J. Math. Anal. Appl. 369 (2010) 205–213

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Automorphisms of $B(H)$ with respect to minus partial order ☆

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ARTICLE INFO

Article history:
Received 17 February 2010
Available online 3 March 2010
Submitted by Richard M. Aron

Keywords:
Linear operator
Minus partial order
Matrix partial order
Generalized inverse
Idempotent operator

ABSTRACT

Let H be an infinite-dimensional Hilbert space and $B(H)$ the algebra of all bounded linear operators on H. We discuss possible extensions of the concept of the minus partial order from matrix to operator algebras. In particular, we show that Mitra's unified theory of matrix partial orders based on generalized inverses can be modified in such a way that we get a unified approach to partial orders on $B(H)$ (or even more general algebras and rings). Then we choose the most natural among possible definitions of minus partial order on $B(H)$ and describe the structure of corresponding automorphisms.

1. Introduction

Let M_n be the algebra of all $n \times n$ complex matrices. The minus partial order on M_n introduced by Hartwig [3] is defined by $A \leqslant B \iff \operatorname{rank}(B - A) = \operatorname{rank} B - \operatorname{rank} A$. Recall that rank is subadditive, that is, $\operatorname{rank}(A + B) \leqslant \operatorname{rank} A + \operatorname{rank} B$. It is then easy to verify that $\leqslant$ is indeed a partial order.

A map $\phi : M_n \to M_n$ is an order automorphism of the poset M_n (with the minus partial order) if ϕ is bijective and for every pair $A, B \in M_n$ we have

$$A \leqslant B \quad \iff \quad \phi(A) \leqslant \phi(B).$$

In [4] Legiša proved that every order automorphism of M_n is either of the form $A \mapsto TA_\varphi S$, $A \in M_n$, or of the form $A \mapsto TA_\varphi^t S$, $A \in M_n$, where $T, S \in M_n$ are invertible matrices, φ is an automorphism of the complex field, and A_φ is a matrix obtained from A by applying φ entrywise, $[a_{ij}]_\varphi = [\varphi(a_{ij})]$.

Our aim is to generalize this result to the infinite-dimensional case. First we need to extend the notion of the minus partial order to bounded linear operators acting on an infinite-dimensional Hilbert space. This turns out to be the most interesting part of our problem. For operators A, B acting on a finite-dimensional Hilbert space we have $A \leqslant B \iff \operatorname{rank} B = \operatorname{rank} A + \operatorname{rank}(B - A)$. As we shall see later this rank additivity condition is equivalent to the condition $\operatorname{Im} B = \operatorname{Im} A \oplus \operatorname{Im}(B - A)$ that makes sense in the infinite-dimensional case as well. However, one may argue that in the infinite-dimensional case it is more natural to replace images of operators by their closures (because we prefer to work with closed subspaces rather than with just linear subspaces). As it is not obvious which of these two possibilities is the right one we may look at other equivalent definitions of the minus partial order on matrices. It is well known (see the second section) that for matrices $A, B \in M_n$ we have $A \leqslant B$ if and only if $\mathcal{G}(B) \subset \mathcal{G}(A)$. This is further equivalent to the condition that $AT = BT$ and $TB = TA$ for some $T \in \mathcal{G}(A)$. Here, $\mathcal{G}(A)$ denotes the set of all inner generalized inverses of A, $\mathcal{G}(A) = \{C \in M_n\colon\ ACA = A\}$. However, it is not natural to use any of these two equivalent conditions to define $\leqslant$ when dealing with

☆ This work was supported by a grant from the Ministry of Science of Slovenia.
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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2010.02.059

operators acting on an infinite-dimensional Hilbert space. The reason is that there are many operators having no inner generalized inverses.

In the second section we will describe how to overcome these difficulties. Among all the above mentioned equivalent definitions of the minus partial order on matrices Mitra [6] choose the following one: $A \leqslant B$ if and only if $AT = BT$ and $TB = TA$ for some $T \in \mathcal{G}(A)$. Based on this definition he developed a unified theory of matrix partial orders including the minus partial order as well as the star order introduced by Drazin [2] and the sharp order to mention just the few most important ones. We will modify Mitra's approach in such a way that we will get a natural definition of the minus partial order on $B(H)$. At the same time we will describe the idea how to develop a unified theory of partial orders on rather general operator algebras or even more general rings. It would be interesting to work out the details of the proposed unified theory.

In the last section the description of the general form of poset automorphisms of $B(H)$ will be given. Note that poset automorphisms are not assumed to be linear. Nevertheless, the main theorem of the last section yields that in the infinite-dimensional case such maps are automatically real-linear and bounded.

2. The definition of the minus partial order

In order to explain our choice of the definition of the minus partial order on the algebra of bounded linear operators on a Hilbert space we first survey some known results in the matrix case. For the sake of completeness we will give short and simple proofs of all the statements. Most of them differ from those that one can find in the existing literature.

One of the basic facts in linear algebra is that for every $A \in M_n$ one can find invertible matrices S and T such that

$$SAT = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix},$$

where I is the $r \times r$ identity matrix and $r = \operatorname{rank} A$. It is then natural to define that $A \leqslant B$ if and only if there exist invertible matrices S and T such that

$$SAT = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad SBT = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{1}$$

In particular, if $A \leqslant B$ then

$$\operatorname{rank} B = \operatorname{rank} A + \operatorname{rank}(B - A). \tag{2}$$

Identifying matrices with operators acting on $\mathbb{C}^n$ we first observe that

$$\operatorname{Im} B \subset \operatorname{Im} A + \operatorname{Im}(B - A)$$

is true for any pair of $A, B \in M_n$. So, if (2), then

$$\operatorname{Im} B = \operatorname{Im} A \oplus \operatorname{Im}(B - A). \tag{3}$$

