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Contractivity of Leader type and fixed points in uniform spaces with generalized pseudodistances

Kazimierz Włodarczyk *, Robert Plebaniak

Department of Nonlinear Analysis, Faculty of Mathematics and Computer Science, University of Łódź, Banacha 22, 90-238 Łódź, Poland

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ABSTRACT

Recently, Jachymski and Jóźwik proved that among various classes of contractions which are introduced and studied in the metric fixed point theory, the Leader contractions are greatest general contractions. In this article, we want to show how generalized pseudodistances in uniform spaces can be used to obtain new and general results of Leader type without complete graph assumptions about maps and without sequentially complete assumptions about spaces, which was not done in the previous publications on this subject. The definitions, results and methods presented here are new for maps in uniform and locally convex spaces and even in metric spaces. Examples showing a difference between our results and the well-known ones are given.

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1. Introduction

The studies of contractive fixed points in metric spaces inspired by Banach [1] and Caccioppoli [2] were developed substantially by Burton [3], Rakotch [4], Geraghty [5,6], Matkowski [7–9], Walter [10], Dugundji [11], Tasković [12], Dugundji and Granas [13], Browder [14], Krasnosel'skiĭ et al. [15], Boyd and Wong [16], Mukherjea [17], Meir and Keeler [18], Leader [19], Jachymski [20,21], Jachymski and Jóźwik [22] and many others not mentioned in this paper.

It is worth noticing that some of the results of the papers of Jachymski [20,21] and Jachymski and Jóźwik [22], concerning discussions, comparisons and corrections, are in fact essential tools in the proofs that among various classes of contractions which are introduced and studied in the above mentioned papers the Leader contractions are the greatest general contractions. In the complete metric spaces with τ-distances, beautiful generalizations of Leader's result [19, Theorem 3] are established by Suzuki [23, Theorem 4] and [24]. The above are some of the reasons why in metric spaces the study of Leader contractions plays a particularly important part in the metric fixed point theory.

Recall, that the maps satisfying the following conditions (L1) and (L2) are called in literature *Leader contractions* and *weak Leader contractions*, respectively.

Theorem 1.1. *(See Leader [19, Theorem 3].) Let (X, d) be a metric space and let $T : X \to X$ be a map with a complete graph (i.e. closed in Y^2 where Y is the completion of X). The following hold:*

(a) *T has a contractive fixed point if and only if* (**L1**) $\forall_{x,y\in X}\forall_{\varepsilon>0}\exists_{\eta>0}\exists_{r\in\mathbb{N}}\forall_{i,j\in\mathbb{N}}\{d(T^{[i]}(x), T^{[j]}(y)) < \varepsilon + \eta \Rightarrow d(T^{[i+r]}(x), T^{[j+r]}(y)) < \varepsilon\}$.

* Corresponding author.
E-mail addresses: wlkzxa@math.uni.lodz.p (K. Włodarczyk), robpleb@math.uni.lodz.pl (R. Plebaniak).

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(b) *T has a fixed point if and only if* (**L2**) $\exists_{x\in X}\forall_{\varepsilon>0}\exists_{\eta>0}\exists_{r\in\mathbb{N}}\forall_{i,j\in\mathbb{N}}\{d(T^{[i]}(x),T^{[j]}(x))<\varepsilon+\eta\Rightarrow d(T^{[i+r]}(x),T^{[j+r]}(x))<\varepsilon\}$. *Moreover, if x, ε, η and r are as in* (L2) *and if* $\lim_{m\to\infty}T^{[m]}(x)=w$, *then* $\forall_{i\in\mathbb{N}}\{d(T^{[i]}(x),T^{[i+r]}(x))\leqslant\eta\Rightarrow d(T^{[i+r]}(x),w)\leqslant\varepsilon\}$.

By a *contractive fixed point* of $T:X\to X$ we mean a fixed point w of T in X such that, for each $w^0\in X$, $\lim_{m\to\infty}T^{[m]}(w^0)=w$.

Recently, Włodarczyk and Plebaniak in [25] have studied among others the $\mathcal{J}$-families of generalized pseudodistances in uniform spaces which generalize distances of Tataru [27], w-distances of Kada et al. [28], τ-distances of Suzuki [29] and τ-functions of Lin and Du [30] in metric spaces and distances of Vályi [31] in uniform spaces. Motivated by works reported in [19,23,24,20–22], our main interest in this paper is the following

Question 1.1. If the spaces X are uniform with $\mathcal{J}$-families of generalized pseudodistances, under what conditions does the fixed point theorem of Leader type for maps $T:X\to X$ exist even in the case when the spaces X are not sequentially complete and the maps T do not have complete graphs?

In this paper, in the uniform spaces, to answer affirmatively this question, we give the definition of the $\mathcal{J}$-family of generalized pseudodistances, we apply it to construct $\mathcal{J}$-contractions of Leader type on X and we provide the conditions guaranteeing the existence and uniqueness of fixed points of these contractions and the convergence to these fixed points of all iterative sequences of these contractions. Also we construct weak $\mathcal{J}$-contractions of Leader type on X and study the existence of their fixed points. Our contractions essentially extend Leader type contractions introduced and studied in the literature. Examples showing a fundamental difference between our results and the well-known ones are given. The results and methods of investigation presented here are new for maps in uniform and locally convex spaces and even in metric spaces.

2. Definitions, notations and statement of results

Let X be a Hausdorff uniform space with uniformity defined by a saturated family $\mathcal{D}=\{d_\alpha:\ \alpha\in\mathcal{A}\}$ of pseudometrics d_α, $\alpha\in\mathcal{A}$, uniformly continuous on X^2. If $T:X\to X$, then, for each $w^0\in X$, we define a sequence $(w^m:\ m\in\{0\}\cup\mathbb{N})$ starting with w^0 as follows $\forall_{m\in\{0\}\cup\mathbb{N}}\{w^m=T^{[m]}(w^0)\}$ where $T^{[m]}=T\circ T\circ\cdots\circ T$ (m-times) and $T^{[0]}=I_X$ is an identity map on X. Denote by $Fix(T)$ the set of all *fixed points* of T, i.e. $Fix(T)=\{w\in X:\ w=T(w)\}$.

We start by defining the notions of $\mathcal{J}$-family of generalized pseudodistances on X and $\mathcal{J}$-contractions and weak $\mathcal{J}$-contractions of Leader type on X.

Definition 2.1. (See [25,26].) Let X be a uniform space. The family $\mathcal{J}=\{J_\alpha:\ X^2\to[0,\infty),\ \alpha\in\mathcal{A}\}$ is said to be a *$\mathcal{J}$-family of generalized pseudodistances J_α, $\alpha\in\mathcal{A}$, on X* (*$\mathcal{J}$-family*, for short) if the following two conditions hold:

($\mathcal{J}$1) $\forall_{\alpha\in\mathcal{A}}\forall_{x,y,z\in X}\{J_\alpha(x,z)\leqslant J_\alpha(x,y)+J_\alpha(y,z)\}$; and

($\mathcal{J}$2) For any sequence $(x_m:\ m\in\mathbb{N})$ in X such that

$$\forall_{\alpha\in\mathcal{A}}\Big\{\lim_{n\to\infty}\sup_{m>n}J_\alpha(x_n,x_m)=0\Big\},\tag{2.1}$$

if there exists a sequence $(y_m:\ m\in\mathbb{N})$ in X satisfying

$$\forall_{\alpha\in\mathcal{A}}\Big\{\lim_{m\to\infty}J_\alpha(x_m,y_m)=0\Big\},\tag{2.2}$$

then

$$\forall_{\alpha\in\mathcal{A}}\Big\{\lim_{m\to\infty}d_\alpha(x_m,y_m)=0\Big\}.\tag{2.3}$$

In the following remark, we list some basic properties of $\mathcal{J}$-families.

