

Existence of solutions for nonlinear parabolic problem with  $p(x)$ -growth

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## ABSTRACT

In this paper, we study the existence of solutions for nonlinear parabolic initial boundary value problem with  $p(x)$ -growth conditions in the space  $W^{1,x}L^{p(x)}(Q) \cap L^\infty(0, T; L^2(\Omega))$ , and give an existence theorem of weak solutions for the following equation

$$\frac{\partial u}{\partial t} + A(u) = f,$$

where  $A(u) = -\operatorname{div} a(x, t, u, \nabla u) + a_0(x, t, u, \nabla u)$ ,  $a(x, t, u, \nabla u)$  and  $a_0(x, t, u, \nabla u)$  satisfy  $p(x)$ -growth conditions with respect to  $u$  and  $\nabla u$ , and  $f \in W^{-1,x}L^{q(x)}(Q)$ .

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## 1. Introduction

Let  $N \geq 2$  be an integer and  $\Omega$  be a bounded Lipschitz domain in  $R^N$ . Let  $Q$  be  $\Omega \times (0, T)$  where  $T > 0$  is given. We consider the problem

$$\frac{\partial u}{\partial t} + A(u) = f, \quad \text{in } Q, \quad (1.1)$$

$$u(x, t) = 0, \quad \text{on } \partial\Omega \times (0, T), \quad (1.2)$$

$$u(x, 0) = \psi(x), \quad \text{in } \Omega, \quad (1.3)$$

and

$$f \in W^{-1,x}L^{q(x)}(Q), \quad (1.4)$$

where  $\psi(x)$  is a given function in  $L^2(\Omega)$  and  $A : W_0^{1,x}L^{p(x)}(Q) \rightarrow W^{-1,x}L^{q(x)}(Q)$  is an elliptic operator of the form  $A(u) = -\operatorname{div} a(x, t, u, \nabla u) + a_0(x, t, u, \nabla u)$  with the coefficients  $a$  and  $a_0$  satisfying the classical Leray–Lions conditions. The spaces  $W_0^{1,x}L^{p(x)}(Q)$  and  $W^{-1,x}L^{q(x)}(Q)$  will be introduced in Section 2.

If  $a$  and  $a_0$  are assumed to satisfy polynomial growth conditions with respect to  $u$  and  $\nabla u$ , the above equation was studied in space  $L^p(0, T; W_0^{1,p}(\Omega))$ , where  $1 < p < \infty$ . It is well known that the problem was solved by J.L. Lions [14] and H. Brezis and F.E. Browder [5] in the case  $p \geq 2$ , and by R. Landes [15] and R. Landes and V. Mustonen [16] in the case  $1 < p < 2$ .

If  $a$  and  $a_0$  are satisfied more general growth conditions with respect to  $u$  and  $\nabla u$ , the above problem was discussed in the inhomogeneous Orlicz–Sobolev space  $W^{1,x}L_M(Q)$ . T. Donaldson [8] considered the case that the function  $M$  is related to the growth of  $a$  and  $a_0$ . A. Elmahl and D. Meskine [10] studied the above problem in Orlicz space without assuming any growth restriction on function  $M$ .

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The study of variational problems with nonstandard growth conditions is an interesting topic in recent years.  $p(x)$ -growth problems can be regarded as a kind of nonstandard growth problems and  $p(x)$ -growth problems appear in nonlinear elastic, electrorheological fluids and other physics phenomena. Many results have been obtained on this kind of problems, for example [1–3,17–19,24].

In Section 5 of [6], the authors studied the following problem

$$\begin{aligned} \frac{\partial u}{\partial t} - \operatorname{div}(\phi_r(x, Du)) + \lambda(u - I) &= 0, \quad \text{in } \Omega \times [0, T], \\ \frac{\partial u}{\partial \bar{n}}(x, t) &= 0, \quad \text{on } \partial\Omega \times [0, T], \\ u(0) &= I, \quad \text{in } \Omega, \end{aligned}$$

where  $\lambda \geq 0$  is a constant,  $I = u + \text{noise}$ ,  $\bar{n}$  denotes the outward normal unit vector at  $x \in \partial\Omega$ ,  $\operatorname{div}(\phi_r(x, Du)) = |\nabla u|^{p(x)-2}[(p(x)-1)\Delta u + (2-p(x))|\nabla u| \operatorname{div}(\frac{\nabla u}{|\nabla u|}) + \nabla p(x) \cdot \nabla u \log |\nabla u|]$ . In this equation,  $1 \leq p(x) \leq 2$  only depends on  $x$ . Consider that the space  $W_0^{1,x}L^{p(x)}(Q)$  was introduced and discussed in [21], we think that the space  $W_0^{1,x}L^{p(x)}(Q)$  is a reasonable framework to discuss the above  $p(x)$ -growth problem. So in this paper, we study the existence of the weak solutions in the space  $W_0^{1,x}L^{p(x)}(Q)$  under  $p(x)$ -growth conditions, where  $p(x)$  is only dependent on the space variable  $x$ . This space is similar to the space  $W^{1,x}L_M(Q)$ . In [4], the authors studied the similar equation that discussed in this paper. The difference is that in [4] the coefficient of nonlinearity is allowed to depend on  $x$  and  $t$  and is assumed to be continuous, and that equation is the evolutionary  $p(x, t)$ -Laplacian.

In this paper, let  $a: Q \times R \times R^N \rightarrow R^N$  and  $a_0: Q \times R \times R^N \rightarrow R$  be the operators such that for any  $s \in R$  and  $\xi \in R^N$ ,  $a(x, t, s, \xi)$  and  $a_0(x, t, s, \xi)$  are both continuous in  $(t, s, \xi)$  for a.e.  $x \in \Omega$  and measurable in  $x$  for all  $(t, s, \xi) \in (0, T) \times R \times R^N$ . They also satisfy that for a.e.  $(x, t) \in Q$ , any  $s \in R$  and  $\xi \neq \xi^* \in R^N$ :

$$|a(x, t, s, \xi)| \leq \alpha(C(x, t) + |s|^{p(x)-1} + |\xi|^{p(x)-1}), \quad (1.5)$$

$$|a_0(x, t, s, \xi)| \leq \alpha(C(x, t) + |s|^{p(x)-1} + |\xi|^{p(x)-1}), \quad (1.6)$$

$$[a(x, t, s, \xi) - a(x, t, s, \xi^*)](\xi - \xi^*) > 0, \quad (1.7)$$

$$a(x, t, s, \xi)\xi + a_0(x, t, s, \xi)s \geq \beta|\xi|^{p(x)} + \gamma|s|^{p(x)}, \quad (1.8)$$

where  $C(x, t) \in L^q(Q)$ ,  $\alpha, \beta, \gamma > 0$  are constants. Throughout this paper, unless special statement, we always suppose that  $p(x)$  is  $*$ -continuous on  $\overline{\Omega}$ , i.e.,  $\lim_{y \rightarrow x, y \in \overline{\Omega}} p(y) = p(x)$  for every  $x \in \overline{\Omega}$ , and satisfy

$$1 < p^- = \inf_{\Omega} p(x) \leq p(x) \leq \sup_{\Omega} p(x) = p^+ < \infty. \quad (1.9)$$

$q(x)$  is the conjugate function of  $p(x)$  (see Section 2).

We will prove the following existence theorem.

**Theorem 1.** Assume that (1.4)–(1.8) hold, then there exists at least one weak solution  $u \in W_0^{1,x}L^{p(x)}(Q) \cap L^\infty(0, T; L^2(\Omega))$  of (1.1)–(1.3) in the following sense:

$$-\int_Q u \frac{\partial \varphi}{\partial t} dx dt + \int_\Omega u(t)\varphi(t) dx \Big|_0^T + \int_Q [a(x, t, u, \nabla u) \nabla \varphi + a_0(x, t, u, \nabla u) \varphi] dx dt = \langle f, \varphi \rangle$$

for all  $\varphi \in C^1(0, T; C_0^\infty(\Omega))$ .

## 2. Preliminaries

We first recall some facts on spaces  $L^{p(x)}(\Omega)$ ,  $W^{m,p(x)}(\Omega)$  and  $W^{m,x}L^{p(x)}(Q)$ . For the details see [11–13,21].

Although we assume (1.9) holds in this paper, in this section we introduce the general spaces  $L^{p(x)}(\Omega)$ ,  $W^{m,p(x)}(\Omega)$  and  $W^{m,x}L^{p(x)}(Q)$ .

Denote

$$E = \{\omega: \omega \text{ is a measurable function on } \Omega\},$$

where  $\Omega \in R^N$  is an open subset.