Assume finally that (3) is satisfied. We will show that then (1) holds for some invertible matrices $S, T \in M_n$. Thus, all the above three conditions are equivalent and the relation $\leqslant$ on M_n defined by any of these equivalent conditions is called the minus partial order. Let S_1 and T_1 be invertible matrices such that

$$S_1 B T_1 = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix},$$

where I is the identity matrix of the appropriate size, say $p \times p$. It follows from (3) and $Bx = Ax + (B - A)x$ that for every $x \in \mathbb{C}^n$ the implication $Bx = 0 \Rightarrow Ax = 0$ holds. As $\operatorname{Im} A \subset \operatorname{Im} B$ we have

$$S_1 A T_1 = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix}.$$

We can consider from now on only the upper-left $p \times p$ corners of A and B. So, we may, and we will assume that B is the identity matrix. In order to prove (1) we only need to show that A is idempotent. We have

$$\operatorname{Im} B = \operatorname{Im} I = \mathbb{C}^p = \operatorname{Im} A \oplus \operatorname{Im}(I - A). \tag{4}$$

For $x \in \operatorname{Im} A$ we have $x = Ax + (I - A)x$. On the other hand, with respect to the direct sum decomposition (4) we have $x = x + 0$ and since this decomposition of x is unique, we conclude that $Ax = x$ for every $x \in \operatorname{Im} A$. Similarly, if $x \in \operatorname{Im}(I - A)$, then $(I - A)x = x$, and consequently, $Ax = 0$ for every $x \in \operatorname{Im}(I - A)$. It follows that A is idempotent, as desired.

Let H be a Hilbert space and $B(H)$ the algebra of all bounded linear operators on H. Of course, in the infinite-dimensional setting we cannot deal with the rank of operators (except in the special case of finite rank operators). In view of the above discussion it would be tempting to define the minus partial order on $B(H)$ by

$$A \leqslant B \quad \Longleftrightarrow \quad \operatorname{Im} B = \operatorname{Im} A \oplus \operatorname{Im}(B - A). \tag{5}$$

But one may argue that it would be more natural to replace the condition $\operatorname{Im} B = \operatorname{Im} A \oplus \operatorname{Im}(B - A)$ by

$$\overline{\operatorname{Im} B} = \overline{\operatorname{Im} A} \oplus \overline{\operatorname{Im}(B - A)} \tag{6}$$

or even by

$$\overline{\operatorname{Im} B} = \overline{\overline{\operatorname{Im} A} \oplus \overline{\operatorname{Im}(B - A)}}. \tag{7}$$

It is not clear which one of these possibilities should be chosen (see [5] for some comments on the relations between these conditions).

Let $A \in M_n$. If

$$SAT = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix},$$

then it is straightforward to check that $\mathcal{G}(A) = T\mathcal{M}S$, where $\mathcal{M} \subset M_n$ is the set of all matrices of the form

$$\begin{bmatrix} I & * \\ * & * \end{bmatrix}.$$

Here, the $*$'s stand for arbitrary matrices of the appropriate sizes. It is then obvious that $A \leqslant B$ if and only if $\mathcal{G}(B) \subset \mathcal{G}(A)$. So the next idea that one may try is to define the minus partial order on $B(H)$ by comparing the sets of generalized inner inverses of operators. Once again we do not find this approach satisfactory. Namely, it is well known that $A \in B(H)$ has a generalized inner inverse if and only if its image is closed [7]. And we do not want to restrict our attention to closed range operators only. For example, if H is infinite-dimensional, then we can identify H with $H \oplus H$ (and then $B(H \oplus H)$ can be identified with the algebra of all 2×2 matrices with entries from $B(H)$). If we choose two compact non-finite rank operators K_1 and K_2, then it is natural to define that

$$A = \begin{bmatrix} K_1 & 0 \\ 0 & 0 \end{bmatrix} \leqslant \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix} = B. \tag{8}$$

However, the images of A and B are not closed and therefore both A and B are without generalized inner inverses.

Hartwig [3] observed that for $A, B \in M_n$ we have $A \leqslant B$ if and only if there exists a reflexive generalized inverse C of A such that $CA = CB$ and $AC = BC$. Recall that $C \in M_n$ is a reflexive generalized inverse of A if $ACA = A$ and $CAC = C$. In fact, it turns out that $A \leqslant B$ if and only if there exists an inner generalized inverse C of A such that $CA = CB$ and $AC = BC$. Indeed, if $A \leqslant B$, then we may assume with no loss of generality that

$$A = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} I & 0 \\ 0 & B_1 \end{bmatrix}$$

for some square matrix B_1 of the appropriate size. Then $C = A$ is a generalized inner inverse of A with the property that $CA = CB$ and $AC = BC$. To prove the other direction assume that $A, B, C \in M_n$ satisfy $ACA = A$, and $CA = CB$ and $AC = BC$. Replacing A, B, C by SAT, SBT, and $T^{-1}CS^{-1}$, respectively, we may assume that

$$A = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}.$$

It follows that

$$C = \begin{bmatrix} I & X \\ Y & Z \end{bmatrix}$$

for some matrices X, Y, Z of the appropriate sizes (in the trivial case when A is invertible these matrices are absent). Thus,

$$[\,I \quad X\,]\, W = 0 \quad \text{and} \quad W \begin{bmatrix} I \\ Y \end{bmatrix} = 0,$$

where $W = B - A$. We need to show that $\operatorname{Im} B = \operatorname{Im} A \oplus \operatorname{Im} W$. From $B = A + W$ we infer that $\operatorname{Im} B \subset \operatorname{Im} A + \operatorname{Im} W$. To prove the opposite inclusion it is enough to see that $\operatorname{Im} A \subset \operatorname{Im} B$, since then $\operatorname{Im} W = \operatorname{Im}(B - A) \subset \operatorname{Im} B + \operatorname{Im} A \subset \operatorname{Im} B$. So, take any $z \in \operatorname{Im} A$. Then

$$z = \begin{bmatrix} u \\ 0 \end{bmatrix} = A \begin{bmatrix} u \\ Yu \end{bmatrix} = A \begin{bmatrix} u \\ Yu \end{bmatrix} + W \begin{bmatrix} I \\ Y \end{bmatrix} u = B \begin{bmatrix} u \\ Yu \end{bmatrix}.$$

It remains to show that the intersection $\operatorname{Im} A \cap \operatorname{Im}(B - A)$ is trivial. If

$$\begin{bmatrix} u \\ v \end{bmatrix} \in \operatorname{Im} A,$$

then $v = 0$. From

$$\begin{bmatrix} u \\ v \end{bmatrix} \in \operatorname{Im} W \quad \text{and} \quad [\,I \quad X\,]\,W = 0$$

we conclude that $u + Xv = 0$. Hence, $u = 0$. Thus, $\operatorname{Im} B = \operatorname{Im} A \oplus \operatorname{Im} W$, as desired.