Remark 2.1. Let X be a Hausdorff uniform space and let $\mathcal{J}=\{J_\alpha:X^2\to[0,\infty),\ \alpha\in\mathcal{A}\}$ be a $\mathcal{J}$-family on X.

(a) From ($\mathcal{J}$1) and ($\mathcal{J}$2) it follows that if $x\neq y$, $x,y\in X$, then $\exists_{\alpha\in\mathcal{A}}\{J_\alpha(x,y)>0\vee J_\alpha(y,x)>0\}$. Indeed, if $\forall_{\alpha\in\mathcal{A}}\{J_\alpha(x,y)=J_\alpha(y,x)=0\}$, then $\forall_{\alpha\in\mathcal{A}}\{J_\alpha(x,x)=0\}$, since, by ($\mathcal{J}$1), we get $\forall_{\alpha\in\mathcal{A}}\{J_\alpha(x,x)\leqslant J_\alpha(x,y)+J_\alpha(y,x)=0\}$. Now, defining $x_m=x$ and $y_m=y$ for $m\in\mathbb{N}$, we conclude that (2.1) and (2.2) hold. Consequently, by ($\mathcal{J}$2), we get (2.3) which implies $\forall_{\alpha\in\mathcal{A}}\{d_\alpha(x,y)=0\}$. However, X is a Hausdorff and hence, since $x\neq y$, we have $\exists_{\alpha\in\mathcal{A}}\{d_\alpha(x,y)\neq0\}$. Contradiction.

(b) If $\forall_{\alpha\in\mathcal{A}}\forall_{x\in X}\{J_\alpha(x,x)=0\}$, then, for each $\alpha\in\mathcal{A}$, J_α is quasi-pseudometric. Examples of $\mathcal{J}$-families such that the maps J_α, $\alpha\in\mathcal{A}$, are not quasi-pseudometrics are given in Section 6.

(c) The family $\mathcal{D}$ is a $\mathcal{J}$-family on X.

Definition 2.2. Let X be a uniform space and let the family $\mathcal{J} = \{J_\alpha : X^2 \to [0, \infty),\ \alpha \in \mathcal{A}\}$ be a $\mathcal{J}$-family on X. We say that:

(i) $T : X \to X$ is a *$\mathcal{J}$-contraction of Leader type on X* (in short, *J-contraction on X*) if
 (C1) $\forall_{x,y\in X}\forall_{\alpha\in\mathcal{A}}\forall_{\varepsilon>0}\exists_{\eta>0}\exists_{r\in\mathbb{N}}\forall_{s,l\in\mathbb{N}}\{J_\alpha(T^{[s]}(x), T^{[l]}(y)) < \varepsilon + \eta \Rightarrow J_\alpha(T^{[s+r]}(x), T^{[l+r]}(y)) < \varepsilon\}$.
(ii) $T : X \to X$ is a *weak $\mathcal{J}$-contraction of Leader type on X* (in short, *weak J-contraction on X*) if
 (C2) $\exists_{x\in X}\forall_{\alpha\in\mathcal{A}}\forall_{\varepsilon>0}\exists_{\eta>0}\exists_{r\in\mathbb{N}}\forall_{s,l\in\mathbb{N}}\{J_\alpha(T^{[s]}(x), T^{[l]}(x)) < \varepsilon + \eta \Rightarrow J_\alpha(T^{[s+r]}(x), T^{[l+r]}(x)) < \varepsilon\}$.

Definition 2.3. Let X be a uniform space and let $\mathcal{J} = \{J_\alpha : X^2 \to [0, \infty),\ \alpha \in \mathcal{A}\}$ be a $\mathcal{J}$-family on X. We say that $T : X \to X$ is *$\mathcal{J}$-admissible* if for each $u^0 \in X$ satisfying $\forall_{\alpha\in\mathcal{A}}\{\lim_{n\to\infty} \sup_{m>n} J_\alpha(u^n, u^m) = 0\}$ there exists $w \in X$ such that $\forall_{\alpha\in\mathcal{A}}\{\lim_{m\to\infty} J_\alpha(u^m, w) = 0\}$.

Remark 2.2. Let X be a Hausdorff uniform space and let $T : X \to X$. If X is sequentially complete, then T is $\mathcal{D}$-admissible.

Definition 2.4. Let X be a uniform space and let $T : X \to X$. We say that T is *closed* on X, if whenever $(x_m\colon m \in \mathbb{N})$ is a sequence in X converging to $x \in X$ and $(y_m\colon m \in \mathbb{N})$ is a sequence converging to $y \in X$ such that $y_m = T(x_m)$ for all $m \in \mathbb{N}$, then $y = T(x)$.

Basing on ideas from [25,26,32–34] we will present an affirmative answer to Question 1.1. More precisely, we will prove the following three stronger than [1–19], [23, Theorem 4] and [24] results.

Theorem 2.1. *Let X be a Hausdorff uniform space and let $\mathcal{J} = \{J_\alpha : X^2 \to [0, \infty),\ \alpha \in \mathcal{A}\}$ be the $\mathcal{J}$-family on X. Let a map $T : X \to X$ be $\mathcal{J}$-admissible and let it satisfy one of the following conditions*:

(D1) $\forall_{w^0,w\in X}\{\{\lim_{m\to\infty} w^m = w\} \Rightarrow \{T \text{ is continuous at } w\}\}$;
(D2) *T is closed on X.*

The following hold:

(a) *If T is a $\mathcal{J}$-contraction on X, then*: (i) *T has a unique fixed point in X, say w*; (ii) *for each $w^0 \in X$, the sequence $(w^m\colon m \in \{0\} \cup \mathbb{N})$ converges to w; and* (iii) $\forall_{\alpha\in\mathcal{A}}\{J_\alpha(w, w) = 0\}$.
(b) *If T is a weak $\mathcal{J}$-contraction on X, then*: (i) *there exist $w^0, w \in X$ such that the sequence $(w^m\colon m \in \{0\} \cup \mathbb{N})$ converges to w; and* (ii) $w \in Fix(T)$.

Theorem 2.2. *Let X be a Hausdorff uniform space and let $\mathcal{J} = \{J_\alpha : X^2 \to [0, \infty),\ \alpha \in \mathcal{A}\}$ be the $\mathcal{J}$-family on X. Let a map $T : X \to X$ satisfy one of the conditions* (D1) *or* (D2) *and, in addition, the condition*

(D3) $\exists_{w^0,w\in X}\forall_{\alpha\in\mathcal{A}}\{\lim_{m\to\infty} J_\alpha(w^m, w) = 0\}$.

If T is a $\mathcal{J}$-contraction on X, then: (i) *there exist $w^0, w \in X$ such that the sequence $(w^m\colon m \in \{0\} \cup \mathbb{N})$ converges to w*; (ii) $Fix(T) = \{w\}$; *and* (iii) $\forall_{\alpha\in\mathcal{A}}\{J_\alpha(w, w) = 0\}$.

Theorem 2.3. *Let X be a Hausdorff sequentially complete uniform space and let the family $\mathcal{J} = \{J_\alpha : X^2 \to [0, \infty),\ \alpha \in \mathcal{A}\}$ be a $\mathcal{J}$-family on X. Let $T : X \to X$ satisfy one of the conditions* (D1) *or* (D2)*. The following hold*:

(a) *If T is a $\mathcal{J}$-contraction on X, then*: (i) *T has a unique fixed point in X, say w*; (ii) *for each $w^0 \in X$, the sequence $(w^m\colon m \in \{0\} \cup \mathbb{N})$ converges to w; and* (iii) $\forall_{\alpha\in\mathcal{A}}\{J_\alpha(w, w) = 0\}$.
(b) *If T is a weak $\mathcal{J}$-contraction on X, then*: (i) *there exist $w^0, w \in X$ such that the sequence $(w^m\colon m \in \{0\} \cup \mathbb{N})$ converges to w; and* (ii) $w \in Fix(T)$.

