutation such that  $\sum_{i \in \omega} a_{\pi(i)} = s + 1$  and  $C = \{n \in \omega : \pi(n) \neq n\} \in \mathcal{I}$ . Let  $r = \sum_{i \in C} a_i = \sum_{i \in C} a_{\pi(i)}$ . Then

$$s - r = \sum_{i \in \omega} a_i - \sum_{i \in C} a_i = \sum_{i \in \omega \setminus C} a_i = \sum_{i \in \omega} a_{\pi(i)} - \sum_{i \in C} a_{\pi(i)} = s + 1 - r,$$

a contradiction.  $\square$

**Theorem 3.3.** *Let  $\mathcal{I}$  be an ideal on  $\omega$ . The following are equivalent.*

- (1)  $\mathcal{I}$  has the (R) property.
- (2)  $\mathcal{I}$  cannot be extended to a summable ideal.
- (3)  $\mathcal{I}$  has the (W) property.

**Proof.** (1)  $\Rightarrow$  (2). It is easy to see that if a series  $\sum_{n \in \omega} a_n$  witnesses that  $\mathcal{I}$  does not have the (R) property then it also witnesses that  $\mathcal{I}$  does not have the (R) property for each  $\mathcal{I} \subset \mathcal{J}$ . Hence, if  $\mathcal{I}$  can be extended to a summable ideal, then by Lemma 3.2 it cannot have the (R) property.

(2)  $\Rightarrow$  (3). Suppose that  $\mathcal{I}$  does not satisfies (W). Let  $\sum_{n \in \omega} a_n$  be a series which witnesses this fact.

We claim that then either  $\sum_{n \in A} a_n^+ < +\infty$  for every  $A \in \mathcal{I}$ , or  $\sum_{n \in A} a_n^- > -\infty$  for every  $A \in \mathcal{I}$ . Indeed, suppose that there is  $A \in \mathcal{I}$  with  $\sum_{n \in A} a_n^+ = +\infty$  and there is  $B \in \mathcal{I}$  with  $\sum_{n \in B} a_n^- = -\infty$ . Then  $C = A \cup B \in \mathcal{I}$  and  $\sum_{n \in C} a_n^+ = +\infty$  and  $\sum_{n \in C} a_n^- = -\infty$  as well. Then there exists  $D \subset C$  such that  $\sum_{n \in D} a_n$  is conditionally convergent, a contradiction.

Without loss of generality, suppose that  $\sum_{n \in A} a_n^+ < +\infty$  for every  $A \in \mathcal{I}$ . Let  $f: \omega \rightarrow \mathbb{R}$  be given by  $f(n) = a_n^+$  for every  $n \in \omega$ . Then the summable ideal  $\mathcal{I}_f$  extends  $\mathcal{I}$ .

(3)  $\Rightarrow$  (1). Obvious.  $\square$

**Remark.** There is a well-known conjecture by Erdős and Turán which says that the van der Waerden ideal  $\mathcal{W}$  is contained in the ideal  $\mathcal{I}_1$ . Thus, if the ideal  $\mathcal{W}$  had the (R) property then that conjecture would be false.

**Corollary 3.4.** Every maximal ideal has the (R) property.

**Proof.** Since there is no proper extension of a maximal ideal, it has the (R) property iff it is not a summable ideal. Summable ideals are measurable (in fact  $F_\sigma$ ), but maximal ideals are non-measurable (see e.g. [3]).  $\square$

**Corollary 3.5.** If an ideal  $\mathcal{I}$  is not BW then it has the (R) property.

**Proof.** Let  $\mathcal{I}$  be an ideal which does not have the (R) property. Then by Theorem 3.3 there is a summable ideal  $\mathcal{I}_f \supset \mathcal{I}$ . Since every summable ideal is  $F_\sigma$  ideal and every  $F_\sigma$  ideal is Fin-BW [9, Proposition 3.4] we have that  $\mathcal{I}$  is also Fin-BW [9, Proposition 4.1] hence BW.  $\square$

For more examples of ideals which are not BW see [9]. For instance, we know that  $\mathcal{I}_d$  is not BW, hence it has the (R) property—so we get a different proof of Wilczyński's theorem [28].