We have shown that $A \leqslant B$ if and only if there exists an inner generalized inverse C of A such that $CA = CB$ and $AC = BC$. One may now argue that this observation makes no sense since we are looking for an appropriate definition of the minus partial order on $B(H)$ and by one of the previous remarks such a definition should not depend on generalized inner inverses. But we will develop this idea a little bit further. We observe that if C is an inner generalized inverse of a matrix A, then

$$\operatorname{rank} A = \operatorname{rank}(ACA) \leqslant \operatorname{rank}(AC) \leqslant \operatorname{rank} A,$$

and similarly, $\operatorname{rank}(CA) = \operatorname{rank} A$. From $ACA = A$ we see that both AC and CA are idempotents. Clearly, $\operatorname{Im}(AC) \subset \operatorname{Im} A$ and $\operatorname{Ker} A \subset \operatorname{Ker} CA$. By the above rank equalities, we have

$$\operatorname{Im} P = \operatorname{Im} A \quad \text{and} \quad \operatorname{Ker} A = \operatorname{Ker} Q$$

where $P = AC$ and $Q = CA$ are idempotent matrices. Clearly, if $CA = CB$, then $PA = PB$, and similarly, if $AC = BC$, then $AQ = BQ$.

Assume now that $P, Q \in M_n$ are idempotent matrices with $\operatorname{Im} P = \operatorname{Im} A$, $\operatorname{Ker} A = \operatorname{Ker} Q$, $PA = PB$, and $AQ = BQ$. From $P(B - A) = 0$ we conclude that $\operatorname{Im}(B - A) \subset \operatorname{Ker} P$, and therefore, $\operatorname{Im}(B - A) \cap \operatorname{Im} A = \operatorname{Im}(B - A) \cap \operatorname{Im} P = \{0\}$. It follows that

$$\operatorname{Im} B \subset \operatorname{Im} A \oplus \operatorname{Im}(B - A).$$

To see that $A \leqslant B$ we only need to show that $\operatorname{Im} A \subset \operatorname{Im} B$ since then also $\operatorname{Im}(B - A) \subset \operatorname{Im} B + \operatorname{Im} A \subset \operatorname{Im} B$. From $\operatorname{Im}(I - Q) = \operatorname{Ker} Q = \operatorname{Ker} A$ we see that $A(I - Q) = 0$. Consequently,

$$\operatorname{Im} A \subset \operatorname{Im}(AQ) + \operatorname{Im}\bigl(A(I - Q)\bigr) = \operatorname{Im}(AQ) = \operatorname{Im}(BQ) \subset \operatorname{Im} B,$$

as desired.

We have shown that $A \leqslant B$ if and only if there exist idempotent matrices P, Q such that $\operatorname{Im} P = \operatorname{Im} A$, $\operatorname{Ker} A = \operatorname{Ker} Q$, $PA = PB$, and $AQ = BQ$. And when passing to the infinite-dimensional case we deal with bounded linear operators. Since the image of a bounded idempotent operator is always closed we have no other choice but to replace $\operatorname{Im} A$ in the first of the four equations by its closure. Hence, we have arrived at the following definition of the minus partial order on $B(H)$.

Definition 1. Let H be a Hilbert space and $B(H)$ the algebra of all bounded linear operators on H. For $A, B \in B(H)$ we write $A \leqslant B$ if and only if there exist idempotent operators $P, Q \in B(H)$ such that

1. $\operatorname{Im} P = \overline{\operatorname{Im} A}$,
2. $\operatorname{Ker} A = \operatorname{Ker} Q$,
3. $PA = PB$, and
4. $AQ = BQ$.

The order $\leqslant$ is called the minus partial order on $B(H)$.

We should now justify the name by proving that $\leqslant$ is indeed a partial order. But we will first make a remark on a unified theory of partial orders on $B(H)$ and more general algebras and rings. Recall that we can define a generalized inverse of a matrix A in many different ways. There is a unique matrix $C \in M_n$ such that $ACA = A$, $CAC = C$, $(CA)^* = CA$, and $(AC)^* = AC$. It is called the Moore–Penrose inverse of A. We can define new classes of generalized inverses by requiring that only a certain subset of these four conditions is satisfied (but one has to assume that at least one of the first two conditions is satisfied in order to get a reasonable definition). It should be mentioned that there are few other important classes of generalized inverses that are not defined using the above four Moore–Penrose equations. Mitra [6] introduced a unified theory for matrix partial orders through generalized inverses. One can take any definition of a generalized inverse. Denote by $\mathcal{G}'(A)$ the set of all generalized inverses of A with respect to this definition. Following Mitra we can then define an ordering on M_n by $A \leqslant_{\mathcal{G}'} B \iff AC = BC$ and $CA = CB$ for some $C \in \mathcal{G}'(A)$. Mitra and his followers created a well-developed theory of such matrix orders. It would be interesting to develope an analogous theory for orders on $B(H)$ (or even more general $*$-rings having enough idempotents, for example, C^*-algebras with real rank zero). We denote by $P(H) \subset B(H)$ the set of all bounded idempotent operators on a Hilbert space H. We start with a pair of maps $\phi, \psi : B(H) \to \mathcal{P}(P(H))$ which map every operator $A \in B(H)$ to two specified sets of idempotents $\phi(A), \psi(A) \subset P(H)$. Having in mind the four conditions from the definition of the Moore–Penrose generalized inverse it is natural to choose $\phi(A)$ to be the set of idempotents $Q \in P(H)$ that satisfy either one, or two, or even all three of the following conditions

$$\operatorname{Ker} A \subset \operatorname{Ker} Q,$$

$$\operatorname{Ker} Q \subset \operatorname{Ker} A,$$

$$Q = Q^*,$$

and to choose $\psi(A)$ to be the set of idempotents $P \in P(H)$ that satisfy either one, or two, or even all three of the following conditions

$$\operatorname{Im} A \subset \operatorname{Im} P,$$

$$\operatorname{Im} P \subset \overline{\operatorname{Im} A},$$

$$P = P^*.$$

Then we define an ordering $\leqslant_{\phi,\psi}$ on $B(H)$ by $A \leqslant_{\phi,\psi} B$ if and only if $AQ = BQ$ and $PA = PB$ for some $Q \in \phi(A)$ and some $P \in \psi(A)$. Of course, instead of $B(H)$ we can study such orderings on more general algebras or rings, thus extending and unifying the work of Drazin, Hartwig [2,3] and the followers. Moreover, we can consider different conditions defining the sets $\phi(A)$ and $\psi(A)$ that are related to other classes of generalized inverses.