3. Proof of Theorem 2.1

For each $w^0, v^0 \in X$, $\alpha \in \mathcal{A}$ and $k \in \mathbb{N}$, we define

$$\delta_{\mathcal{J};\alpha,k}(w^0, v^0) = \inf\{\Delta_{\mathcal{J};\alpha,k}(w^0, v^0, n)\colon n \in \mathbb{N}\}, \tag{3.1}$$

$$\gamma_{\mathcal{J};\alpha,k}(w^0, v^0) = \inf\{\Gamma_{\mathcal{J};\alpha,k}(w^0, v^0, n)\colon n \in \mathbb{N}\}, \tag{3.2}$$

$$\Delta_{\mathcal{J};\alpha,k}(w^0, v^0, n) = \max\{J_\alpha(w^s, v^l)\colon n \leqslant s, l \leqslant n + k\}, \quad n \in \mathbb{N}, \tag{3.3}$$

$$\Gamma_{\mathcal{J};\alpha,k}(w^0, v^0, n) = \max\{J_\alpha(v^s, w^l)\colon n \leqslant s, l \leqslant n + k\}, \quad n \in \mathbb{N}. \tag{3.4}$$

Proof of Theorem 2.1(a). Assume that the condition (C1) holds. The proof will be broken into nine steps.

Step 1. *The following property holds*

$$\forall_{w^0,v^0\in X}\forall_{\alpha\in\mathcal{A}}\forall_{\varepsilon>0}\exists_{\eta>0}\bigl\{\exists_{r_1\in\mathbb{N}}\forall_{s,l\in\mathbb{N}}\bigl\{J_\alpha\bigl(w^s,v^l\bigr)<\varepsilon+\eta\Rightarrow J_\alpha\bigl(w^{s+r_1},v^{l+r_1}\bigr)<\varepsilon\bigr\} \\ \wedge\exists_{r_2\in\mathbb{N}}\forall_{s,l\in\mathbb{N}}\bigl\{J_\alpha\bigl(v^s,w^l\bigr)<\varepsilon+\eta\Rightarrow J_\alpha\bigl(v^{s+r_2},w^{l+r_2}\bigr)<\varepsilon\bigr\}\bigr\}. \tag{3.5}$$

Indeed, let $w^0, v^0\in X$ be arbitrary and fixed. If we define the sequences $(w^m\colon m\in\{0\}\cup\mathbb{N})$ and $(v^m\colon m\in\{0\}\cup\mathbb{N})$ (remember that $w^m=T^{[m]}(w^0)$ and $v^m=T^{[m]}(v^0)$, $m\in\{0\}\cup\mathbb{N}$) and assume that $\alpha\in\mathcal{A}$ and $\varepsilon>0$ are arbitrary and fixed, then, using (C1) for $x=w^0$ and $y=v^0$, we obtain $\exists_{\eta_1>0}\exists_{r_1\in\mathbb{N}}\forall_{s,l\in\mathbb{N}}\{J_\alpha(w^s,v^l)<\varepsilon+\eta_1\Rightarrow J_\alpha(w^{s+r_1},v^{l+r_1})<\varepsilon\}$ and, using (C1) for $x=v^0$ and $y=w^0$, we obtain $\exists_{\eta_2>0}\exists_{r_2\in\mathbb{N}}\forall_{s,l\in\mathbb{N}}\{J_\alpha(v^s,w^l)<\varepsilon+\eta_2\Rightarrow J_\alpha(v^{s+r_2},w^{l+r_2})<\varepsilon\}$. Hence, putting $\eta=\min\{\eta_1,\eta_2\}$, we have $\exists_{r_1\in\mathbb{N}}\forall_{s,l\in\mathbb{N}}\{J_\alpha(w^s,v^l)<\varepsilon+\eta\Rightarrow J_\alpha(w^{s+r_1},v^{l+r_1})<\varepsilon\}$ and $\exists_{r_2\in\mathbb{N}}\forall_{s,l\in\mathbb{N}}\{J_\alpha(v^s,w^l)<\varepsilon+\eta\Rightarrow J_\alpha(v^{s+r_2},w^{l+r_2})<\varepsilon\}$. This gives (3.5).

Step 2. *We show that*

$$\forall_{w^0,v^0\in X}\forall_{\alpha\in\mathcal{A}}\forall_{k\in\mathbb{N}}\bigl\{\delta_{\mathcal{J};\alpha,k}\bigl(w^0,v^0\bigr)=0\bigr\} \tag{3.6}$$

and

$$\forall_{w^0,v^0\in X}\forall_{\alpha\in\mathcal{A}}\forall_{k\in\mathbb{N}}\bigl\{\gamma_{\mathcal{J};\alpha,k}\bigl(w^0,v^0\bigr)=0\bigr\}. \tag{3.7}$$

Indeed, suppose that (3.6) does not hold; that is,

$$\exists_{u^0,z^0\in X}\exists_{\alpha_0\in\mathcal{A}}\exists_{k_0\in\mathbb{N}}\exists_{\varepsilon_0>0}\bigl\{\delta_{\mathcal{J};\alpha_0,k_0}\bigl(u^0,z^0\bigr)=\varepsilon_0\bigr\}. \tag{3.8}$$

With this choice of u^0, z^0, α_0 and ε_0 we can use (3.5) and then there exist $\eta_0>0$ and $r_0\in\mathbb{N}$, such that

$$\forall_{s,l\in\mathbb{N}}\bigl\{J_{\alpha_0}\bigl(u^s,z^l\bigr)<\varepsilon_0+\eta_0\Rightarrow J_{\alpha_0}\bigl(u^{s+r_0},z^{l+r_0}\bigr)<\varepsilon_0\bigr\}. \tag{3.9}$$

Additionally, (3.8) and (3.1) imply that there exists $n_0\in\mathbb{N}$ such that $\Delta_{\mathcal{J};\alpha_0,k_0}(u^0,z^0,n_0)<\varepsilon_0+\eta_0$ which, by (3.3), gives $\forall_{n_0\leqslant s,l\leqslant n_0+k_0}\{J_{\alpha_0}(u^s,z^l)<\varepsilon_0+\eta_0\}$. Consequently, by (3.9), we get $\forall_{n_0\leqslant s,l\leqslant n_0+k_0}\{J_{\alpha_0}(u^{s+r_0},z^{l+r_0})<\varepsilon_0\}$ which we can write as $\forall_{n_0+r_0\leqslant s,l\leqslant n_0+r_0+k_0}\{J_{\alpha_0}(u^s,z^l)<\varepsilon_0\}$. This, by (3.3), gives that $\Delta_{\mathcal{J};\alpha_0,k_0}(u^0,z^0,n_0+r_0)<\varepsilon_0$. However, hence and from (3.8) and (3.1) it follows that $\varepsilon_0=\delta_{\mathcal{J};\alpha_0,k_0}(u^0,z^0)=\inf\{\Delta_{\mathcal{J};\alpha_0,k_0}(u^0,z^0,n)\colon n\in\mathbb{N}\}\leqslant\Delta_{\mathcal{J};\alpha_0,k_0}(u^0,z^0,n_0+r_0)<\varepsilon_0$ which is impossible. Therefore, (3.6) holds. Now, suppose that (3.7) does not hold, i.e.

$$\exists_{u^0,z^0\in X}\exists_{\alpha_0\in\mathcal{A}}\exists_{k_0\in\mathbb{N}}\exists_{\varepsilon_0>0}\bigl\{\gamma_{\mathcal{J};\alpha_0,k_0}\bigl(u^0,z^0\bigr)=\varepsilon_0\bigr\}. \tag{3.10}$$

