



The inviscid limit for an inflow problem of compressible viscous gas in presence of both shocks and boundary layers

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ABSTRACT

In this paper, we study the inviscid limit problem for the Navier–Stokes equations of one-dimensional compressible viscous gas on half plane. We prove that if the solution of the inviscid Euler system on half plane is piecewise smooth with a single shock satisfying the entropy condition, then there exist solutions to Navier–Stokes equations which converge to the inviscid solution away from the shock discontinuity and the boundary at an optimal rate of ε^1 as the viscosity ε tends to zero.

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1. Introduction

In the theory of compressible fluids, the basic physics issue motivating the mathematical problem is the asymptotic equivalence between the viscous flows and the associated inviscid flows in the limit of small viscosity. This problem is particularly important and of great significance in many physical phenomena and their numerical computations in the presence of boundaries and shock discontinuities. When the underlying inviscid flow is smooth, the Cauchy problem can be solved by classical methods. However, in the presence of boundaries or shock discontinuities, the solutions near the boundaries or shock discontinuities exhibit very singular behavior as the viscosity is small, see the studies [1–3,7,12–17] and the references therein. The rigorous mathematical justification of this asymptotic equivalence poses challenging problems in many important cases. Here we consider the case in presence of both boundary and shock discontinuity for an “inflow problem” for the isentropic fluids of one-dimensional compressible viscous gas on half plane, which was recently proposed by Matsumura [8]. The goal is to understand the evolution and structure of viscous boundary layers and shock layers and their interactions with interior inviscid hyperbolic flows, and to show the uniform convergence of the viscous solutions to the smooth inviscid flow away from the boundary and shock discontinuity. This inflow problem is described by the following Navier–Stokes equations in Euler coordinates

$$\begin{cases} \tilde{Q}_t + (\tilde{Q}\tilde{u})_{\tilde{x}} = 0, & \tilde{x} > 0, t > 0, \\ (\tilde{Q}\tilde{u})_t + (\tilde{Q}(\tilde{u})^2 + p(\tilde{Q}))_{\tilde{x}} = \varepsilon \tilde{u}_{\tilde{x}\tilde{x}}, & \tilde{x} > 0, t > 0, \\ \tilde{Q}(0, t) = \tilde{Q}(t), & t > 0, \\ \tilde{u}(0, t) = \tilde{u}(t) > 0, & t > 0, \end{cases} \quad (1.1)$$

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where $\bar{\varrho}$ and \bar{u} are smooth functions with $0 < \bar{\varrho} < +\infty$. The corresponding inviscid flow is expressed as

$$\begin{cases} \bar{\varrho}_t + (\bar{\varrho}\bar{u})_{\tilde{x}} = 0, & \tilde{x} > 0, t > 0, \\ (\bar{\varrho}\bar{u})_t + (\bar{\varrho}(\bar{u})^2 + p(\bar{\varrho}))_{\tilde{x}} = 0, & \tilde{x} > 0, t > 0, \end{cases} \quad (1.2)$$

which is called Euler system. Here \bar{u} is the velocity, $\bar{\varrho} > 0$ is the density, $p(\bar{\varrho}) = A\bar{\varrho}^\gamma$ is the pressure, $A > 0$ is the gas constant, $\gamma \geq 1$ is the adiabatic constant, and $\varepsilon > 0$ is the viscosity constant. We impose (1.2) with the following discontinuous initial data

$$(\bar{\varrho}, \bar{u})(\tilde{x}, 0) = \begin{cases} (\bar{\varrho}_-, \bar{u}_-)(\tilde{x}), & \tilde{x} < \tilde{x}_0, \\ (\bar{\varrho}_+, \bar{u}_+)(\tilde{x}), & \tilde{x} > \tilde{x}_0, \end{cases} \quad (1.3)$$

for some large $\tilde{x}_0 > 0$, where $\bar{\varrho}_\pm$ and \bar{u}_\pm are smooth functions satisfying

$$-\sqrt{p'(\bar{\varrho}_-)} < \bar{u}_- < \sqrt{p'(\bar{\varrho}_-)}. \quad (1.4)$$

By this condition, we know that for sufficiently small time T , the two eigenvalues of (1.2) have the following property

$$\lambda_1 = \bar{u} - c < 0 \quad \text{and} \quad \lambda_2 = \bar{u} + c > 0, \quad (1.5)$$

on the boundary $\{(\tilde{x}, t): \tilde{x} = 0, t \in [0, T]\}$, where $c = \sqrt{p'(\bar{\varrho})}$. So we only need to impose one boundary condition to the Euler system (1.2). Here we apply (1.2) with the following boundary condition

$$(\bar{\varrho}\bar{u})(0, t) = \bar{\varrho}(t)\bar{u}(t), \quad t > 0. \quad (1.6)$$

Now as in [10], we transform the compressible Navier–Stokes equations (1.1) to the problems in Lagrangian coordinates:

$$\begin{cases} v_t - u_x = 0, & x > X(t), t > 0, \\ u_t + p(v)_x = \varepsilon \left(\frac{u_x}{v} \right)_x, & x > X(t), t > 0, \\ (v, u)(X(t), t) = (\bar{v}(t), \bar{u}(t)), & \bar{v}(t) = \frac{1}{\bar{\varrho}(t)}, \quad \bar{u}(t) > 0, t > 0, \end{cases} \quad (1.7)$$

and the associated Euler system is reduced to

$$\begin{cases} v_t - u_x = 0, & x > X(t), t > 0, \\ u_t + p(v)_x = 0, & x > X(t), t > 0, \\ \frac{u}{v}(X(t), t) = \frac{\bar{u}(t)}{\bar{v}(t)}, & t > 0, \end{cases} \quad (1.8)$$

where $X(t) = -\int_0^t \bar{\varrho}(t)\bar{u}(t)dt$, $v = \frac{1}{\bar{\varrho}}$ denotes the specific volume, and x is the Lagrangian coordinate so that $x = \text{constant}$ corresponds to a particle path. And the initial conditions (1.3)–(1.4) are transformed into

$$(v, u)(x, 0) = \begin{cases} (v_-, u_-)(x), & x < x_0, \\ (v_+, u_+)(x), & x > x_0, \end{cases} \quad (1.9)$$

where $x_0 = \int_0^{\tilde{x}_0} \bar{\varrho}_-(\tilde{x})d\tilde{x}$, and (v_-, u_-) satisfies

$$u_-^2 < -v_-^2 p'(v_-). \quad (1.10)$$

The aim of this paper is to study the asymptotic equivalence between the solutions of the problems (1.7) and (1.8). We consider the case that there is a 2-shock solution (v, u) to the problem (1.8)–(1.9) with the shock issuing from the point x_0 .

Definition 1.1. A function $(v(x, t), u(x, t))$ is called a 2-shock solution of (1.8) up to time T if

- (i) $(v(x, t), u(x, t))$ is a distributional solution of the hyperbolic system (1.8) in the region $[X(t), \infty) \times [0, T]$.
- (ii) There is a smooth curve, the shock, $x = s(t)$, $s(0) = x_0$, $0 \leq t \leq T$, so that $(v(x, t), u(x, t))$ is sufficiently smooth at any point $x \neq s(t)$.
- (iii) The limits

$$\begin{aligned} \partial_x^l(v, u)(s(t) - 0, t) &= \lim_{x \rightarrow s(t)-} \partial_x^l(v, u)(x, t), \\ \partial_x^l(v, u)(s(t) + 0, t) &= \lim_{x \rightarrow s(t)+} \partial_x^l(v, u)(x, t), \end{aligned}$$

exist and are finite for $t \leq T$ and $0 \leq l \leq 5$.

(iv) The Lax geometrical entropy condition [6] is satisfied at $x = s(t)$, that is,

$$\lambda_1(u(s(t) - 0, t)) < \dot{s} < \lambda_2(u(s(t) - 0, t)) \quad \text{and} \quad \lambda_2(u(s(t) + 0, t)) < \dot{s}, \quad (1.11)$$

where $\dot{s} = \frac{d}{dt}s(t)$ and $\lambda_1 = -\sqrt{-p'(v)}$, $\lambda_2 = \sqrt{-p'(v)}$ are characteristic speeds of the hyperbolic system (1.8).

The Lax's shock condition implies that $\dot{s} > 0$. And by (1.10), we know that for sufficiently small time T , the 2-shock solution (v, u) satisfies

$$v(x, t)^2 < -v(x, t)^2 p'(v(x, t)), \quad \text{for } (x, t) \in [X(t), s(t)] \times [0, T]. \quad (1.12)$$

Here we also impose the following requirement on the 2-shock solution (v, u)

$$v_* \leq v(x, t) \leq v^*, \quad (x, t) \in [X(t), \infty) \times [0, T], \quad (1.13)$$

for some positive constants v_* and v^* .

The main results of this paper are as follows:

Theorem 1.2. Suppose that (v, u) is a 2-shock solution up to time T of system (1.8) with the initial data (1.9) satisfying (1.10) such that (1.12) and (1.13) hold, and

$$\sum_{1 \leq k \leq 8} \int_0^T \int_{\{x > X(t), x \neq s(t)\}} |\partial_x^k(v(x, t), u(x, t))|^2 dx dt < \infty. \quad (1.14)$$

Then, there exist constants $\varepsilon_0 > 0$, $\sigma_0 > 0$ and $\mu_0 > 0$, such that if $|\bar{u}(t)| \leq \sigma_0$, $|\bar{v}(t) - v(X(t), t)| \leq \mu_0$ and

$$(\gamma - 1)(v(s(t) + 0, t) - v(s(t) - 0, t)) \leq 2\gamma v(s(t) - 0, t), \quad (1.15)$$

on $[0, T]$, there is a smooth solution $(v^\varepsilon, u^\varepsilon)$ to (1.7) with the same initial data as the approximate solution, constructed in (2.45), for each $\varepsilon \in [0, \varepsilon_0]$. Moreover, it holds for any given $\eta \in (0, 1)$ that

$$\sup_{0 \leq t \leq T} \int_{X(t)}^\infty |(v^\varepsilon, u^\varepsilon)(x, t) - (v, u)(x, t)|^2 dx \leq C_\eta \varepsilon^\eta, \quad (1.16)$$

and

$$\sup_{0 \leq t \leq T, x - X(t) \geq \varepsilon^\eta, |x - s(t)| \geq \varepsilon^\eta} |(v^\varepsilon, u^\varepsilon)(x, t) - (v, u)(x, t)| \leq C_\eta \varepsilon, \quad (1.17)$$

where C_η is a positive constant depending only on η .

Remark 1.3. The main methods of the proof are a matched asymptotic analysis and energy estimates related to the stability theories for the viscous shock profiles and the leading order boundary layer profiles. One of the difficulty of this problem is to control the value $\Psi(0, \tau)$ on the boundary due to the bad sign of $\frac{d}{dt}X(t)$. That is, if we multiply the second equation of (3.5) by Ψ , then the bad term $\frac{d}{dt}X(t)\Psi^2(0, \tau)$, which is due to $\dot{X}(\varepsilon\tau)\Psi_y$ (see (3.5)), comes out after integration by parts. To overcome this difficulty, we use a new unknown variable $\tilde{\Psi} = \Psi - \dot{X}\Phi$ instead of Ψ so that in the new system (see (3.10)), the similar bad term $\dot{X}(\varepsilon\tau)\tilde{\Psi}_y$ does not appear anymore. Thus we can avoid to estimate the boundary term $\Psi^2(0, \tau)$ in the lower order estimate. Of course, the boundary term $\Psi^2(0, \tau)$ can be estimated finally after the a priori estimates are obtained. This idea is originally from [4]. Roughly speaking, such difficulty is caused by the fact that the gas flows into the right-hand side on the boundary, and does not exist for the outflow problem and impermeable wall problem. For higher order derivative estimates, we use temporal derivatives to bound spatial derivatives. The reason is that the boundary values of the spatial derivatives for the error terms may not vanish.

Throughout this paper, we use $O(1)$ to denote any positive bounded function which is independent of ε .