A careful reader was probably not satisfied with the formulation of the last two conditions in Definition 1. Namely, if we restrict our attention to the minus partial order only, then it would be more natural to replace these two conditions by the equivalent ones: $A = PB$ and $A = BQ$. This would give an equivalent definition of the minus partial order. Indeed, if $\operatorname{Im} P = \overline{\operatorname{Im} A}$, then $PA = A$. And if $\operatorname{Ker} Q = \operatorname{Ker} A$, then the restrictions of both AQ and A to $\operatorname{Ker} Q$ are zero operators. Of course, AQ and A coincide on $\operatorname{Im} Q$, and therefore, $AQ = A$. However, our choice was more suitable having in mind a general approach to partial orders on $B(H)$ as explained above.

Let us now turn back to the minus partial order on $B(H)$. Is it indeed a partial order? Is $A \leqslant B$ equivalent to any of the conditions (5), (6), or (7)? The following theorem will help us to answer these questions.

Theorem 2. *Let H be a Hilbert space and $B(H)$ the algebra of all bounded linear operators on H. For every pair $A, B \in B(H)$ the following statements are equivalent*:

(i) $A \leqslant B$;

(ii) *there exist direct sum decompositions $H = H_1 \oplus H_2$ and $H = H_3 \oplus H_4$ (here, H_1, H_2, H_3, and H_4 are closed subspaces) such that $A, B : H_1 \oplus H_2 \to H_3 \oplus H_4$ have matrix representations*

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} A_1 & 0 \\ 0 & B_1 \end{bmatrix},$$

where $A_1 : H_1 \to H_3$ and $B_1 : H_2 \to H_4$ are bounded linear operators and A_1 is injective with $\overline{\operatorname{Im} A_1} = H_3$;

(iii) $\overline{\operatorname{Im} B} = \overline{\operatorname{Im} A} \oplus \overline{\operatorname{Im}(B - A)}$ *and* $\overline{\operatorname{Im} B^*} = \overline{\operatorname{Im} A^*} \oplus \overline{\operatorname{Im}(B^* - A^*)}$.

Proof. Assume first that (i) holds true. Let $P, Q \in B(H)$ be idempotent operators such that all four conditions from Definition 1 are satisfied. Set $H_1 = \operatorname{Im} Q$, $H_2 = \operatorname{Ker} Q$, $H_3 = \operatorname{Im} P$, and $H_4 = \operatorname{Ker} P$. Then clearly, $H = H_1 \oplus H_2$ and $H = H_3 \oplus H_4$. If we now represent A and B as 2×2 operator matrices with respect to these direct sum decompositions, then $\operatorname{Im} P = \overline{\operatorname{Im} A}$ and $\operatorname{Ker} A = \operatorname{Ker} Q$ yield that

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix}$$

with $A_1 : H_1 \to H_3$ being an injective operator with a dense image. It follows from $(B - A)Q = 0$ that the first column of $B - A$ is zero and from $P(B - A) = 0$ that $\operatorname{Im}(B - A) \subset \operatorname{Ker} P$. Hence, B is of the desired form as well.

It is trivial to verify that (ii) yields (i) and also that (ii) implies

$$\overline{\operatorname{Im} B} = \overline{\operatorname{Im} A} \oplus \overline{\operatorname{Im}(B - A)}.$$

Thus, in order to show that (ii) $\Rightarrow$ (iii) it is enough to see that $A \leqslant B \Rightarrow A^* \leqslant B^*$. So, assume that $A \leqslant B$ and let P, Q be as in Definition 1. Then P^* and Q^* are idempotents with $\operatorname{Ker} P^* = \operatorname{Im} P^{\perp} = \operatorname{Im} A^{\perp} = \operatorname{Ker} A^*$ and $\operatorname{Im} Q^* = \overline{\operatorname{Im} A^*}$. Moreover, $A^* P^* = B^* P^*$ and $Q^* A^* = Q^* B^*$. Thus, $A^* \leqslant B^*$, as desired.

It remains to show that (iii) implies (i). Assume that we have (iii). Then there exists an idempotent operator $P \in B(H)$ such that $\operatorname{Im} P = \overline{\operatorname{Im} A}$ and $\operatorname{Im}(B - A) \subset \operatorname{Ker} P$. Then $P(B - A) = 0$. Similarly, we can find an idempotent operator $R \in B(H)$ such that $\operatorname{Im} R = \overline{\operatorname{Im} A^*}$ and $R(B^* - A^*) = 0$. Set $Q = R^*$ to conclude the proof. $\square$

Corollary 3. *For all $A, B \in B(H)$ we have $A \leqslant B \iff A^* \leqslant B^*$. The relation $\leqslant$ is a partial order on $B(H)$.*

Proof. The first statement has been already proved. It is clear that $A \leqslant A$ for every $A \in B(H)$. If $A \leqslant B$ and $B \leqslant A$ for some $A, B \in B(H)$, then, by Theorem 2(ii), $\operatorname{Im} A \subset \operatorname{Im} B$ and $\operatorname{Im} B \subset \operatorname{Im} A$. Hence, $\operatorname{Im} A = \operatorname{Im} B$, and thus, Theorem 2(iii) yields that $\operatorname{Im}(B - A) = \{0\}$. Consequently, $A = B$, as desired. Finally, assume that $A \leqslant B$ and $B \leqslant C$ for some $A, B, C \in B(H)$. We need to show that $\overline{\operatorname{Im} C} = \overline{\operatorname{Im} A} \oplus \overline{\operatorname{Im}(C - A)}$ and $\overline{\operatorname{Im} C^*} = \overline{\operatorname{Im} A^*} \oplus \overline{\operatorname{Im}(C^* - A^*)}$. As $A^* \leqslant B^*$ and $B^* \leqslant C^*$ it is enough to show that only the first equation is satisfied. From $C = A + (C - A)$ we get $\operatorname{Im} C \subset \operatorname{Im} A + \operatorname{Im}(C - A) \subset \overline{\operatorname{Im} A} + \overline{\operatorname{Im}(C - A)}$. Hence,

$$\overline{\operatorname{Im} C} \subset \overline{\overline{\operatorname{Im} A} + \overline{\operatorname{Im}(C - A)}}.$$