Of course, for this u^0, z^0, α_0 and ε_0, by (3.5), there exist $\eta_0>0$ and $r_0\in\mathbb{N}$, such that

$$\forall_{s,l\in\mathbb{N}}\bigl\{J_{\alpha_0}\bigl(z^s,u^l\bigr)<\varepsilon_0+\eta_0\Rightarrow J_{\alpha_0}\bigl(z^{s+r_0},u^{l+r_0}\bigr)<\varepsilon_0\bigr\}. \tag{3.11}$$

In addition, by (3.10) and (3.2), there exists $n_0\in\mathbb{N}$ such that $\Gamma_{\mathcal{J};\alpha_0,k_0}(u^0,z^0,n_0)<\varepsilon_0+\eta_0$. Hence, using (3.4), we conclude that $\forall_{n_0\leqslant s,l\leqslant n_0+k_0}\{J_{\alpha_0}(z^s,u^l)<\varepsilon_0+\eta_0\}$ and this, using (3.11), gives that $\forall_{n_0\leqslant s,l\leqslant n_0+k_0}\{J_{\alpha_0}(z^{s+r_0},u^{l+r_0})<\varepsilon_0\}$, i.e. that $\forall_{n_0+r_0\leqslant s,l\leqslant n_0+r_0+k_0}\{J_{\alpha_0}(z^s,u^l)<\varepsilon_0\}$. This means, by (3.4), that $\Gamma_{\mathcal{J};\alpha_0,k_0}(u^0,z^0,n_0+r_0)<\varepsilon_0$. Consequently, $\varepsilon_0=\gamma_{\mathcal{J};\alpha_0,k_0}(u^0,z^0)=\inf\{\Gamma_{\mathcal{J};\alpha_0,k_0}(u^0,z^0,n)\colon n\in\mathbb{N}\}\leqslant\Gamma_{\mathcal{J};\alpha_0,k_0}(u^0,z^0,n_0+r_0)<\varepsilon_0$ which is impossible. Thus (3.7) holds.

Step 3. *Let $w^0, v^0\in X$, $\alpha\in\mathcal{A}$ and $\varepsilon>0$ be arbitrary and fixed and let $\eta>0$ and $r_1, r_2\in\mathbb{N}$ satisfy* (3.5)*. Denote $r=\max\{r_1,r_2\}$. We show that if there exists $n_0\in\mathbb{N}$ such that*

$$\max\bigl\{\Delta_{\mathcal{J};\alpha,r}\bigl(w^0,v^0,n_0\bigr),\Gamma_{\mathcal{J};\alpha,r}\bigl(w^0,v^0,n_0\bigr)\bigr\}<\min\{\varepsilon,\eta/2\}, \tag{3.12}$$

then

$$\forall_{s,l\geqslant n_0}\bigl\{J_\alpha\bigl(w^s,v^l\bigr)<3\varepsilon\bigr\}. \tag{3.13}$$

Let n_0 satisfy (3.12) and let us write $\Delta^i=\Delta_{\mathcal{J};\alpha,r_i}(w^0,v^0,n_0)$ and $\Gamma^i=\Gamma_{\mathcal{J};\alpha,r_i}(w^0,v^0,n_0)$, $i=1,2$. Then, by (3.3), (3.4) and definition of r, we obtain that $\max\{\Delta_{\mathcal{J};\alpha,r_1}(w^0,v^0,n_0),\Delta_{\mathcal{J};\alpha,r_2}(w^0,v^0,n_0)\}\leqslant\Delta_{\mathcal{J};\alpha,r}(w^0,v^0,n_0)$ and $\max\{\Gamma_{\mathcal{J};\alpha,r_1}(w^0,v^0,n_0),\Gamma_{\mathcal{J};\alpha,r_2}(w^0,v^0,n_0)\}\leqslant\Gamma_{\mathcal{J};\alpha,r}(w^0,v^0,n_0)$ and, taking this into account, we see that (3.12) implies

$$\max\bigl\{\Delta^1,\Delta^2,\Gamma^1,\Gamma^2\bigr\}<\min\{\varepsilon,\eta/2\}. \tag{3.14}$$

To establish

$$\forall_{l\geqslant n_0}\bigl\{J_\alpha\bigl(w^{n_0+r_1},v^l\bigr)<\varepsilon\bigr\} \tag{3.15}$$

it suffices to show that

$$L=\emptyset \tag{3.16}$$

where $L=\{l\in\mathbb{N}\colon\ l\geqslant n_0\wedge J_\alpha(w^{n_0+r_1},v^l)\geqslant\varepsilon\}$. Suppose that

$$L\neq\emptyset \tag{3.17}$$

and let $l_0=\min L$; of course $l_0\geqslant n_0$. It is clear then that (3.17) implies

$$\forall_{n_0\leqslant l<l_0}\big\{J_\alpha\big(w^{n_0+r_1},v^l\big)<\varepsilon\big\}. \tag{3.18}$$

Now, we see that $l_0>n_0+r_1$. Otherwise, $l_0\leqslant n_0+r_1$ and, by virtue of (3.3) and (3.14), we get $J_\alpha(w^{n_0+r_1},v^{l_0})\leqslant\max\{J_\alpha(w^i,v^j)\colon\ n_0\leqslant i,j\leqslant n_0+r_1\}=\Delta_{\mathcal{J};\alpha,r_1}(w^0,v^0,n_0)<\min\{\varepsilon,\eta/2\}\leqslant\varepsilon$, which, by the definitions of l_0 and L, is impossible. Hence it follows that $n_0<l_0-r_1<l_0$ and, consequently, using (3.18), we conclude that

$$J_\alpha\big(w^{n_0+r_1},v^{l_0-r_1}\big)<\varepsilon. \tag{3.19}$$

Next, using ($\mathcal{J}$1), (3.3), (3.4), (3.19) and (3.14), we get $J_\alpha(w^{n_0},v^{l_0-r_1})\leqslant J_\alpha(w^{n_0},v^{n_0})+J_\alpha(v^{n_0},w^{n_0+r_1})+J_\alpha(w^{n_0+r_1},v^{l_0-r_1})<\Delta_{\mathcal{J};\alpha,r_1}(w^0,v^0,n_0)+\Gamma_{\mathcal{J};\alpha,r_1}(w^0,v^0,n_0)+\varepsilon<\eta/2+\eta/2+\varepsilon=\varepsilon+\eta$. Hence, since, by assumption, r_1 satisfies (3.5), we get $J_\alpha(w^{n_0+r_1},v^{l_0})<\varepsilon$, which, by definitions of l_0 and L, is impossible. Consequently, (3.16) holds which implies (3.15).

We can show in a similar way that

$$\forall_{s\geqslant n_0}\big\{J_\alpha\big(w^s,v^{n_0+r_2}\big)<\varepsilon\big\}. \tag{3.20}$$

In fact, suppose that

$$S\neq\emptyset \tag{3.21}$$

where $S=\{s\in\mathbb{N}\colon\ s\geqslant n_0\wedge J_\alpha(w^s,v^{n_0+r_2})\geqslant\varepsilon\}$ and let $s_0=\min S$; of course $s_0\geqslant n_0$. Then, by (3.21),

$$\forall_{n_0\leqslant s<s_0}\big\{J_\alpha\big(w^s,v^{n_0+r_2}\big)<\varepsilon\big\}. \tag{3.22}$$

We see that $s_0>n_0+r_2$. Indeed, if $s_0\leqslant n_0+r_2$, then, since $s_0\geqslant n_0$, we see that $J_\alpha(w^{s_0},v^{n_0+r_2})\leqslant\max\{J_\alpha(w^s,v^l)\colon\ n_0\leqslant s,l\leqslant n_0+r_2\}=\Delta_{\mathcal{J};\alpha,r_2}(w^0,v^0,n_0)<\min\{\varepsilon,\eta/2\}\leqslant\varepsilon$ which, by (3.21) and definition of s_0, is impossible. Therefore, $n_0<s_0-r_2<s_0$, and, by (3.22),

$$J_\alpha\big(w^{s_0-r_2},v^{n_0+r_2}\big)<\varepsilon. \tag{3.23}$$

Consequently, using ($\mathcal{J}$1), (3.23), (3.4), (3.3) and (3.14), we have $J_\alpha(w^{s_0-r_2},v^{n_0})\leqslant J_\alpha(w^{s_0-r_2},v^{n_0+r_2})+J_\alpha(v^{n_0+r_2},w^{n_0+r_2})+J_\alpha(w^{n_0+r_2},v^{n_0})<\varepsilon+\Gamma_{\mathcal{J};\alpha,r_2}(w^0,v^0,n_0)+\Delta_{\mathcal{J};\alpha,r_2}(w^0,v^0,n_0)<\varepsilon+\eta/2+\eta/2=\varepsilon+\eta$. Hence, using (3.5) we get $J_\alpha(w^{s_0},v^{n_0+r_2})<\varepsilon$. This, by the definitions of s_0 and S, is impossible. Consequently, $S=\emptyset$ which gives (3.20).