2. Construction of the approximate solutions

In this section, we construct the approximate solutions (v^a, u^a) through different scaling and asymptotic expansions in the regions near and away from the shock and the boundary respectively, such that (v^a, u^a) approximates the piecewise smooth inviscid solution (v, u) away from the shock and the boundary and has a sharp change near the shock and the boundary. Considering the shock is like a free boundary, we first construct the shock layer and inner expansions and then establish the boundary layer expansions.

2.1. Shock layer and inner expansions

2.1.1. Shock layer and inner expansions and the matching conditions

Let $W_i = (v_i, u_i)^t$ and $W_s^i = (V_s^i, U_s^i)^t$, $i = 0, 1, 2, \dots$. In the region away from the shock, $x = s(t)$, and the boundary, $x = X(t)$, we approximate the solution of (1.7) by truncating the formal series

$$w^\varepsilon(x, t) \sim W^{IN}(x, t) \equiv W_0(x, t) + \varepsilon W_1(x, t) + \varepsilon^2 W_2(x, t) + \dots \quad (2.1)$$

Substituting this into (1.7) and comparing the coefficients of powers of ε , we get, for $x \neq s(t)$, that

$$O(1): \quad \begin{cases} v_{0t} - u_{0x} = 0, \\ u_{0t} + p(v_0)_x = 0, \end{cases} \quad (2.2)$$

$$O(\varepsilon): \quad \begin{cases} v_{1t} - u_{1x} = 0, \\ u_{1t} + (p'(v_0)v_1)_x = \left(\frac{u_{0x}}{v_0}\right)_x, \end{cases} \quad (2.3)$$

$$O(\varepsilon^2): \quad \begin{cases} v_{2t} - u_{2x} = 0, \\ u_{2t} + (p'(v_0)v_2)_x = \left(\frac{u_{1x}}{v_0} - \frac{u_{0x}v_1}{v_0^2}\right)_x - \left(\frac{1}{2}p''(v_0)v_1^2\right)_x, \end{cases} \quad (2.4)$$

and etc. The inner functions W_0, W_1, \dots , are generally discontinuous at the shock, $x = s(t)$, but smooth up to the shock.

Near the shock, w^ε is represented by the following shock layer expansion:

$$w^\varepsilon \sim W^S(x, t) \equiv W_s^0(\xi, t) + \varepsilon W_s^1(\xi, t) + \varepsilon^2 W_s^2(\xi, t) + \dots, \quad (2.5)$$

where

$$\xi = \frac{x - s(t)}{\varepsilon} + \delta(t, \varepsilon), \quad (2.6)$$

and $\delta(t, \varepsilon)$ is a perturbation of the shock position to be determined later. We assume that $\delta(t, \varepsilon)$ has the form

$$\delta(t, \varepsilon) = \delta_0(t) + \varepsilon \delta_1(t) + \varepsilon^2 \delta_2(t) + \dots \quad (2.7)$$

Substitute (2.5)–(2.7) into (1.7) to obtain

$$O\left(\frac{1}{\varepsilon}\right): \quad \begin{cases} -\dot{s}\partial_\xi V_s^0 - \partial_\xi U_s^0 = 0, \\ -\dot{s}\partial_\xi U_s^0 + \partial_\xi p(V_s^0) = \partial_\xi \left(\frac{\partial_\xi U_s^0}{V_s^0}\right), \end{cases} \quad (2.8)$$

$$O(1): \quad \begin{cases} -\dot{s}\partial_\xi V_s^1 - \partial_\xi U_s^1 = -\dot{\delta}_0\partial_\xi V_s^0 - \partial_t V_s^0, \\ -\dot{s}\partial_\xi U_s^1 + \partial_\xi(p'(V_s^0)V_s^1) = \partial_\xi \left(\frac{\partial_\xi U_s^1}{V_s^0} - \frac{V_s^1\partial_\xi U_s^0}{(V_s^0)^2}\right) - \dot{\delta}_0\partial_\xi U_s^0 - \partial_t U_s^0, \end{cases} \quad (2.9)$$

$$O(\varepsilon): \quad \begin{cases} -\dot{s}\partial_\xi V_s^2 - \partial_\xi U_s^2 = -\dot{\delta}_1\partial_\xi V_s^0 - \dot{\delta}_0\partial_\xi V_s^1 - \partial_t V_s^1, \\ -\dot{s}\partial_\xi U_s^2 + \partial_\xi(p'(V_s^0)V_s^2) \\ = \partial_\xi \left(\frac{\partial_\xi U_s^2}{V_s^0} - \frac{V_s^2\partial_\xi U_s^0}{(V_s^0)^2}\right) - \partial_\xi \left(\frac{V_s^1\partial_\xi U_s^1}{(V_s^0)^2} - \frac{\partial_\xi U_s^0(V_s^1)^2}{(V_s^0)^3}\right) - \frac{1}{2}\partial_\xi(p''(V_s^0)(V_s^1)^2) \\ - \dot{\delta}_1\partial_\xi U_s^0 - \dot{\delta}_0\partial_\xi U_s^1 - \partial_t U_s^1, \end{cases} \quad (2.10)$$

and etc., where $\dot{s} = ds/dt$, $\dot{\delta}_0 = d\delta_0/dt$, etc. The inner approximation is supposed to be valid in a small zone of size $O(\varepsilon)$ near the shock $x = s(t)$.

In a matching zone, we expect that the outer and the inner expansion agree with each other. Using the Taylor's series to express the outer solutions in terms of ξ , we obtain the following "matching conditions" as $\xi \rightarrow \pm\infty$:

$$W_s^0(\xi, t) = W_0(s(t) \pm 0, t) + o(1), \quad (2.11)$$

$$W_s^1(\xi, t) = W_1(s(t) \pm 0, t) + (\xi - \delta_0)\partial_x W_0(s(t) \pm 0, t) + o(1), \quad (2.12)$$

$$W_s^2(\xi, t) = W_2(s(t) \pm 0, t) + (\xi - \delta_0)\partial_x W_1(s(t) \pm 0, t) - \delta_1\partial_x W_0(s(t) \pm 0, t) \\ + \frac{1}{2}(\xi - \delta_0)^2\partial_x^2 W_0(s(t) \pm 0, t) + o(1), \quad (2.13)$$

and etc.

2.1.2. The structure of viscous shock profiles

Our construction of the approximate solutions depends on the properties of the forward traveling waves $\Phi_s = (V_s, U_s)^t$, which are the solutions of the following ordinary differential equations

$$\begin{cases} -\sigma V'_s - U'_s = 0, \\ -\sigma U'_s + p(V_s)' = \left(\frac{U'_s}{V_s}\right)', \end{cases}$$

with the boundary conditions

$$\Phi_s(\xi) \rightarrow \begin{cases} \phi_l \equiv (v_l, u_l)^t, & \text{as } \xi \rightarrow -\infty, \\ \phi_r \equiv (v_r, u_r)^t, & \text{as } \xi \rightarrow +\infty, \end{cases}$$

and moving with speed σ satisfying

$$\begin{cases} -\sigma(v_l - v_r) - (u_l - u_r) = 0, \\ -\sigma(u_l - u_r) + (p_l - p_r) = 0, \end{cases} \quad (2.14)$$

and the Lax's shock condition $\lambda_{2r} < \sigma < \lambda_{2l}$, where $p_l = p(v_l)$ and $p_r = p(v_r)$. Integrate the differential equations to get

$$\begin{cases} -\sigma V_s - U_s = a_1, \\ \frac{U'_s}{V_s} = -\sigma U_s + P_s + a_2, \end{cases}$$

where $P_s = p(V_s)$, $a_1 = -\sigma v_l - u_l$ and $a_2 = \sigma u_l - p_l$. This system is transformed into

$$\begin{cases} U_s = -\sigma V_s - a_1, \\ \frac{\sigma V'_s}{V_s} = -\left\{P_s + \sigma^2\left(V_s - \frac{b_1}{\sigma^2}\right)\right\} \equiv h(V_s), \end{cases} \quad (2.15)$$

where $b_1 = -\sigma a_1 - a_2 = \sigma^2 v_l + p_l$.

It is well known that there exists a unique shock profile Φ_s up to shift, which connects the states ϕ_l and ϕ_r and satisfies

$$0 < v_l < V_s < v_r, \quad u_r < U_s < u_l, \quad (2.16)$$

$$h(V_s) > 0, \quad V'_s = \frac{V_s h(V_s)}{\sigma} > 0. \quad (2.17)$$

Moreover, as $\xi \rightarrow -\infty$,

$$|\Phi_s(\xi) - \phi_l| = O(1)|v_r - v_l|e^{-\alpha|\xi|}, \quad (2.18)$$

and as $\xi \rightarrow +\infty$,

$$|\Phi_s(\xi) - \phi_r| = O(1)|v_r - v_l|e^{-\alpha|\xi|}, \quad (2.19)$$

where the constant α depends only on ϕ_l .

2.1.3. Solutions of the shock layer and inner problems

Now we can construct W_j and W_s^j order by order.

The leading order outer function, W_0 , is taken to be the single-shock solution in Theorem 1.2. For any fixed t , the leading order inner solution $W_s^0(\xi, t)$ is exactly the viscous shock profile with $w_l(t) \equiv (v_l(t), u_l(t))^t = W(s(t) - 0, t)$, $w_r(t) \equiv (v_r(t), u_r(t))^t = W(s(t) + 0, t)$ and $\sigma = \dot{s}(t)$. So

$$W_s^0(\xi, t) = \Phi_s(\xi, w_l(t), \dot{s}(t)). \quad (2.20)$$

Here we take the shift to be zero since it can be absorbed into $\delta_0(t)$.

Next, we smoothly extend W_0 to the whole space R , still denoted by W_0 , such that

$$\sum_{1 \leq k \leq 8} \int_0^T \int_{\{x \neq s(t)\}} |\partial_x^k W_0(x, t)|^2 dx dt < +\infty.$$

Then as in [14], we can determine W_1 , W_s^1 and δ_0 together.

Proposition 2.1. $W_1(x, t)$, $W_s^1(\xi, t)$ and $\delta_0(t)$ can be determined such that

(i) $W_1(x, t)$ and its derivatives are uniformly continuous up to $x = s(t)$, and

$$\sum_{0 \leq k \leq 6} \int_0^T \int_{\{x \neq s(t)\}} |\partial_x^k W_1(x, t)|^2 dx dt < \infty. \quad (2.21)$$

(ii) $W_s^1(\xi, t)$ and $\delta_0(t)$ are smooth functions, and there is an $\alpha > 0$ such that as $\xi \rightarrow \pm\infty$,

$$W_s^1(\xi, t) = W_1(s(t) \pm 0, t) + (\xi - \delta_0) \partial_x W_0(s(t) \pm 0, t) + O(1) \exp\{-\alpha|\xi|\}. \quad (2.22)$$

And the construction procedure can be carried out to any order. In particular, W_2 , W_s^2 , δ_1 ; W_3 , W_s^3 and δ_2 can be established and the similar results as in Proposition 2.1 hold for them.

