J. Math. Anal. Appl. 390 (2012) 151–164

Contents lists available at SciVerse ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

On the cometary flow equations with force fields

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ARTICLE INFO

Article history:
Received 24 June 2011
Available online 18 January 2012
Submitted by D. Wang

Keywords:
Cometary flow equation
Lorentz field
Electrostatic field
Global existence
Weak solution

ABSTRACT

The Cauchy problem of a nonlinear kinetic equation modeling the time evolution of a cometary flow interacting with a force field is discussed, two kinds of existence results for weak solutions are established for initial data having finite mass and finite kinetic energy. The first one concerns a given force field which is assumed to be divergence free with respect to the velocity variable, it is shown that there exists a nonnegative weak solution to the Cauchy problem when the initial datum and the force field have reasonable integrability. As a special case, we also consider a Lorentz field and give another type of existence result. The second one deals with self-consistent electrostatic field, we show that when the initial datum has an L^2 integrability the system has a global nonnegative solution which extends a previous result obtained by one of the authors.

1. Introduction

Consider a cometary flow contained in the three-dimensional Euclid space $\mathbb{R}^3$, the physical state of which is determined by the one-particle distributional function, namely by the microscopic density $f(t,x,\xi) \geqslant 0$ of particles in the cometary flow at time $t \geqslant 0$ and position $x \in \mathbb{R}^3$, moving with velocity $\xi \in \mathbb{R}^3$. In view of the theory of statistical physics, the macroscopic density $\rho_f(t,x)$ and the bulk velocity $u_f(t,x)$ of the fluid are respectively defined by

$$\begin{pmatrix} \rho_f \\ \rho_f u_f \end{pmatrix}(t,x) = \int_{\mathbb{R}^3} \begin{pmatrix} 1 \\ \xi \end{pmatrix} f(t,x,\xi)\,d\xi, \quad t \geqslant 0,\ x \in \mathbb{R}^3. \tag{1.1}$$

Let $F(t,x,\xi)$ be a force field imposed on the cometary flow, then the time evolution of the microscopic density $f(t,x,\xi)$ is governed by (see, e.g., [12,30–32])

$$\begin{cases} \partial_t f + \xi \cdot \nabla_x f + F(t,x,\xi) \cdot \nabla_\xi f = Q_{u_f}(f), \\ f(0,x,\xi) = f_0(x,\xi), \end{cases} \tag{1.2}$$

where $f_0(x,\xi) \geqslant 0$ is the initial microscopic density of the cometary flow which is assumed to be known. The nonlinear operator $Q_{u_f}(f)$ is a simplified version of the collision integral modeling wave-particle interactions in the cometary flow and is defined by $Q_{u_f}(f)(t,x,\xi) = P_{u_f}(f)(t,x,\xi) - f(t,x,\xi)$ with

$$P_{u_f}(f) = \begin{cases} \frac{1}{4\pi} \int_{\mathbb{S}^2} f(t,x,u_f + |\xi - u_f|\omega)\,d\omega, & \rho_f \neq 0, \\ 0, & \rho_f = 0, \end{cases} \tag{1.3}$$

where $\int_{\mathbb{S}^2} \cdots d\omega$ denotes the Lebesgue integral on the unit sphere $\mathbb{S}^2$ of $\mathbb{R}^3$.

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2012.01.030

For the time being by ignoring the precise description of the collision integral for general Lebesgue measurable functions, which will be specified in Lemma A.2 and Remark A.3, we only point out that the operator $P_{u_f}(f)$ is a nonlinear projector. Consequently, the structure of Eq. (1.2) is similar to that of the classical BGK model of the Boltzmann equation (see, e.g., [4,25,33]), but it is fully nonlinear with infinitely many collision invariants [5,6].

The nonlinear evolutional equation (1.2) is an important kinetic model in the theory of astrophysical plasmas. A mathematical investigation of this model is initiated by P. Degond, J.L. López and P.F. Peyrard [5,6], more specifically, they were devoted to the derivation of the equations governing the macroscopic regime at the level of the Hilbert expansion and the Chapman–Enskog expansion.

Assuming that $F(t,x,\xi)\equiv 0$ and that the initial datum $f_0\in L^1\cap L^\infty(\mathbb{R}^3\times\mathbb{R}^3)$ has finite velocity moment of order two and has no vacuum regions, P. Degond, J.L. López and F. Poupaud [7] established, for the first time, the existence of a nonnegative solution to the Cauchy problem (1.2), furthermore, conservation laws for mass, momentum and energy, as well as an entropy dissipation law and the propagation of higher order moments, were derived. Those results were extended in [13] to a bounded domain with reflecting boundary and to initial datum f_0 permitting vacuum regions such that $0\leqslant(1+|\xi|^r)f_0\in L^1(\mathbb{R}^3\times\mathbb{R}^3)$ and $f_0\in L^p(\mathbb{R}^3\times\mathbb{R}^3)$ for any $r,\ p>1$, and existence and trends towards equilibria in weak topology were also established. For more information on equilibrium solutions of Eq. (1.2) in the case of $F(t,x,\xi)\equiv 0$ and their links to the explicit solutions of the compressible Euler equations for monatomic gases, as well as perturbation theory of global equilibria, we refer the readers to Refs. [14,15].

In this paper, we consider effects of force fields in several cases and establish global existence results of nonnegative weak solutions. Firstly, we study the situation that the force field $F(t,x,\xi)$ in (1.2) is given and incompressible with respect to the velocity variable ξ, i.e., $\nabla_\xi\cdot F(t,x,\xi)\equiv 0$. We show that if $F(t,x,\xi)\in L^q([0,T]\times\mathbb{R}^3_x\times\mathbb{R}^3_\xi)$ and $f_0(x,\xi)\in L^p(\mathbb{R}^3_x\times\mathbb{R}^3_\xi)$ for $p,q>1$ and $\frac{1}{p}+\frac{1}{q}<1$, then the Cauchy problem (1.2) has a nonnegative solution $f(t,x,\xi)\in L^\infty([0,T],L^p(\mathbb{R}^3_x\times\mathbb{R}^3_\xi))$ as long as the second order velocity moment of the initial datum f_0 is finite (Theorem 3.2).