Let  $p(x): \Omega \rightarrow [1, \infty]$  be an element in  $E$ . Denote  $\Omega_\infty = \{x \in \Omega: p(x) = \infty\}$ . For  $u \in E$ , we define

$$\rho(u) = \int_{\Omega \setminus \Omega_\infty} |u(x)|^{p(x)} dx + \operatorname{ess\,sup}_{x \in \Omega_\infty} |u(x)|.$$

The space  $L^{p(x)}(\Omega)$  is

$$L^{p(x)}(\Omega) = \{u \in E: \exists \lambda > 0, \rho(\lambda u) < \infty\}$$

endowed with the norm

$$\|u\|_{L^{p(x)}(\Omega)} = \inf \left\{ \lambda > 0: \rho\left(\frac{u}{\lambda}\right) \leq 1 \right\}.$$

We define the conjugate function  $q(x)$  of  $p(x)$  by

$$q(x) = \begin{cases} \infty, & \text{if } p(x) = 1; \\ 1, & \text{if } p(x) = \infty; \\ \frac{p(x)}{p(x)-1}, & \text{if } 1 < p(x) < \infty. \end{cases}$$

**Lemma 2.1.** (See [13].)

- (1) The dual space of  $L^{p(x)}(\Omega)$  is  $L^{q(x)}(\Omega)$ , if  $1 \leq p(x) < \infty$ .
- (2) The space  $L^{p(x)}(\Omega)$  is reflexive if and only if (1.9) is satisfied.

**Lemma 2.2.** (See [13].) *If  $1 \leq p(x) < \infty$ ,  $C_0^\infty(\Omega)$  is dense in the space  $L^{p(x)}(\Omega)$  and  $L^{p(x)}(\Omega)$  is separable.*

**Lemma 2.3.** (See [13].) *Let  $1 \leq p(x) \leq \infty$ . For every  $u(x) \in L^{p(x)}(\Omega)$  and  $v(x) \in L^{q(x)}(\Omega)$ , we have*

$$\int_{\Omega} |u(x)v(x)| dx \leq C \|u(x)\|_{L^{p(x)}(\Omega)} \|v(x)\|_{L^{q(x)}(\Omega)},$$

where  $C$  is only dependent on  $p(x)$  and  $\Omega$ , not dependent on  $u(x)$ ,  $v(x)$ .

**Lemma 2.4.** (See [11].) *Let  $1 \leq p(x) < \infty$ . We have*

- (1)  $\|u\|_{L^{p(x)}(\Omega)} < 1$  ( $= 1, > 1$ )  $\Leftrightarrow \rho(u) < 1$  ( $= 1, > 1$ ).
- (2) If  $\|u\|_{L^{p(x)}(\Omega)} \geq 1$ ,  $\|u\|_{L^{p(x)}(\Omega)}^{p^-} \leq \rho(u) \leq \|u\|_{L^{p(x)}(\Omega)}^{p^+}$ .
- (3) If  $\|u\|_{L^{p(x)}(\Omega)} \leq 1$ ,  $\|u\|_{L^{p(x)}(\Omega)}^{p^+} \leq \rho(u) \leq \|u\|_{L^{p(x)}(\Omega)}^{p^-}$ .

**Lemma 2.5.** (See [12].) *Suppose that  $1 \leq p(x) < \infty$ . Let  $\{u_k\}_{k=1}^\infty$  be bounded in  $L^{p(x)}(\Omega)$ . If  $u_k \rightarrow u$  a.e. on  $\Omega$ , then  $u_k \rightharpoonup u$  weakly in  $L^{p(x)}(\Omega)$ .*

Next let  $m > 0$  be an integer. For each  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ ,  $\alpha_i$  are nonnegative integers and  $|\alpha| = \sum_{i=1}^n \alpha_i$ , denote by  $D^\alpha$  the distributional derivative of order  $\alpha$  with respect to the variable  $x$ .

We now introduce the generalized Lebesgue–Sobolev space  $W^{m,p(x)}(\Omega)$  which defined as

$$W^{m,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega): D^\alpha u \in L^{p(x)}(\Omega), |\alpha| \leq m\}.$$

$W^{m,p(x)}(\Omega)$  is a Banach space endowed with the norm

$$\|u\| = \sum_{|\alpha| \leq m} \|D^\alpha u\|_{L^{p(x)}(\Omega)}.$$

The space  $W_0^{m,p(x)}(\Omega)$  is defined as the closure of  $C_0^\infty(\Omega)$  in  $W^{m,p(x)}(\Omega)$ . The dual space  $(W_0^{m,p(x)}(\Omega))^*$  is denoted by  $W^{-m,q(x)}(\Omega)$  equipped with the norm

$$\|f\|_{W^{-m,q(x)}(\Omega)} = \inf \sum_{|\alpha| \leq m} \|f_\alpha\|_{L^{q(x)}(\Omega)},$$

where infimum is taken on all possible decompositions

$$f = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha f_\alpha, \quad f_\alpha \in L^{q(x)}(\Omega).$$

**Lemma 2.6.** (See [13].)

- (1)  $W^{m,p(x)}(\Omega)$  and  $W_0^{m,p(x)}(\Omega)$  are separable if  $1 \leq p(x) < \infty$ .  
 (2)  $W^{m,p(x)}(\Omega)$  and  $W_0^{m,p(x)}(\Omega)$  are reflexive if (1.9) holds.

**Lemma 2.7.** (See [13].)

- (1) If  $p_2(x) \leq p_1(x)$  for a.e.  $x \in \Omega$ , then the embedding  $W^{m,p_1(x)}(\Omega) \hookrightarrow W^{m,p_2(x)}(\Omega)$  is continuous.  
 (2) If  $p(x)$  is  $*$ -continuous on  $\overline{\Omega}$ , then the embedding  $W_0^{m,p(x)}(\Omega) \hookrightarrow L^{p(x)}(\Omega)$  is compact.

We define the space  $W^{m,x}L^{p(x)}(Q)$  as the following:

$$W^{m,x}L^{p(x)}(Q) = \{u \in L^{p(x)}(Q) : D^\alpha u \in L^{p(x)}(Q), |\alpha| \leq m\}.$$

$W^{m,x}L^{p(x)}(Q)$  is a Banach space with the norm  $\|u\| = \sum_{|\alpha| \leq m} \|D^\alpha u\|_{L^{p(x)}(Q)}$ , where  $p(x)$  is independent of  $t$ .

The space  $W_0^{m,x}L^{p(x)}(Q)$  is defined as the closure of  $C_0^\infty(Q)$  in  $W^{m,x}L^{p(x)}(Q)$  and  $W_0^{m,x}L^{p(x)}(Q) \hookrightarrow L^{p(x)}(Q)$  is continuous embedding. Let  $\tilde{M}$  be the number of multiindexes  $\alpha$  which satisfies  $0 \leq |\alpha| \leq m$ , then the space  $W_0^{m,x}L^{p(x)}(Q)$  can be considered as a close subspace of the product space  $\prod_{i=1}^{\tilde{M}} L^{p(x)}(Q)$ . So if  $1 < p(x) < \infty$ ,  $\prod_{i=1}^{\tilde{M}} L^{p(x)}(Q)$  is reflexive and further we can get that the space  $W_0^{m,x}L^{p(x)}(Q)$  is reflexive. The dual space  $(W_0^{m,x}L^{p(x)}(Q))^*$  is denoted by  $W^{-m,x}L^{q(x)}(Q)$  equipped with the norm

$$\|f\|_{W^{-m,x}L^{q(x)}(Q)} = \sup_{\|u\|_{W_0^{m,x}L^{p(x)}(Q)} \leq 1} |\langle f, u \rangle| = \inf \sum_{|\alpha| \leq m} \|f_\alpha\|_{L^{q(x)}(Q)},$$

where infimum is taken on all possible decompositions

$$f = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D_x^\alpha f_\alpha, \quad f_\alpha \in L^{q(x)}(Q).$$

**Lemma 2.8.** (See [21].) Let  $p^- > 1$ . Then for any  $f \in W^{-1,x}L^{q(x)}(Q)$ , there exists a sequence  $\{f_n\} \subset C_0^\infty(Q)$  such that  $f_n \rightharpoonup f$  weakly in  $W^{-1,x}L^{q(x)}(Q)$ , in the sense that

$$\int_Q \varphi f_n dx dt \rightarrow \langle f, \varphi \rangle, \quad \forall \varphi \in W^{1,x}L^{p(x)}(Q).$$

**Remark 2.9.** (See [22].) Let  $1 \leq p < \infty$  be a constant and  $X$  be a Banach space, the space  $L^p(0, T; X)$  denotes the space of  $L^p$ -integrable functions from  $[0, T]$  into  $X$  with the norm:  $\|u\|_{L^p(0,T;X)} = (\int_0^T \|u\|_X^p dt)^{\frac{1}{p}}$ . If  $p = \infty$ , the space  $L^\infty(0, T; X)$  is the space of essentially bounded functions from  $[0, T]$  into  $X$  with the norm:  $\|u\|_{L^\infty(0,T;X)} = \text{ess sup}_{[0,T]} \|u\|_X$ .