Let now $s,l\geqslant n_0$ be arbitrary and fixed. Then, by ($\mathcal{J}$1), (3.20), (3.15), (3.3) and (3.12), we obtain $J_\alpha(w^s,v^l)\leqslant J_\alpha(w^s,v^{n_0+r_2})+J_\alpha(v^{n_0+r_2},w^{n_0+r_1})+J_\alpha(w^{n_0+r_1},v^l)<\varepsilon+\max\{J_\alpha(v^s,w^l)\colon\ n_0\leqslant s,l\leqslant n_0+r\}+\varepsilon=2\varepsilon+\Gamma_{\mathcal{J};\alpha,r}(w^0,v^0,n_0)<3\varepsilon$. Therefore, (3.13) holds.

Step 4. *We show that*

$$\forall_{w^0\in X}\forall_{\alpha\in\mathcal{A}}\forall_{\varepsilon>0}\exists_{n_0\in\mathbb{N}}\forall_{s,l\geqslant n_0}\big\{J_\alpha\big(w^s,w^l\big)<\varepsilon/2\big\}. \tag{3.24}$$

Indeed, let $w^0\in X$ be arbitrary and fixed and let $(v^m\colon\ m\in\{0\}\cup\mathbb{N})$ be a sequence defined by formulae $v^m=w^m$, $m\in\{0\}\cup\mathbb{N}$. We see that for sequences $(w^m\colon\ m\in\{0\}\cup\mathbb{N})$ and $(v^m\colon\ m\in\{0\}\cup\mathbb{N})$ the property (3.5) holds, i.e.

$$\forall_{\alpha\in\mathcal{A}}\forall_{\varepsilon>0}\exists_{\eta>0,r\in\mathbb{N}}\forall_{s,l\in\mathbb{N}}\big\{J_\alpha\big(w^s,w^l\big)<\varepsilon+\eta\Rightarrow J_\alpha\big(w^{s+r},w^{l+r}\big)<\varepsilon\big\} \tag{3.25}$$

and, by (3.3) and (3.4), we have

$$\forall_{\alpha\in\mathcal{A}}\forall_{k,n\in\mathbb{N}}\big\{\Delta_{\mathcal{J};\alpha,k}\big(w^0,w^0,n\big)=\Gamma_{\mathcal{J};\alpha,k}\big(w^0,w^0,n\big)\big\}. \tag{3.26}$$

Moreover, by Step 2, (3.1), (3.2) and (3.26), we have

$$\forall_{\alpha\in\mathcal{A}}\forall_{k\in\mathbb{N}}\big\{\delta_{\mathcal{J};\alpha,k}\big(w^0,w^0\big)=\gamma_{\mathcal{J};\alpha,k}\big(w^0,w^0\big)=0\big\}. \tag{3.27}$$

Let now $w^0\in X$, $\alpha_0\in\mathcal{A}$ and $\varepsilon_0>0$ be arbitrary and fixed. By (3.25) there exist $\eta_0>0$ and $r_0\in\mathbb{N}$ such that $\forall_{s,l\in\mathbb{N}}\{J_{\alpha_0}(w^s,w^l)<\varepsilon_0+\eta_0\Rightarrow J_\alpha(w^{s+r_0},w^{l+r_0})<\varepsilon_0\}$ and, in particular, (3.27) implies

$$\delta_{\mathcal{J};\alpha_0,r_0}\big(w^0,w^0\big)=\gamma_{\mathcal{J};\alpha_0,r_0}\big(w^0,w^0\big)=0\}. \tag{3.28}$$

By (3.28), using (3.26), (3.1) and (3.2), there exists $n_0\in\mathbb{N}$, such that

$$\Delta_{\mathcal{J};\alpha_0,r_0}\big(w^0, w^0, n_0\big) = \Gamma_{\mathcal{J};\alpha_0,r_0}\big(w^0, w^0, n_0\big) < \min\{\varepsilon_0/6, \eta_0/2\}. \tag{3.29}$$

From (3.29), using Step 3, we get $\forall_{s,l\geqslant n_0}\{J_{\alpha_0}(w^s, w^l) < \varepsilon_0/2\}$. This proved that (3.24) holds.

Step 5. *We show that*

$$\forall_{w^0\in X}\forall_{\alpha\in\mathcal{A}}\Big\{\lim_{n\to\infty}\sup_{m>n} J_\alpha\big(w^n, w^m\big) = 0\Big\}. \tag{3.30}$$

Indeed, (3.24) implies, in particular, that $\forall_{w^0\in X}\forall_{\alpha\in\mathcal{A}}\forall_{\varepsilon>0}\exists_{n_0\in\mathbb{N}}\forall_{m>n\geqslant n_0}\{J_\alpha(w^n, w^m) < \varepsilon/2\}$. This implies $\forall_{w^0\in X}\forall_{\alpha\in\mathcal{A}}\forall_{\varepsilon>0}$ $\exists_{n_0\in\mathbb{N}}\forall_{n\geqslant n_0}\{\sup_{m>n} J_\alpha(w^n, w^m) \leqslant \varepsilon/2 < \varepsilon\}$. Therefore, (3.30) holds.

Step 6. *For each* $w^0 \in X$, *there exists a point* $w \in X$ *such that* $\lim_{m\to\infty} w^m = w$ *and* $w \in Fix(T)$. Indeed, let $w^0 \in X$ be arbitrary and fixed. Since T is $\mathcal{J}$-admissible, (3.30) implies that there exists $w \in X$ such that

$$\forall_{\alpha\in\mathcal{A}}\Big\{\lim_{m\to\infty} J_\alpha\big(w^m, w\big) = 0\Big\}. \tag{3.31}$$

From properties (3.30) and (3.31), defining $x_m = w^m$ and $y_m = w$ for $m \in \mathbb{N}$, we conclude that for sequences $(x_m\colon m \in \mathbb{N})$ and $(y_m\colon m \in \mathbb{N})$ in X the conditions (2.1) and (2.2) hold. Consequently, by $(\mathcal{J}2)$, we get (2.3) which implies $\forall_{\alpha\in\mathcal{A}}\{\lim_{m\to\infty} d_\alpha(w^m, w) = \lim_{m\to\infty} d_\alpha(x_m, y_m) = 0\}$, i.e. the limit $\lim_{m\to\infty} w^m = w$ holds.

If (D1) holds, then we have that T is a continuous map at w and, consequently, $w = \lim_{m\to\infty} w^{m+1} = \lim_{m\to\infty} T(w^m) = T(\lim_{m\to\infty} w^m) = T(w)$. If (D2) holds, then, since $\lim_{m\to\infty} w^m = w$ and $w^{m+1} = T(w^m)$ for all $m \in \mathbb{N}$, we get $w \in Fix(T)$.