In realistic applications, probably the most important case is that the force $F(t,x,\xi)$ is an electromagnetic or Lorentz field (see, e.g., [6,12,30]), namely $F(t,x,\xi)=E(t,x)+\xi\times B(t,x)$, where $E(t,x)$ and $B(t,x)$ are given electric intensity and magnetic intensity respectively. In this circumstance, (1.2) is reduced to

$$\begin{cases}\partial_t f+\xi\cdot\nabla_x f+\big[E(t,x)+\xi\times B(t,x)\big]\cdot\nabla_\xi f=Q_{u_f}(f),\\ f(0,x,\xi)=f_0(x,\xi).\end{cases}\tag{1.4}$$

Our second aim is to find another set of reasonable conditions for the Lorentz field and the initial datum that ensure the existence of a nonnegative solution to Eq. (1.4). As a matter of fact, we will show that if $E(t,x)\in L^q([0,T]\times\mathbb{R}^3)$, $B(t,x)\in L^{p'}([0,T]\times\mathbb{R}^3)$ and $f_0\in L^p(\mathbb{R}^3\times\mathbb{R}^3)$ with $p>1$ and $q>3+p'$ and if $(1+|\xi|^2)f_0\in L^1(\mathbb{R}^3\times\mathbb{R}^3)$, then there exists a nonnegative solution to (1.4) in the function class $L^\infty([0,T],L^p(\mathbb{R}^3_x\times\mathbb{R}^3_\xi))$ (Theorem 3.3).

Finally, we deal with the situation that the force field $F(t,x,\xi)$ is self-consistent, namely it is generated by particles themselves in the cometary flow. We only consider a simplified case $F(t,x,\xi)=E(t,x)$, where $E(t,x)$ is the electrostatic field induced by the particles themselves, this means that we ignore the possibly existing magnetic field. It is well known that $E(t,x)=-\nabla_x U(t,x)$ and $-\triangle_x U(t,x)=\rho_f(t,x)$ with $\lim_{|x|\to\infty}U(t,x)=0$ (see, e.g., [16,28]). Hence, we obtain the following kinetic model:

$$\partial_t f+\xi\cdot\nabla_x f+E\cdot\nabla_\xi f=Q_{u_f}(f),\qquad f(0,x,\xi)=f_0(x,\xi),\tag{1.5}$$

$$-\triangle_x U(t,x)=\rho_f(t,x),\qquad \lim_{|x|\to\infty}U(t,x)=0,\tag{1.6}$$

$$E(t,x)=-\nabla_x U(t,x).\tag{1.7}$$

Obviously, if we replace the right-hand side of Eq. (1.5) with 0, then the system (1.5)–(1.7) is just the classical Vlasov–Poisson system, which has already received a great deal of discussion (see, e.g., [1–3,8,19–24,26,27,29]).

Recently, assuming that the initial datum f_0 belongs to $L^1\cap L^\infty(\mathbb{R}^3\times\mathbb{R}^3)$ and has finite velocity moment of order two, the existence of a global nonnegative solution to the Cauchy problem (1.5)–(1.7) has been built in [34]. The L^∞ regularity condition on the initial datum is a quite strong constraint compared with known results for the classical Vlasov–Poisson system [21], and it is probable that much lower integrability of the initial datum should guarantee global existence of a nonnegative solution. Hence, the third aim of the present paper is to weaken the condition on the integrability of the initial datum required in [34]. Actually we shall show that the L^2 integrability is sufficient for the global existence of a nonnegative solution (Theorem 4.1).

We remark that in this paper we shall only study weak solutions, i.e., solutions in the sense of distributions. For example, a nonnegative function $f(t,x,\xi)\in L^1([0,T);L^1(\mathbb{R}^3\times\mathbb{R}^3))$ is said to be a weak solution on $[0,T)$ to (1.4) if $f(t,x,\xi)$ verifies

$$\int_0^T dt\int_{\mathbb{R}^3\times\mathbb{R}^3} f\big[\partial_t\phi+\xi\cdot\nabla_x\phi+(E+\xi\times B)\cdot\nabla_\xi\phi\big]\,dx\,d\xi+\int_{\mathbb{R}^3\times\mathbb{R}^3} f_0\phi|_{t=0}\,dx\,d\xi=-\int_0^T dt\int_{\mathbb{R}^3\times\mathbb{R}^3} Q_{u_f}(f)\phi\,dx\,d\xi\tag{1.8}$$

for any test function $\phi(t,x,\xi)\in C^1_c([0,T)\times\mathbb{R}^3\times\mathbb{R}^3)$. If in addition $f(t,x,\xi)\in L^1_{\mathrm{loc}}([0,\infty);L^1(\mathbb{R}^3\times\mathbb{R}^3))$ and (1.8) is valid for all $T>0$, then f is said to be a global weak solution to (1.4).

In the following of this paper, $\|\cdot\|_p$ always denotes the norm of the space $L^p(\mathbb{R}^3\times\mathbb{R}^3)$ for $1\leqslant p\leqslant\infty$. Let $f(t,x,\xi)$ be a weak solution to the system (1.5)–(1.7), its kinetic energy and potential energy at time t are respectively defined by [19–21,26]

$$\mathcal{E}_k(f)(t)=\int\limits_{\mathbb{R}^3\times\mathbb{R}^3}|\xi|^2 f(t,x,\xi)\,dx\,d\xi \quad\text{and}\quad \mathcal{E}_p(f)(t)=\int\limits_{\mathbb{R}^3}\left|\nabla_x U(t,x)\right|^2 dx. \tag{1.9}$$

2. Regularizing effects of velocity averages

Regularizing effect of velocity averages has been proven to be an extremely powerful tool in kinetic theory (see, e.g., [17,18,9,11]). In this section, we shall deduce, from a general velocity averaging lemma obtained in [11] (Lemma 2.1 below), an L^1 compactness result for linear transport equations with external force fields.