The above corollary cannot be reversed. All maximal ideals are BW (see e.g. [9]) and they also have the (R) property.

The above corollary cannot be reversed even in the class of all analytic ideals. For instance, Mazur has given an example of  $F_\sigma$  ideal (hence BW) which cannot be extended to a summable ideal (see [17, Theorem 1.9]). The ideal introduced by Mazur is not a P-ideal. In the following example we show how to modify it to get an  $F_\sigma$  P-ideal which cannot be extended to a summable ideal.

**Example 3.6.** By [17, Lemma 1.8] for every  $n > 0$  there exists a finite set  $K_n$  and a family  $\mathcal{S}_n \subset \mathcal{P}(K_n)$  such that:

- (1)  $\forall w_1, \dots, w_n \in \mathcal{S}_n$  ( $w_1 \cup \dots \cup w_n \neq K_n$ );
- (2) if  $P$  is a probability distribution on  $K_n$  then there exists a  $w \in \mathcal{S}_n$  such that  $P(w) \geq 1/2$ .

Assume that  $\{K_n: n \in \omega\}$  is a partition of  $\omega$  into intervals, and define  $\phi_n: \mathcal{P}(K_n) \rightarrow [0, \infty)$  by

$$\phi_n(A) = \min \left\{ |S|: S \subset \mathcal{S}_n \text{ and } A \subset \bigcup S \right\}$$

for any  $A \subset K_n$ . For any  $B \subset \omega$  let

$$\phi(B) = \sum_{n=1}^{\infty} \frac{1}{n} \cdot \frac{\phi_n(B \cap K_n)}{\phi_n(K_n)}.$$

Let  $\mathcal{I} = \text{Fin}(\phi) = \text{Exh}(\phi)$ . We claim that  $\mathcal{I} \not\subset \mathcal{I}_f$  for any summable ideal  $\mathcal{I}_f$ . Indeed, let  $f: \omega \rightarrow [0, \infty)$  with  $\sum_{i=0}^{\infty} f(i) = \infty$ . For each  $n$  define a probability distribution  $P_f^n$  on  $K_n$  by  $P_f^n(\{i\}) = f(i) / \sum_{j \in K_n} f(j)$ . Pick  $w_n \in \mathcal{S}_n$  with  $P_f^n(w_n) \geq 1/2$ , and  $w = \bigcup_{n=1}^{\infty} w_n$ . Then  $\sum_{i \in w} f(i) \geq 1/2 \cdot \sum_{i=0}^{\infty} f(i) = \infty$ . Since  $\phi_n(w_n) = 1$  and  $\phi_n(K_n) \geq n$ ,  $\phi(w) \leq \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$ . Thus  $w \in \mathcal{I} \setminus \mathcal{I}_f$ .

Below we show that there is a subclass of analytic P-ideals in which the property (R) and negation of BW are equivalent.

**Proposition 3.7.** A density ideal has the (R) property if and only if it is not BW ideal.

**Proof.** Let  $\mathcal{I} = \text{Exh}(\phi)$  where  $\phi = \sup_n \mu_n$  and  $\mu_n$  is a measure on an interval  $I_n$ . We have two cases:

- (1) There is  $\delta > 0$  such that  $\{n \in \omega: \phi(\{n\}) > \delta\}$  is infinite.
- (2)  $\lim_{n \rightarrow \infty} \phi(\{n\}) = 0$ .

In the first case ideal  $\mathcal{I}$  is not dense (see e.g. [12, Lemma 1.4]), so by Lemma 3.1 it does not have the (R) property. Moreover  $\mathcal{I}$  is BW ideal.

Assume the second case. Here we have two subcases:

- (1)  $\sup_n \mu_n(I_n) < \infty$ .
- (2)  $\sup_n \mu_n(I_n) = \infty$ .

