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Multipliers of a wandering subspace for a shift invariant subspace ☆

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ABSTRACT

For a shift-invariant subspace M of the two variable Hardy space H^2, we consider the associated wandering subspace $M_0 = M \ominus zM$. Then there exists a nonconstant function ϕ in H^∞ such that $\phi M_0 \subseteq M_0$ if and only if $M = qH^2$ for some inner function q.

Let Γ^2 be the torus that is the Cartesian product of two unit circles Γ in $\mathbb{C}$. For $1 \leqslant p \leqslant \infty$, the usual Lebesgue spaces, with respect to the Lebesgue measure m on Γ^2, are denoted by $L^p = L^p(\Gamma^2)$, and $H^p = H^p(\Gamma^2)$ is the space of all f in L^p whose Fourier coefficients

$$\hat{f}(j,\ell) = \int_{\Gamma^2} f(z,w)\bar{z}^j \bar{w}^\ell \, dm(z,w)$$

are zero as soon as at least one component of (j,ℓ) is negative. Then H^p is called a Hardy space. As $\Gamma^2 = \Gamma_z \times \Gamma_w$, $H^p(\Gamma_z)$ and $H^p(\Gamma_w)$ denote the one variable Hardy spaces.

A closed subspace $M \subseteq H^2$ is called (shift) invariant if $zM \subseteq M$ and $wM \subseteq M$. We write $M_0 = M \ominus zM$ for the associated wandering subspace. We call $\mathcal{M}(M_0) = \{\phi \in H^\infty\colon\ \phi M_0 \subseteq M_0\}$ the multipliers of M_0. When $w \in \mathcal{M}(M_0)$, the author [2] showed that $M = qH^2$ for some inner function q, that is, q is a unimodular function in H^∞. In this paper, we show $\mathcal{M}(M_0) = H^\infty(\Gamma_w)$ if $\mathcal{M}(M_0) \neq \mathbb{C}$.

Lemma 1. $\mathcal{M}(M_0) \subseteq H^\infty$.

Proof. If $\phi \in \mathcal{M}(M_0)$ then $\phi M_0 \subseteq M_0$ and so $\phi M \subseteq M$ because $M = \sum_{j=0}^\infty \oplus z^j M_0$. It is known that ϕ belongs to H^∞. □

Lemma 2. *If ϕ is a nonzero function in $\mathcal{M}(M_0)$ then $\phi(z,w) = q(z,w)h(w)$ where q is inner in H^∞ and h is outer in $H^\infty(\Gamma_w)$.*

Proof. For any g in M_0, $|g|^2 \perp zH^\infty$ and so $|g|^2 \in L^1(\Gamma_w)$. Hence if $\phi \in \mathcal{M}(M_0)$ and $f \in M_0$ then $|\phi f|^2 \in L^1(\Gamma_w)$.

If f is nonzero then $|f|^2 \in L^1(\Gamma_w)$ and $|f|^2 > 0$ a.e., and so $|\phi|^2$ belongs to $L^\infty(\Gamma_w)$. Hence there exists an outer function h in $H^\infty(\Gamma_w)$ such that $|\phi|^2 = |h|^2$. Therefore we can put $q(z,w) = \phi(z,w)/h(w)$. □

Lemma 3. *If qk is in $\mathcal{M}(M_0)$ and q is inner in H^∞ then k belongs to $\mathcal{M}(M_0)$.*

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Proof. If $f \in M_0$ is nonzero then $qkf \in M_0$ and so $qkf \perp zqM$. This shows $kf \perp zM$ and so $k \in \mathcal{M}(M_0)$.