2.2. Boundary layer expansions

Since the boundary condition at $x = X(t)$ makes the boundary non-characteristic for the inviscid hyperbolic problem (1.8), we will approximate the viscous solution to the Navier–Stokes equations (1.7) uniformly up to the boundary by the following two-scale expansion

$$w^\varepsilon(x, t) \sim W^{IN}(x, t) + W^B(x, t) \equiv \sum_{i \geq 0} \varepsilon^i W_i(x, t) + \sum_{i \geq 0} \varepsilon^i W_b^i(y, t), \quad x \geq X(t), \quad (2.23)$$

where $y = \frac{x-X(t)}{\varepsilon}$, $W_b^i = (v_b^i, u_b^i)^t$, $i = 0, 1, 2, \dots$, and W^{IN} is the inner solution constructed in the previous subsection. Substituting it into (1.7), according to the power of ε , we thus require that the boundary layer functions W_b^i solve the following ordinary differential equations (regarding t as a parameter):

$$\begin{cases} -\dot{X}(t) \partial_y v_b^0(y, t) - \partial_y u_b^0(y, t) = 0, \\ -\dot{X}(t) \partial_y u_b^0(y, t) + p(v_0(X(t), t) + v_b^0(y, t))_y = \left(\frac{\partial_y u_b^0(y, t)}{v_0(X(t), t) + v_b^0(y, t)} \right)_y, \end{cases} \quad (2.24)$$

$$\begin{cases} -\dot{X}(t) \partial_y v_b^1(y, t) - \partial_y u_b^1(y, t) = -\partial_t v_b^0(y, t), \\ -\dot{X}(t) \partial_y u_b^1(y, t) + (p'(v_0(X(t), t) + v_b^0(y, t)) v_b^1(y, t))_y \\ = \left(\frac{\partial_y u_b^1(y, t)}{v_0(X(t), t) + v_b^0(y, t)} - \frac{\partial_y u_b^0(y, t) v_b^1(y, t)}{(v_0(X(t), t) + v_b^0(y, t))^2} \right)_y + S^1(y, t; v_b^0, u_b^0), \end{cases} \quad (2.25)$$

and

$$\begin{cases} -\dot{X}(t) \partial_y v_b^i(y, t) - \partial_y u_b^i(y, t) = -\partial_t v_b^{i-1}(y, t), \\ -\dot{X}(t) \partial_y u_b^i(y, t) + (p'(v_0(X(t), t) + v_b^0(y, t)) v_b^i(y, t))_y \\ = \left(\frac{\partial_y u_b^i(y, t)}{v_0(X(t), t) + v_b^0(y, t)} - \frac{\partial_y u_b^0(y, t) v_b^i(y, t)}{(v_0(X(t), t) + v_b^0(y, t))^2} \right)_y + S^i(y, t; v_b^0, u_b^0; \dots; v_b^{i-1}, u_b^{i-1}), \quad i \geq 2, \end{cases} \quad (2.26)$$

where $\dot{X}(t) = \frac{d}{dt} X(t)$ and we used the following notations

$$P(y, t, \varepsilon) = p(V^{IN} + V^B) - p(V^{IN}), \quad F(y, t, \varepsilon) = \frac{\partial_y(U^{IN} + U^B)}{(V^{IN} + V^B)} - \frac{\partial_y U^{IN}}{V^{IN}}, \quad (2.27)$$

$$\begin{aligned} S^1(y, t; v_b^0, u_b^0) = & -\left\{ \partial_t u_b^0(y, t) + v_1(X(t), t) p'(v_0(X(t), t) + v_b^0(y, t))_y \right. \\ & \left. - v_1(X(t), t) \left(\frac{\partial_y u_b^0(y, t)}{(v_0(X(t), t) + v_b^0(y, t))^2} \right)_y \right\}, \end{aligned} \quad (2.28)$$

and

$$\begin{aligned}
& S^i(y, t; v_b^0, u_b^0; \dots; v_b^{i-1}, u_b^{i-1}) \\
&= -\partial_t u_b^{i-1}(y, t) - \left\{ \frac{1}{i!} \frac{\partial^i}{\partial \varepsilon^i} P(y, t, \varepsilon) \Big|_{\varepsilon=0} - p'(v_0(X(t), t) + v_b^0(y, t)) v_b^i(y, t) \right\}_y \\
&+ \left\{ \frac{1}{i!} \frac{\partial^i}{\partial \varepsilon^i} F(y, t, \varepsilon) \Big|_{\varepsilon=0} - \left(\frac{\partial_y u_b^i(y, t)}{v_0(X(t), t) + v_b^0(y, t)} - \frac{\partial_y u_b^0(y, t) v_b^i(y, t)}{(v_0(X(t), t) + v_b^0(y, t))^2} \right) \right\}_y, \quad i \geq 2.
\end{aligned} \quad (2.29)$$

Denote $(v(X(t), t), u(X(t), t))$ by $(l_1(t), l_2(t))$. We are looking for the solutions to Eqs. (2.24)–(2.26) with the following boundary conditions

$$\begin{aligned}
& (v_b^0, u_b^0)^t(y=0, t) = (\bar{v}(t) - l_1(t), \bar{u}(t) - l_2(t))^t, \\
& (v_b^0, u_b^0)^t(y, t) \rightarrow 0 \quad \text{as } y \rightarrow \infty,
\end{aligned} \quad (2.30)$$

and

$$(v_b^i, u_b^i)^t(y=0, t) = -(v_i, u_i)^t(X(t), t), \quad (v_b^i, u_b^i)^t(y, t) \rightarrow 0 \quad \text{as } y \rightarrow \infty. \quad (2.31)$$

Lemma 2.2. For each $0 \leq i \leq 3$ and $t \in [0, T]$, if there is a sufficiently small $\mu_0(t) > 0$ such that $|v_b^0(y=0, t)| \leq \mu_0(t)$, then the problems (2.24) ((2.25) or (2.26)) with the boundary conditions (2.30) ((2.31)) has a unique smooth solution $(v_b^i, u_b^i)^t$. Furthermore, there exists a positive constant $\alpha_i > 0$, such that

$$|(v_b^i, u_b^i)^t(y, t)| \leq O(1) \left(\sum_{j=0}^i |v_b^j(0, t)| \right) e^{-\alpha_i y}, \quad (y, t) \in [0, \infty) \times [0, T]. \quad (2.32)$$

Proof. Step 1. We first consider the nonlinear case (2.24) and (2.30). Insert $(2.24)_1$ into $(2.24)_2$ to give

$$\partial_y^2 v_b^0 + \frac{1}{\dot{X}(t)} (l_1(t) + v_b^0) (\dot{X}(t)^2 + p'(l_1(t) + v_b^0)) \partial_y v_b^0 = \frac{(\partial_y v_b^0)^2}{l_1(t) + v_b^0}. \quad (2.33)$$

Set $\omega_1 = v_b^0$ and $\omega_2 = \partial_y v_b^0$. We can rewrite (2.33) as

$$\partial_y \omega = \Lambda(\omega) \omega + G(\omega),$$

where $\omega = (\omega_1, \omega_2)$, $G(\omega) = O(|\omega|^2)$ and

$$\Lambda(\omega) = \begin{pmatrix} 0 & 1 \\ 0 & -\frac{1}{\dot{X}(t)} (l_1(t) + \omega_1) (\dot{X}(t)^2 + p'(l_1(t) + \omega_1)) \end{pmatrix}.$$

Consider the matrix

$$\Lambda(0) = \begin{pmatrix} 0 & 1 \\ 0 & -\frac{1}{\dot{X}(t)} l_1(t) (\dot{X}(t)^2 + p'(l_1(t))) \end{pmatrix},$$

whose eigenvalues are

$$\lambda_1 = 0 \quad \text{and} \quad \lambda_2 = -\frac{l_1(t)}{\dot{X}(t)} (\dot{X}(t)^2 + p'(l_1(t))) < 0,$$

for $t \in [0, T]$ provided that $\mu_0(t)$ is sufficiently small. Therefore, for any boundary condition $\omega(0)$, which is in a small neighborhood of 0, the center-stable manifold theorem implies that in a neighborhood of 0, there exists a smooth solution ω with exponential decay property at infinity. Hence, for each $t \in [0, T]$, if there is a sufficiently small $\mu_0(t) > 0$ such that $|v_b^0(y=0, t)| \leq \mu_0(t)$, then (2.33) has a smooth solution v_b^0 and there is a positive constant α_0 , such that

$$|v_b^0(y, t)| \leq O(1) |v_b^0(0, t)| e^{-\alpha_0 y}. \quad (2.34)$$

The exponential decay property of the profile u_b^0 is a direct consequence of $(2.24)_1$ and (2.34). In addition, one checks that (2.24), (2.30) and (2.32) for $i=0$ imply

$$|\partial_t \partial_y^n (v_b^0, u_b^0)(y, t)| \leq O(1) e^{-\alpha_0 y}, \quad (2.35)$$

for any integer $0 \leq n \leq 5$.

Step 2. Now we consider (2.25) with the boundary condition (2.31) for $i=1$. It follows from (2.28) and (2.35) that

$$|S^1(y, t; v_b^0, u_b^0)| \leq O(1) e^{-\alpha_0 y}. \quad (2.36)$$

Insert (2.25)₁ into (2.25)₂ to get

$$\begin{aligned} \left(\frac{\dot{X}(t) \partial_y v_b^1}{l_1(t) + v_b^0} \right)_y &= - \left\{ \left(\dot{X}(t)^2 + p'(l_1(t) + v_b^0) + \frac{\partial_y u_b^0}{(l_1(t) + v_b^0)^2} \right) v_b^1 \right\}_y \\ &\quad + \dot{X}(t) \partial_t v_b^0 + \left(\frac{\partial_t v_b^0}{l_1(t) + v_b^0} \right)_y + S^1(y, t; v_b^0, u_b^0). \end{aligned} \quad (2.37)$$

Then the unique solution of (2.37) with the boundary condition (2.31) for v_b^1 is given explicitly by

$$v_b^1(y, t) = v_b^1(0, t) \exp \left\{ - \int_0^y f(\zeta, t) d\zeta \right\} - \int_0^y \left(\exp \left\{ \int_z^y f(\zeta, t) d\zeta \right\} \int_z^{+\infty} g(\zeta, t) d\zeta \right) dz, \quad (2.38)$$

where

$$f(y, t) = \frac{l_1(t) + v_b^0(y, t)}{\dot{X}(t)} \left\{ \dot{X}(t)^2 + p'(l_1(t) + v_b^0(y, t)) + \frac{\partial_y u_b^0(y, t)}{(l_1(t) + v_b^0(y, t))^2} \right\}, \quad (2.39)$$

and

$$g(y, t) = \dot{X}(t) \partial_t v_b^0(y, t) + \left(\frac{\partial_t v_b^0(y, t)}{l_1(t) + v_b^0(y, t)} \right)_y + S^1(y, t; v_b^0, u_b^0). \quad (2.40)$$

By (2.35)–(2.36) and the fact $\dot{X}(t) < 0$ for any $t \in [0, T]$, we know that for sufficiently small $\mu_0(t)$, there is an α_1 such that

$$|v_b^1(y, t)| \leq O(1) (|v_b^0(0, t)| + |v_b^1(0, t)|) e^{-\alpha_1 y}. \quad (2.41)$$

Then the estimate for u_b^1 comes from (2.25)₁ and (2.41). To prove the existence of higher order of boundary layer functions, one may derive from (2.29) by direct calculations that S^i depends only on $v_b^0, u_b^0, \dots; v_b^{i-1}, u_b^{i-1}$. Then, (2.26) is a linear second order system of ordinary differential equations about (v_b^i, u_b^i) with a source term S^i . Thus, by induction, one shows that

$$|S^i(y, t; v_b^0, u_b^0; \dots; v_b^{i-1}, u_b^{i-1})| \leq O(1) \left(\sum_{j=0}^{i-1} |v_b^j(0, t)| \right) e^{-\alpha_{i-1} y},$$

for sufficiently small $\mu_0(t), t \in [0, T]$ and some constant α_{i-1} . As a consequence, one can solve (2.26) and (2.31) for $2 \leq i \leq 3$ exactly by the same way as for the case $i = 1$ to obtain a similar solution formula as in (2.38)–(2.40). Then (2.32) for $2 \leq i \leq 3$ also follows provided that $\mu_0(t), t \in [0, T]$ is sufficiently small. The lemma is proved. \square

Indeed, the above construction procedure can be applied to any order.