As $A \leqslant B$ and $B \leqslant C$ we have $\operatorname{Im} A \subset \operatorname{Im} B \subset \operatorname{Im} C \subset \overline{\operatorname{Im} C}$. Consequently, $\operatorname{Im}(C - A) \subset \operatorname{Im} C + \operatorname{Im} A \subset \overline{\operatorname{Im} C}$. It follows that

$$\overline{\operatorname{Im} C} \supset \overline{\operatorname{Im} A} + \overline{\operatorname{Im}(C - A)}.$$

Since

$$\overline{\operatorname{Im} C} = \overline{\operatorname{Im} B} \oplus \overline{\operatorname{Im}(C - B)} = \overline{\operatorname{Im} A} \oplus \overline{\operatorname{Im}(B - A)} \oplus \overline{\operatorname{Im}(C - B)}$$

and

$$\operatorname{Im}(C - A) \subset \overline{\operatorname{Im}(C - B)} + \overline{\operatorname{Im}(B - A)} = \overline{\operatorname{Im}(C - B)} \oplus \overline{\operatorname{Im}(B - A)} = \overline{\overline{\operatorname{Im}(C - B)} \oplus \overline{\operatorname{Im}(B - A)}}$$

we conclude that $\overline{\operatorname{Im} A} + \overline{\operatorname{Im}(C - A)} = \overline{\overline{\operatorname{Im} A} + \overline{\operatorname{Im}(C - A)}}$ is actually a direct sum. $\square$

We have asked before which of the conditions (5), (6), or (7) would be the most natural choice when defining the minus partial order on $B(H)$. Our first guess when starting to work on this question was that (6) is the right choice. To explain that our guess was wrong we first observe that the two equations appearing in Theorem 2(iii) are equivalent in the finite-dimensional case. Indeed, all we need to recall is that in the finite-dimensional case all the subspaces are closed, the conditions (2) and (3) are equivalent, and $\operatorname{rank} A = \operatorname{rank} A^*$ for every $A \in M_n$. We will conclude this section by showing that in the infinite-dimensional case it may happen that $\overline{\operatorname{Im} B} = \overline{\operatorname{Im} A} \oplus \overline{\operatorname{Im}(B - A)}$ but $\overline{\operatorname{Im} B^*} \neq \overline{\operatorname{Im} A^*} \oplus \overline{\operatorname{Im}(B^* - A^*)}$. So, let H be an infinite-dimensional separable Hilbert space and $K : H \to H$ a self-adjoint compact operator such that $\overline{\operatorname{Im} K} = H$. Define $A, B : H \oplus H \to H \oplus H$ by $A(u, v) = (u, 0)$ and $B(u, v) = (u, u + Kv)$, $(u, v) \in H \oplus H$. It is clear that $\operatorname{Im} A = H \oplus \{0\}$, $\operatorname{Im}(B - A) = \{0\} \oplus H$, and $\overline{\operatorname{Im} B} = H \oplus H$. Thus, $\overline{\operatorname{Im} B} = \overline{\operatorname{Im} A} \oplus \overline{\operatorname{Im}(B - A)}$. Clearly, $A^* = A$ and $B^*(u, v) = (u + v, Kv)$, $(u, v) \in H \oplus H$. Choose a sequence of unit vectors $z_n \in H$ such that $\lim K z_n = 0$. Then we have $A^*(z_n, 0) = (z_n, 0)$ and $(B^* - A^*)(0, -z_n) = (-z_n, -K z_n)$. If

$$\overline{\operatorname{Im} B^*} = \overline{\operatorname{Im} A^*} \oplus \overline{\operatorname{Im}\bigl(B^* - A^*\bigr)}$$

was true, then we would conclude from

$$(z_n, 0) + (-z_n, -K z_n) \to 0$$

that $z_n \to 0$, a contradiction. To conclude, we believe that the condition (ii) in Theorem 2 shows that our definition of the minus partial order is the most natural one. We have shown that this is not equivalent to the condition (6). Obviously, it is not equivalent to any of the conditions (5) and (7). As we have seen, we have to assume that (6) is satisfied for both the pair of operators A, B and their adjoints A^*, B^*. In fact, this is quite natural as the first condition in Theorem 2(iii) can be interpreted as the condition on the images of A, B, and $B - A$, while taking orthogonal complements the second condition in Theorem 2(iii) can be interpreted as the assumption on the kernels of A, B, and $B - A$.

3. $\leqslant$-automorphisms of $B(H)$

Throughout this section $B(H)$ denotes the algebra of all bounded linear operators on an infinite-dimensional Hilbert space H. We will start with some simple lemmas.

Lemma 4. *Let $A, P \in B(H)$ such that $A \leqslant P$. Assume that P is an idempotent. Then A is an idempotent and $AP = PA = A$.*

Proof. Since P is an idempotent operator we have $H = \operatorname{Im} P \oplus \operatorname{Ker} P$. We already know that $A \leqslant P$ yields $\operatorname{Ker} P \subset \operatorname{Ker} A$. Moreover, we have $\operatorname{Im} P = \overline{\operatorname{Im} A} \oplus \overline{\operatorname{Im}(P - A)}$. For $x \in \overline{\operatorname{Im} A} \subset \operatorname{Im} P$ we have $x = Px = Ax + (P - A)x$ and at the same time $x = x + 0 \in \overline{\operatorname{Im} A} \oplus \overline{\operatorname{Im}(P - A)}$. Hence, $Ax = x$ for every $x \in \overline{\operatorname{Im} A}$. Similarly, $(P - A)x = x - Ax = x$ for every $x \in \overline{\operatorname{Im}(P - A)}$. Hence, $\operatorname{Im} A = \overline{\operatorname{Im} A}$, $\operatorname{Im}(P - A) = \overline{\operatorname{Im}(P - A)}$, and A is an idempotent with $H = \operatorname{Im} A \oplus \operatorname{Ker} A$, where $\operatorname{Ker} A = \operatorname{Im}(P - A) \oplus \operatorname{Ker} P$. In particular, $AP = PA = A$. $\square$

Let $x, y \in H$ be nonzero vectors. We denote by $x \otimes y^* \in B(H)$ a rank one operator defined by $(x \otimes y^*)z = \langle z, y\rangle x$, $z \in H$. Note that every rank one operator in $B(H)$ can be written in this form.