**Lemma 2.10.** For  $1 \leq p(x) < \infty$ ,  $1 \leq p'(x) < \infty$ , the following continuous embedding holds:  $W_0^{1,x}L^{p(x)}(Q) \hookrightarrow L^1(0, T; W_0^{1,p(x)}(\Omega))$  and  $W^{-1,x}L^{q'(x)}(Q) \hookrightarrow L^1(0, T; W^{-1,q'(x)}(\Omega))$ ,  $q'(x)$  is the conjugate function of  $p'(x)$ .

**Proof.** For every  $\lambda > 1$ , we have  $\lambda \rho(u) \leq \rho(\lambda u)$  and therefore  $\lambda \rho(\nabla u / \lambda) \leq \rho(\nabla u)$ . By the fact  $\|\frac{\nabla u}{\|\nabla u\|_{L^{p(x)}(\Omega)}}\|_{L^{p(x)}(\Omega)} = 1$  and Lemma 2.4, we can get  $\int_\Omega |\frac{\nabla u}{\|\nabla u\|_{L^{p(x)}(\Omega)}}|^{p(x)} dx = 1$ .

$$\text{If } \frac{\|\nabla u\|_{L^{p(x)}(\Omega)}}{\|\nabla u\|_{L^{p(x)}(Q)}} \geq 1,$$

$$\begin{aligned} \int_\Omega \left| \frac{|\nabla u|}{\|\nabla u\|_{L^{p(x)}(Q)}} \right|^{p(x)} dx &= \int_\Omega \left| \frac{\|\nabla u\|_{L^{p(x)}(\Omega)}}{\|\nabla u\|_{L^{p(x)}(Q)}} \frac{|\nabla u|}{\|\nabla u\|_{L^{p(x)}(\Omega)}} \right|^{p(x)} dx \\ &\geq \frac{\|\nabla u\|_{L^{p(x)}(\Omega)}}{\|\nabla u\|_{L^{p(x)}(Q)}} \int_\Omega \left| \frac{|\nabla u|}{\|\nabla u\|_{L^{p(x)}(\Omega)}} \right|^{p(x)} dx \\ &= \frac{\|\nabla u\|_{L^{p(x)}(\Omega)}}{\|\nabla u\|_{L^{p(x)}(Q)}}. \end{aligned}$$

Thus we obtain

$$\frac{\|\nabla u\|_{L^{p(x)}(\Omega)}}{\|\nabla u\|_{L^{p(x)}(Q)}} \leq \int_{\Omega} \left| \frac{|\nabla u|}{\|\nabla u\|_{L^{p(x)}(Q)}} \right|^{p(x)} dx + 1,$$

i.e.,

$$\int_0^T \frac{\|\nabla u\|_{L^{p(x)}(\Omega)}}{\|\nabla u\|_{L^{p(x)}(Q)}} dt \leq \int_Q \left| \frac{|\nabla u|}{\|\nabla u\|_{L^{p(x)}(Q)}} \right|^{p(x)} dx dt + T = 1 + T,$$

which implies  $\int_0^T \|\nabla u\|_{L^{p(x)}(\Omega)} dt \leq (1+T)\|\nabla u\|_{L^{p(x)}(Q)}$ .

It follows that  $W_0^{1,x}L^{p(x)}(Q) \subset L^1(0, T; W_0^{1,p(x)}(\Omega))$ , thus we can get

$$W_0^{1,x}L^{p(x)}(Q) \hookrightarrow L^1(0, T; W_0^{1,p(x)}(\Omega)).$$

Similarly, for  $u = \sum_{|\alpha| \leq 1} (-1)^{|\alpha|} D^\alpha u_\alpha \in W^{-1,x}L^{q'(x)}(Q)$  where  $u_\alpha \in L^{q'(x)}(Q)$ , we have

$$\int_0^T \|u_\alpha\|_{L^{q'(x)}(\Omega)} dt \leq (1+T)\|u_\alpha\|_{L^{q'(x)}(Q)}.$$

Then

$$\sum_{|\alpha| \leq 1} \int_0^T \|u_\alpha\|_{L^{q'(x)}(\Omega)} dt \leq (1+T) \sum_{|\alpha| \leq 1} \|u_\alpha\|_{L^{q'(x)}(Q)}.$$

$\forall v \in W_0^{1,p'(x)}(\Omega)$ , we have

$$\begin{aligned} \|u\|_{W^{-1,q'(x)}(\Omega)} &= \sup_{\|v\|_{W_0^{1,p'(x)}(\Omega)} \leq 1} |\langle u, v \rangle| = \sup_{\|v\|_{W_0^{1,p'(x)}(\Omega)} \leq 1} \left| \sum_{|\alpha| \leq 1} \int_{\Omega} D^\alpha v u_\alpha dx \right| \\ &\leq \sup_{\|v\|_{W_0^{1,p'(x)}(\Omega)} \leq 1} \sum_{|\alpha| \leq 1} \|D^\alpha v\|_{L^{p'(x)}(\Omega)} \|u_\alpha\|_{L^{q'(x)}(\Omega)} \\ &\leq \sup_{\|v\|_{W_0^{1,p'(x)}(\Omega)} \leq 1} \|v\|_{W^{1,p'(x)}(\Omega)} \sum_{|\alpha| \leq 1} \|u_\alpha\|_{L^{q'(x)}(\Omega)} \\ &\leq \sum_{|\alpha| \leq 1} \|u_\alpha\|_{L^{q'(x)}(\Omega)}. \end{aligned}$$

So

$$\int_0^T \|u\|_{W^{-1,q'(x)}(\Omega)} dt \leq \int_0^T \sum_{|\alpha| \leq 1} \|u_\alpha\|_{L^{q'(x)}(\Omega)} dt \leq (1+T) \sum_{|\alpha| \leq 1} \|u_\alpha\|_{L^{q'(x)}(Q)},$$

that is

$$\int_0^T \|u\|_{W^{-1,q'(x)}(\Omega)} dt \leq (1+T) \inf_{|\alpha| \leq 1} \sum_{|\alpha| \leq 1} \|u_\alpha\|_{L^{q'(x)}(Q)} = (1+T)\|u\|_{W^{-1,x}L^{q'(x)}(Q)},$$

where the infimum is taken on all possible decompositions. Furthermore

$$W^{-1,x}L^{q'(x)}(Q) \hookrightarrow L^1(0, T; W^{-1,q'(x)}(\Omega)). \quad \square$$

**Remark 2.11.** Suppose  $1 \leq p(x) \leq \infty$ , we denote the dual space of  $W^{m,p(x)}(\Omega)$  by  $W^{-m,q(x)}(\Omega)$ , where  $\frac{1}{p(x)} + \frac{1}{q(x)} = 1$ . Since  $W^{1,\infty}(\Omega) \hookrightarrow W^{1,p(x)}(\Omega)$ ,  $W^{-1,q(x)}(\Omega) \hookrightarrow W^{-1,1}(\Omega)$ . By the fact  $L^1(\Omega) \hookrightarrow W^{-1,1}(\Omega)$  is continuous (see [9]), we can get that  $L^1(Q) \hookrightarrow L^1(0, T; W^{-1,1}(\Omega))$  is continuous.

**Remark 2.12.** By [20], We know  $\forall \varepsilon > 0, \exists N < \infty$ , such that  $\forall v \in W_0^{1,p(x)}(\Omega)$

$$\|v\|_{L^1(\Omega)} \leq \varepsilon \|v\|_{W_0^{1,p(x)}(\Omega)} + N \|v\|_Y, \quad (2.1)$$

where  $W_0^{1,p(x)}(\Omega) \hookrightarrow L^1(\Omega)$  is compact,  $Y$  is a Banach space and  $L^{p(x)}(\Omega) \hookrightarrow Y$  is continuous.

**Lemma 2.13.** Let  $Y$  be a Banach space such that  $L^1(\Omega) \hookrightarrow Y$  is continuous. If  $F$  is bounded in  $W_0^{1,x}L^{p(x)}(Q)$  and is relatively compact in  $L^1(0, T; Y)$ , then  $F$  is relatively compact in  $L^1(Q)$ .