2.3. Approximate solutions

Now we can construct an approximate solution to (1.7) by patching the truncated boundary layer, shock layer and inner solutions in the previous discussion. Define

$$W^B(x, t) = \sum_{i=0}^3 \varepsilon^i W_b^i \left(\frac{x - X(t)}{\varepsilon}, t \right), \quad x \geq X(t), \quad (2.42)$$

$$W^S(x, t) = \sum_{i=0}^3 \varepsilon^i W_s^i \left(\frac{x - S(t)}{\varepsilon} + \delta_0(t) + \varepsilon \delta_1(t) + \varepsilon^2 \delta_2(t), t \right), \quad x \geq X(t), \quad (2.43)$$

and

$$W^{IN}(x, t) = W_0(x, t) + \varepsilon W_1(x, t) + \varepsilon^2 W_2(x, t) + \varepsilon^3 W_3(x, t), \quad x \geq X(t). \quad (2.44)$$

Let $m \in C_0^\infty(R)$ satisfy $0 \leq m(y) \leq 1$, and

$$m(y) = \begin{cases} 1, & |y| \leq 1, \\ 0, & |y| \geq 2. \end{cases}$$

Set $\nu \in (\frac{3}{4}, 1)$ to be a constant and ε to be small such that $x_0 > 4\varepsilon^\nu$. Then we define the approximate solution to (1.7) as

$$W^a(x, t) = m \left(\frac{x - X(t)}{\varepsilon^\nu} \right) (W^B + W^{IN})(x, t) + m \left(\frac{x - s(t)}{\varepsilon^\nu} \right) W^S(x, t) \\ + \left(1 - m \left(\frac{x - X(t)}{\varepsilon^\nu} \right) - m \left(\frac{x - s(t)}{\varepsilon^\nu} \right) \right) W^{IN}(x, t) + D(x, t), \quad x \geq X(t), \quad (2.45)$$

where $D(x, t)$ is a higher-order correction term to be determined, and we have used the following notations:

$$W^a = (v^a, u^a)^t, \quad W^B = (v^B, u^B)^t, \quad \text{etc.}, \quad D = (d_1, d_2)^t.$$

Let $m_1(x, t) = m(\frac{x-X(t)}{\varepsilon^\nu})$ and $m_2(x, t) = m(\frac{x-s(t)}{\varepsilon^\nu})$. Using the structures of the various orders of boundary layer, shock layer and inner solutions, we compute that

$$\begin{cases} v_t^a - u_x^a = d_{1t} - d_{2x} + q_1^B(x, t) + q_1^S(x, t), & x > X(t), \quad t > 0, \\ u_t^a + p(v^a)_x - \varepsilon \left(\frac{u_x^a}{v^a} \right)_x \\ \quad = d_{2t} - \varepsilon \left(\frac{d_{2x}}{m_1(v^B + v^{IN}) + m_2 v^S + (1 - m_1 - m_2)v^{IN}} \right)_x \\ \quad \quad + q_2(x, t) + q_3^B(x, t) + q_3^S(x, t) + q_4(x, t) - q_{5x}(x, t) + q_{6x}(x, t), & x > X(t), \quad t > 0, \\ (v^a, u^a)(X(t), t) = (\bar{v}(t), \bar{u}(t)), & t > 0, \end{cases} \quad (2.46)$$

where

$$q_1^B(x, t) = m_{1t} v^B - m_{1x} u^B + m_1 \{ \partial_t (v^B + v^{IN}) - \partial_x (u^B + u^{IN}) \}, \quad (2.47)$$

$$q_1^S(x, t) = m_{2t} (v^S - v^{IN}) - m_{2x} (u^S - u^{IN}) + m_2 (v_t^S - u_x^S), \quad (2.48)$$

$$q_2(x, t) = m_{1t} u^B + m_{2t} (u^S - u^{IN}) + \{ p(m_1(v^B + v^{IN}) + m_2 v^S + (1 - m_1 - m_2)v^{IN}) \\ - m_1 p(v^B + v^{IN}) - m_2 p(v^S) - (1 - m_1 - m_2)p(v^{IN}) \}_x \\ + m_{1x} (p(v^B + v^{IN}) - p(v^{IN})) + m_{2x} (p(v^S) - p(v^{IN})) \\ - \varepsilon m_{1x} \left(\frac{(u^B + u^{IN})_x}{v^B + v^{IN}} - \frac{u_x^{IN}}{v^{IN}} \right) - \varepsilon m_{2x} \left(\frac{u_x^S}{v^S} - \frac{u_x^{IN}}{v^{IN}} \right), \quad (2.49)$$

$$q_3^B(x, t) = m_1 \left\{ \partial_t (u^B + u^{IN}) + p(v^B + v^{IN})_x - \varepsilon \left(\frac{(u^B + u^{IN})_x}{v^B + v^{IN}} \right)_x \right\}, \quad (2.50)$$

$$q_3^S(x, t) = m_2 \left\{ \partial_t u^S + p(v^S)_x - \varepsilon \left(\frac{u_x^S}{v^S} \right)_x \right\}, \quad (2.51)$$

$$q_4(x, t) = (1 - m_1 - m_2) \left\{ \partial_t u^{IN} + p(v^{IN})_x - \varepsilon \left(\frac{u_x^{IN}}{v^{IN}} \right)_x \right\}, \quad (2.52)$$

$$q_5(x, t) = \varepsilon \left\{ \frac{u^a - d_2}{v^a} - m_1 \frac{(u^B + u^{IN})_x}{v^B + v^{IN}} - m_2 \frac{u_x^S}{v^S} - (1 - m_1 - m_2) \frac{u_x^{IN}}{v^{IN}} + \left(\frac{1}{v^a} - \frac{1}{v^a - d_1} \right) d_{2x} \right\}, \quad (2.53)$$

$$q_6(x, t) = p(v^a) - p(v^a - d_1). \quad (2.54)$$

In view of our construction, we have

(i) $\text{supp}(q_1^B, q_3^B) \subseteq \{(x, t): 0 \leq x - X(t) \leq 2\varepsilon^\nu, 0 \leq t \leq T\}$, and

$$\partial_x^l (q_1^B, q_3^B)(x, t) = O(1)\varepsilon^{(3-l)\nu}, \quad l = 0, 1, 2, 3. \quad (2.55)$$

(ii) $\text{supp}(q_1^S, q_3^S) \subseteq \{(x, t): |x - s(t)| \leq 2\varepsilon^\nu, 0 \leq t \leq T\}$, and

$$\partial_x^l (q_1^S, q_3^S)(x, t) = O(1)\varepsilon^{(3-l)\nu}, \quad l = 0, 1, 2, 3. \quad (2.56)$$

(iii) $\text{supp} q_2 \subseteq \{(x, t): \varepsilon^\nu \leq x - X(t) \leq 2\varepsilon^\nu \text{ or } \varepsilon^\nu \leq |x - s(t)| \leq 2\varepsilon^\nu, 0 \leq t \leq T\}$, and

$$\partial_x^l q_2(x, t) = O(1)\varepsilon^{(3-l)\nu}, \quad l = 0, 1, 2, 3. \quad (2.57)$$

(iv) $\text{supp} q_4 \subseteq \{(x, t): x - X(t) \geq \varepsilon^\nu \text{ and } |x - s(t)| \geq \varepsilon^\nu, 0 \leq t \leq T\}$, and

$$\begin{aligned} \partial_x^l q_4(x, t) &= O(1)\varepsilon^{4-l\nu}, \quad \left(\int_0^T \int_{X(t)}^\infty |q_4(x, t)|^2 dx dt \right)^{\frac{1}{2}} \leq O(1)\varepsilon^4, \\ \left(\int_0^T \int_{X(t)}^\infty |\partial_x^l q_4(x, t)|^2 dx dt \right)^{\frac{1}{2}} &\leq O(1)\varepsilon^{4-(l-\frac{1}{2})\nu}, \quad l = 1, 2, 3. \end{aligned} \quad (2.58)$$

We now choose $D(x, t) = (d_1(x, t), d_2(x, t))$ to be the solution of

$$\begin{cases} d_{1t} - d_{2x} = -q_1^B(x, t) - q_1^S(x, t), & x > X(t), t > 0, \\ d_{2t} - \varepsilon \left(\frac{d_{2x}}{m_1(v^B + v^{IN}) + m_2 v^S + (1 - m_1 - m_2)v^{IN}} \right)_x \\ \quad = -(q_2(x, t) + q_3^B(x, t) + q_3^S(x, t) + q_4(x, t)), & x > X(t), t > 0, \\ d_1(X(t), t) = 0, \quad d_2(X(t), t) = 0, & t > 0, \\ d_1(x, 0) = d_2(x, 0) = 0, & x > 0, \end{cases} \quad (2.59)$$

so that W^a satisfies

$$\begin{cases} v_t^a - u_x^a = 0, & x > X(t), t > 0, \\ u_t^a + p(v^a)_x = \varepsilon \left(\frac{u_x^a}{v^a} \right)_x - q_{5x}(x, t) + q_{6x}(x, t), & x > X(t), t > 0, \\ (v^a, u^a)(X(t), t) = (\bar{v}(t), \bar{u}(t)), & t > 0, \\ \text{the initial data of } (v^a, u^a) \text{ is determined by those of } W_i, W_s^i, W_b^i \text{ and } D \text{ for } 0 \leq i \leq 3. \end{cases} \quad (2.60)$$

Using the standard energy estimates for linear parabolic equation and the fact $d_1(x, t) = \int_0^t d_{2x}(x, \tau) d\tau - \int_0^t q_1(x, \tau) d\tau$, we have the following results. Here we omit the proof.

Lemma 2.3. Let $D(x, t)$ be the solution of (2.59). The following estimates hold for all $t \in [0, T]$:

- (i) $\text{esssup}_{x \in [X(t), \infty)} |\partial_x^l d_2(x, t)| \leq O(1)\varepsilon^{(4-l)\nu - \frac{1}{2}}, \quad \text{for } l = 0, 1, 2, 3, 4,$
 $\left(\int_{X(t)}^\infty |\partial_x^l d_2(x, t)|^2 dx \right)^{\frac{1}{2}} \leq O(1)\varepsilon^{(4-l+\frac{1}{2})\nu - \frac{1}{2}}, \quad l = 0, 1, 2, 3, 4.$
- (ii) $\text{esssup}_{x \in [X(t), \infty)} |\partial_x^l d_1(x, t)| \leq O(1)\varepsilon^{(3-l)\nu - \frac{1}{2}}, \quad \text{for } l = 0, 1, 2, 3,$
 $\left(\int_{X(t)}^\infty |\partial_x^l d_1(x, t)|^2 dx \right)^{\frac{1}{2}} \leq O(1)\varepsilon^{(3-l+\frac{1}{2})\nu - \frac{1}{2}}, \quad l = 0, 1, 2, 3.$
- (iii) $\left(\int_{X(t)}^\infty |\partial_x^l (q_5, q_6)(x, t)|^2 dx \right)^{\frac{1}{2}} \leq O(1)\varepsilon^{\frac{7}{2}\nu - l - \frac{1}{2}}, \quad l = 0, 1, 2, 3.$
- (iv) $\left(\int_{X(t)}^\infty |(q_{5t}, q_{6t})(x, t)|^2 dx \right)^{\frac{1}{2}} \leq O(1)\varepsilon^{\frac{7}{2}\nu - \frac{3}{2}}.$

It follows from our construction that W^a has the following property.

Lemma 2.4. Let W^a be defined in (2.45), then

$$W^a(x, t) = \begin{cases} W_0(x, t) + O(1)\varepsilon, & \text{if } x - X(t) \geq \varepsilon^\nu \text{ and } |x - s(t)| \geq \varepsilon^\nu, \\ W_0(X(t), t) + W_b^0(y, t) + O(1)\varepsilon^\nu, & \text{if } 0 \leq x - X(t) \leq 2\varepsilon^\nu, \\ W_s^0(\xi, t) + O(1)\varepsilon^\nu, & \text{if } |x - s(t)| \leq 2\varepsilon^\nu. \end{cases} \quad (2.61)$$

Under the following coordinate transformation

$$y = \frac{x - X(t)}{\varepsilon} \quad \text{and} \quad \tau = \frac{t}{\varepsilon},$$

we have

$$\partial_y^l W^a = m_1 \partial_y^l W_b^0 + m_2 \partial_y^l W_s^0 + O(1)\varepsilon, \quad 1 \leq l \leq 3, \quad (2.62)$$

and

$$\partial_\tau \partial_y^l W^a = m_2 (\dot{X} - \dot{s}) \partial_y^{l+1} W_s^0 + O(1)\varepsilon, \quad 0 \leq l \leq 3. \quad (2.63)$$

3. Stability analysis

We now show that there exists an exact solution to (1.7) in a neighborhood of the approximate solution W^a , and that the asymptotic behavior of the viscous solution is given by W^a for small viscosity ε .