Lemma 5. *Let $x, y \in H$ be nonzero vectors and $A \in B(H)$. Then the following two statements are equivalent*:

- $x \otimes y^* \leqslant A$;
- $x \in \operatorname{Im} A$ *and there exists* $z \in H$ *such that* $\langle x, z\rangle = 1$ *and* $y = A^*z$.

Proof. Assume first that $x \otimes y^* \leqslant A$. Then $x \in \operatorname{Im} x \otimes y^* \subset \operatorname{Im} A$. Furthermore, there exists an idempotent $P \in B(H)$ such that $\operatorname{Im} P$ is the one-dimensional subspace of H spanned by x, and $x \otimes y^* = PA$. Hence, $P = x \otimes z^*$ for some $z \in H$ with $\langle x, z\rangle = 1$. From $x \otimes y^* = (x \otimes z^*)A = x \otimes (A^*z)^*$ we finally conclude that $y = A^*z$.

Assume now that the second condition is satisfied. Set $P = x \otimes z^*$. There exists $u \in H$ such that $Au = x$. We have

$$\langle u, y\rangle = \langle u, A^*z\rangle = \langle Au, z\rangle = \langle x, z\rangle = 1,$$

and therefore, $Q = u \otimes y^*$ is an idempotent with $\operatorname{Ker} Q = \operatorname{Ker} x \otimes y^*$. It is clear that $x \otimes y^* = PA = AQ$, as desired. □

In particular, if $A \in B(H)$ is a nonzero operator and $x \in \operatorname{Im} A$ a nonzero vector, then there exists a rank one operator R such that $\operatorname{Im} R$ is spanned by x and $R \leqslant A$. Indeed, we may, and we will assume that $\|x\| = 1$. Set $W = x + \{x\}^\perp = \{x + u$: $u \in H$ and $\langle x, u\rangle = 0\}$. Since $A^* \neq 0$ there exists $z \in W$ such that $A^*z = y \neq 0$. Then clearly, $x \otimes y^* \leqslant A$. Furthermore, if $B \in B(H)$ is of rank one and if for $R \in B(H)$ we have $R \leqslant B$, then either $R = 0$, or $R = B$. This simple fact follows directly from Theorem 2(iii).

Let $x \otimes y^*, u \otimes v^* \in B(H)$ be two rank one operators. We will write $x \otimes y^* \sim u \otimes v^*$ if x and u are linearly dependent or y and v are linearly dependent. In other words, for two rank one operators $A, B \in B(H)$ we write $A \sim B$ if they have the same image or the same kernel.

Lemma 6. *Let $A, B \in B(H)$ be rank one operators, $A \neq B$. Then the following two statements are equivalent*:

- $A \sim B$;
- *for every rank two operator C we have $A \leqslant C \Rightarrow B \not\leqslant C$, or there exist rank two operators $D, E \in B(H)$ and a rank three operator $F \in B(H)$ such that $D \neq E$, and $A, B \leqslant D \leqslant F$, and $A, B \leqslant E \leqslant F$.*

Proof. Throughout the proof we will use the obvious fact that if $T, S \in B(H)$ are invertible operators then for every pair $R, K \in B(H)$ we have

$$R \leqslant K \quad \iff \quad TRS \leqslant TKS.$$

Another rather simple statement that we will need in the proof of this lemma as well as later is the following one: if for a rank one operator R and an operator $K \in B(H)$ we have $R \leqslant K$, then $\lambda R \not\leqslant K$ for each $\lambda \in \mathbb{C} \setminus \{0, 1\}$. In order to prove this statement set $R = x \otimes y^*$ and choose any $\lambda \in \mathbb{C} \setminus \{0, 1\}$. Assume that $\lambda x \otimes y^* = x \otimes (\bar{\lambda} y)^* \leqslant K$. By Lemma 5 we can find $u, z_1, z_2 \in H$ such that

$$x = Au, \qquad \langle x, z_1\rangle = \langle x, z_2\rangle = 1, \qquad y = A^*z_1, \quad \text{and} \quad \bar{\lambda} y = A^*z_2.$$

But then

$$1 = \langle x, z_1\rangle = \langle Au, z_1\rangle = \langle u, A^*z_1\rangle = \langle u, y\rangle$$

and at the same time

$$1 = \langle u, \bar{\lambda} y\rangle = \lambda\langle u, y\rangle,$$

contradicting the fact that $\lambda \neq 1$.

Assume first that $A \sim B$. We will distinguish two cases. If A and B are linearly dependent, then because they are both nonzero and $A \neq B$, we have $A = \lambda B$ for some $\lambda \in \mathbb{C} \setminus \{0, 1\}$. By the previous step we see that for every rank two operator C we have $A \leqslant C \Rightarrow B \not\leqslant C$. So, we may assume that A and B are linearly independent and that they have either the same image or the same kernel. We will consider just the first case. Then $A = x \otimes y^*$ and $B = x \otimes u^*$ for some nonzero $x \in H$ and some linearly independent $y, u \in H$. Let $e_1, e_2, e_3 \in H$ be pairwise orthogonal vectors of norm one and $T, S \in B(H)$ invertible operators such that $Tx = e_1$, $S^*y = e_1$, and $S^*u = e_1 + e_2$. Replacing A and B by TAS and TBS, respectively, we may, and we will assume that $A = e_1 \otimes e_1^*$ and $B = e_1 \otimes (e_1 + e_2)^*$. Set $D = e_1 \otimes e_1^* + e_2 \otimes e_2^*$, $E = e_1 \otimes e_1^* + e_2 \otimes e_2^* + e_3 \otimes e_2^*$, and $F = e_1 \otimes e_1^* + e_2 \otimes e_2^* + e_3 \otimes e_3^*$. One can then easily verify that $A, B \leqslant D \leqslant F$, and $A, B \leqslant E \leqslant F$.