In the first subcase the ideal  $\mathcal{I}$  is an Erdős–Ulam ideal (by the result of Farah [7, Lemma 1.13.9]). On the other hand, it is known that no Erdős–Ulam ideal is BW (see [9]), so by Corollary 3.5 the ideal  $\mathcal{I}$  has the (R) property.

Now assume the second subcase. In [9] it is proved that in this case the ideal  $\mathcal{I}$  is a BW ideal. So we have to show that  $\mathcal{I}$  does not have the (R) property.

It is not difficult to prove that there is a sequence  $(k_n)_{n \in \omega}$  and sets  $(A_n)_{n \in \omega}$  such that:

- (1)  $k_0 < k_1 < \dots$ ,
- (2)  $A_n \subset I_{k_n}$  for every  $n \in \omega$ , and
- (3)  $n < \mu_{k_n}(A_n) < n + 1$ .

Let  $A = \bigcup_{n \in \omega} A_n$ . Let

$$a_i = (-1)^n \cdot \frac{\mu_{k_n}(\{i\})}{n^2}$$

if  $i \in A_n$  for some  $n \in \omega$ , and  $a_i = 0$  otherwise.

Since

$$\sum_{i \in A_n} |a_i| = \sum_{i \in A_n} \frac{\mu_{k_n}(\{i\})}{n^2} = \frac{\mu_{k_n}(A_n)}{n^2}$$

and

$$\frac{n}{n^2} < \frac{\mu_{k_n}(A_n)}{n^2} < \frac{n+1}{n^2}$$

we get that the series  $\sum_{n \in \omega} a_n$  is conditionally convergent.

Now we will show that for every  $B \in \mathcal{I}$  the series  $\sum_{n \in B} a_n$  is absolutely convergent. Let  $B \in \mathcal{I}$ . Since

$$0 = \|\mathbf{B}\|_\phi = \lim_{k \rightarrow \infty} \phi(B \setminus k) = \lim_{k \rightarrow \infty} \left( \sup_n \mu_n((B \setminus k) \cap I_n) \right),$$

there is  $N \in \omega$  such that for every  $n \geq N$ ,  $\mu_n(B \cap I_n) < 1$ . On the other hand,

$$\begin{aligned} \sum_{i \in B} |a_i| &= \sum_{n \in \omega} \left( \sum_{i \in I_n \cap B} |a_i| \right) = \sum_{n \in \omega} \left( \sum_{i \in I_n \cap B} \frac{\mu_{k_n}(\{i\})}{n^2} \right) = \sum_{n \in \omega} \frac{\mu_{k_n}(A_n \cap B)}{n^2} \\ &= \sum_{n < N} \frac{\mu_{k_n}(A_n \cap B)}{n^2} + \sum_{n \geq N} \frac{\mu_{k_n}(A_n \cap B)}{n^2} < \sum_{n < N} \frac{\mu_{k_n}(A_n \cap B)}{n^2} + \sum_{n \geq N} \frac{1}{n^2} < \infty. \end{aligned}$$

So  $\sum_{n \in B} a_n$  is absolutely convergent. It follows that  $\mathcal{I} \notin (W)$ , and by Theorem 3.3  $\mathcal{I}$  does not have the (R) property.  $\square$

An example of analytic P-ideal which is not a density ideal is the Louveau–Veličković ideal. This ideal is BW (see [9]) and as we show below this ideal does not have the (R) property.

**Proposition 3.8.** *A Louveau–Veličković ideal does not have the (R) property.*

**Proof.** Let  $\mathcal{I} = \mathcal{LV}(n_i)$  be a Louveau–Veličković ideal. Let  $a_n = (-1)^i / (i \cdot 2^{n_i})$  for  $n \in I_i$  and  $i \in \omega$ ,  $a_n = 0$  otherwise. Since

$$\sum_{n \in I_i} a_n = (-1)^i \cdot |I_i| \cdot \frac{1}{i \cdot 2^{n_i}} = \frac{(-1)^i \cdot 2^{n_i}}{i \cdot 2^i} = \frac{(-1)^i}{i},$$

so the series  $\sum_{n \in \omega} a_n$  is non-absolutely convergent.