Lemma 2.1. *Let $p\in(1,\infty)$, let $G\in L^p(\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{R})$ and let $f\in L^p(\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{R})$ satisfy*

$$\partial_t f+\xi\cdot\nabla_x f=g \quad \text{in } D'\big(\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^N\big) \tag{2.1}$$

with $g=(I-\Delta_x)^{\tau/2}(I-\Delta_\xi)^{m/2}G$, $\tau\in[0,1)$, $m\geqslant 0$. Let $\psi\in L^\infty(\mathbb{R}^N)$ has compact support. Then $\int_{\mathbb{R}^N} f(t,x,\xi)\psi(\xi)\,d\xi\in B^{s,p}_t(\mathbb{R}^N\times\mathbb{R})$, where $B^{s,p}_t(\mathbb{R}^N\times\mathbb{R})$ denotes the usual Besov space and where

$$t=\max\{p,2\},\qquad s=\frac{(1-\tau)}{(m+1)}\min\left\{\frac{1}{p},\frac{1}{p'}\right\}.$$

According to Lemma 2.1, we have the following L^1 velocity averaging result for linear transport equations with external force fields.

Theorem 2.2. *Let the sequence $\{F_n(t,x,\xi)\colon n=1,2,\dots\}\subset C([0,T];C^1_b(\mathbb{R}^3_x\times\mathbb{R}^3_\xi))$ be bounded in $L^q_{\mathrm{loc}}((0,T)\times\mathbb{R}^3\times\mathbb{R}^3)$ for $q>1$ and satisfy $\nabla_\xi\cdot F_n=0$ ($n=1,2,\dots$). Suppose that the sequences $\{f_n\colon n=1,2,\dots\}$ is weakly compact in $L^1((0,T)\times\mathbb{R}^3\times\mathbb{R}^3)$ and $\{g_n\colon n=1,2,\dots\}$ is weakly compact in $L^1_{\mathrm{loc}}((0,T)\times\mathbb{R}^3\times\mathbb{R}^3)$ such that*

$$\partial_t f_n+\xi\cdot\nabla_x f_n+F_n(t,x,\xi)\cdot\nabla_\xi f_n=g_n \tag{2.2}$$

in the distributional sense. Then for any bounded sequence $\{\psi_n(t,x,\xi)\colon n=1,2,\dots\}\subset L^\infty([0,T]\times\mathbb{R}^3\times\mathbb{R}^3)$ that converges almost everywhere, the sequence

$$\int\limits_{\mathbb{R}^3} f_n(t,x,\xi)\psi_n(t,x,\xi)\,d\xi,\quad n=1,2,\dots$$

is compact in $L^1((0,T)\times\mathbb{R}^3)$. Here and in the following, $C^1_b(\mathbb{R}^N)$ denotes the space of continuously differentiable functions having bounded derivatives on $\mathbb{R}^N$ up to order one.

Proof. Step 1: We assume that f_n (and consequently g_n) are supported in a common compact set $K\Subset(0,T)\times\mathbb{R}^3_x\times\mathbb{R}^3_\xi$ and $\psi_n(t,x,\xi)=\psi(t,x,\xi)$ for all $n\geqslant 1$, where $\psi(t,x,\xi)$ is a cutoff function such that $\psi|_K=1$. For any $M>1$, define h^1_n and h^2_n by

$$\begin{cases}\partial_t h^1_n+\xi\cdot\nabla_x h^1_n+F_n\cdot\nabla_\xi h^1_n=g_n\cdot\chi_{\{(t,x,\xi)\colon |g_n|\leqslant M\}},\\ h^1_n(0,x,\xi)=0\end{cases} \tag{2.3}$$

and

$$\begin{cases}\partial_t h^2_n+\xi\cdot\nabla_x h^2_n+F_n\cdot\nabla_\xi h^2_n=g_n\cdot\chi_{\{(t,x,\xi)\colon |g_n|\geqslant M\}},\\ h^2_n(0,x,\xi)=0.\end{cases} \tag{2.4}$$

Since $f_n(0,x,\xi)=0$ (due to the assumption on its support) and f_n is the unique solution of (2.2), we have $f_n=h^1_n+h^2_n$. Let $(X_n(s),\Xi_n(s))=(X_n(s,t),\Xi_n(s,t))$ be the characteristics flow determined by the vector field $(\xi,F_n(t,x,\xi))$, namely $(X_n(s),\Xi_n(s))$ is the unique solution of the characteristic equation

$$\begin{cases}\dot X_n(s)=\Xi_n(s), & X_n(t)=x;\\ \dot\Xi_n(s)=F_n\big(s,X_n(s),\Xi_n(s)\big), & \Xi_n(t)=\xi,\end{cases}$$

then h^2_n can be written as

$$h_n^2(t,x,\xi)=\int_0^t g_n\cdot\chi_{\{|g_n|\geqslant M\}}\bigl(s,X_n(s),\Xi_n(s)\bigr)\,ds.$$

Consequently, we have

$$\int_0^T\int_{\mathbb{R}_x^3\times\mathbb{R}_\xi^3}\bigl|h_n^2(t,x,\xi)\bigr|\,dt\,dx\,d\xi\leqslant T\int_{|g_n|\geqslant M}\bigl|g_n(t,x,\xi)\bigr|\,dt\,dx\,d\xi.$$