Suppose that $W^\varepsilon = (v^\varepsilon, u^\varepsilon)$ is the exact solution to (1.7) with the initial data $W^\varepsilon(x, 0) = W^a(x, 0)$. We decompose the solution as

$$v^\varepsilon(x, t) = v^a(x, t) + \phi(x, t), \quad u^\varepsilon(x, t) = u^a(x, t) + \psi(x, t), \quad (3.1)$$

for $(x, t) \in [X(t), \infty) \times [0, T]$. Then using the relation (2.60) for W^a , we obtain that

$$\begin{cases} \phi_t - \psi_x = 0, & x > X(t), t > 0, \\ \psi_t + (p'(v^a)\phi)_x + Q(v^a, \phi)_x = \varepsilon \left(\frac{u_x^\varepsilon}{v^\varepsilon} - \frac{u_x^a}{v^a} \right)_x + (q_5 - q_6)_x, & x > X(t), t > 0, \\ \phi(X(t), t) = \psi(X(t), t) = 0, & t > 0, \\ \phi(x, 0) = \psi(x, 0) = 0, & x > 0, \end{cases} \quad (3.2)$$

where

$$Q(v^a, \phi) = p(v^\varepsilon) - p(v^a) - p'(v^a)\phi \quad \text{satisfies } |Q| \leq O(1)\phi^2.$$

To exploit the fact that a shock satisfying the entropy condition is compressive, we need to integrate the system (3.2) once. Thus we set $(\phi, \psi)(x, t) = (\bar{\phi}_x, \bar{\psi}_x)(x, t)$. Substitute these quantities into (3.2) and integrate the resulting equation with respect to x to obtain

$$\begin{cases} \bar{\phi}_t - \bar{\psi}_x = 0, & x > X(t), t > 0, \\ \bar{\psi}_t + p'(v^a)\bar{\phi}_x = \varepsilon \left(\frac{u_x^\varepsilon}{v^\varepsilon} - \frac{u_x^a}{v^a} \right) - Q(v^a, \bar{\phi}_x) + q_5 - q_6, & x > X(t), t > 0, \\ (\bar{\phi}_x, \bar{\psi}_x)(X(t), t) = 0, & t > 0, \\ \bar{\phi}(x, 0) = \bar{\psi}(x, 0), & x > 0. \end{cases} \quad (3.3)$$

By making the following rescalings,

$$(\bar{\phi}, \bar{\psi})(x, t) = \varepsilon(\Phi, \Psi)(y, \tau), \quad y = \frac{x - X(t)}{\varepsilon}, \quad \tau = \frac{t}{\varepsilon}, \quad (3.4)$$

we transform (3.3) into

$$\begin{cases} \Phi_\tau - \dot{X}(\varepsilon\tau)\Phi_y - \Psi_y = 0, & y > 0, \tau > 0, \\ \Psi_\tau - \dot{X}(\varepsilon\tau)\Psi_y - g(W^a)\Phi_y \\ \quad = \frac{\Psi_{yy}}{v^a} - \frac{\Phi_y\Psi_{yy}}{v^\varepsilon v^a} + \frac{u_y^a\Phi_y^2}{(v^a)^2 v^\varepsilon} - Q(v^a, \Phi_y) + q_5 - q_6, & y > 0, \tau > 0, \\ (\Phi_y, \Psi_y)(0, \tau) = 0, & \tau > 0, \\ \Phi(y, 0) = \Psi(y, 0) = 0, & y > 0, \end{cases} \quad (3.5)$$

where $g(W^a) = -(p'(v^a) + u_y^a/(v^a)^2)$, and

$$Q(v^a, \Phi_y) = p(v^a + \Phi_y) - p(v^a) - p'(v^a)\Phi_y \quad \text{satisfies } |Q| \leq O(1)\Phi_y^2. \quad (3.6)$$

Then we only need to show that for suitably small ε , (3.5) has a unique “small” smooth solution up to T/ε . Now we give some notations. In what follows, we use $H^l(\Omega)$ ($l \geq 1$) to denote the usual Sobolev space on the domain $\Omega \subset \mathbb{R} = (-\infty, +\infty)$ with the norm $\|\cdot\|_l \equiv \|\cdot\|_{H^l(\Omega)}$ and $\|\cdot\| = \|\cdot\|_0$ denotes the corresponding L_2 -norm. The domain Ω will be often abbreviated without confusion. We set $\mu = \sup_{0 \leq t \leq T} |\bar{v}(t) - v(X(t), t)|$. We also use c to denote any positive constant which is independent of ε, y and τ ; and \bar{c} to denote any positive constant which is independent of ε and μ . By the standard

existence and uniqueness theory, and the continuous induction argument for hyperbolic-parabolic equations [5], it suffices to close the following a priori estimate

$$N(\tau) \equiv \|(\Phi, \Psi)(\cdot, \tau)\|_2 + \|\partial_y^3 \Psi\| \leq \varpi, \quad (3.7)$$

where ϖ is a positive small constant depending only on the initial data, the strength of the shock and the difference of the boundary values for the Navier–Stokes equations and Euler equations. In fact, we have the following result.

Proposition 3.1. *Suppose that the initial boundary value problem (3.5) has a smooth solution (Φ, Ψ) with $\Phi \in C([0, \tau_0] : H^2(0, +\infty))$ and $\Psi \in C([0, \tau_0] : H^3(0, +\infty))$ for some $\tau_0 \in (0, T/\varepsilon]$. Then there exist positive constants $\varepsilon_1, \sigma_1, \mu_1$, and C , which are independent of ε and τ_0 , such that if $0 < \varepsilon < \varepsilon_1$, $0 < \tilde{u}(\varepsilon\tau) \leq \sigma_1$, $N(\tau) + \mu \leq \mu_1$ and $(\gamma - 1)(v_r - v_l)^2 < 2\gamma v_l$, then*

$$\sup_{0 \leq \tau \leq \tau_0} N(\tau)^2 + \int_0^{\tau_0} (\|\partial_y \Phi(\cdot, \tau)\|_1^2 + \|\partial_y \Psi(\cdot, \tau)\|_2^2) d\tau \leq C\varepsilon^{7\nu-4}, \quad (3.8)$$

where ν is defined in Section 2.3.

The proof of Proposition 3.1 occupies the rest of this section. The key point is the following lower order estimate. We set $\varepsilon \leq 1$.

3.1. Lower order estimate

Lemma 3.2. *Suppose that the conditions in Proposition 3.1 are satisfied. Then*

$$\|(\Phi, \Psi)(\cdot, \tau)\|_1^2 + \int_0^\tau (\|\Phi_y(\cdot, \tau)\|^2 + \|\Psi_y(\cdot, \tau)\|_1^2) d\tau + \int_0^\tau \int_0^\infty m_2 \partial_y V_s \Psi^2 dy d\tau \leq c\varepsilon^{7\nu-4}, \quad (3.9)$$

for all $\tau \in [0, \tau_0]$, where the constant c is independent of τ_0 and ε .

Proof. Step 1. Set $\tilde{\Psi} = \Psi - \dot{X}\Phi$. Then (3.5) is transformed into

$$\begin{cases} \Phi_\tau - 2\dot{X}(\varepsilon\tau)\Phi_y - \tilde{\Psi}_y = 0, & y > 0, \tau > 0, \\ \tilde{\Psi}_\tau - (g(W^a) - \dot{X}^2(\varepsilon\tau))\Phi_y + \varepsilon\ddot{X}(\varepsilon\tau)\Phi \\ \quad = \frac{\tilde{\Psi}_{yy}}{v^a} + \frac{\dot{X}(\varepsilon\tau)\Phi_{yy}}{v^a} - \frac{\Phi_y(\tilde{\Psi}_{yy} + \dot{X}(\varepsilon\tau)\Phi_{yy})}{v^\varepsilon v^a} + \frac{u_y^a \Phi_y^2}{v^\varepsilon (v^a)^2} - Q(v^a, \Phi_y) + q_5 - q_6, & y > 0, \tau > 0, \\ (\Phi_y, \tilde{\Psi}_y)(0, \tau) = 0, & \tau > 0, \\ \Phi(y, 0) = \tilde{\Psi}(y, 0) = 0, & y > 0, \end{cases} \quad (3.10)$$

where $\ddot{X} = \frac{d^2}{dt^2} X(t)$. Multiplying (3.10)₁ and (3.10)₂ by Φ and $\frac{\tilde{\Psi}}{g(W^a) - \dot{X}^2}$, respectively, then integrating over $[0, +\infty)$, and adding the resulting equations, we obtain after integration by parts that

$$\begin{aligned} & \frac{1}{2} \frac{d}{d\tau} \int_0^\infty \left(\Phi^2 + \frac{\tilde{\Psi}^2}{g(W^a) - \dot{X}^2} \right) (y, \tau) dy \\ & + \int_0^\infty \left\{ -\frac{1}{2} \left(\frac{1}{g(W^a) - \dot{X}^2} \right)_\tau \tilde{\Psi}^2 + \left(\frac{1}{v^a(g(W^a) - \dot{X}^2)} \right)_y \tilde{\Psi} \tilde{\Psi}_y + \frac{1}{v^a(g(W^a) - \dot{X}^2)} \tilde{\Psi}_y^2 \right\} dy \\ & = -\varepsilon \ddot{X} \int_0^\infty \frac{1}{g(W^a) - \dot{X}^2} \Phi \tilde{\Psi} dy - \dot{X} \int_0^\infty \frac{1}{v^a(g(W^a) - \dot{X}^2)} \Phi_y \tilde{\Psi}_y dy \\ & - \dot{X} \int_0^\infty \left(\frac{1}{v^a(g(W^a) - \dot{X}^2)} \right)_y \Phi_y \tilde{\Psi} dy - \int_0^\infty \frac{\tilde{\Psi} \Phi_y (\tilde{\Psi}_{yy} + \dot{X} \Phi_{yy})}{v^\varepsilon v^a (g(W^a) - \dot{X}^2)} dy \\ & + \int_0^\infty \left\{ \frac{u_y^a \tilde{\Psi} \Phi_y^2}{v^\varepsilon (v^a)^2 (g(W^a) - \dot{X}^2)} - \frac{Q \tilde{\Psi}}{g(W^a) - \dot{X}^2} \right\} dy + \int_0^\infty \frac{\tilde{\Psi} (q_5 - q_6)}{g(W^a) - \dot{X}^2} dy, \end{aligned} \quad (3.11)$$

where we have used the fact $\Phi(0, \tau) = 0$, which comes from (3.10)₁ and the boundary conditions. We denote the second term on the left of (3.11) by I , and the terms on the right-hand side of (3.11) in order by J_i , $1 \leq i \leq 6$. Now we estimate them separately as follows.