**Proof.** Integrating on  $[0, T]$ , from (2.1) we can get

$$\int_0^T \|v\|_{L^1(\Omega)} dt \leq \varepsilon \int_0^T \|v\|_{W_0^{1,p(x)}(\Omega)} dt + N \|v\|_{L^1(0,T;Y)},$$

i.e.,

$$\|v\|_{L^1(Q)} \leq \varepsilon \int_0^T \|v\|_{W_0^{1,p(x)}(\Omega)} dt + N \|v\|_{L^1(0,T;Y)}.$$

Moreover there exist  $u_1, u_2, \dots, u_m$  in  $F$  satisfying

$$\forall u \in F, \exists u_n, 1 \leq n \leq m, \text{ such that } \|u_n - u\|_{L^1(0,T;Y)} \leq \varepsilon,$$

then

$$\|u_n - u\|_{L^1(Q)} \leq \varepsilon \int_0^T \|u_n - u\|_{W_0^{1,p(x)}(\Omega)} dt + N \|u_n - u\|_{L^1(0,T;Y)}.$$

By Lemma 2.10 and  $F$  is bounded in  $W_0^{1,x}L^{p(x)}(Q)$ , we can get  $\int_0^T \|u_n - u\|_{W_0^{1,p(x)}(\Omega)} dt$  is bounded, therefore  $\|u_n - u\|_{L^1(Q)} \leq \varepsilon$  and  $F$  is relatively compact in  $L^1(Q)$ .  $\square$

For each  $h > 0$ , define the usual translation  $\tau_h u$  of the function  $u$  by  $\tau_h u = u(t+h)$  (see [9]).

**Theorem 2.14.** If  $F$  is bounded in  $W_0^{1,x}L^{p(x)}(Q)$  and  $\{\frac{\partial u}{\partial t} : u \in F\}$  is bounded in  $W^{-1,x}L^{q(x)}(Q)$ , then  $F$  is relatively compact in  $L^1(Q)$ .

**Proof.**  $\forall u \in F$ , there exists  $\{u_n\} \in C_0^\infty(Q)$  such that  $u_n \rightarrow u$  in  $W_0^{1,p(x)}(\Omega)$ . For  $0 < t_1 < t_2 < T$ , since  $\|\int_{t_1}^{t_2} u_n dt\|_{W_0^{1,p(x)}(\Omega)} \leq \int_0^T \|u_n\|_{W_0^{1,p(x)}(\Omega)} dt$ , by Lemma 2.10 we have

$$\left\| \int_{t_1}^{t_2} u dt \right\|_{W_0^{1,p(x)}(\Omega)} \leq \int_0^T \|u\|_{W_0^{1,p(x)}(\Omega)} dt \leq C \|u\|_{W_0^{1,x}L^{p(x)}(Q)}.$$

By Lemma 2.7,  $W_0^{1,p(x)}(\Omega) \hookrightarrow L^1(\Omega)$  is compact and then we deduce that  $(\int_{t_1}^{t_2} u dt)_{u \in F}$  is relatively compact in  $L^1(\Omega)$  and  $W^{-1,1}(\Omega)$ .

On the other hand,  $\{\frac{\partial u}{\partial t} : u \in F\}$  is bounded in  $W^{-1,x}L^{q(x)}(Q)$  and also bounded in  $L^1(0, T; W^{-1,q(x)}(\Omega))$ , then by Remark 2.11 it is bounded in  $L^1(0, T; W^{-1,1}(\Omega))$  too.

As it is easy to get

$$L^1(0, T; W^{1,p(x)}(\Omega)) \hookrightarrow L^1(0, T; L^1(\Omega)) \hookrightarrow L^1(0, T; W^{-1,1}(\Omega)).$$

Thus  $F \subset L^1(0, T; W^{-1,1}(\Omega))$ .

By Remark 3 in [9], we deduce that  $\|\tau_h u - u\|_{L^1(0,T-h;W^{-1,1}(\Omega))} \rightarrow 0$  uniformly with respect to  $u \in F$  as  $h \rightarrow 0$ . By Theorem 2 in [9],  $F$  is relatively compact in  $L^1(0, T; W^{-1,1}(\Omega))$ .

Since  $L^1(\Omega) \hookrightarrow W^{-1,1}(\Omega)$  is continuous, we can apply Lemma 2.13 to conclude that  $F$  is relatively compact in  $L^1(Q)$ .  $\square$

### 3. Galerkin method

As stated in [15], we choose a sequence of functions  $\{w_j\}_{j=1}^\infty \subset C_0^\infty(\Omega)$  such that the closure of  $\bigcup_{n=1}^\infty V_n$  with  $V_n = \text{span}\{w_1, w_2, \dots, w_n\}$  contains  $C_0^\infty(\Omega)$  in  $C^1(\overline{\Omega})$ .

Then we define  $W_n = C^1([0, T]; V_n)$  endowed with the norm

$$\|\varpi\|_{W_n} = \sup_{0 \leq t \leq T} \|\varpi(x, t)\|_{V_n} + \sup_{0 \leq t \leq T} \left\| \frac{\partial \varpi(x, t)}{\partial t} \right\|_{V_n},$$

where  $\|\omega\|_{V_n} = \sup_{x \in \overline{\Omega}} |\omega(x)| + \sup_{x \in \overline{\Omega}} |\nabla \omega(x)|$ , for  $\omega \in V_n$ .

By Lemma 2.8, we know for  $f \in W^{-1,x}L^{q(x)}(Q)$  there exists  $\{f_n\} \subset C_0^\infty(Q)$ , such that  $f_n \rightharpoonup f$  weakly in  $W^{-1,x}L^{q(x)}(Q)$ , i.e.,

$$\int_Q \varphi f_n dx dt \rightarrow \langle f, \varphi \rangle,$$

for all  $\varphi \in W_0^{1,x}L^{p(x)}(Q)$ .

For any  $\psi(x) \in L^2(\Omega)$ , there is a sequence  $\{\psi_n(x)\} \in \bigcup_{n=1}^\infty V_n$  such that  $\psi_n(x) \rightarrow \psi(x)$  in  $L^2(\Omega)$ .

**Definition 3.1.** A function  $u_n \in W_n$  is called a Galerkin solution of (1.1)–(1.3) if

$$\int_{Q_\tau} \varphi \frac{\partial u_n}{\partial t} dx dt + \int_{Q_\tau} a(x, t, u_n, \nabla u_n) \nabla \varphi dx dt + \int_{Q_\tau} a_0(x, t, u_n, \nabla u_n) \varphi dx dt = \int_{Q_\tau} f_n \varphi dx dt,$$

for all  $\tau \in [0, T]$  and  $\varphi \in C^1([0, T], V_n)$ , where  $Q_\tau = \Omega \times (0, \tau)$ ,  $u_n(0) = \psi_n(x)$ .

We define a vector valued function  $p_n(t, v) : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  by

$$(p_n(t, v))_i = \int_\Omega a \left( x, t, \sum_{j=1}^n v_j w_j, \sum_{j=1}^n v_j \nabla w_j \right) \nabla w_i + a_0 \left( x, t, \sum_{j=1}^n v_j w_j, \sum_{j=1}^n v_j \nabla w_j \right) w_i dx,$$

for  $v = (v_1, \dots, v_n)$ , where  $p_n(t, v)$  is continuous about  $t$  and  $v$  because  $a$  and  $a_0$  are both continuous in  $(t, s, \xi)$  for a.e.  $x \in \Omega$ .

Then we study

$$\begin{cases} \eta' + p_n(t, \eta) = F_n, \\ \eta(0) = U_n(0), \end{cases}$$

where  $(F_n)_i = \int_\Omega f_n w_i dx$ ,  $(U_n(0))_i = \int_\Omega \psi_n(x) w_i dx$ ,  $\psi_n(x) = u_n(0)$ .

Multiplying both sides of the ordinary differential equation by  $\eta$ , we can get  $\eta' \eta \leq F_n \eta$ , i.e.,

$$\frac{1}{2} \frac{\partial}{\partial t} |\eta(t)|^2 \leq |F_n| |\eta| \leq \frac{1}{2} |F_n|^2 + \frac{1}{2} |\eta(t)|^2,$$

since  $p_n(t, \eta) \eta \geq 0$ . Further by Gronwall's lemma, we get  $|\eta(t)| \leq C_n(T)$ .

Thus  $|\eta(t) - \eta(0)| \leq 2C_n(T)$ . Let  $L_n = \max_{t \in [0, T]} |F_n - p_n(t, \eta)|$  and  $q = \min\{T, \frac{2C_n(T)}{L_n}\}$ , then we can get a local solution in  $[0, q]$ . Let  $q = t_1$ . Next let  $t_1$  be the initial value, we can get the ordinary differential equation has a local solution in  $[t_1, t_2]$ ,  $t_2 = t_1 + q, \dots$ . So we can divide  $[0, T]$  into  $[0, t_1], [t_1, t_2], \dots, [t_{l-1}, t_l], t_i = t_{i-1} + q, i = 1, 2, \dots, l-1, t_l = T$  and there exists a local solution in  $[t_{i-1}, t_i]$  and  $[t_{l-1}, t_l]$ . In this way we can get that there exists a solution  $\eta_n$  in  $C^1[0, T]$ .