Because g_n is weakly compact and hence bounded in L^1, we have

$$\mu\bigl(\bigl\{|g_n|\geqslant M\bigr\}\bigr)\leqslant\frac{1}{M}\int_0^T\int_{\mathbb{R}_x^3\times\mathbb{R}_\xi^3}\bigl|g_n(t,x,\xi)\bigr|\,dt\,dx\,d\xi\leqslant\frac{C}{M},$$

where C is the L^1 bound of the set $\{g_n\colon n=1,2,\dots\}$ and $\mu(\{|g_n|\geqslant M\})$ is the Lebesgue measure of the set $\{|g_n|\geqslant M\}$. The last inequality and the Dunford–Pettis theorem for weak compactness of subset of L^1 imply that

$$\sup_{n\geqslant1}\int_{|g_n|\geqslant M}\bigl|g_n(t,x,\xi)\bigr|\,dt\,dx\,d\xi\to0,\quad M\to\infty.$$

Hence, we obtain

$$\sup_{n\geqslant1}\int_0^T\int_{\mathbb{R}_x^3\times\mathbb{R}_\xi^3}\bigl|h_n^2(t,x,\xi)\bigr|\,dt\,dx\,d\xi\to0,\quad M\to\infty. \tag{2.5}$$

On the other hand, h_n^1 solves

$$\partial_t h_n^1+\xi\cdot\nabla_x h_n^1+F_n\cdot\nabla_\xi h_n^1=g_n\cdot\chi_{\{(t,x,\xi)\colon |g_n|\leqslant M\}}. \tag{2.6}$$

It can be written as

$$h_n^1(t,x,\xi)=\int_0^t g_n\cdot\chi_{\{|g_n|\leqslant M\}}\bigl(s,X_n(s),\Xi_n(s)\bigr)\,ds.$$

We define $g_n^1=g_n\cdot\chi_{\{(t,x,\xi)\colon |g_n|\leqslant M\}}$, $g_n^2=-F_nh_n^1$, then

$$\partial_t h_n^1+\xi\cdot\nabla_x h_n^1=g_n^1+\nabla_\xi g_n^2, \tag{2.7}$$

where h_n^1, g_n^1 and g_n^2 are bounded sequences in $L^q_{\mathrm{loc}}((0,T)\times\mathbb{R}_x^3\times\mathbb{R}_\xi^3)$. Lemma 2.1 implies that $\int_{\mathbb{R}_\xi^3}h_n^1\psi\,d\xi$ is bounded in $B_t^{s,q}(\mathbb{R}^3\times\mathbb{R})$ where $t=\max(q,2)$, $s=(2\max(q,q'))^{-1}$ (notice that we could let $q<\infty$). Consequently, $\int_{\mathbb{R}_\xi^3}h_n^1\psi\,d\xi$ compact in $L^1((0,T)\times\mathbb{R}_x^3)$. Combining this with (2.5), $\int_{\mathbb{R}_\xi^3}f_n\psi\,d\xi$ has a compact ε-net $\int_{\mathbb{R}_\xi^3}h_n^1\psi\,d\xi$. So, $\int_{\mathbb{R}_\xi^3}\psi f_n\,d\xi$ is compact in $L^1((0,T)\times\mathbb{R}_x^3)$.

Step 2: We assume that f_n (and consequently g_n) are supported in a common compact set $K\Subset(0,T)\times\mathbb{R}_x^3\times\mathbb{R}_\xi^3$ and $\psi_n=\psi\in C^\infty((0,T)\times\mathbb{R}_x^3\times\mathbb{R}_\xi^3)$ also has compact support contained in K for all $n\geqslant1$. Then $f_n\psi$ satisfies

$$\bigl(\partial_t+\xi\cdot\nabla_x+F_n(t,x,\xi)\cdot\nabla_\xi\bigr)(f_n\psi)=g_n\psi+f_n\bigl(\partial_t+\xi\cdot\nabla_x+F_n(t,x,\xi)\cdot\nabla_\xi\bigr)\psi.$$

Let $\tilde f_n=f_n\psi$, $\tilde g_n=g_n\psi+f_n(\partial_t+\xi\cdot\nabla_x+F_n(t,x,\xi)\cdot\nabla_\xi)\psi$. Obviously, $\tilde f_n$ and $\tilde g_n$ satisfy the same hypotheses as f_n and g_n in step 1. Then $\int_{\mathbb{R}_\xi^3}f_n\psi\,d\xi$ is compact in $L^1((0,T)\times\mathbb{R}_x^3)$.

Step 3: We assume that f_n (and consequently g_n) are supported in a common compact set $K\Subset(0,T)\times\mathbb{R}_x^3\times\mathbb{R}_\xi^3$ and $\psi_n=\psi\in L^\infty((0,T)\times\mathbb{R}_x^3\times\mathbb{R}_\xi^3)$ also has compact support contained in K for all $n\geqslant1$. We can approximate ψ by C^∞ functions ϕ_k such that

$$\|\phi_k-\psi\|_1\to0,\qquad\sup_{k\geqslant1}\|\phi_k\|_\infty<\infty.$$

Then by the a.e. convergence of ϕ_k to ψ (choosing a subsequence if necessary), the Egorov's theorem and the weak compactness of f_n in $L^1((0,T)\times\mathbb{R}_x^3\times\mathbb{R}_\xi^3)$

$$\sup_{n\geqslant 1}\int_0^T dt\int_{\mathbb{R}_x^3} dx\left|\int_{\mathbb{R}_\xi^3}\phi_k f_n\,d\xi-\int_{\mathbb{R}_\xi^3}\psi f_n\,d\xi\right|\leqslant \sup_{n\geqslant 1}\int_{(0,T)\times\mathbb{R}_x^3\times\mathbb{R}_\xi^3}|\phi_k-\psi||f_n|\,dt\,dx\,d\xi\to 0,\quad k\to\infty.$$

This means by step 2 that for every $\varepsilon>0$, $\int_{\mathbb{R}_\xi^3}\psi f_n\,d\xi$ has a compact ε-net $\int_{\mathbb{R}_\xi^3}\phi_k f_n\,d\xi$. So, $\int_{\mathbb{R}_\xi^3} f_n\psi\,d\xi$ is compact in $L^1((0,T)\times\mathbb{R}_x^3)$.