First, Using Lemma 2.2 and Lemma 2.4, we have $g(W^a) - \dot{X}^2 = -p'(v^a) - \dot{X}^2 - \frac{u_y^a}{(v^a)^2} = -p'(v^a) - \frac{m_2 \partial_y U_s}{(v^a)^2} - \dot{X}^2 + O(1) \times (\mu + \varepsilon) > 0$ for sufficiently small $|\dot{X}|$, μ and ε . Then it follows from Young's inequality that

$$I \geq (1 - \eta_1) \int_0^\infty \frac{1}{v^a(g(W^a) - \dot{X}^2)} \tilde{\psi}_y^2 dy \\ + \int_0^\infty \left\{ -\frac{1}{2} \left(\frac{1}{g(W^a) - \dot{X}^2} \right)_\tau - \frac{1}{4\eta_1} v^a(g(W^a) - \dot{X}^2) \left(\left(\frac{1}{v^a(g(W^a) - \dot{X}^2)} \right)_y \right)^2 \right\} \tilde{\psi}^2 dy,$$

for any $\eta_1 \in (0, 1)$, which will be determined later. Denote the second term by $\int_0^\infty z(W^a) \tilde{\psi}^2 dy$. Then due to (2.15), Lemma 2.4 and the fact

$$|\partial_y(V_s, U_s)| = O(1)\varepsilon \quad \text{on} \quad \left| y - \frac{s(t) - X(t)}{\varepsilon} \right| \geq \varepsilon^{\nu-1}, \quad (3.12)$$

we get

$$z(W^a) = -\frac{1}{2} \left(\frac{1}{g(W^a) - \dot{X}^2} \right)_\tau - \frac{1}{4\eta_1} v^a(g(W^a) - \dot{X}^2) \left(\left(\frac{1}{v^a(g(W^a) - \dot{X}^2)} \right)_y \right)^2 \\ = -\frac{1}{2} m_2 (\dot{X} - \dot{s}) \left(\frac{1}{g(W_s^0) - \dot{X}^2} \right)_y - \frac{1}{4\eta_1} m_2 V_s (g(W_s^0) - \dot{X}^2) \left(\left(\frac{1}{V_s(g(W_s^0) - \dot{X}^2)} \right)_y \right)^2 \\ + O(1)\varepsilon^\nu m_2 \partial_y V_s + O(1) |\partial_y v_b^0|^2 + O(1)\varepsilon,$$

where

$$g(W_s^0) = -p'(V_s) - \frac{\partial_y U_s}{V_s^2} = \frac{\dot{s} \partial_y V_s}{V_s^2} - p'(V_s) = \frac{h(V_s) - V_s p'(V_s)}{V_s} \equiv \frac{K(V_s)}{V_s},$$

and so

$$z(W^a) = \frac{1}{2} m_2 \dot{s} \left(\frac{V_s}{K(V_s) - \dot{X}^2 V_s} \right)_y - \frac{1}{4\eta_1} m_2 V_s \frac{K(V_s) - \dot{X}^2 V_s}{V_s} \left(\left(\frac{1}{K(V_s) - \dot{X}^2 V_s} \right)_y \right)^2 \\ - \frac{1}{2} m_2 \dot{X} \left(\frac{V_s}{K(V_s) - \dot{X}^2 V_s} \right)_y + O(1)\varepsilon^\nu m_2 \partial_y V_s + O(1) |\partial_y v_b^0|^2 + O(1)\varepsilon \\ = \left\{ \frac{\dot{s}}{2} \frac{K(V_s) - V_s K'(V_s)}{(K(V_s) - \dot{X}^2 V_s)^2} - \frac{1}{4\eta_1} \frac{(K'(V_s) - \dot{X}^2)^2 \partial_y V_s}{(K(V_s) - \dot{X}^2 V_s)^3} \right\} m_2 \partial_y V_s - \frac{1}{2} \dot{X} \frac{K(V_s) - V_s K'(V_s)}{(K(V_s) - \dot{X}^2 V_s)^2} m_2 \partial_y V_s \\ + O(1)\varepsilon^\nu m_2 \partial_y V_s + O(1) |\partial_y v_b^0|^2 + O(1)\varepsilon \\ = \left\{ \frac{\dot{s}}{2} \frac{K(V_s) - V_s K'(V_s)}{K(V_s)^2} - \frac{1}{4\eta_1} \frac{K'(V_s)^2 \partial_y V_s}{K(V_s)^3} \right\} m_2 \partial_y V_s \\ + O(1)(\dot{X}^2 + \dot{X} + \varepsilon^\nu) m_2 \partial_y V_s + O(1) |\partial_y v_b^0|^2 + O(1)\varepsilon \\ \equiv z(V_s) m_2 \partial_y V_s + O(1)(\dot{X} + \varepsilon^\nu) m_2 \partial_y V_s + O(1) |\partial_y v_b^0|^2 + O(1)\varepsilon,$$

for bounded $|\dot{X}|$. As in [9], a tedious but straightforward computation gives

$$z(V_s) = \frac{1}{4\dot{s}K(V_s)^3} \left\{ 2\dot{s}^2 [\gamma^3 p(V_s)^2 + h(V_s)^2 + \gamma \dot{s}^2 V_s p(V_s)] + 2\dot{s}^2 \gamma [(\gamma + 1) - (2\eta_1)^{-1}(\gamma - 1)] p(V_s) h(V_s) \right. \\ \left. - \frac{\gamma^2 (\gamma - 1)^2}{\eta_1 V_s} h(V_s) p(V_s)^2 + 2\dot{s}^4 [1 - (2\eta_1)^{-1}] V_s h(V_s) \right\} \\ \geq \frac{1}{4\dot{s}K(V_s)^3} \left\{ 2\dot{s}^2 [h(V_s)^2 + \gamma \dot{s}^2 V_s p(V_s)] + 2\dot{s}^2 \gamma^2 \left[\gamma - \frac{(\gamma - 1)^2 (v_r - v_l)}{2\eta_1 v_l} \right] p(V_s)^2 \right\}$$

$$+ 2\dot{s}^2\gamma[(\gamma+1) - (2\eta_1)^{-1}(\gamma-1)]p(V_s)h(V_s) + 2\dot{s}^4[1 - (2\eta_1)^{-1}]V_sh(V_s)\Big\} \\ \geq \underline{c} > 0,$$

for some constant $\underline{c} > 0$, provided that $\max\{\frac{(\gamma-1)(v_r-v_l)^2}{2\gamma v_l}, \frac{1}{2}\} \leq \eta_1 < 1$. Hence

$$I \geq (\underline{c} + O(1)(|\dot{X}| + \varepsilon^\nu)) \int_0^\infty m_2 \partial_y V_s \tilde{\Psi}^2 dy + (1 - \eta_1) \int_0^\infty \frac{1}{v^a(g(W^a) - \dot{X}^2)} \tilde{\Psi}_y^2 dy \\ - c \int_0^\infty |\partial_y v_b^0|^2 \tilde{\Psi}^2 dy - c\varepsilon \|\tilde{\Psi}(\cdot, \tau)\|^2.$$

Next we estimate the terms J_i , $1 \leq i \leq 6$, respectively. First Cauchy–Schwarz inequality gives

$$J_1 \leq c\varepsilon (\|\Phi(\cdot, \tau)\|^2 + \|\tilde{\Psi}(\cdot, \tau)\|^2),$$

and Young's inequality deduces that

$$J_2 \leq \frac{1 - \eta_1}{2} \int_0^\infty \frac{1}{v^a(g(W^a) - \dot{X}^2)} \tilde{\Psi}_y^2 dy + c\dot{X}^2 \|\Phi_y(\cdot, \tau)\|^2.$$

Using (2.62) and Young's inequality again, one finds

$$J_3 = O(1)|\dot{X}| \int_0^\infty (\partial_y v^a + \partial_y^2 u^a) \tilde{\Psi} \Phi_y dy \\ = O(1)|\dot{X}| \int_0^\infty (m_1 \partial_y v_b^0 + m_2 \partial_y V_s + \varepsilon) \tilde{\Psi} \Phi_y dy \\ \leq \frac{1}{2}\underline{c} \int_0^\infty m_2 \partial_y V_s \tilde{\Psi}^2 dy + c \int_0^\infty |\partial_y v_b^0|^2 \tilde{\Psi}^2 dy + c\varepsilon \|\tilde{\Psi}(\cdot, \tau)\|^2 + c(\dot{X}^2 + \varepsilon) \|\Phi_y(\cdot, \tau)\|^2.$$

Continuing, it follows from Sobolev's inequality that

$$J_4 = - \int_0^\infty \frac{\tilde{\Psi} \Phi_y \Psi_{yy}}{v^\varepsilon v^a(g(W^a) - \dot{X}^2)} dy \leq cN(\tau) (\|\Phi_y(\cdot, \tau)\|^2 + \|\Psi_{yy}(\cdot, \tau)\|^2),$$

and

$$J_5 \leq cN(\tau) \|\Phi_y(\cdot, \tau)\|^2.$$

Finally, by Young's inequality and Lemma 2.3, we obtain

$$J_6 \leq c\varepsilon \|\tilde{\Psi}(\cdot, \tau)\|^2 + c\varepsilon^{-1} \int_0^\infty (q_5^2 + q_6^2) dy \leq c\varepsilon \|\tilde{\Psi}(\cdot, \tau)\|^2 + c\varepsilon^{7\nu-3}.$$

Collecting all the estimates we have obtained, we get

$$\frac{d}{d\tau} \int_0^\infty \left(\Phi^2 + \frac{\tilde{\Psi}^2}{g(W^a) - \dot{X}^2} \right) (y, \tau) dy + (1 - \eta_1) \int_0^\infty \frac{1}{v^a(g(W^a) - \dot{X}^2)} \tilde{\Psi}_y^2 dy + (\underline{c} - c(|\dot{X}| + \varepsilon^\nu)) \int_0^\infty m_2 \partial_y V_s \tilde{\Psi}^2 dy \\ \leq c \int_0^\infty |\partial_y v_b^0|^2 \tilde{\Psi}^2 dy + c(|\dot{X}|^2 + \varepsilon) \|\Phi_y\|^2 + cN(\tau) (\|\Phi_y\|^2 + \|\Psi_{yy}\|^2) \\ + c\varepsilon (\|\Phi(\cdot, \tau)\|^2 + \|\tilde{\Psi}(\cdot, \tau)\|^2) + c\varepsilon^{7\nu-3}. \quad (3.13)$$

Now we estimate the term $\int_0^\tau \int_0^\infty |\partial_y v_b^0|^2 \tilde{\Psi}^2 dy d\tau$ by the idea of Kawashima and Nikkuni [11]. Since

$$\tilde{\Psi}(y, \tau) = \tilde{\Psi}(0, \tau) + \int_0^y \tilde{\Psi}(\zeta, \tau) d\zeta,$$

we have

$$|\tilde{\Psi}(y, \tau)| \leq |\tilde{\Psi}(0, \tau)| + y^{\frac{1}{2}} \|\tilde{\Psi}(\cdot, \tau)\|,$$

which yields

$$\int_0^\infty |\partial_y v_b^0|^2 \tilde{\Psi}^2 dy \leq \int_0^\infty |\partial_y v_b^0|^2 (\tilde{\Psi}(0, \tau)^2 + |y| \|\tilde{\Psi}(\cdot, \tau)\|^2) dy \leq \bar{c} \mu^2 (\tilde{\Psi}(0, \tau)^2 + \|\tilde{\Psi}_y(\cdot, \tau)\|^2). \quad (3.14)$$

Substituting (3.14) into (3.13) and integrating with respect to τ , and then using the fact $\tilde{\Psi}^2 \geq \frac{1}{2} \Psi^2 - \dot{X}^2 \Phi^2$, we have

$$\begin{aligned} & \int_0^\infty ((1 - c\dot{X}^2) \Phi^2 + \Psi^2) dy + (1 - \bar{c} \mu^2) \int_0^\tau \|\Psi_y(\cdot, \tau)\|^2 d\tau + \int_0^\tau (\bar{c} - c(|\dot{X}| + \varepsilon^\nu)) \int_0^\infty m_2 \partial_y V_s \Psi^2 dy d\tau \\ & \leq \bar{c} \mu^2 \int_0^\tau \Psi(0, \tau)^2 d\tau + c \int_0^\tau |\dot{X}|^2 \int_0^\infty m_2 \partial_y V_s \Phi^2 dy d\tau + c \int_0^\tau (N(\tau) + |\dot{X}|^2 + \varepsilon) \|\Phi_y(\cdot, \tau)\|^2 d\tau \\ & \quad + cN(\tau) \int_0^\tau \|\Psi_{yy}(\cdot, \tau)\|^2 d\tau + c\varepsilon \int_0^\tau (\|\Phi(\cdot, \tau)\|^2 + \|\Psi(\cdot, \tau)\|^2) d\tau + c\varepsilon^{7\nu-4}. \end{aligned} \quad (3.15)$$