By the definition of  $p_n(t, v)$ , we know that the function  $u_n(t, x) = \sum_{j=1}^n (\eta_n(t))_j w_j(x)$  is a Galerkin solution of (1.1)–(1.3).

### 4. Proof of the main theorem

We shall prove the theorem in several steps.

**Step 1.** Weak convergence of  $u_n$ ,  $a$  and  $a_0$ .

By Section 3, we first get a Galerkin solution  $u_n$  of (1.1)–(1.3), i.e.,

$$\int_{Q_\tau} \varphi \frac{\partial u_n}{\partial t} dx dt + \int_{Q_\tau} a(x, t, u_n, \nabla u_n) \nabla \varphi dx dt + \int_{Q_\tau} a_0(x, t, u_n, \nabla u_n) \varphi dx dt = \int_{Q_\tau} f_n \varphi dx dt \quad (4.1)$$

for all  $\varphi \in W_n$  and  $\tau \in [0, T]$ . Let  $\varphi = u_n$  in (4.1), we get

$$\int_{Q_\tau} u_n \frac{\partial u_n}{\partial t} dx dt + \int_{Q_\tau} a(x, t, u_n, \nabla u_n) \nabla u_n dx dt + \int_{Q_\tau} a_0(x, t, u_n, \nabla u_n) u_n dx dt = \int_{Q_\tau} f_n u_n dx dt.$$

By (1.8), we know

$$\begin{aligned} \int_{Q_\tau} u_n \frac{\partial u_n}{\partial t} dx dt + \int_{Q_\tau} \alpha |\nabla u_n|^{p(x)} + \gamma |u_n|^{p(x)} dx dt &\leq \int_{Q_\tau} f_n u_n dx dt \\ &\leq \|f_n\|_{W^{-1,x}L^{q(x)}(Q_\tau)} \|u_n\|_{W^{1,x}L^{p(x)}(Q_\tau)} \\ &\leq C \|u_n\|_{W^{1,x}L^{p(x)}(Q_\tau)}, \end{aligned}$$

where  $C > 0$  is a constant.

Since

$$\int_{Q_\tau} u_n \frac{\partial u_n}{\partial t} dx dt = \frac{1}{2} \int_{\Omega} u_n^2(x, \tau) dx - \frac{1}{2} \int_{\Omega} \psi_n^2(x) dx,$$

we can get

$$\int_{Q_\tau} \alpha |\nabla u_n|^{p(x)} + \gamma |u_n|^{p(x)} dx dt \leq C \left( \|u_n\|_{L^{p(x)}(Q_\tau)} + \|\nabla u_n\|_{L^{p(x)}(Q_\tau)} + \int_{\Omega} \psi_n^2(x) dx \right).$$

Since  $\psi_n(x) \rightarrow \psi(x)$  in  $L^2(\Omega)$ , we know  $\int_{\Omega} \psi_n^2(x) dx \leq C$ .

If  $\|u_n\|_{L^{p(x)}(Q_\tau)} \leq 1$  and  $\|\nabla u_n\|_{L^{p(x)}(Q_\tau)} \leq 1$ ,

$$\|u_n\|_{L^{p(x)}(Q_\tau)} + \|\nabla u_n\|_{L^{p(x)}(Q_\tau)} \leq C, \quad \forall \tau \in [0, T].$$

If  $\|u_n\|_{L^{p(x)}(Q_\tau)} > 1$  and  $\|\nabla u_n\|_{L^{p(x)}(Q_\tau)} \leq 1$ , by Lemma 2.4 and Young's inequality,

$$\begin{aligned} \|u_n\|_{L^{p(x)}(Q_\tau)}^{p^-} &\leq \int_{Q_\tau} |u_n|^{p(x)} dx dt \\ &\leq C \left( \|u_n\|_{L^{p(x)}(Q_\tau)} + \|\nabla u_n\|_{L^{p(x)}(Q_\tau)} + \int_{\Omega} \psi_n^2(x) dx \right) \\ &\leq C (\|u_n\|_{L^{p(x)}(Q_\tau)} + 1) \\ &\leq \varepsilon \|u_n\|_{L^{p(x)}(Q_\tau)}^{p^-} + C(\varepsilon), \end{aligned}$$

where  $C(\varepsilon)$  depends on  $\varepsilon$ . We choose  $\varepsilon = \frac{1}{2}$ , then there exists a constant  $C$  such that  $\|u_n\|_{L^{p(x)}(Q_\tau)} \leq C$  and further  $\|u_n\|_{L^{p(x)}(Q_\tau)} + \|\nabla u_n\|_{L^{p(x)}(Q_\tau)} \leq C, \forall \tau \in [0, T]$ .

If  $\|u_n\|_{L^{p(x)}(Q_\tau)} \leq 1$  and  $\|\nabla u_n\|_{L^{p(x)}(Q_\tau)} > 1$ , applying Lemma 2.4 and Young's inequality again we have

$$\begin{aligned} \|\nabla u_n\|_{L^{p(x)}(Q_\tau)}^{p^-} &\leq \int_{Q_\tau} |\nabla u_n|^{p(x)} dx dt \\ &\leq C \left( \|u_n\|_{L^{p(x)}(Q_\tau)} + \|\nabla u_n\|_{L^{p(x)}(Q_\tau)} + \int_{\Omega} \psi_n^2(x) dx \right) \\ &\leq C (\|\nabla u_n\|_{L^{p(x)}(Q_\tau)} + 1) \leq \varepsilon \|\nabla u_n\|_{L^{p(x)}(Q_\tau)}^{p^-} + C(\varepsilon), \end{aligned}$$

where  $C(\varepsilon)$  depends on  $\varepsilon$ . We choose  $\varepsilon = \frac{1}{2}$ , then there exists a constant  $C$  such that  $\|\nabla u_n\|_{L^{p(x)}(Q_\tau)} \leq C$  and further

$$\|u_n\|_{L^{p(x)}(Q_\tau)} + \|\nabla u_n\|_{L^{p(x)}(Q_\tau)} \leq C, \quad \forall \tau \in [0, T].$$



If  $\|u_n\|_{L^{p(x)}(Q_\tau)} > 1$  and  $\|\nabla u_n\|_{L^{p(x)}(Q_\tau)} > 1$ , by Lemma 2.4 and Young's inequality once more we can also get

$$\begin{aligned} \frac{1}{2^{p^-}} (\|u_n\|_{L^{p(x)}(Q_\tau)} + \|\nabla u_n\|_{L^{p(x)}(Q_\tau)})^{p^-} &\leq \|u_n\|_{L^{p(x)}(Q_\tau)}^{p^-} + \|\nabla u_n\|_{L^{p(x)}(Q_\tau)}^{p^-} \\ &\leq \int_{Q_\tau} |u_n|^{p(x)} dx dt + \int_{Q_\tau} |\nabla u_n|^{p(x)} dx dt \\ &\leq C \left( \|u_n\|_{L^{p(x)}(Q_\tau)} + \|\nabla u_n\|_{L^{p(x)}(Q_\tau)} + \int_{\Omega} \psi_n^2(x) dx \right) \\ &\leq \varepsilon (\|u_n\|_{L^{p(x)}(Q_\tau)} + \|\nabla u_n\|_{L^{p(x)}(Q_\tau)})^{p^-} + C(\varepsilon), \end{aligned}$$

where  $C(\varepsilon)$  depends on  $\varepsilon$ . We choose  $\varepsilon = \frac{1}{2^{p^-+1}}$ , then there exists a constant  $C$  such that

$$\|u_n\|_{L^{p(x)}(Q_\tau)} + \|\nabla u_n\|_{L^{p(x)}(Q_\tau)} \leq C, \quad \forall \tau \in [0, T].$$

From the discussion above, we conclude

$$\|u_n\|_{L^{p(x)}(Q_\tau)} + \|\nabla u_n\|_{L^{p(x)}(Q_\tau)} \leq C, \quad \forall \tau \in [0, T],$$

i.e.

$$\|u_n\|_{W_0^{1,x}L^{p(x)}(Q)} \leq C.$$

We can also get

$$\|u_n\|_{L^\infty(0,T;L^2(\Omega))} \leq C,$$

and

$$\int_{Q_\tau} a(x, t, u_n, \nabla u_n) \nabla u_n dx dt + \int_{Q_\tau} a_0(x, t, u_n, \nabla u_n) u_n dx dt \leq C.$$