Let  $A \in \mathcal{I}$ . Then there is  $N \in \omega$  such that for every  $i \geq N$ ,  $|A \cap I_i| < 2^{n_i} / i$ . Indeed, if there were infinitely many  $i$ 's with  $|A \cap I_i| \geq 2^{n_i} / i$  then for infinitely many  $i$

$$\phi_i(A) = \frac{\log_2(|A \cap I_i| + 1)}{n_i} \geq \frac{\log_2 \frac{2^{n_i}}{i}}{n_i} = 1 - \frac{\log_2 i}{n_i} \rightarrow 1,$$

so  $\sup_i \phi_i(A) > 0$ . Thus  $A \notin \mathcal{I}$ , a contradiction.

On the other hand,

$$\begin{aligned} \sum_{n \in A} |a_n| &= \sum_{i \in \omega} \left( \sum_{n \in I_i \cap A} |a_n| \right) = \sum_{i \in \omega} |A \cap I_i| \cdot \frac{1}{i \cdot 2^{n_i}} \\ &= \sum_{i < N} |A \cap I_i| \cdot \frac{1}{i \cdot 2^{n_i}} + \sum_{i \geq N} |A \cap I_i| \cdot \frac{1}{i \cdot 2^{n_i}} \\ &\leq \sum_{i < N} |A \cap I_i| \cdot \frac{1}{i \cdot 2^{n_i}} + \sum_{i \geq N} \frac{2^{n_i}}{i} \cdot \frac{1}{i \cdot 2^{n_i}} = \sum_{i < N} |A \cap I_i| \cdot \frac{1}{i \cdot 2^{n_i}} + \sum_{i \geq N} \frac{1}{i^2} < \infty. \end{aligned}$$

So  $\sum_{n \in A} a_n$  is absolutely convergent for every  $A \in \mathcal{I}$ . Thus  $\mathcal{I} \notin (W)$ , so it does not have the (R) property.  $\square$

#### 4. Sierpiński-like theorems

Sierpiński [23–25] proved the following versions of Riemann's theorem:

- (1) For every conditionally convergent series  $\sum_{n \in \omega} a_n$  and every  $r \in \overline{\mathbb{R}}$ , there is a permutation  $\pi : \omega \rightarrow \omega$  such that  $\sum_{n \in \omega} a_{\pi(n)} = r$  and
 
$$a_n < 0 \iff a_{\pi(n)} < 0$$
 for every  $n \in \omega$ .
- (2a) For every conditionally convergent series  $\sum_{n \in \omega} a_n = s$  and every  $r \leq s$ , there is a permutation  $\pi : \omega \rightarrow \omega$  such that  $\sum_{n \in \omega} a_{\pi(n)} = r$  and
 
$$a_n < 0 \implies \pi(n) = n$$
 for every  $n \in \omega$ .
- (2b) For every conditionally convergent series  $\sum_{n \in \omega} a_n = s$  and every  $r \geq s$ , there is a permutation  $\pi : \omega \rightarrow \omega$  such that  $\sum_{n \in \omega} a_{\pi(n)} = r$  and
 
$$a_n > 0 \implies \pi(n) = n$$
 for every  $n \in \omega$ .

In [28], Wilczyński proved that in Sierpiński's theorems one can require that  $\{n \in \omega : \pi(n) \neq n\} \in \mathcal{I}_d$ . Below we show that ideals for which ideal version of Sierpiński's theorems hold are exactly ideals which have the (R) property.