Step 2. Multiplying (3.5)₁ and (3.5)₂ by $g(W^a)\Phi$ and Ψ , respectively, and then summing them up, and integrating over $[0, +\infty)$, we obtain after integration by parts that

$$\begin{aligned} & -\frac{1}{2} \dot{X} \Psi(0, \tau)^2 + \frac{1}{2} \int_0^\infty (g(W^a)_\tau - \dot{X} g(W^a)_y) \Phi^2 dy + \int_0^\infty g(W^a)_y \Phi \Psi dy \\ & = \frac{1}{2} \frac{d}{d\tau} \int_0^\infty (g(W^a) \Phi^2 + \Psi^2) dy + \int_0^\infty \frac{\Psi_y^2}{v^a} dy + \int_0^\infty \left(\frac{1}{v^a} \right)_y \Psi \Psi_y dy \\ & \quad + \int_0^\infty \frac{\Psi \Phi_y \Psi_{yy}}{v^\varepsilon v^a} dy - \int_0^\infty \frac{u_y^a}{v^\varepsilon (v^a)^2} \Psi \Phi_y^2 dy + \int_0^\infty \Psi Q(v^a, \Phi_y) dy - \int_0^\infty \Psi (q_5 - q_6) dy \end{aligned} \quad (3.16)$$

Denote the second term on the left by ϑ . Due to Lemma 2.4, we estimate ϑ as follows

$$\begin{aligned} \vartheta & = \frac{1}{2} \int_0^\infty \{(\dot{X} - \dot{s}) m_2 g(W_s^0)_y - \dot{X} m_2 g(W_s^0)_y\} \Phi^2 dy + O(1) \int_0^\infty |\partial_y v_b^0| \Phi^2 dy \\ & \quad + O(1) \varepsilon^\nu \int_0^\infty m_2 \partial_y V_s \Phi^2 dy + c\varepsilon \|\Phi(\cdot, \tau)\|^2 \\ & = -\frac{1}{2} \dot{s} \int_0^\infty \left(\frac{K(V_s)}{V_s} \right)' m_2 \partial_y V_s \Phi^2 dy + O(1) \int_0^\infty |\partial_y v_b^0| \Phi^2 dy \\ & \quad + O(1) \varepsilon^\nu \int_0^\infty m_2 \partial_y V_s \Phi^2 dy + c\varepsilon \|\Phi(\cdot, \tau)\|^2. \end{aligned} \quad (3.17)$$

A direct calculation gives

$$\begin{aligned} \left(\frac{K(V_s)}{V_s} \right)' &= \frac{h'(V_s)V_s - h(V_s)}{V_s^2} - p''(V_s) \leq \frac{h'(V_s)}{V_s} - p''(V_s) \leq -\frac{p'(V_s)}{V_s} - p''(V_s) \\ &= A\gamma V_s^{-(\gamma+2)} - A\gamma(\gamma+1)V_s^{-(\gamma+2)} = -A\gamma^2 V_s^{-(\gamma+2)} < 0. \end{aligned}$$

Insert this into (3.17) to give

$$\begin{aligned} \vartheta &\geq (\underline{c} - c\varepsilon^\nu) \int_0^\infty m_2 \partial_y V_s \Phi^2 dy - c \int_0^\infty |\partial_y v_b^0| \Phi^2 dy - c\varepsilon \|\Phi(\cdot, \tau)\|^2 \\ &\geq (\underline{c} - c\varepsilon^\nu) \int_0^\infty m_2 \partial_y V_s \Phi^2 dy - \bar{c}\mu \|\Phi_y(\cdot, \tau)\|^2 - c\varepsilon \|\Phi(\cdot, \tau)\|^2, \end{aligned}$$

for some constant $\underline{c} > 0$, where we have used the similar estimate as in (3.14). Using Lemma 2.4 and Young's inequality, we have

$$\begin{aligned} \int_0^\infty g(W^a)_y \Phi \Psi dy &= O(1) \int_0^\infty (m_2 \partial_y V_s + |\partial_y v_b^0| + \varepsilon) \Phi \Psi dy \\ &\leq \frac{\underline{c}}{2} \int_0^\infty m_2 \partial_y V_s \Phi^2 dy + c \int_0^\infty m_2 \partial_y V_s \Psi^2 dy \\ &\quad + c \int_0^\infty |\partial_y v_b^0| (\Phi^2 + \Psi^2) dy + c\varepsilon (\|\Phi(\cdot, \tau)\|^2 + \|\Psi(\cdot, \tau)\|^2) \\ &\leq \frac{\underline{c}}{2} \int_0^\infty m_2 \partial_y V_s \Phi^2 dy + c \int_0^\infty m_2 \partial_y V_s \Psi^2 dy + \bar{c}\mu (\Psi(0, \tau)^2 + \|\Phi_y(\cdot, \tau)\|^2 + \|\Psi_y(\cdot, \tau)\|^2) \\ &\quad + c\varepsilon (\|\Phi(\cdot, \tau)\|^2 + \|\Psi(\cdot, \tau)\|^2), \end{aligned}$$

and

$$\begin{aligned} \int_0^\infty \left(\frac{1}{v^a} \right)_y \Psi \Psi_y dy &\leq c \|\Psi_y\|^2 + c \int_0^\infty m_2 \partial_y V_s \Psi^2 dy + c \int_0^\infty |\partial_y v_b^0| \Psi^2 dy + c\varepsilon \|\Psi(\cdot, \tau)\|^2 \\ &\leq c \|\Psi_y\|^2 + c \int_0^\infty m_2 \partial_y V_s \Psi^2 dy + \bar{c}\mu \Psi(0, \tau)^2 + c\varepsilon \|\Psi(\cdot, \tau)\|^2. \end{aligned}$$

By Sobolev's inequality, we obtain

$$\int_0^\infty \frac{\Psi \Phi_y \Psi_{yy}}{v^\varepsilon v^a} dy - \int_0^\infty \frac{u_y^a}{v^\varepsilon (v^a)^2} \Psi \Phi_y^2 dy + \int_0^\infty \Psi Q(v^a, \Phi_y) dy \leq cN(\tau) (\|\Phi_y(\cdot, \tau)\|^2 + \|\Psi_{yy}(\cdot, \tau)\|^2).$$

Finally,

$$-\int_0^\infty \Psi(q_5 - q_6) dy \leq c\varepsilon \|\Psi(\cdot, \tau)\|^2 + c\varepsilon^{7\nu-3}.$$

Collecting all the estimates and integrating with respect to τ , we get

$$\begin{aligned} &\int_0^\tau |\dot{X}| \Psi(0, \tau)^2 d\tau + (\underline{c} - c\varepsilon^\nu) \int_0^\tau \int_0^\infty m_2 \partial_y V_s \Phi^2 dy d\tau \\ &\leq \int_0^\tau (g(W^a) \Phi^2 + \Psi^2) dy + c \left(\int_0^\tau \|\Psi_y(\cdot, \tau)\|^2 d\tau + \int_0^\tau \int_0^\infty m_2 \partial_y V_s \Psi^2 dy d\tau \right) \end{aligned}$$

$$\begin{aligned}
& + \bar{c}\mu \int_0^\tau (\psi(0, \tau)^2 + \|\Phi_y(\cdot, \tau)\|^2) d\tau + cN(\tau) \int_0^\tau (\|\Phi_y(\cdot, \tau)\|^2 + \|\Psi_{yy}(\cdot, \tau)\|^2) d\tau \\
& + c\varepsilon \int_0^\tau (\|\Phi(\cdot, \tau)\|^2 + \|\Psi(\cdot, \tau)\|^2) d\tau + c\varepsilon^{7\nu-4}.
\end{aligned} \tag{3.18}$$

Step 3. We rewrite (3.5)₂ as

$$\Psi_\tau - \dot{X}\Psi_y + p(v^\varepsilon) - p(v^a) = \frac{\Psi_{yy}}{v^\varepsilon} - \frac{u_y^a}{v^\varepsilon v^a} \Phi_y + q_5 - q_6. \tag{3.19}$$

Differentiating (3.19) with respect to y , multiplying the resulting equation by Ψ_y , and integrating on $[0, \infty)$, we have

$$\begin{aligned}
\frac{1}{2} \frac{d}{d\tau} \|\Psi_y(\cdot, \tau)\|^2 + \int_0^\infty \frac{1}{v^a} \Psi_{yy}^2 dy d\tau &= \int_0^\infty (p(v^\varepsilon) - p(v^a)) \Psi_{yy} dy + \int_0^\infty \frac{\Phi_y}{v^\varepsilon v^a} \Psi_{yy}^2 dy \\
&+ \int_0^\infty \frac{u_y^a}{v^\varepsilon v^a} \Phi_y \Psi_{yy} dy - \int_0^\infty \Psi_{yy} (q_5 - q_6) dy.
\end{aligned}$$

Using Young's inequality, Sobolev's inequality and Lemma 2.3, and integrating with respect to τ , we get

$$\|\Psi_y(\cdot, \tau)\|^2 + \int_0^\tau \int_0^\infty \frac{1}{v^a} \Psi_{yy}^2 dy d\tau \leq c \int_0^\tau \|\Phi_y(\cdot, \tau)\|^2 d\tau + c(N(\tau) + \varepsilon) \int_0^\tau \|\Psi_{yy}(\cdot, \tau)\|^2 d\tau + c\varepsilon^{7\nu-4}. \tag{3.20}$$

We denote the first constant on the right by c_1 .

Step 4. Multiply both sides of (3.5)₂ by Φ_y and integrate over $[0, +\infty)$ to obtain

$$\begin{aligned}
& \int_0^\infty \Phi_y \Psi_\tau dy - \dot{X} \int_0^\infty \Phi_y \Psi_y dy + \int_0^\infty p'(v^a) \Phi_y^2 dy \\
&= \int_0^\infty \frac{\Phi_y \Psi_{yy}}{v^a} dy - \int_0^\infty \frac{\Phi_y^2 \Psi_{yy}}{v^\varepsilon v^a} dy \\
&- \int_0^\infty \frac{u_y^a}{v^\varepsilon v^a} \Phi_y^2 dy - \int_0^\infty \Phi_y Q(v^a, \Phi_y) dy + \int_0^\infty \Phi_y (q_5 - q_6) dy.
\end{aligned} \tag{3.21}$$

The first term on the left can be written as

$$\int_0^\infty \Phi_y \Psi_\tau dy = \frac{d}{d\tau} \int_0^\infty \Phi_y \Psi dy + \int_0^\infty \Phi_\tau \Psi_y dy = -\frac{d}{d\tau} \int_0^\infty \Phi \Psi_y dy + \int_0^\infty \Psi_y^2 dy + \dot{X} \int_0^\infty \Phi_y \Psi_y dy,$$

and the first term on the right reads

$$\begin{aligned}
\int_0^\infty \frac{\Phi_y \Psi_{yy}}{v^a} dy &= \int_0^\infty \frac{\Phi_y (\Phi_{y\tau} - \dot{X} \Phi_{yy})}{v^a} dy \\
&= \frac{1}{2} \frac{d}{d\tau} \int_0^\infty \frac{\Phi_y^2}{v^a} dy + \frac{1}{2} \int_0^\infty \frac{\Phi_y^2}{(v^a)^2} (v_\tau^a - \dot{X} v_y^a) dy \\
&= \frac{1}{2} \frac{d}{d\tau} \int_0^\infty \frac{\Phi_y^2}{v^a} dy + \frac{1}{2} \int_0^\infty \frac{u_y^a}{(v^a)^2} \Phi_y^2 dy,
\end{aligned}$$

where we have used (3.5)₁ and (2.60)₁. Substituting them into (3.21), we get

$$\begin{aligned} \frac{d}{d\tau} \int_0^\infty \left(\frac{\Phi_y^2}{2v^a} + \Phi \Psi_y \right) dy &= \int_0^\infty p'(v^a) \Phi_y^2 dy + \int_0^\infty \Psi_y^2 dy + \int_0^\infty \frac{\Phi_y^2 \Psi_{yy}}{v^\varepsilon v^a} dy - \int_0^\infty \frac{u_y^a \Phi_y^3}{v^\varepsilon (v^a)^2} dy + \int_0^\infty \frac{u_y^a}{2(v^a)^2} \Phi_y^2 dy \\ &\quad + \int_0^\infty \Phi_y Q(v^a, \Phi_y) dy - \int_0^\infty \Phi_y (q_5 - q_6) dy. \end{aligned}$$