Step 4: We assume that f_n (and consequently g_n) are supported in a common compact set $K\Subset(0,T)\times\mathbb{R}_x^3\times\mathbb{R}_\xi^3$ and $\psi_n\in L^\infty((0,T)\times\mathbb{R}_x^3\times\mathbb{R}_\xi^3)$ for all $n\geqslant 1$ with compact support contained in K, and we also assume that ψ_n is bounded in $L^\infty((0,T)\times\mathbb{R}_x^3\times\mathbb{R}_\xi^3)$ and a.e. convergence to $\psi\in L^\infty((0,T)\times\mathbb{R}_x^3\times\mathbb{R}_\xi^3)$. Then

$$\int_0^T dt\int_{\mathbb{R}_x^3} dx\left|\int_{\mathbb{R}_\xi^3}\psi_n f_n\,d\xi-\int_{\mathbb{R}_\xi^3}\psi f_n\,d\xi\right|\leqslant\int_{(0,T)\times\mathbb{R}_x^3\times\mathbb{R}_\xi^3}|\psi_n-\psi||f_n|\,dt\,dx\,d\xi\to 0,\quad n\to\infty.$$

By step 3, $\int_{\mathbb{R}_\xi^3} f_n\psi_n\,d\xi$ is compact in $L^1((0,T)\times\mathbb{R}_x^3)$.

Step 5: The general case. Because of the weak compactness of $f_n(t,x,\xi)$ in $L^1((0,T)\times\mathbb{R}_x^3\times\mathbb{R}_\xi^3)$ and the boundedness of ψ_n in $L^\infty((0,T)\times\mathbb{R}_x^3\times\mathbb{R}_\xi^3)$, for any $\varepsilon>0$ there exists a compact set $K\subset(0,T)\times\mathbb{R}_x^3\times\mathbb{R}_\xi^3$ such that

$$\sup_{n\geqslant 1}\int_0^T\int_{\mathbb{R}_x^3\times\mathbb{R}_\xi^3}|\psi_n f_n|\chi_{K^c}(t,x,\xi)\,dt\,dx\,d\xi\leqslant\varepsilon.$$

Taking a cutoff function ϕ such that $0\leqslant\phi\leqslant 1$, $\phi=1$ on K and $\operatorname{supp}\phi\subset K_\delta$, where K_δ is the δ neighborhood of the set K. By step 4, $\int_{\mathbb{R}_\xi^3}(\phi f_n)(\phi\psi_n)\,d\xi$ is compact in $L^1((0,T)\times\mathbb{R}_x^3)$ since ϕf_n and $\phi\psi_n$ have the same compact support K_δ and $f_n\psi$ satisfies

$$\big(\partial_t+\xi\cdot\nabla_x+F_n(t,x,\xi)\cdot\nabla_\xi\big)(f_n\phi)=g_n\psi+f_n\big(\partial_t+\xi\cdot\nabla_x+F_n(t,x,\xi)\cdot\nabla_\xi\big)\phi.$$

On the other hand, we have

$$\int_0^T dt\int_{\mathbb{R}_x^3} dx\left|\int_{\mathbb{R}_\xi^3}\psi_n f_n\,d\xi-\int_{\mathbb{R}_\xi^3}(\phi\psi_n)(\phi f_n)\,d\xi\right|\leqslant\int_0^T\int_{\mathbb{R}_x^3\times\mathbb{R}_\xi^3}|\psi_n f_n|\chi_{K^c}(t,x,\xi)\,dt\,dx\,d\xi\leqslant\varepsilon.$$

Then, $\int_{\mathbb{R}_\xi^3} f_n\psi_n\,d\xi$ has a compact ε-net $\int_{\mathbb{R}_\xi^3}(\phi f_n)(\phi\psi_n)\,d\xi$, hence $\int_{\mathbb{R}_\xi^3} f_n\psi_n\,d\xi$ is compact in $L^1((0,T)\times\mathbb{R}_x^3)$. $\square$

By Theorem 2.2 and a further cutoff argument, we can easily show the following result (see, e.g., [33]).

Corollary 2.3. *Let the sequence $\{F_n(t,x,\xi)\colon n=1,2,\ldots\}\subset C([0,T];C_b^1(\mathbb{R}_x^3\times\mathbb{R}_\xi^3))$ be bounded in $L^q_{\mathrm{loc}}([0,T]\times\mathbb{R}^3\times\mathbb{R}^3)$ for $q>1$ and satisfy $\nabla_\xi\cdot F_n=0$ $(n=1,2,\ldots)$. Suppose that the sequence $\{f_n\colon n=1,2,\ldots\}$ is weakly compact in $L^1((0,T)\times W_x\times\mathbb{R}^3)$ for any compact set $W_x\Subset\mathbb{R}^3$, and the sequence $\{g_n\colon n=1,2,\ldots\}$ is also weakly compact in $L^1_{\mathrm{loc}}((0,T)\times\mathbb{R}^3\times\mathbb{R}^3)$ such that*

$$\partial_t f_n+\xi\cdot\nabla_x f_n+F_n(t,x,\xi)\cdot\nabla_\xi f_n=g_n \tag{2.8}$$

in the distributional sense. Then for any compact set $K_x\Subset\mathbb{R}^3$ and any bounded sequence $\psi_n(t,x,\xi)\in L^\infty([0,T]\times K_x\times\mathbb{R}^3)$ that converges almost everywhere, the sequence $\{\int_{\mathbb{R}^3} f_n(t,x,\xi)\psi_n(t,x,\xi)\,d\xi\colon n=1,2,\ldots\}$ is compact in $L^1((0,T)\times K_x)$.

Next, we review another type of regularizing result concerning velocity averages for any microscopic density, which is useful when we deal with a given Lorentz field.