To prove the converse we assume that $A \nsim B$. Then $A = x \otimes y^*$ and $B = u \otimes v^*$ with x and u linearly independent and y and v linearly independent. Clearly, $C = x \otimes y^* + u \otimes v^*$ is rank two operator such that $A, B \leqslant C$. So in order to complete the proof we have to take any $F \in B(H)$ of rank three and any two operators $D, E \in B(H)$ of rank two with $A, B \leqslant D, E \leqslant F$ and show that $D = E$. Multiplying all these operators by suitable invertible operators on the left- and on the right-hand side we may, and we will assume that F is an idempotent of rank three. Then, by Lemma 4, all operators A, B, D, E are idempotents. Recall that an idempotent is uniquely determined by its image and its kernel. The images of D and E are two-dimensional spaces and both of them contain linearly independent one-dimensional images of A and B. Thus, the images of D and E are the same. The kernels of D and E are subspaces of codimension two with $\operatorname{Ker} D, \operatorname{Ker} E \subset \operatorname{Ker} A \cap \operatorname{Ker} B$. As $\operatorname{Ker} A \neq \operatorname{Ker} B$ are of codimension one, we actually have $\operatorname{Ker} D = \operatorname{Ker} E = \operatorname{Ker} A \cap \operatorname{Ker} B$. Thus, $D = E$, and this completes the proof. $\square$

Lemma 7. *Let $A, B \in B(H)$. Assume that for every rank one operator $C \in B(H)$ we have*

$$C \leqslant A \quad \Longleftrightarrow \quad C \leqslant B.$$

Then $A = B$.

Proof. Denote by $B_1(H)$ the subset of $B(H)$ consisting of all operators of rank one. For any $D \in B(H)$ we set $D_- = \{C \in B_1(H)\colon C \leqslant D\}$. We already know (see Lemma 5 and the remarks following this lemma) that $0_- = \emptyset$, $D_- = \{D\}$ for every $D \in B_1(H)$, and $\operatorname{card} D_- = \infty$ whenever $D \notin B_1(H) \cup \{0\}$. Hence, our statement holds true if any of operators A and B belongs to $B_1(H) \cup \{0\}$.

So, suppose from now on that $A, B \notin B_1(H) \cup \{0\}$ satisfy $A_- = B_-$. Assuming that $A \neq B$ we will arrive at a contradiction. We will first consider the special case when $A = \lambda B$, $\lambda \in \mathbb{C} \setminus \{0, 1\}$. Then

$$A_- = (\lambda B)_- = \lambda B_- = \lambda A_-,$$

contradicting what we have proved in the previous lemma. Hence, this possibility cannot occur and we may assume from now on that A and B are linearly independent. By [1, Theorem 2.3], there exists $x \in H$ such that Ax and Bx are linearly independent. From $A_- = B_-$ and the remark following Lemma 5 we conclude that $\operatorname{Im} A = \operatorname{Im} B$. Hence, there exists $y \in H$ with $Ay = Bx$. Let e_1, e_2 be orthogonal vectors of norm one. There exist invertible operators $T, S_1 \in B(H)$ such that $S_1 e_1 = x$, $S_1 e_2 = y$, $TAx = e_1$, and $TAy = e_2$. Then $TAS_1 e_1 = e_1$ and $TAS_1 e_2 = e_2$. It follows that with respect to the direct sum decomposition $H = \operatorname{span}\{e_1, e_2\} \oplus (\operatorname{span}\{e_1, e_2\})^\perp$ the operator TAS_1 has the matrix representation

$$\begin{bmatrix} I & A_1 \\ 0 & A_2 \end{bmatrix}.$$

Set

$$S_2 = \begin{bmatrix} I & -A_1 \\ 0 & I \end{bmatrix}.$$

Clearly, $S = S_1 S_2$ maps e_1 into x and e_2 into y. Hence, with respect to the direct sum decomposition $H = \operatorname{span}\{e_1\} \oplus \operatorname{span}\{e_1\} \oplus (\operatorname{span}\{e_1, e_2\})^\perp$ the operators TAS and TBS have the matrix representations

$$TAS = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & A_2 \end{bmatrix} \quad \text{and} \quad TBS = \begin{bmatrix} 0 & * & * \\ 1 & * & * \\ 0 & * & * \end{bmatrix}.$$

Obviously, $(TAS)_- = (TBS)_-$. Clearly, $e_2 \otimes (2e_1 + e_2)^* \leqslant TAS$, and therefore, $e_2 \otimes (2e_1 + e_2)^* \leqslant TBS$. By Lemma 5, we can find $z \in H$ such that $\langle e_2, z\rangle = 1$ and $2e_1 + e_2 = (TBS)^* z$. But then

$$1 = \langle e_2, z\rangle = \langle TBSe_1, z\rangle = \bigl\langle e_1, (TBS)^* z\bigr\rangle = \langle e_1, 2e_1 + e_2\rangle = 2,$$

a contradiction. $\square$

We are now ready to prove the main result of this section.

Theorem 8. *Let H be an infinite-dimensional Hilbert space. Assume that $\phi : B(H) \to B(H)$ is a bijective map such that for every pair $A, B \in B(H)$ we have*

$$A \leqslant B \quad \Longleftrightarrow \quad \phi(A) \leqslant \phi(B).$$

Then there exist bounded bijective both linear or both conjugate-linear maps $T, S : H \to H$ such that either

$$\phi(A) = TAS$$

for every $A \in B(H)$, or

$$\phi(A) = TA^* S$$

for every $A \in B(H)$.

Proof. We know that $A = 0$ is the only operator with the property that $B \leqslant A \Rightarrow B = A$ and that operators A of rank at most one are characterized by the property that $B \leqslant A \Rightarrow (B = 0 \text{ or } B = A)$. Hence, we have $\phi(0) = 0$ and $\phi(B_1(H)) = B_1(H)$. Now, $A \in B(H)$ is of rank two if and only if for every $B \in B(H)$ we have $B \leqslant A$ and $B \neq A \Rightarrow B \in B_1(H) \cup \{0\}$. It follows that ϕ maps the set of all rank two operators onto itself. The same is true for the set of rank three operators. So, by Lemma 6, for every pair $A, B \in B_1(H)$ we have $A \sim B \iff \phi(A) \sim \phi(B)$.