Suppose

$$S_w f = \frac{f(w) - f(0)}{w} \quad \big(f \in H^\infty(\Gamma_w)\big).$$

Then the following lemma is known in [1, Theorem 1]. □

Lemma 4. *Let $\mathcal{B}$ be a nonzero weak $*$ closed subalgebra of $H^\infty(\Gamma_w)$ which contains constants. If $S_w\mathcal{B} \subseteq \mathcal{B}$ then $\mathcal{B} = H^\infty(\Gamma_w)$.*

Theorem. *Let M be an invariant subspace of H^2. Then $\mathcal{M}(M_0) = \mathbb{C}$ or $\mathcal{M}(M_0) = H^\infty(\Gamma_w)$.*

Proof. Suppose $\phi \in \mathcal{M}(M_0)$ and ϕ is nonconstant. By Lemma 2 $\phi(z,w) = q(z,w)h(w)$ where q is inner in H^∞ and h is outer in $H^\infty(\Gamma_w)$. Therefore by Lemma 3, $h \in \mathcal{M}(M_0)$. If h is nonconstant then $h - h(0)$ is nonzero in $\mathcal{M}(M_0)$ because $\mathbb{C} \subseteq \mathcal{M}(M_0)$. By Lemma 3 $S_w h \in \mathcal{M}(M_0)$. Since $\mathcal{M}(M_0) \cap H^\infty(\Gamma_w)$ is a nonzero weak $*$ closed subalgebra in $H^\infty(\Gamma_w)$ which contains constants, $\mathcal{M}(M_0) \cap H^\infty(\Gamma_w) = H^\infty(\Gamma_w)$ by Lemma 4. By [2, Theorem 5] $M = QH^2$ for some inner Q and so $M_0 = QH^2(\Gamma_w)$. Thus $\mathcal{M}(M_0) = H^\infty(\Gamma_w)$. Therefore we may assume that any nonzero functions are scalar multiples of inner functions. If q_1 and q_2 are inner functions in $\mathcal{M}(M_0)$ then $q_1 + q_2$ and $q_1 + iq_2$ are scalar multiples of inner functions. Hence $\mathrm{Re}\,\bar{q}_1 q_2$ and $\mathrm{Im}\,\bar{q}_1 q_2$ are constants. Therefore $q_2 = \alpha q_1$ for some $\alpha \in \mathbb{C}$ with $|\alpha| = 1$. Thus $\mathcal{M}(M_0) = \langle q \rangle$. Since $\mathcal{M}(M_0)$ is an algebra, this shows $\mathcal{M}(M_0) = \mathbb{C}$. □

Corollary. *There exists a nonconstant function ϕ in H^∞ such that $\phi M_0 \subseteq M_0$ if and only if $M = qH^2$ for some inner function q.*

Proof. The 'if' part is clear. The 'only if' part is a result by Theorem and [2, Corollary 1]. □

Remark. Let M be an invariant subspace and ϕ a function in H^∞. Suppose $T_\phi f = \phi f$ $(f \in H^2)$ and $V_\phi f = \phi f$ $(f \in M)$. It is easy to see that $T_z^* T_w = T_w T_z^*$ and if $T_z^* T_\phi = T_\phi T_z^*$ then ϕ belongs to $H^\infty(\Gamma_w)$. On the other hand, $V_z^* V_w = V_w V_z^*$ may not be valid. In fact, it is easy to see that $V_z^* V_\phi = V_\phi V_z^*$ if and only if ϕ belongs to $\mathcal{M}(M_0)$. For if $V_z^* V_\phi = V_\phi V_z^*$ then $V_\phi \operatorname{Ker} V_z^* \subseteq \operatorname{Ker} V_z^*$ and so $\phi \in \mathcal{M}(M_0)$. Conversely if ϕ is in $\mathcal{M}(M_0)$ then $V_z^* V_\phi = V_\phi V_z^*$ on $\operatorname{Ker} V_z^*$. While, $V_z^* V_\phi = V_\phi V_z^*$ holds clearly on zM. Hence $V_z^* V_\phi = V_\phi V_z^*$. Thus if $V_z^* V_\phi = V_\phi V_z^*$ and ϕ is nonconstant then by Theorem $V_z^* V_w = V_w V_z^*$ and $\mathcal{M}(M_0) = H^\infty(\Gamma_w)$.

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