**Theorem 4.1.** *Let  $\mathcal{I}$  be an ideal on  $\omega$ . The following are equivalent:*

- (1)  $\mathcal{I}$  has the (R) property.
- (2a) For every conditionally convergent series  $\sum_{n \in \omega} a_n = s$  and every  $r \leq s$ , there is a permutation  $\pi : \omega \rightarrow \omega$  such that  $\sum_{n \in \omega} a_{\pi(n)} = r$ ,  $\{n \in \omega : \pi(n) \neq n\} \in \mathcal{I}$  and
 
$$a_n < 0 \implies \pi(n) = n$$
 for every  $n \in \omega$ .
- (2b) For every conditionally convergent series  $\sum_{n \in \omega} a_n = s$  and every  $r \geq s$ , there is a permutation  $\pi : \omega \rightarrow \omega$  such that  $\sum_{n \in \omega} a_{\pi(n)} = r$ ,  $\{n \in \omega : \pi(n) \neq n\} \in \mathcal{I}$  and
 
$$a_n > 0 \implies \pi(n) = n$$
 for every  $n \in \omega$ .
- (3) For every conditionally convergent series  $\sum_{n \in \omega} a_n$  and every  $r \in \overline{\mathbb{R}}$ , there is a permutation  $\pi : \omega \rightarrow \omega$  such that  $\sum_{n \in \omega} a_{\pi(n)} = r$ ,  $\{n \in \omega : \pi(n) \neq n\} \in \mathcal{I}$  and
 
$$a_n < 0 \iff a_{\pi(n)} < 0$$
 for every  $n \in \omega$ .

**Proof.** (1)  $\Rightarrow$  (2a). Let  $\sum_{n \in \omega} a_n = s$  be a conditionally convergent series and  $r \leq s$ . Then there is  $A \in \mathcal{I}$  such that  $\sum_{n \in A} a_n = s'$  is conditionally convergent. Now, applying ordinary Sierpiński's theorem to the series  $\sum_{n \in A} a_n = s' \geq r - s + s'$ , there is a permutation  $\sigma : A \rightarrow A$  such that  $\sum_{n \in A} a_{\sigma(n)} = r - s + s'$  and  $a_n < 0 \Rightarrow \sigma(n) = n$  for every  $n \in A$ . Then the permutation  $\pi : \omega \rightarrow \omega$  given by  $\pi(n) = \sigma(n)$  if  $n \in A$  and  $\pi(n) = n$  otherwise is as required.

(2a)  $\Rightarrow$  (2b). Apply (2a) to the series  $\sum_{n \in \omega} (-a_n)$ .

(2b)  $\Rightarrow$  (2a). Apply (2b) to the series  $\sum_{n \in \omega} (-a_n)$ .

(2b)  $\Rightarrow$  (3). Let  $\sum_{n \in \omega} a_n = s$  be a conditionally convergent series and  $r \in \overline{\mathbb{R}}$ . If  $r \leq s$  then we are done by (2a), otherwise we are done by (2b).

(3)  $\Rightarrow$  (1). Obvious.  $\square$

## Acknowledgment

We would like to thank the referee for pointing out that there are  $F_\sigma$   $P$ -ideals with the (R) property.