Lemma 2.4. *Let $1\leqslant p\leqslant\infty$ with $1/p+1/p'=1$, $0\leqslant k'\leqslant k$ and*

$$r=\frac{k+3/p'}{k'+3/p'+(k-k')/p}.$$

If $f \in L^p_+(\mathbb{R}^6)$ *and* $\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |\xi|^k f(x,\xi)\,d\xi\,dx < \infty$ *then* $\int_{\mathbb{R}^3} |\xi|^{k'} f(x,\xi)\,d\xi \in L^r(\mathbb{R}^3)$ *and*

$$\left\| \int_{\mathbb{R}^3} |\xi|^{k'} f(x,\xi)\,d\xi \right\|_r \leqslant C\|f\|_p^{(k-k')/(k+3/p')} \left(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |\xi|^k f(x,\xi)\,d\xi\,dx \right)^{(k'+3/p')/(k+3/p')}$$

where $C = C(k,k',p) > 0$.

For the proof of this lemma, see, e.g., [28].

3. Existence of weak solutions for given fields

In this section, we deal with the Cauchy problems (1.2) and (1.4). Firstly, we consider smooth external force fields, namely $F(t,x,\xi) \in C([0,T]; C_b^1(\mathbb{R}^3_x \times \mathbb{R}^3_\xi))$, or $E(t,x)$ and $B(t,x) \in C([0,T]; C_b^1(\mathbb{R}^3))$. In these cases, the methods used in Refs. [7,13,25,33,34] are applicable, specifically we can show the following lemma.

Lemma 3.1. *Let the initial datum* $f_0(x,\xi)$ *be a nonnegative function such that*

$$\left(1+|\xi|^2\right) f_0 \in L^1\left(\mathbb{R}^3 \times \mathbb{R}^3\right), \qquad f_0 \in L^\infty\left(\mathbb{R}^3 \times \mathbb{R}^3\right). \tag{3.1}$$

(1) *If* $F(t,x,\xi) \in C([0,T]; C_b^1(\mathbb{R}^3_x \times \mathbb{R}^3_\xi))$ *and* $\nabla_\xi \cdot F(t,x,\xi) = 0$, *then there exists a nonnegative weak solution* $f(t,x,\xi)$ *to the Cauchy problem* (1.2) *with*

$$\left\|f(t)\right\|_1 = \|f_0\|_1, \qquad \left\|f(t)\right\|_\infty \leqslant \|f_0\|_\infty, \quad 0 \leqslant t \leqslant T \tag{3.2}$$

and

$$\mathcal{E}_k(f)(t) \leqslant \left(\mathcal{E}_k^{1/2}(f_0) + T\|F\|_\infty \|f_0\|_1^{1/2}\right)^2, \quad 0 \leqslant t \leqslant T. \tag{3.3}$$

(2) *If* $E(t,x) \in C([0,T]; C_b^1(\mathbb{R}^3))$ *and* $B(t,x) \in C([0,T]; C_b^1(\mathbb{R}^3))$, *then there exists a nonnegative weak solution* $f(t,x,\xi)$ *to the Cauchy problem* (1.4) *with*

$$\left\|f(t)\right\|_1 = \|f_0\|_1, \qquad \left\|f(t)\right\|_\infty \leqslant \|f_0\|_\infty, \quad 0 \leqslant t \leqslant T \tag{3.4}$$

and

$$\mathcal{E}_k(f)(t) \leqslant \left(\mathcal{E}_k^{1/2}(f_0) + T\|E\|_\infty \|f_0\|_1^{1/2}\right)^2, \quad 0 \leqslant t \leqslant T. \tag{3.5}$$

To shorten the presentation of this paper, we skip the proof of this lemma (for the details of the proof, we refer the readers to [34]), and turn to describing our first result.

Theorem 3.2. *Let the initial datum* $f_0(x,\xi)$ *be a nonnegative function such that*

$$\left(1+|\xi|^2\right) f_0 \in L^1\left(\mathbb{R}^3 \times \mathbb{R}^3\right), \qquad f_0 \in L^p\left(\mathbb{R}^3 \times \mathbb{R}^3\right), \tag{3.6}$$

and let $F(t,x,\xi) \in L^q([0,T] \times \mathbb{R}^3_x \times \mathbb{R}^3_\xi)$ *such that* $\nabla_\xi \cdot F = 0$ *in the distributional sense, where* $\frac{1}{p} + \frac{1}{q} < 1$. *Then there exists a weak solution* $f(t,x,\xi)$ *to the Cauchy problem* (1.2) *such that*

$$\left\|f(t)\right\|_1 = \|f_0\|_1, \qquad \left\|f(t)\right\|_p \leqslant \|f_0\|_p, \quad 0 \leqslant t \leqslant T. \tag{3.7}$$

Proof. To prove this theorem, we will use asymptotic methods in kinetic theory developed in recent years, the main tools are velocity averaging lemmas and renormalization method. Let $F^\varepsilon = F(t,x,\xi) * \eta_\varepsilon(t,x,\xi)$, where η_ε is the mollifier. Then, $\nabla_\xi \cdot F^\varepsilon = 0$, $F^\varepsilon(t,x,\xi) \in C([0,T]; C_b^1(\mathbb{R}^3_x \times \mathbb{R}^3_\xi))$ and

$$\left\|F^\varepsilon\right\|_{L^q([0,T]\times\mathbb{R}^3_x\times\mathbb{R}^3_\xi)} \leqslant \|F\|_{L^q([0,T]\times\mathbb{R}^3_x\times\mathbb{R}^3_\xi)}, \tag{3.8}$$

$$F^\varepsilon \to F, \quad \text{in } L^q\left([0,T]\times\mathbb{R}^3_x\times\mathbb{R}^3_\xi\right), \quad \text{if } q \neq \infty. \tag{3.9}$$