Noting that $u_y^a = m_2 \partial_y U_s + O(1)(\mu + \varepsilon)$ and $\partial_y U_s < 0$, using the Young's inequality, Sobolev's inequality and Lemma 2.3 again, and integrating with respect to τ , we get

$$\int_0^\infty \left(\frac{\Phi_y^2}{2v^a} + \Phi \Psi_y \right) dy - \int_0^\tau \int_0^\infty p'(v^a) \Phi_y^2 dy \leq \|\Psi_y(\cdot, \tau)\|^2 + (\bar{c}\mu + c(N(\tau) + \varepsilon)) \int_0^\tau \|\Phi_y(\cdot, \tau)\|^2 d\tau + c\varepsilon^{7\nu-4}. \quad (3.22)$$

Step 5. Insert (3.18) into (3.15) to yield

$$\begin{aligned} &\int_0^\infty \left(1 - c|\dot{X}|^2 - \frac{\bar{c}\mu^2}{|\dot{X}| - \bar{c}\mu} \right) (\Phi^2 + \Psi^2) dy + \int_0^\tau \int_0^\infty \left(1 - c|\dot{X}|^2 - \frac{\bar{c}\mu^2}{|\dot{X}| - \bar{c}\mu} - \bar{c}\mu^2 \right) \Psi_y^2 dy d\tau \\ &\quad + \int_0^\tau \int_0^\infty \left(\underline{c} - c(|\dot{X}| + \varepsilon^\nu) - c|\dot{X}|^2 - \frac{\bar{c}\mu^2}{|\dot{X}| - \bar{c}\mu} \right) m_2 \partial_y V_s \Psi^2 dy d\tau \\ &\leq \int_0^\tau \int_0^\infty (\bar{c}\mu + c(N(\tau) + |\dot{X}|^2 + \varepsilon)) \Phi_y^2 dy d\tau + cN(\tau) \int_0^\tau \|\Psi_{yy}(\cdot, \tau)\|^2 d\tau \\ &\quad + c\varepsilon \int_0^\tau (\|\Phi(\cdot, \tau)\|^2 + \|\Psi(\cdot, \tau)\|^2) d\tau + c\varepsilon^{7\nu-4}. \end{aligned} \quad (3.23)$$

Multiplying (3.20) and (3.22) by constants β_1 and $\beta_2 > 0$, respectively, and then adding the resulting inequalities to (3.23), we obtain the following inequality

$$\begin{aligned} &\int_0^\infty \left\{ \left(1 - c|\dot{X}|^2 - \frac{\bar{c}\mu^2}{|\dot{X}| - \bar{c}\mu} \right) (\Phi^2 + \Psi^2) + \beta_2 \frac{\Phi_y^2}{v^a} + 2\beta_2 \Phi \Psi_y + \beta_1 \Psi_y^2 \right\} dy \\ &\quad - \int_0^\tau \int_0^\infty (\beta_2 p'(v^a) + c_1 \beta_1 + c|\dot{X}|^2 + (\bar{c}\mu + c(N(\tau) + \varepsilon))(1 + \beta_2)) \Phi_y^2 dy \\ &\quad + \int_0^\tau \int_0^\infty \left(1 - c|\dot{X}|^2 - \frac{\bar{c}\mu^2}{|\dot{X}| - \bar{c}\mu} - \bar{c}\mu^2 - \beta_2 \right) \Psi_y^2 dy d\tau \\ &\quad + \int_0^\tau \int_0^\infty \left(\frac{\beta_1}{v^a} - cN(\tau)(1 + \beta_1) - c\beta_1 \varepsilon \right) \Psi_{yy}^2 dy d\tau \\ &\quad + \int_0^\tau \int_0^\infty \left(\underline{c} - c(|\dot{X}| + \varepsilon^\nu) - c|\dot{X}|^2 - \frac{\bar{c}\mu^2}{|\dot{X}| - \bar{c}\mu} \right) m_2 \partial_y V_s \Psi^2 dy d\tau \\ &\leq c\varepsilon \int_0^\tau (\|\Phi(\cdot, \tau)\|^2 + \|\Psi(\cdot, \tau)\|^2) d\tau + c\varepsilon^{7\nu-4}. \end{aligned}$$

To get desired signs, we first choose $\beta_1 = 2\beta_2^2 > 0$ such that

$$\Phi^2 + 2\beta_2 \Phi \Psi_y + \beta_1 \Psi_y^2 \geq C_{\beta_2} (\Phi^2 + \Psi_y^2),$$

and then choose $|\dot{X}|$, $N(\tau)$, μ and ε sufficiently small, by the Gronwall type inequality, we obtain

$$\|(\Phi, \Psi)(\cdot, \tau)\|_1^2 + \int_0^\tau (\|\Phi_y(\cdot, \tau)\|^2 + \|\Psi_y(\cdot, \tau)\|_1^2) d\tau + \int_0^\tau \int_0^\infty m_2 \partial_y V_s \Psi^2 dy d\tau \leq c\varepsilon^{7\nu-4}.$$

This completes the proof of Lemma 3.2. \square

3.2. Higher order estimate

Lemma 3.3. Suppose that the conditions in Proposition 3.1 are satisfied. Then

$$\|(\Phi_\tau, \Psi_\tau)(\cdot, \tau)\|_1^2 + \int_0^\tau (\|\Phi_{y\tau}(\cdot, \tau)\|^2 + \|\Psi_{y\tau}(\cdot, \tau)\|_1^2) d\tau \leq c\varepsilon^{7\nu-4}, \quad (3.24)$$

with some constant c independent of τ_0 and ε .

Differentiating (3.5) with respect to τ and then using the similar argument to the proof of Lemma 3.2, we can obtain (3.24). Here we omit the proof.

Lemma 3.4. Suppose that the conditions in Proposition 3.1 are satisfied. Then

$$\|\partial_y^2 \Phi(\cdot, \tau)\|^2 + \|\partial_y^2 \Psi(\cdot, \tau)\|_1^2 + \int_0^\tau (\|\partial_y^2 \Phi(\cdot, \tau)\|^2 + \|\partial_y^3 \Psi(\cdot, \tau)\|_1^2) d\tau \leq c\varepsilon^{7\nu-4}, \quad (3.25)$$

with some constant c independent of τ_0 and ε .

Proof. Using (3.19) and Lemmas 3.2 and 3.3, we have

$$\|\Psi_{yy}(\cdot, \tau)\|^2 \leq c\|(\Psi_\tau, \Phi_y, \Psi_y)(\cdot, \tau)\|^2 + c \int_0^\infty (q_{5y}^2 + q_{6y}^2) dy \leq c\varepsilon^{7\nu-4}.$$

By (3.5)₁, we obtain

$$\|\Phi_{yy}(\cdot, \tau)\|^2 \leq c\|(\Phi_{y\tau}, \Psi_{yy})(\cdot, \tau)\|^2 \leq c\varepsilon^{7\nu-4},$$

and

$$\int_0^\tau \|\Phi_{yy}(\cdot, \tau)\|^2 d\tau \leq c \int_0^\tau \|(\Phi_{y\tau}, \Psi_{yy})(\cdot, \tau)\|^2 d\tau \leq c\varepsilon^{7\nu-4}.$$

Differentiate (3.19) to give

$$\Psi_{y\tau} + (p(v^\varepsilon) - p(v^a))_y = \frac{\partial_y^3 \Psi}{v^\varepsilon} + \frac{\Phi_{yy} + v_y^a}{(v^\varepsilon)^2} \Psi_{yy} - \left(\frac{u_y^a}{v^\varepsilon v^a} \Phi_y \right)_y + q_{5y} - q_{6y}.$$

Then

$$\|\partial_y^3 \Psi(\cdot, \tau)\|^2 \leq c(\|\Psi_{y\tau}(\cdot, \tau)\|^2 + \|\Phi_y\|_1^2) + c \int_0^\infty (q_{5y}^2 + q_{6y}^2) dy \leq c\varepsilon^{7\nu-4},$$

and

$$\int_0^\tau \|\partial_y^3 \Psi(\cdot, \tau)\|^2 d\tau \leq c \int_0^\tau (\|\Psi_{y\tau}(\cdot, \tau)\|^2 + \|\Phi_y\|_1^2) d\tau + c \int_0^\tau \int_0^\infty (q_{5y}^2 + q_{6y}^2) dy d\tau \leq c\varepsilon^{7\nu-4}.$$

Up to now, we finish the proof of Lemma 3.4. \square

Combining the results of Lemma 3.2 and Lemma 3.4, we complete the proof of Proposition 3.1.

4. Proof of Theorem 1.2

Using Proposition 3.1 and the standard continuous induction argument, we conclude that

Proposition 4.1. *There exist positive constants $\varepsilon_0, \sigma_0, \mu_0$ and C , which are independent of ε such that if $0 < \varepsilon < \varepsilon_0, 0 \leq \mu \leq \mu_0$ and $\bar{u}(t) \leq \sigma_0$ for $t \in [0, T]$, then the initial boundary value problem (3.5) has a unique smooth solution (Φ, Ψ) with $\Phi \in C([0, T/\varepsilon] : H^2(X(t), +\infty))$ and $\Psi \in C([0, T/\varepsilon] : H^3(X(t), +\infty))$. Furthermore, the following inequality holds*

$$\sup_{0 \leq \tau \leq T/\varepsilon} (\|\Phi(\cdot, \tau)\|_2^2 + \|\Psi(\cdot, \tau)\|_3^2) + \int_0^{T/\varepsilon} (\|\partial_y \Phi(\cdot, \tau)\|_1^2 + \|\partial_y \Psi(\cdot, \tau)\|_2^2) d\tau \leq C\varepsilon^{7\nu-4}. \quad (4.1)$$

Proof of Theorem 1.2. Now we choose $\nu \in (\eta, 1) \cap (\frac{6}{7}, 1)$. In view of (4.1) and Sobolev's inequality, we have

$$\begin{aligned} \sup_{0 \leq t \leq T} \|(v^\varepsilon - v^a, u^\varepsilon - u^a)(\cdot, t)\|^2 &= \sup_{0 \leq t \leq T} \|(\bar{\Phi}_x, \bar{\Psi}_x)(\cdot, t)\|^2 \\ &= \varepsilon \sup_{0 \leq \tau \leq T/\varepsilon} \|(\Phi_y, \Psi_y)(\cdot, \tau)\|^2 \\ &\leq C\varepsilon^{7\nu-3} \leq C\varepsilon^3. \end{aligned}$$

On the other hand, it follows from Lemma 2.4 that

$$\sup_{0 \leq t \leq T} \|(v^a - v_0, u^a - u_0)(\cdot, t)\|^2 \leq C\varepsilon^\nu \leq C\varepsilon^\eta.$$

Consequently,

$$\begin{aligned} \sup_{0 \leq t \leq T} \|(v^\varepsilon - v_0, u^\varepsilon - u_0)(\cdot, t)\|^2 &\leq \sup_{0 \leq t \leq T} \|(v^\varepsilon - v^a, u^\varepsilon - u^a)(\cdot, t)\|^2 + \sup_{0 \leq t \leq T} \|(v^a - v_0, u^a - u_0)(\cdot, t)\|^2 \\ &\leq C\varepsilon^\eta, \end{aligned}$$

which gives (1.16). Finally,

$$\begin{aligned} \|(v^\varepsilon - v^a, u^\varepsilon - u^a)(\cdot, t)\|_{L^\infty} &= \|(\Phi_y, \Psi_y)(\cdot, t)\|_{L^\infty} \\ &\leq c \|(\Phi_y, \Psi_y)(\cdot, t)\|^{\frac{1}{2}} \|(\Phi_{yy}, \Psi_{yy})(\cdot, t)\|^{\frac{1}{2}} \\ &\leq c\varepsilon^{(7\nu-4)/2} \leq c\varepsilon. \end{aligned}$$

This gives (1.17) by using Lemma 2.4 again.

We complete the proof of Theorem 1.2. \square

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