Because

$$\begin{aligned} \int_{Q_\tau} |a_0(x, t, u_n, \nabla u_n)|^{q(x)} dx dt &\leq \int_{Q_\tau} |\alpha(C(x, t) + |\nabla u_n|^{p(x)-1} + |u_n|^{p(x)-1})|^{q(x)} dx dt \\ &\leq \int_{Q_\tau} |\alpha C(x, t)|^{q(x)} + \alpha^{q(x)} |\nabla u_n|^{p(x)} + \alpha^{q(x)} |u_n|^{p(x)} dx dt \leq C, \end{aligned}$$

we can get

$$\|a_0(x, t, u_n, \nabla u_n)\|_{L^{q(x)}(Q)} \leq C.$$

In the same way

$$\|a(x, t, u_n, \nabla u_n)\|_{L^{q(x)}(Q)} \leq C.$$

Therefore there exists a subsequence of  $\{u_n\}$ , still denoted by  $\{u_n\}$ , such that  $u_n \rightharpoonup u$  weakly in  $W_0^{1,x}L^{p(x)}(Q)$ , and  $a(x, t, u_n, \nabla u_n) \rightharpoonup h$  weakly in  $(L^{q(x)}(Q))^N$ , where  $h \in (L^{q(x)}(Q))^N$ , and  $a_0(x, t, u_n, \nabla u_n) \rightharpoonup h_0$  weakly in  $L^{q(x)}(Q)$ , where  $h_0 \in L^{q(x)}(Q)$ .

We know that  $\{u_n\}$  is bounded in  $L^\infty(0, T; L^2(\Omega))$ , so there exists a subsequence of  $\{u_n\}$ , still denoted by  $\{u_n\}$ , such that  $u_n \xrightarrow{*} u$  in  $L^\infty(0, T; L^2(\Omega))$ , thus  $u \in L^\infty(0, T; L^2(\Omega))$  (by Remark 2.9).

**Step 2.** Almost everywhere convergence of  $\nabla u_n$ .

For each  $k > 0$ , we define

$$T_k(s) = \begin{cases} s, & \text{if } |s| \leq k, \\ \frac{ks}{|s|}, & \text{if } |s| > k. \end{cases}$$

Since  $u_n \in W_0^{1,x}L^{p(x)}(Q)$ , we know  $u_n \in L^{p(x)}(Q)$  and  $\nabla u_n \in (L^{p(x)}(Q))^N$ . By the fact  $\nabla T_k(u_n) = \frac{\partial T_k}{\partial s} \nabla u_n$  and  $\frac{\partial T_k}{\partial s}$  is bounded, it is easy to get  $\nabla T_k(u_n) \in (L^{p(x)}(Q))^N$  and further  $T_k(u_n) \in W^{1,x}L^{p(x)}(Q)$ . In the same way,  $T_k(u) \in W^{1,x}L^{p(x)}(Q)$ .

Fix a compact set  $M$  with  $M \subset Q$  and a function  $\varphi_M \in C_0^\infty(Q)$  such that  $0 \leq \varphi_M \leq 1$  in  $Q$  and  $\varphi_M = 1$  on  $M$ . Let  $v_n = \varphi_M(T_k(u_n) - T_k(u)) \in W_0^{1,x}L^{p(x)}(Q) \cap L^\infty(Q)$  and using  $v_n$  as test function in (4.1) yields:

$$\begin{aligned} \int_Q f_n \varphi_M (T_k(u_n) - T_k(u)) dx dt &= \int_Q \varphi_M (T_k(u_n) - T_k(u)) \frac{\partial u_n}{\partial t} dx dt + \int_Q a(x, t, u_n, \nabla u_n) \varphi_M \nabla (T_k(u_n) - T_k(u)) dx dt \\ &\quad + \int_Q a(x, t, u_n, \nabla u_n) (T_k(u_n) - T_k(u)) \nabla \varphi_M dx dt \\ &\quad + \int_Q a_0(x, t, u_n, \nabla u_n) \varphi_M (T_k(u_n) - T_k(u)) dx dt \\ &= J_1 + J_2 + J_3 + J_4. \end{aligned}$$

Now we prove  $J_1 \rightarrow 0$ .

$$\forall \varrho \in W_0^{1,x}L^{p(x)}(Q),$$

$$\begin{aligned} |(-\operatorname{div} a(x, t, u_n, \nabla u_n), \varrho)| &= \left| \int_Q a(x, t, u_n, \nabla u_n) \nabla \varrho dx dt \right| \\ &\leq \|a(x, t, u_n, \nabla u_n)\|_{L^{q(x)}(Q)} \|\nabla \varrho\|_{L^{p(x)}(Q)} \\ &\leq C \|\varrho\|_{W_0^{1,x}L^{p(x)}(Q)}. \end{aligned}$$

Since

$$\|\operatorname{div} a(x, t, u_n, \nabla u_n)\|_{W^{-1,x}L^{q(x)}(Q)} = \sup_{\varrho \in W_0^{1,x}L^{p(x)}(Q)} \frac{|(\operatorname{div} a(x, t, u_n, \nabla u_n), \varrho)|}{\|\varrho\|_{W_0^{1,x}L^{p(x)}(Q)}} \leq C,$$

we know that  $\operatorname{div} a(x, t, u_n, \nabla u_n)$  is bounded in  $W^{-1,x}L^{q(x)}(Q)$ .

$\frac{\partial u_n}{\partial t}$  is the sum of bounded terms in  $W^{-1,x}L^{q(x)}(Q)$  and  $\{u_n\}$  is bounded in  $W_0^{1,x}L^{p(x)}(Q)$ , then by Theorem 2.14 we can get a subsequence, still denoted by  $\{u_n\}$ , such that  $u_n \rightarrow u$  in  $L^1(Q)$ , moreover  $u_n \rightarrow u$  a.e. in  $Q$ . Because  $T_k(s)$  is continuous,  $T_k(u_n) \rightarrow T_k(u)$  a.e. in  $Q$ . On the other hand  $|T_k(u_n) - T_k(u)|^{p(x)}$  is bounded, by Lebesgue's theorem  $T_k(u_n) \rightarrow T_k(u)$  in  $L^{p(x)}(Q)$ .

Let  $S_k(s) = \int_0^s T_k(\tau) d\tau$ . For all  $w \in W_0^{1,x}L^{p(x)}(Q)$ ,  $\nabla S_k(w) = T_k(w) \nabla w \in (L^{p(x)}(Q))^N$  and  $S_k(w) \in W^{1,x}L^{p(x)}(Q)$ . By the fact  $u_n \rightarrow u$  in  $L^1(Q)$  and

$$\int_Q |S_k(u_n) - S_k(u)| dx dt = \int_Q \left| \int_u^{u_n} T_k(\tau) d\tau \right| dx dt = \int_Q |T_k(\xi)(u_n - u)| dx dt \leq \|T_k(\xi)\|_{L^\infty(Q)} \|u_n - u\|_{L^1(Q)},$$

where  $\xi$  is between  $u$  and  $u_n$ , we can get  $S_k(u_n) \rightarrow S_k(u)$  in  $L^1(Q)$ .

We define  $W^{1,t}L^{p(x)}(Q) = \{u \mid u \in L^{p(x)}, \frac{\partial u}{\partial t} \in L^{p(x)}(Q)\}$ . For  $u_n \in C^1(0, T; V_n) \subset W^{1,t}L^{p(x)}(Q)$ ,  $T_k(u_n) \in W^{1,t}L^{p(x)}(Q)$ . Thus

$$\int_Q \frac{\partial \varphi_M}{\partial t} S_k(u_n) dx dt = - \int_Q \frac{\partial S_k(u_n)}{\partial t} \varphi_M dx dt = - \int_Q T_k(u_n) \frac{\partial u_n}{\partial t} \varphi_M dx dt$$

and further letting  $n \rightarrow \infty$ ,

$$\int_Q \left| \frac{\partial \varphi_M}{\partial t} (S_k(u_n) - S_k(u)) \right| dx dt \leq \left\| \frac{\partial \varphi_M}{\partial t} \right\|_{L^\infty(Q)} \|S_k(u_n) - S_k(u)\|_{L^1(Q)}.$$

Since  $\frac{\partial \varphi_M}{\partial t} \in C_0^\infty(Q)$ , we obtain  $\int_Q \frac{\partial \varphi_M}{\partial t} S_k(u_n) dx dt \rightarrow \int_Q \frac{\partial \varphi_M}{\partial t} S_k(u) dx dt$ , that is  $-\int_Q T_k(u_n) \frac{\partial u_n}{\partial t} \varphi_M dx dt \rightarrow \int_Q \frac{\partial \varphi_M}{\partial t} S_k(u) dx dt$ .