## References

- [1] Ralph Palmer Agnew, On rearrangements of series, *Bull. Amer. Math. Soc.* 46 (1940) 797–799.
- [2] Noga Alon, Lech Drewnowski, Tomasz Łuczak, Stable Kneser hypergraphs and ideals in  $\mathbb{N}$  with the Nikodým property, *Proc. Amer. Math. Soc.* 137 (2) (2009) 467–471.
- [3] Tomek Bartoszyński, Haim Judah, *Set Theory: On the Structure of the Real Line*, A K Peters Ltd., Wellesley, MA, 1995.
- [4] M. Dindoš, T. Šalát, V. Toma, Statistical convergence of infinite series, *Czechoslovak Math. J.* 53 (128) (4) (2003) 989–1000.
- [5] L. Drewnowski, Some remarks on rearrangements of series, *Comment. Math. Prace Mat.* 18 (1974/75) 7–9.
- [6] Lech Drewnowski, Pedro J. Paúl, The Nikodým property for ideals of sets defined by matrix summability methods, *Rev. R. Acad. Cienc. Exactas Fis. Nat. (Esp.)* 94 (4) (2000) 485–503, Perspectives in mathematical analysis (in Spanish).
- [7] Ilijas Farah, Analytic quotients: Theory of liftings for quotients over analytic ideals on the integers, *Mem. Amer. Math. Soc.* 148 (702) (2000), xvi+177.
- [8] Rafał Filipów, Nikodem Mrozek, Ireneusz Reclaw, Piotr Szuca, Ideal version of Ramsey's theorem, preprint.
- [9] Rafał Filipów, Nikodem Mrozek, Ireneusz Reclaw, Piotr Szuca, Ideal convergence of bounded sequences, *J. Symbolic Logic* 72 (2) (2007) 501–512.
- [10] F. Garibay, P. Greenberg, L. Reséndis, J.J. Rivaud, The geometry of sum-preserving permutations, *Pacific J. Math.* 135 (2) (1988) 313–322.
- [11] U.C. Guha, On Levi's theorem on rearrangement of convergent series, *Indian J. Math.* 9 (1967) 91–93.
- [12] Fernando Hernández-Hernández, Michael Hrušák, Cardinal invariants of analytic  $P$ -ideals, *Canad. J. Math.* 59 (3) (2007) 575–595.
- [13] Winfried Just, Adam Krawczyk, On certain Boolean algebras  $\mathcal{P}(\omega)/I$ , *Trans. Amer. Math. Soc.* 285 (1) (1984) 411–429.
- [14] Menachem Kojman, van der Waerden spaces, *Proc. Amer. Math. Soc.* 130 (3) (2002) 631–635 (electronic).
- [15] F.W. Levi, Rearrangement of convergent series, *Duke Math. J.* 13 (1946) 579–585.
- [16] Alain Louveau, Boban Veličković, A note on Borel equivalence relations, *Proc. Amer. Math. Soc.* 120 (1) (1994) 255–259.
- [17] Krzysztof Mazur,  $F_\sigma$ -ideals and  $\omega_1\omega_1^*$ -gaps in the Boolean algebras  $\mathcal{P}(\omega)/I$ , *Fund. Math.* 138 (2) (1991) 103–111.
- [18] C.W. McArthur, Some applications of theorems on uniform Cauchy points to infinite series, *Proc. Amer. Math. Soc.* 6 (1955) 603–612.
- [19] C.St.J.A. Nash-Williams, D.J. White, An application of network flows to rearrangement of series, *J. London Math. Soc.* (2) 59 (2) (1999) 637–646.
- [20] M. Bhaskara Rao, K.P.S. Bhaskara Rao, B.V. Rao, Remarks on subsequences, subseries and rearrangements, *Proc. Amer. Math. Soc.* 67 (2) (1977) 293–296.
- [21] Bernhard Riemann, *Gesammelte mathematische Werke*, Leipzig, 1892.
- [22] H.M. Sengupta, On rearrangements of series, *Proc. Amer. Math. Soc.* 1 (1950) 71–75.
- [23] Wacław Sierpiński, Contribution à la théorie des séries divergentes, *Comp. Rend. Soc. Sci. Varsovie* 3 (1910) 89–93 (in Polish).
- [24] Wacław Sierpiński, Remarque sur la théorie de Riemann relatif aux séries semi-convergentes, *Prac. Mat. Fiz.* XXI (1910) 17–20 (in Polish).
- [25] Wacław Sierpiński, Sur une propriété des séries qui ne sont pas absolument convergentes, *Bull. Intern. Acad. Sci.: Cracovie A* (1911) 149–158.
- [26] John Howard Smith, Rearrangements of conditionally convergent real series with preassigned cycle type, *Proc. Amer. Math. Soc.* 47 (1975) 167–170.
- [27] Sławomir Solecki, Analytic ideals and their applications, *Ann. Pure Appl. Logic* 99 (1–3) (1999) 51–72.
- [28] Władysław Wilczyński, On Riemann derangement theorem, *Stup. Pr. Mat.-Fiz.* 4 (2007) 79–82.