Step 7. *For* $w \in X$ *satisfying* $w \in Fix(T)$, *the following holds* $\forall_{\alpha\in\mathcal{A}}\{J_\alpha(w, w) = 0\}$. Indeed, if we assume that there exists $\alpha_0 \in \mathcal{A}$ such that $J_{\alpha_0}(w, w) > 0$, i.e. $\varepsilon_0 = J_{\alpha_0}(w, w) > 0$, then, by (C1), there exist $\eta_0 > 0$ and $r_0 \in \mathbb{N}$, such that

$$\forall_{s,l\in\mathbb{N}}\Big\{\big\{J_{\alpha_0}\big(T^{[s]}(w), T^{[l]}(w)\big) < \varepsilon_0 + \eta_0\big\} \Rightarrow \big\{J_{\alpha_0}\big(T^{[s+r_0]}(w), T^{[l+r_0]}(w)\big) < \varepsilon_0\big\}\Big\}. \tag{3.32}$$

However, for each $s, l \in \mathbb{N}$, we have $J_{\alpha_0}(T^{[s]}(w), T^{[l]}(w)) = J_{\alpha_0}(w, w) = \varepsilon_0 < \varepsilon_0 + \eta_0$. Thus, using (3.32), we obtain that $0 < \varepsilon_0 = J_{\alpha_0}(w, w) = J_{\alpha_0}(T^{[s+r_0]}(w), T^{[l+r_0]}(w)) < \varepsilon_0$, which is impossible.

Step 8. *The map* T *has* a *unique fixed point in* X. Otherwise $u, v \in Fix(T)$ and $u \neq v$ for some $u, v \in X$. Then, by Remark 2.1(a), there exists $\alpha_0 \in \mathcal{A}$ such that $J_{\alpha_0}(u, v) > 0$ or $J_{\alpha_0}(v, u) > 0$. Suppose $J_{\alpha_0}(u, v) > 0$. Then, for $\varepsilon_0 = J_{\alpha_0}(u, v) > 0$, by (C1), there exist $\eta_0 > 0$ and $r_0 \in \mathbb{N}$, such that

$$\forall_{s,l\in\mathbb{N}}\Big\{\big\{J_{\alpha_0}\big(T^{[s]}(u), T^{[l]}(v)\big) < \varepsilon_0 + \eta_0\big\} \Rightarrow \big\{J_{\alpha_0}\big(T^{[s+r_0]}(u), T^{[l+r_0]}(v)\big) < \varepsilon_0\big\}\Big\}. \tag{3.33}$$

However, for each $s, l \in \mathbb{N}$, we have $J_{\alpha_0}(T^{[s]}(u), T^{[l]}(v)) = J_{\alpha_0}(u, v) = \varepsilon_0 < \varepsilon_0 + \eta_0$ and thus, by (3.33), we get $0 < \varepsilon_0 = J_{\alpha_0}(u, v) = J_{\alpha_0}(T^{[s+r_0]}(u), T^{[l+r_0]}(v)) < \varepsilon_0$, which is impossible. We obtain a similar conclusion in the case when $J_{\alpha_0}(v, u) > 0$. Therefore, $Fix(T) = \{w\}$ for some $w \in X$.

Step 9. *The assertions* (i)–(iii) *hold.* Indeed, this is a consequence of Steps 6–8.

Proof of Theorem 2.1(b). Assume that the condition (C2) holds. Denoting $(w^m\colon m \in \{0\} \cup \mathbb{N})$, where $w^0 = x \in X$ and x is such as in condition (C2), and, by using a similar argumentation as in the proof of Theorem 2.1(a) for this sequence $(w^m\colon m \in \{0\} \cup \mathbb{N})$, we have that there exists a point $w \in X$ such that $\lim_{m\to\infty} w^m = w$ and $w \in Fix(T)$. □

4. Proof of Theorem 2.2

Assume that the condition (C1) holds. Let $w^0, w \in X$ and let the sequence $(w^m\colon m \in \{0\} \cup \mathbb{N})$ be such as in (D3), i.e.

$$\forall_{\alpha\in\mathcal{A}}\Big\{\lim_{m\to\infty} J_\alpha\big(w^m, w\big) = 0\Big\}. \tag{4.1}$$

By similar considerations as in the proof of Theorem 2.1(a), we obtain that this sequence $(w^m\colon m \in \{0\} \cup \mathbb{N})$ satisfies

$$\forall_{\alpha\in\mathcal{A}}\Big\{\lim_{n\to\infty}\sup_{m>n} J_\alpha\big(w^n, w^m\big) = 0\Big\}. \tag{4.2}$$

Now, defining $x_m = w^m$ and $y_m = w$ for $m \in \mathbb{N}$, we conclude, by (4.1) and (4.2), that for sequences $(x_m\colon m \in \mathbb{N})$ and $(y_m\colon m \in \mathbb{N})$ in X the conditions (2.1) and (2.2) hold. Consequently, by $(\mathcal{J}2)$, we get (2.3) which implies $\forall_{\alpha\in\mathcal{A}}\{\lim_{m\to\infty} d_\alpha(w^m, w) = \lim_{m\to\infty} d_\alpha(x_m, y_m) = 0\}$, i.e. the limit $\lim_{m\to\infty} w^m = w$ holds. If (D1) holds, then we have that T is a continuous map at w and, consequently, $w = \lim_{m\to\infty} w^{m+1} = \lim_{m\to\infty} T(w^m) = T(\lim_{m\to\infty} w^m) = T(w)$. If (D2) holds, then, since $\lim_{m\to\infty} w^m = w$ and $w^{m+1} = T(w^m)$ for all $m \in \mathbb{N}$, we get $w \in Fix(T)$. Finally, using similar argumentations as in Steps 7 and 8 of the proof of Theorem 2.1(a), we conclude that $Fix(T) = \{w\}$ and $\forall_{\alpha\in\mathcal{A}}\{J_\alpha(w, w) = 0\}$. □

5. Proof of Theorem 2.3

Proof of Theorem 2.3(a). Assume that condition (C1) holds and let $w^0 \in X$ be arbitrary and fixed. By similar considerations as in the proof of Theorem 2.1(a), we obtain that the sequence $(w^m\colon m \in \{0\} \cup \mathbb{N})$ satisfies

$$\forall_{\alpha\in\mathcal{A}}\Big\{\lim_{n\to\infty}\sup_{m>n} J_\alpha\big(w^n, w^m\big) = 0\Big\}. \tag{5.1}$$

The proof will be broken into three steps.

Step 1. *For each* $w^0 \in X$, *the sequence* $(w^m\colon m \in \{0\} \cup \mathbb{N})$ satisfies

$$\forall_{w^0\in X}\forall_{\alpha\in\mathcal{A}}\forall_{\varepsilon>0}\exists_{n_0\in\mathbb{N}}\forall_{s,l\in\mathbb{N},s>l>n_0}\big\{d_\alpha\big(w^s, w^l\big) < \varepsilon\big\}. \tag{5.2}$$

Indeed, let $w^0 \in X$ be arbitrary and fixed. By (5.1), $\forall_{\alpha\in\mathcal{A}}\forall_{\varepsilon>0}\exists_{n_1=n_1(\alpha,\varepsilon)\in\mathbb{N}}\forall_{n>n_1}\{\sup\{J_\alpha(w^n, w^m)\colon m > n\} < \varepsilon\}$ and, in particular,

$$\forall_{\alpha\in\mathcal{A}}\forall_{\varepsilon>0}\exists_{n_1=n_1(\alpha,\varepsilon)\in\mathbb{N}}\forall_{n>n_1}\forall_{q\in\mathbb{N}}\big\{J_\alpha\big(w^n, w^{q+n}\big) < \varepsilon\big\}. \tag{5.3}$$