Because  $\frac{\partial u_n}{\partial t}$  is the sum of bounded terms in  $W^{-1,x}L^{q(x)}(Q)$ , we have  $\frac{\partial u_n}{\partial t} \rightharpoonup \eta$  in  $W^{-1,x}L^{q(x)}(Q)$ . On the other hand  $\forall \phi \in C_0^\infty(Q)$ ,

$$\int_Q \phi \frac{\partial u_n}{\partial t} dx dt = - \int_Q u_n \frac{\partial \phi}{\partial t} dx dt \rightarrow - \int_Q u \frac{\partial \phi}{\partial t} dx dt = - \int_Q \eta \phi dx dt,$$

then  $\eta = \frac{\partial u}{\partial t}$ . Furthermore  $\varphi_M T_k(u)$  belongs to  $W^{1,x}L^{p(x)}(Q)$ , we obtain

$$\int_Q \varphi_M T_k(u) \frac{\partial u_n}{\partial t} dx dt \rightarrow \int_Q \varphi_M T_k(u) \frac{\partial u}{\partial t} dx dt = \int_Q \frac{\partial \varphi_M}{\partial t} S_k(u) dx dt.$$

Thus  $J_1 \rightarrow 0$ .

$J_3 = \int_Q a(x, t, u_n, \nabla u_n)(T_k(u_n) - T_k(u)) \nabla \varphi_M dx dt \rightarrow 0$  as  $n \rightarrow \infty$ , because  $\|a(x, t, u_n, \nabla u_n)\|_{L^{q(x)}(Q)} \leq C$ . In the same way  $J_4 = \int_Q a_0(x, t, u_n, \nabla u_n)(T_k(u_n) - T_k(u)) \varphi_M dx dt \rightarrow 0$ .

Moreover  $\int_Q f_n \varphi_M (T_k(u_n) - T_k(u)) dx dt \rightarrow 0$  because  $f_n \rightarrow f$  in  $W^{-1,x}L^{q(x)}(Q)$  and  $T_k(u_n) \rightarrow T_k(u)$  in  $L^{p(x)}(Q)$ .

We have thus proved that

$$J_2 = \int_Q a(x, t, u_n, \nabla u_n) \varphi_M \nabla (T_k(u_n) - T_k(u)) dx dt \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (4.2)$$

Fix a real number  $s > 0$ , set  $Q_{(s)} = \{(x, t) \in Q : |\nabla T_k(u)| \leq s\}$  and denote by  $\chi_s$  the characteristic function of  $Q_{(s)}$ . Taking  $s \geq r$ ,

$$\begin{aligned} 0 &\leq \int_{Q_{(r)}} \varphi_M [a(x, t, u_n, \nabla T_k(u_n)) - a(x, t, u_n, \nabla T_k(u))] [\nabla T_k(u_n) - \nabla T_k(u)] dx dt \\ &\leq \int_{Q_{(s)}} \varphi_M [a(x, t, u_n, \nabla T_k(u_n)) - a(x, t, u_n, \nabla T_k(u))] [\nabla T_k(u_n) - \nabla T_k(u)] dx dt \\ &= \int_{Q_{(s)}} \varphi_M [a(x, t, u_n, \nabla T_k(u_n)) - a(x, t, u_n, \nabla T_k(u)) \chi_s] [\nabla T_k(u_n) - \nabla T_k(u) \chi_s] dx dt \\ &\leq \int_Q \varphi_M [a(x, t, u_n, \nabla T_k(u_n)) - a(x, t, u_n, \nabla T_k(u) \chi_s)] [\nabla T_k(u_n) - \nabla T_k(u) \chi_s] dx dt \\ &= \int_Q \varphi_M a(x, t, u_n, \nabla u_n) [\nabla T_k(u_n) - \nabla T_k(u)] dx dt \\ &\quad - \int_Q \varphi_M [a(x, t, u_n, \nabla u_n) - a(x, t, u_n, \nabla T_k(u_n))] [\nabla T_k(u_n) - \nabla T_k(u) \chi_s] dx dt \\ &\quad + \int_Q \varphi_M a(x, t, u_n, \nabla u_n) (\nabla T_k(u) - \nabla T_k(u) \chi_s) dx dt \\ &\quad - \int_Q \varphi_M a(x, t, u_n, \nabla T_k(u) \chi_s) (\nabla T_k(u_n) - \nabla T_k(u) \chi_s) dx dt \\ &= I_1 + I_2 + I_3 + I_4. \end{aligned}$$

By (4.2), we know that  $I_1$  tends to 0 as  $n \rightarrow \infty$ .

Denoting by  $\chi_{G_n}$  the characteristic function of  $G_n = \{(x, t) \in Q : |u_n(x, t)| > k\}$ . If  $|u_n(x, t)| > k$ , we have  $\nabla T_k(u_n) = 0$ , thus we can write

$$I_2 = \int_Q \varphi_M [a(x, t, u_n, \nabla u_n) - a(x, t, u_n, 0)] \chi_{G_n} \nabla T_k(u) \chi_s dx dt.$$

If  $|u(x, t)| \geq k$ ,  $\nabla T_k(u) = 0$ . Then  $\chi_{G_n} \nabla T_k(u) \chi_s = 0$  and  $I_2 = 0$ . If  $|u(x, t)| < k$ , there exists  $n_0$  such that for all  $n > n_0$ ,  $|u_n(x, t)| < k$ . In this case,  $x \notin G_n$  and  $\chi_{G_n} \nabla T_k(u) \chi_s \rightarrow 0$ , a.e. in  $Q$ , as  $n \rightarrow \infty$ . Then by Lebesgue's theorem  $\{\chi_{G_n} \nabla T_k(u) \chi_s\}$  converges strongly in  $(L^{p(x)}(Q))^N$  to 0. So by the fact that  $\{a(x, t, u_n, \nabla u_n) - a(x, t, u_n, 0)\}$  is bounded in  $(L^{q(x)}(Q))^N$  we can get  $I_2$  tends to 0.

Because  $a(x, t, u_n, \nabla u_n) \rightarrow h$  in  $(L^{q(x)}(Q))^N$ ,  $I_3$  tends to

$$\int_{Q \setminus Q_{(s)}} \varphi_M h \nabla T_k(u) dx dt.$$

Since  $u_n \rightarrow u$  a.e. in  $Q$ ,  $a(x, t, u_n, \nabla T_k(u) \chi_s)$  tends to  $a(x, t, u, \nabla T_k(u) \chi_s)$  a.e. in  $Q$ . Then by the fact that  $a(x, t, u_n, \nabla T_k(u) \chi_s)$  is bounded in  $(L^{q(x)}(Q))^N$ , we can get  $a(x, t, u_n, \nabla T_k(u) \chi_s) \rightharpoonup a(x, t, u, \nabla T_k(u) \chi_s)$  weakly in  $(L^{q(x)}(Q))^N$ . As

$$\begin{aligned} |a(x, t, T_k(u_n), \nabla T_k(u) \chi_s)|^{q(x)} &\leq C(x, t)^{q(x)} + |T_k(u_n)|^{p(x)} + |\nabla T_k(u) \chi_s|^{p(x)} \\ &\leq C(x, t)^{q(x)} + k^{p(x)} + |\nabla T_k(u) \chi_s|^{p(x)}, \end{aligned}$$

we know that  $a(x, t, T_k(u_n), \nabla T_k(u) \chi_s)$  tends to  $a(x, t, T_k(u), \nabla T_k(u) \chi_s)$  strongly in  $(L^{q(x)}(Q))^N$  by Lebesgue's theorem.

On the other hand, because  $\{\nabla T_k(u_n)\}$  is bounded in  $(L^{p(x)}(Q))^N$ , we can get  $\nabla T_k(u_n) \rightharpoonup \eta'$  in  $(L^{p(x)}(Q))^N$ .  $\forall \phi \in C_0^\infty(Q)$ ,  $\int_Q \nabla T_k(u_n) \phi \, dx \, dt = - \int_Q \nabla \phi T_k(u_n) \, dx \, dt \rightarrow - \int_Q \nabla \phi T_k(u) \, dx \, dt = \int_Q \phi \eta' \, dx \, dt$ , therefore  $\eta' = \nabla T_k(u)$ . Then we know that  $\nabla T_k(u_n) - \nabla T_k(u) \chi_s$  tends to  $\nabla T_k(u) - \nabla T_k(u) \chi_s$  weakly in  $(L^{p(x)}(Q))^N$ .