By Lemma 3.1, we know that there is a solution f^ε to

$$\begin{cases} \partial_t f^\varepsilon + \xi \cdot \nabla_x f^\varepsilon + F^\varepsilon \cdot \nabla_\xi f^\varepsilon = Q_{u_{f^\varepsilon}}\left(f^\varepsilon\right), \\ f_0^\varepsilon(x,\xi) = \min\left\{ f_0(x,\xi), \dfrac{1}{\varepsilon} \right\} + \varepsilon \exp\left(-\left(|x|^2+|\xi|^2\right)\right) \end{cases} \tag{3.10}$$

such that

$$\|f^\varepsilon(t)\|_1 = \|f_0^\varepsilon\|_1 \leqslant \|f_0\|_1 + \pi^3, \quad 0 \leqslant t \leqslant T. \tag{3.11}$$

Next, we show that $\|f^\varepsilon(t)\|_p$ is uniformly bounded. Let $\beta(\cdot) = |\cdot|^p$, for any fixed ε, we have

$$\beta(f^\varepsilon), \quad F^\varepsilon \cdot \beta(f^\varepsilon), \quad \beta'(f^\varepsilon) \cdot Q_{u_{f^\varepsilon}}(f^\varepsilon) \in L^1_{\mathrm{loc}}([0,T] \times \mathbb{R}^3 \times \mathbb{R}^3),$$

and $\beta(u)$ is Lipschitz continuous on any bounded interval. Following from the discuss about mild solution and distributional solution in [10], we have that $\beta(f^\varepsilon)$ satisfy

$$\partial_t \beta(f^\varepsilon) + \xi \cdot \nabla_x \beta(f^\varepsilon) + F^\varepsilon \cdot \nabla_\xi \beta(f^\varepsilon) = \beta'(f^\varepsilon) Q_{u_{f^\varepsilon}}(f^\varepsilon)$$

in the sense of distributions. In view of Lemma A.2, we have

$$\|f^\varepsilon(t)\|_p \leqslant \|f_0^\varepsilon\|_p \leqslant \|f_0\|_p + \pi^3, \quad 0 \leqslant t \leqslant T. \tag{3.12}$$

Notice that the lower bound on the initial datum f_0^ε, it is easy to get the lower bound of ρ_{f^ε}, which yields that

$$\|P_{u_{f^\varepsilon}}(f^\varepsilon)(t)\|_p \leqslant \|f^\varepsilon(t)\|_p \leqslant \|f_0\|_p + \pi^3, \quad 0 \leqslant t \leqslant T. \tag{3.13}$$

On the other hand, we have the uniform boundedness of $\int_{\mathbb{R}^3\times\mathbb{R}^3} |\xi|^{1+\delta} f^\varepsilon \, dx\, d\xi$, if

$$0 < \delta \leqslant \min\left\{\left(1 - \frac{1}{p} - \frac{1}{q}\right)q, 1\right\}. \tag{3.14}$$

Actually, we can use Hölder's inequality and Lemma A.4 to obtain that

$$\begin{aligned}
&\frac{d}{dt} \int_{\mathbb{R}^3\times\mathbb{R}^3} |\xi|^{1+\delta} f^\varepsilon \, dx\, d\xi \\
&\quad = (1+\delta) \int_{\mathbb{R}^3\times\mathbb{R}^3} |\xi|^{\delta-1} \xi \cdot F^\varepsilon f^\varepsilon \, dx\, d\xi + \int_{\mathbb{R}^3\times\mathbb{R}^3} |\xi|^{1+\delta} Q_{u_{f^\varepsilon}}(f^\varepsilon) \, dx\, d\xi \\
&\quad \leqslant (1+\delta) \int_{\mathbb{R}^3\times\mathbb{R}^3} |\xi|^{\delta} |F^\varepsilon| f^\varepsilon \, dx\, d\xi + (C_p + 1) \int_{\mathbb{R}^3\times\mathbb{R}^3} |\xi|^{1+\delta} f^\varepsilon \, dx\, d\xi \\
&\quad = (1+\delta) \int_{\mathbb{R}^3\times\mathbb{R}^3} \left(|\xi|^{1+\delta} f^\varepsilon\right)^{\frac{\delta}{1+\delta}} f^{\varepsilon\,\frac{1}{1+\delta}} |F^\varepsilon| \, dx\, d\xi + (C_p + 1) \int_{\mathbb{R}^3\times\mathbb{R}^3} |\xi|^{1+\delta} f^\varepsilon \, dx\, d\xi \\
&\quad \leqslant (1+\delta) \left(\int_{\mathbb{R}^3\times\mathbb{R}^3} |\xi|^{1+\delta} f^\varepsilon \, dx\, d\xi\right)^{\frac{\delta}{1+\delta}} \|f^\varepsilon(t)\|_{p_1}^{\frac{1}{1+\delta}} \|F^\varepsilon(t)\|_q + (C_p + 1) \int_{\mathbb{R}^3\times\mathbb{R}^3} |\xi|^{1+\delta} f^\varepsilon \, dx\, d\xi,
\end{aligned}$$

where $\frac{1}{p_1} = 1 - \frac{1}{q} - \frac{\delta}{q}$. (In view of (3.14), we deduce that $1 < p_1 \leqslant p$.) Following from (3.6), (3.11) and (3.12), we obtain that $\int_{\mathbb{R}^3\times\mathbb{R}^3} |\xi|^{1+\delta} f_0^\varepsilon \, dx\, d\xi$ and $\|f^\varepsilon(t)\|_{p_1}$ are uniformly bounded. Then, Gronwall's lemma implies that there exists a positive constant C independent of ε such that

$$\int_{\mathbb{R}^3\times\mathbb{R}^3} |\xi|^{1+\delta} f^\varepsilon(t) \, dx\, d\xi \leqslant C, \quad 0 \leqslant t \leqslant T. \tag{3.15}$$