Let $i_0, j_0 \in \mathbb{N}$, $i_0 > j_0$, be arbitrary and fixed. If we define

$$x_m = w^{i_0+m} \quad \text{and} \quad y_m = w^{j_0+m} \quad \text{for } m \in \mathbb{N}, \tag{5.4}$$

then (5.3) gives

$$\forall_{\alpha\in\mathcal{A}}\Big\{\lim_{m\to\infty} J_\alpha\big(w^m, x_m\big) = \lim_{m\to\infty} J_\alpha\big(w^m, y_m\big) = 0\Big\}. \tag{5.5}$$

Therefore, by (5.1), (5.5) and $(\mathcal{J}2)$,

$$\forall_{\alpha\in\mathcal{A}}\Big\{\lim_{m\to\infty} d_\alpha\big(w^m, x_m\big) = \lim_{m\to\infty} d_\alpha\big(w^m, y_m\big) = 0\Big\}. \tag{5.6}$$

From (5.4) and (5.6) we then claim that

$$\forall_{\alpha\in\mathcal{A}}\forall_{\varepsilon>0}\exists_{n_2=n_2(\alpha,\varepsilon)\in\mathbb{N}}\forall_{m>n_2}\big\{d_\alpha\big(w^m, w^{i_0+m}\big) < \varepsilon/2\big\} \tag{5.7}$$

and

$$\forall_{\alpha\in\mathcal{A}}\forall_{\varepsilon>0}\exists_{n_3=n_3(\alpha,\varepsilon)\in\mathbb{N}}\forall_{m>n_3}\big\{d_\alpha\big(w^m, w^{j_0+m}\big) < \varepsilon/2\big\}. \tag{5.8}$$

Let now $\alpha_0 \in \mathcal{A}$ and $\varepsilon_0 > 0$ be arbitrary and fixed, let $n_0 = \max\{n_2(\alpha_0, \varepsilon_0), n_3(\alpha_0, \varepsilon_0)\} + 1$ and let $s, l \in \mathbb{N}$ be arbitrary and fixed such that $s > l > n_0$. Then $s = i_0 + n_0$ and $l = j_0 + n_0$ for some $i_0, j_0 \in \mathbb{N}$ such that $i_0 > j_0$ and, using (5.7) and (5.8), we get $d_{\alpha_0}(w^s, w^l) = d_{\alpha_0}(w^{i_0+n_0}, w^{j_0+n_0}) \leqslant d_{\alpha_0}(w^{n_0}, w^{i_0+n_0}) + d_{\alpha_0}(w^{n_0}, w^{j_0+n_0}) < \varepsilon_0/2 + \varepsilon_0/2 = \varepsilon_0$. Hence, we conclude that $\forall_{\alpha\in\mathcal{A}}\forall_{\varepsilon>0}\exists_{n_0=n_0(\alpha,\varepsilon)\in\mathbb{N}}\forall_{s,l\in\mathbb{N},s>l>n_0}\{d_\alpha(w^s, w^l) < \varepsilon\}$. The proof of (5.2) is complete.

Step 2. *For each* $w^0 \in X$, *there exists a unique* $w \in X$ *such that* $\lim_{m\to\infty} w^m = w$ *and* $w \in Fix(T)$. Indeed, let $w^0 \in X$ be arbitrary and fixed. Since X is a Hausdorff and sequentially complete space and, by Step 1, the sequence $(w^m\colon m \in \{0\} \cup \mathbb{N})$ is a Cauchy sequence on X, thus there exists a unique $w \in X$ such that $\lim_{m\to\infty} w^m = w$. If (D1) holds, then we have that T is a continuous map at w and, consequently, $w = \lim_{m\to\infty} w^{m+1} = \lim_{m\to\infty} T(w^m) = T(\lim_{m\to\infty} w^m) = T(w)$. If (D2) holds, then, since $\lim_{m\to\infty} w^m = w$ and $w^{m+1} = T(w^m)$ for all $m \in \mathbb{N}$, we get $w \in Fix(T)$.

Step 3. *The following hold*: $Fix(T) = \{w\}$ *and* $\forall_{\alpha\in\mathcal{A}}\{J_\alpha(w, w) = 0\}$. We obtain this using similar argumentations as in Sections 7 and 8 of the proof of Theorem 2.1(a).

Proof of Theorem 2.3(b). Assume that the condition (C2) holds. Denoting $(w^m\colon m \in \{0\} \cup \mathbb{N})$, where $w^0 = x \in X$ and x is as in condition (C2), and, by using the similar argumentation as in the proof of Theorem 2.3(a) for this sequence $(w^m\colon m \in \{0\} \cup \mathbb{N})$, we have that there exists a point $w \in X$ such that $\lim_{m\to\infty} w^m = w$ and $w \in Fix(T)$. □

6. Examples, comparisons and remarks

In this section we present some examples illustrating the concepts introduced so far. First, we present example of J-generalized pseudodistances.

Example 6.1. Let X be a metric space with metric d. Let the set $E \subset X$, containing at least two different points, be arbitrary and fixed and let $c > 0$ satisfy $\delta(E) < c$ where $\delta(E) = \sup\{d(x, y)\colon x, y \in E\}$. Let $J : X^2 \to [0, \infty)$ be defined by the formulae: $J(x, y) = d(x, y)$ if $E \cap \{x, y\} = \{x, y\}$ and $J(x, y) = c$ if $E \cap \{x, y\} \neq \{x, y\}$, $x, y \in X$. The family $\mathcal{J} = \{J\}$ is $\mathcal{J}$-family on X (see [25, Example 6.1]).

Now, we present two examples which illustrate Theorem 2.1(b).

Example 6.2. Let $X = (0, 1)$ be a metric space with a metric $d : X^2 \to [0, \infty)$, $d(x, y) = |x - y|$, $x, y \in X$. Let $T : X \to X$ be a map given by formula

$$T(x) = \begin{cases} -(3/8)x + 5/8 & \text{if } x \in (0, 1/3], \\ f(x) & \text{if } x \in (1/3, 1/2], \\ [-x^2 + 2x - (3/4)]^{1/2} + 1/2 & \text{if } x \in (1/2, 1), \end{cases}$$

where $f : \mathbb{R} \to \mathbb{R}$ is of the form $f(x) = (3/2)x - 1/4$.

We prove that the condition (D1) is satisfied. Indeed, if $w^0, w \in X$, then $\lim_{m\to\infty} w^m = w$ only when $w^0 \in S = \{s_k\colon f^{[k]}(s_k) = 1/3,\ k \in \{0\} \cup \mathbb{N}\} \cup \{1/2\}$ and $w = 1/2$. We see also that T is continuous in $w = 1/2$.

Note that, for each $k \in \mathbb{N}$ and $x \in \mathbb{R}$, $f^{[k]}(x) = (3/2)^k(x - 1/2) + 1/2$. Therefore, $f^{[k]}(s_k) = 1/3$ for $k \in \mathbb{N}$, implies $\lim_{k\to\infty}(1/2 - s_k) = \lim_{k\to\infty}(2/3)^k(1/6) = 0$. Hence, $\forall_{k\in\{0\}\cup\mathbb{N}}\{s_k < 1/2\}$, the sequence $(s_k\colon k \in \{0\} \cup \mathbb{N})$ is increasing and $\lim_{k\to\infty} s_k = 1/2$. In particular, $s_0 = 1/3$, $s_1 = 7/18$, $s_2 = 23/54$ and $s_3 = 73/162$.

Now, let $E = S$ and let

$$J(x, y) = \begin{cases} d(x, y) & \text{if } \{x, y\} \cap E = \{x, y\}, \\ 2 & \text{if } \{x, y\} \cap E \neq \{x, y\}. \end{cases} \tag{6.1}$$

By Example 6.1, $\mathcal{J} = \{J\}$ is a $\mathcal{J}$-family on X.