By the fact that if  $|u_n| > k$ ,  $\nabla T_k(u_n) = 0$ ,

$$\begin{aligned} I_4 &= \int_Q \varphi_M a(x, t, u_n, \nabla T_k(u) \chi_s) \nabla T_k(u) \chi_s \chi_{G_n} \, dx \, dt \\ &\quad - \int_Q \varphi_M a(x, t, T_k(u_n), \nabla T_k(u) \chi_s) (\nabla T_k(u_n) - \nabla T_k(u) \chi_s) (1 - \chi_{G_n}) \, dx \, dt. \end{aligned}$$

Similar to  $I_2$ , we can get  $\int_Q \varphi_M a(x, t, u_n, \nabla T_k(u) \chi_s) \nabla T_k(u) \chi_s \chi_{G_n} \, dx \, dt \rightarrow 0$  and  $\int_Q \varphi_M a(x, t, T_k(u_n), \nabla T_k(u) \chi_s) (\nabla T_k(u_n) - \nabla T_k(u) \chi_s) \chi_{G_n} \, dx \, dt \rightarrow 0$ .

Since

$$\begin{aligned} &\int_Q \varphi_M a(x, t, T_k(u_n), \nabla T_k(u) \chi_s) (\nabla T_k(u_n) - \nabla T_k(u) \chi_s) \, dx \, dt \\ &\rightarrow \int_Q \varphi_M a(x, t, T_k(u), \nabla T_k(u) \chi_s) (\nabla T_k(u) - \nabla T_k(u) \chi_s) \, dx \, dt, \end{aligned}$$

we have

$$\begin{aligned} I_4 &\rightarrow - \int_Q \varphi_M a(x, t, T_k(u), \nabla T_k(u) \chi_s) (\nabla T_k(u) - \nabla T_k(u) \chi_s) \, dx \, dt \\ &= \int_{Q \setminus Q_s} \varphi_M a(x, t, T_k(u), 0) \nabla T_k(u) \, dx \, dt. \end{aligned}$$

Then we prove that

$$\begin{aligned} 0 &\leq \lim_{n \rightarrow \infty} \int_{Q(r)} \varphi_M [a(x, t, u_n, \nabla T_k(u_n)) - a(x, t, u_n, \nabla T_k(u))] [\nabla T_k(u_n) - \nabla T_k(u)] \, dx \, dt \\ &\leq \int_{Q \setminus Q(s)} \varphi_M [h - a(x, t, T_k(u), 0)] \nabla T_k(u) \, dx \, dt. \end{aligned}$$

Using the fact that  $[h - a(x, t, T_k(u), 0)] \nabla T_k(u) \in L^1(Q)$  and letting  $s \rightarrow \infty$ , we have

$$\lim_{n \rightarrow \infty} \int_{Q(r)} \varphi_M [a(x, t, u_n, \nabla T_k(u_n)) - a(x, t, u_n, \nabla T_k(u))] [\nabla T_k(u_n) - \nabla T_k(u)] \, dx \, dt = 0,$$

since  $|Q \setminus Q(s)| \rightarrow 0$ .

Consequently

$$\lim_{n \rightarrow \infty} \int_{Q(r) \cap M} [a(x, t, u_n, \nabla T_k(u_n)) - a(x, t, u_n, \nabla T_k(u))] [\nabla T_k(u_n) - \nabla T_k(u)] \, dx \, dt = 0$$

and we can get a subsequence of  $\{u_n\}$ , still denoted by  $\{u_n\}$ , such that

$$[a(x, t, u_n, \nabla T_k(u_n)) - a(x, t, u_n, \nabla T_k(u))] [\nabla T_k(u_n) - \nabla T_k(u)] \rightarrow 0, \quad \text{a.e. in } Q(r) \cap M.$$

For  $(x, t) \in Q_{(r)} \cap M$ ,

$$\begin{aligned} & [a(x, t, u_n, \nabla T_k(u_n)) - a(x, t, u_n, \nabla T_k(u))] [\nabla T_k(u_n) - \nabla T_k(u)] \\ & \geq \alpha |\nabla T_k(u_n)|^{p(x)} - C(1 + |\nabla T_k(u_n)|^{p(x)-1} + |\nabla T_k(u_n)|) \end{aligned}$$

which shows that  $\nabla T_k(u_n)$  is bounded in  $Q_{(r)} \cap M$ . Then there exists a subsequence of  $\{T_k(u_n)\}$ , still denoted by  $\{T_k(u_n)\}$ , such that  $\nabla T_k(u_n) \rightarrow \xi$  in  $Q_{(r)} \cap M$ . So

$$\begin{aligned} & [a(x, t, u_n, \nabla T_k(u_n)) - a(x, t, u_n, \nabla T_k(u))] (\nabla T_k(u_n) - \nabla T_k(u)) \\ & \rightarrow [a(x, t, u, \xi) - a(x, t, u, \nabla T_k(u))] (\xi - \nabla T_k(u)), \quad \text{in } Q_{(r)} \cap M \text{ as } n \rightarrow \infty. \end{aligned}$$

Then  $\xi = \nabla T_k(u)$  and  $\nabla T_k(u_n) \rightarrow \nabla T_k(u)$  a.e. in  $Q_{(r)} \cap M$ . Since  $r, k$  and  $M$  are arbitrary, we can construct a subsequence such that

$$\nabla u_n \rightarrow \nabla u \quad \text{a.e. in } Q.$$

So we get  $a(x, t, u_n, \nabla u_n) \rightarrow a(x, t, u, \nabla u)$  a.e. in  $Q$ . Since  $\{a(x, t, u_n, \nabla u_n)\}$  is bounded in  $(L^{q(x)}(Q))^N$ ,  $a(x, t, u_n, \nabla u_n) \rightharpoonup a(x, t, u, \nabla u)$  in  $(L^{q(x)}(Q))^N$  by Lemma 2.5. On the other hand  $a(x, t, u_n, \nabla u_n) \rightarrow h$  in  $(L^{q(x)}(Q))^N$ , so  $h = a(x, t, u, \nabla u)$ . Similarly,  $a_0(x, t, u_n, \nabla u_n) \rightarrow a_0(x, t, u, \nabla u)$  in  $L^{q(x)}(Q)$  and  $h_0 = a_0(x, t, u, \nabla u)$ .

### Step 3. Existence of solutions.

For  $\varphi \in C^1([0, T]; C_0^\infty(\Omega))$ , since  $u_n \rightharpoonup u$  in  $L^{p(x)}(Q)$  and  $u_n \rightarrow u$  in  $L^2(\Omega)$ , we have

$$\lim_{n \rightarrow \infty} \int_Q \frac{\partial u_n}{\partial t} \varphi \, dx \, dt = \lim_{n \rightarrow \infty} \left( \int_Q u_n \varphi \, dx \Big|_0^T - \int_Q u_n \frac{\partial \varphi}{\partial t} \, dx \, dt \right) = \int_Q u \varphi \, dx \Big|_0^T - \int_Q u \frac{\partial \varphi}{\partial t} \, dx \, dt.$$

So for all  $\varphi \in C^1([0, T]; C_0^\infty(\Omega))$ ,

$$- \int_Q u \frac{\partial \varphi}{\partial t} \, dx \, dt + \int_Q u(t) \varphi(t) \, dx \Big|_0^T + \int_Q a(x, t, u, \nabla u) \nabla \varphi \, dx \, dt + \int_Q a_0(x, t, u, \nabla u) \varphi \, dx \, dt = \langle f, \varphi \rangle.$$

Now we complete the proof.

## 5. Example

Now we consider the following equation

$$\begin{aligned} \frac{\partial u}{\partial t} &= \operatorname{div}(|\nabla u|^{p(x)-2} \nabla u) - \lambda |u|^{p(x)-2} u, \quad \text{in } Q, \\ u(x, t) &= 0, \quad \text{on } \partial \Omega \times (0, T), \\ u(x, 0) &= u_0(x), \quad x \in \Omega, \end{aligned}$$

where  $\lambda \geq 0$ ,  $a = |\nabla u|^{p(x)-2} \nabla u$  and  $a_0 = |u|^{p(x)-2} u$ . It satisfies the conditions (1.5)–(1.8). By the main theorem in this paper, we know the equation exists at least one weak solution. Let  $p(x) \equiv p > 1$ ,  $\lambda = 0$ , the above equation becomes

$$\frac{\partial u}{\partial t} = \operatorname{div}(|\nabla u|^{p-2} \nabla u), \quad \text{in } Q,$$

which is non-Newton filtration equation and had been discussed in [7,23]. In [7], the author discussed the boundary estimates for solutions of nonlinear degenerate parabolic systems and studied the regularity of the weak solution. In [23], the author studied Cauchy problem for the above equation and discussed some properties of the solutions for  $\frac{2N}{N+1} < p < 2$ .

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