Next, we will show that ϕ maps invertible operators into invertible operators. For nonzero vectors $x, y \in H$ we denote

$$L_x = \{x \otimes u^*\colon\ u \in H \setminus \{0\}\} \quad \text{and} \quad R_y = \{w \otimes y^*\colon\ w \in H \setminus \{0\}\}.$$

Both L_x and R_y consist of rank one operators. For an arbitrary pair of operators $A, B \in L_x$ we have $A \sim B$ and the same holds true for R_y. Moreover, if $\mathcal{V} \subset B(H)$ is a subset consisting of rank one operators such that $A \sim B$ for every pair $A, B \in \mathcal{V}$ then there exists a nonzero $x \in H$ such that $\mathcal{V} \subset L_x$ or there exists a nonzero $y \in H$ such that $\mathcal{V} \subset R_y$. It follows that for every nonzero x either there exists a nonzero $u \in H$ such that $\phi(L_x) = L_u$, or there exists a nonzero $y \in H$ such that $\phi(L_x) = R_y$. Similarly, every R_y is mapped onto one of such sets. Our next observation is that A is invertible if and only if for every nonzero $x \in H$ and every nonzero $y \in H$ there are operators $B \in L_x$ and $C \in R_y$ such that $B, C \leqslant A$. Indeed, assume first that A is invertible. The existence of B is guaranteed by the remark following Lemma 5. The operator A^* is invertible as well. So, one can find a nonzero $z \in H$ with $y \otimes z^* \leqslant A^*$ which is equivalent to $z \otimes y^* \leqslant A$. If, on the other hand, for every nonzero $x \in H$ and every nonzero $y \in H$ there are operators $B \in L_x$ and $C \in R_y$ such that $B, C \leqslant A$, then A as well as A^* are surjective which yields that A is invertible. From all these observations we conclude that $A \in B(H)$ is invertible if and only if $\phi(A)$ is invertible.

In particular, $\phi(I)$ is invertible, and after replacing ϕ by $A \mapsto \phi(I)^{-1}\phi(A)$, we may assume that $\phi(I) = I$. By Lemma 4, ϕ maps the set of idempotents onto itself. Moreover, for any two idempotent operators P, Q we have $P \preccurlyeq Q \iff \phi(P) \preccurlyeq \phi(Q)$ where the partial order $\preccurlyeq$ on the set of all idempotents is defined by $P \preccurlyeq Q \iff PQ = QP = P$ (note that on the set of idempotent operators this order coincide with the minus partial order). By [8], there exists a bounded bijective linear or conjugate-linear map $T : H \to H$ such that either $\phi(P) = TPT^{-1}$ for every idempotent operator $P \in B(H)$, or $\phi(P) = TP^*T^{-1}$ for every idempotent operator $P \in B(H)$. Replacing ϕ by $A \mapsto T^{-1}\phi(A)T$, $A \in B(H)$, in the first case and by $A \mapsto T^{-1}\phi(A^*)T$, $A \in B(H)$, in the second case, we may, and we will assume that

$$\phi(P) = P$$

for every idempotent $P \in B(H)$.

If $P \in B(H)$ is a finite rank projection (self-adjoint idempotent) of rank $n \geqslant 3$, then the set $PB(H)P = \{PAP\colon\ A \in B(H)\}$ can be identified with M_n. We will show that $\phi(PB(H)P) \subset PB(H)P$ (and then we actually have $\phi(PB(H)P) = PB(H)P$ because ϕ^{-1} has the same properties as ϕ). Assume for a moment that we have already proved this. Then the restriction of ϕ to $PB(H)P$ can be considered as a bijective map on M_n with the property that $A \leqslant B \iff \phi(A) \leqslant \phi(B)$, $A, B \in M_n$. Moreover, $\phi(Q) = Q$ for every idempotent matrix $Q \in M_n$. It follows from the result of Legiša [4] that $\phi(A) = A$ for every $A \in M_n$. In other words, ϕ acts like the identity on $PB(H)P$, where P is any finite rank projection of rank at least three. In particular, $\phi(R) = R$ for every $R \in B_1(H)$. Consequently, for every $A \in B(H)$ we have $A_- = \phi(A_-) = \phi(A)_-$, and thus, by Lemma 7, $\phi(A) = A$, as desired.

So, in order to complete the proof we have to show that $\phi(PB(H)P) \subset PB(H)P$ for every finite rank projection P. It is enough to show $\phi(A) \in PB(H)P$ for every $A \in PB(H)P$ of rank one. Indeed, assume that ϕ maps rank one operators from $PB(H)P$ into $PB(H)P$. If $B \in PB(H)P$ is an arbitrary operator then $\operatorname{Im} B = \bigcup\{\operatorname{Im} R\colon\ R \in B_1(H) \text{ and } R \leqslant B\}$. Observe that $R \in B_1(H)$ and $R \leqslant B$ yields that $R \in PB(H)P$. Since $\operatorname{Im}\phi(B) = \bigcup\{\operatorname{Im}\phi(R)\colon\ R \in B_1(H) \text{ and } R \leqslant B\}$ and $\operatorname{Im}\phi(R) \subset \operatorname{Im} P$ for every rank one $R \leqslant B$, we see that $\operatorname{Im}\phi(B) \subset \operatorname{Im} P$. We prove in the same way that $\operatorname{Im}\phi(B)^* \subset \operatorname{Im} P$. Thus, $\phi(B) \subset PB(H)P$.

Hence, the last step of the proof is to show that for every $A \in PB(H)P$ of rank one we have $\phi(A) \in PB(H)P$. Let $A = x \otimes y^*$. Then $Px = x$ and $Py = y$. We can find linearly independent $u, v \in \operatorname{Im} P$ such that $x \otimes u^*$ and $x \otimes v^*$ are idempotents of rank one. Then $\phi(x \otimes u^*) = x \otimes u^*$, $\phi(x \otimes v^*) = x \otimes v^*$, and $\phi(A) \sim x \otimes u^*$ and $\phi(A) \sim x \otimes v^*$. It follows that $\phi(A) \in L_x$. Similarly, $\phi(A) \in R_y$. Consequently, $\phi(A) \in PB(H)P$. $\square$

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