We observe that T is $\mathcal{J}$-admissible on X. Indeed, let $u^0 \in X$ be arbitrary and fixed and such that for a sequence $(u^m\colon m \in \{0\} \cup \mathbb{N})$ the following holds

$$\lim_{n\to\infty} \sup_{m>n} J(u^n, u^m) = 0. \tag{6.2}$$

Then, by (6.1) (i.e. since $J(x, y) = 2$ if $\{x, y\} \cap E \neq \{x, y\}$), we see that (6.2) holds only when $u^0 \in S$ and, consequently, then $\lim_{m\to\infty} u^m = 1/2$ and $\forall_{m\in\{0\}\cup\mathbb{N}}\{u^m \in S\}$. Hence it follows that, for each $u^0 \in S$, $\lim_{m\to\infty} J(u^m, 1/2) = \lim_{m\to\infty} d(u^m, 1/2) = 0$. This proved that T is $\mathcal{J}$-admissible.

We show that, for each $x \in S$, the condition (C2) is satisfied. Indeed, if $x \in S$ is arbitrary and fixed, then denoting $x^0 = x$ we see that the sequence $(x^m\colon m \in \{0\} \cup \mathbb{N})$ is convergent to $w = 1/2$; we note that if $x = s_k$ for some $k \in \{0\} \cup \mathbb{N}$, then we have $\forall_{m\geqslant 1}\{x^{m+k+1} = T^{[m+k+1]}(s_k) = T^{[m]}(T(T^{[k]}(s_k))) = T^{[m]}(T(f^{[k]}(s_k))) = T^{[m]}(T(1/3)) = T^{[m]}(1/2) = 1/2\}$ and if $x = 1/2$, then we have $\forall_{m\geqslant 1}\{x^m = T^{[m]}(x) = f^{[m]}(x) = 1/2\}$. Hence it follows that this sequence $(x^m\colon m \in \{0\} \cup \mathbb{N})$, convergent in X, is a Cauchy sequence, i.e. $\forall_{\varepsilon>0}\exists_{r\in\mathbb{N}}\forall_{n,m>r}\{d(x^n, x^m) < \varepsilon\}$. Thus, in particular, since $(x^m\colon m \in \{0\} \cup \mathbb{N}) \subset S$, we have $\forall_{\varepsilon>0}\exists_{r\in\mathbb{N}}\forall_{s,l\in\mathbb{N}}\{J(x^{s+r}, x^{l+r}) = d(x^{s+r}, x^{l+r}) < \varepsilon\}$. This implies that the following is true $\forall_{\varepsilon>0}\exists_{\eta>0}\exists_{r\in\mathbb{N}}\forall_{s,l\in\mathbb{N}}\{J(x^s, x^l) < \varepsilon + \eta \Rightarrow J(x^{s+r}, x^{l+r}) < \varepsilon\}$. This means that T is a weak $\mathcal{J}$-contraction on X.

Therefore, all assumptions of Theorem 2.1(b) are satisfied, $Fix(T) = \{w\} = \{1/2\}$ and $\forall_{w^0\in S\subset X}\{\lim_{m\to\infty} w^m = w\}$.

Example 6.3. Let $X = (0, 1)$ be a metric space with a metric $d : X^2 \to [0, \infty)$, $d(x, y) = |x - y|$, $x, y \in X$. Let $T : X \to X$ be a map given by formula

$$T(x) = \begin{cases} -x + 3/4 & \text{for } x \in (0, 1/4], \\ (1/2)x + 1/4 & \text{for } x \in (1/4, 1/2], \\ (3/2)x - (1/4) & \text{for } x \in (1/2, 2/3], \\ 3/4 & \text{for } x \in (2/3, 7/8), \\ -2x + 2 & \text{for } x \in [7/8, 1). \end{cases}$$

We observe that T is $\mathcal{J} = \{d\}$-admissible on X. Indeed, if $w^0 \in (0, 1/4) \cup (1/2, 1)$, then $\lim_{m\to\infty} w^m = w' = 3/4$ and if $w^0 \in [1/4, 1/2]$, then $\lim_{m\to\infty} w^m = w'' = 1/2$.

Moreover, T is continuous in w' and w''. Therefore, the condition (D1) holds.

Next, we observe that the map T is a weak $\mathcal{J} = \{d\}$-contraction on X. Indeed, if $x \in X$ is arbitrary and fixed, then, denoting $w^0 = x$ we have that $(w^m\colon m \in \{0\} \cup \mathbb{N})$ is convergent to w' or w''. Of course, this convergent sequence $(w^m\colon m \in \{0\} \cup \mathbb{N})$ is also a Cauchy sequence, i.e. $\forall_{\varepsilon>0}\exists_{r\in\mathbb{N}}\forall_{n,m>r}\{d(T^{[n]}(x), T^{[m]}(x)) < \varepsilon\}$ which we can write as $\forall_{\varepsilon>0}\exists_{r\in\mathbb{N}}\forall_{s,l\in\mathbb{N}}\{d(T^{[s+r]}(x), T^{[l+r]}(x)) < \varepsilon\}$. Hence $\forall_{\varepsilon>0}\exists_{\eta>0}\exists_{r\in\mathbb{N}}\forall_{s,l\in\mathbb{N}}\{d(T^{[s]}(x), T^{[l]}(x)) < \varepsilon + \eta \Rightarrow d(T^{[s+r]}(x), T^{[l+r]}(x)) < \varepsilon\}$. Therefore, T is a weak $\mathcal{J} = \{d\}$-contraction on X.

All assumptions of Theorem 2.1(b) are satisfied, $Fix(T) = \{w', w''\}$ and, for each $w^0 \in X$, the sequence $(w^m\colon m \in \{0\} \cup \mathbb{N})$ converges to w' or w''.

Finally, we present an example which illustrates Theorems 2.1(a) and 2.2.

Example 6.4. Let $X = (0, 1/3] \cup S \cup (1/2, 1)$ be a metric space with a metric $d : X^2 \to [0, \infty)$, $d(x, y) = |x - y|$, $x, y \in X$, where S is defined in Example 6.2 and let

$$T(x) = \begin{cases} -(3/8)x + 5/8 & \text{if } x \in (0, 1/3], \\ f(x) & \text{if } x \in S, \\ 1/2 & \text{if } x \in (1/2, 1) \end{cases}$$

where $f : \mathbb{R} \to \mathbb{R}$ is of the form $f(x) = (3/2)x - 1/4$. We see that T is $\mathcal{J} = \{d\}$-admissible, T is $\mathcal{J} = \{d\}$-contraction on X, T satisfies (D1) and (D3), $Fix(T) = \{1/2\}$ and $\forall_{w^0 \in X} \{\lim_{m\to\infty} d(w^m, 1/2) = 0\}$.

Remark 6.1. Returning to Examples 6.2–6.4 we see that:

(a) In Example 6.2, the existence of $\mathcal{J} = \{J\}$ such that $\mathcal{J} \neq \{d\}$ and T is $\mathcal{J}$-admissible is essential. Indeed, observe that, for each $w^0 \in X \setminus S$, the sequence $(w^m\colon m \in \{0\} \cup \mathbb{N})$ is not convergent in X since $\lim_{m\to\infty} w^m = w = 1 \notin X$. On the other hand, for each $w^0 \in X \setminus S$, this sequence is Cauchy, i.e. $\lim_{n\to\infty} \sup_{m>n} d(w^n, w^m) = 0$. Hence we conclude that T is not $\mathcal{J} = \{d\}$-admissible.
(b) In Example 6.3 the map T is a weak $\mathcal{J} = \{d\}$-contraction on X.
(c) In Examples 6.2–6.4, X is not complete, T does not have a complete graph, assumptions of some of our theorems are satisfied, but assumptions of [1–19], [23, Theorem 4] and [24] theorems do not hold.

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