From (3.11), (3.12) and (3.15), we get that the sequence f^ε is weakly compact in $L^1((0,T) \times B_R \times \mathbb{R}^3_\xi)$ for any $R > 0$. On the other hand, (3.13), (3.15) and Lemma A.4 imply that $\|P_{u_{f^\varepsilon}}(f^\varepsilon)(t)\|_p$ and $\int_{\mathbb{R}^3\times\mathbb{R}^3} |\xi|^{1+\delta} P_{u_{f^\varepsilon}}(f^\varepsilon) \, dx\, d\xi$ are uniformly bounded. And it is obvious that $\|P_{u_{f^\varepsilon}}(t)\|_1 = \|f^\varepsilon(t)\|_1 = \|f_0^\varepsilon\|_1$, so the sequence $P_{u_{f^\varepsilon}}(f^\varepsilon)$ is weakly compact in $L^1((0,T) \times B_R \times \mathbb{R}^3_\xi)$. According to Theorem 2.2, for any $\varphi \in \mathcal{D}(\mathbb{R}^3)$

$$\int_{\mathbb{R}^3} f^\varepsilon \varphi(\xi) \, d\xi \to \int_{\mathbb{R}^3} f \varphi(\xi) \, d\xi, \quad \text{in } L^1_{\mathrm{loc}}([0,T] \times \mathbb{R}^3), \tag{3.16}$$

where f is the weak limit of f^ε in $L^1((0,T) \times B_R \times \mathbb{R}^3_\xi)$ and

$$f^\varepsilon \to f \quad \text{weakly in } L^s\big((0,T)\times\mathbb{R}^3\times\mathbb{R}^3\big) \quad \text{for } 1<s\leqslant p, \quad \text{if } p<\infty, \tag{3.17}$$

$$f^\varepsilon \to f \quad \text{weakly* in } L^\infty\big((0,T)\times\mathbb{R}^3\times\mathbb{R}^3\big), \quad \text{if } p=\infty. \tag{3.18}$$

Then, we combine (3.15) and (3.16) to get that

$$\rho_{f^\varepsilon}\to\rho_f, \qquad \rho_{f^\varepsilon}u_{f^\varepsilon}\to\rho_f u_f, \quad \text{in } L^1_{\mathrm{loc}}\big([0,T]\times\mathbb{R}^3\big),$$

which yield that

$$Q_{u_{f^\varepsilon}}\big(f^\varepsilon\big)\to Q_{u_f}(f), \quad \text{in } \mathcal{D}'\big((0,T)\times\mathbb{R}^3\times\mathbb{R}^3\big).$$

Furthermore, from (3.9) and the relation between p and q we get that

$$F^\varepsilon\to F, \quad \text{in } L^{p'}\big([0,T]\times B_R\times B_R\big).$$

Combining this and (3.17), we know that

$$F^\varepsilon f^\varepsilon\to Ff \quad \text{in } \mathcal{D}'\big((0,T)\times\mathbb{R}^3\times\mathbb{R}^3\big).$$

Thus, $f(t,x,\xi)$ is a weak solution to the Cauchy problem (1.2), and it is easy to verify the desired estimates. □

Next, we discuss the Cauchy problem (1.4), the main result is the existence of a nonnegative solution under another set of conditions satisfied by the electric intensity $E(t,x)$ and the magnetic intensity $B(t,x)$. Specifically, we have

Theorem 3.3. *Let the initial datum $f_0(x,\xi)$ be a nonnegative function such that*

$$\big(1+|\xi|^2\big)f_0\in L^1\big(\mathbb{R}^3\times\mathbb{R}^3\big), \qquad f_0\in L^p\big(\mathbb{R}^3\times\mathbb{R}^3\big), \quad p>1, \tag{3.19}$$

and let $E(t,x)\in L^q([0,T]\times\mathbb{R}^3)$, $B(t,x)\in L^{p'}([0,T]\times\mathbb{R}^3)$, where $q>3+p'$. Then there exists a weak solution $f(t,x,\xi)$ to the Cauchy problem (1.4) *such that*

$$\big\|f(t)\big\|_1=\|f_0\|_1, \qquad \big\|f(t)\big\|_p\leqslant\|f_0\|_p, \quad 0\leqslant t\leqslant T. \tag{3.20}$$

Proof. Let $E^\varepsilon=E(t,x)*\eta_\varepsilon(t,x)$, $B^\varepsilon=B(t,x)*\eta_\varepsilon(t,x)$, where η_ε is the mollifier. Then,

$$\big\|E^\varepsilon\big\|_{L^q([0,T]\times\mathbb{R}^3)}\leqslant\|E\|_{L^q([0,T]\times\mathbb{R}^3)}, \tag{3.21}$$

$$\big\|B^\varepsilon\big\|_{L^{p'}([0,T]\times\mathbb{R}^3)}\leqslant\|B\|_{L^{p'}([0,T]\times\mathbb{R}^3)} \tag{3.22}$$

and for any $R>0$ (extracting a subsequence if necessary)

$$E^\varepsilon\to E, \qquad B^\varepsilon\to B, \quad \text{in } L^{p'}\big([0,T]\times B_R\big). \tag{3.23}$$

By Lemma 3.1, we know that there is a solution f^ε to

$$\begin{cases} \partial_t f^\varepsilon+\xi\cdot\nabla_x f^\varepsilon+\big(E^\varepsilon+\xi\times B^\varepsilon\big)\cdot\nabla_\xi f^\varepsilon=Q_{u_{f^\varepsilon}}\big(f^\varepsilon\big),\\ f^\varepsilon(0,x,\xi)=\min\left\{f_0(x,\xi),\dfrac{1}{\varepsilon}\right\}+\varepsilon\exp\big(-\big(|x|^2+|\xi|^2\big)\big) \end{cases} \tag{3.24}$$

such that

$$\big\|f^\varepsilon(t)\big\|_1=\big\|f_0^\varepsilon\big\|_1\leqslant\|f_0\|_1+\pi^3, \quad 0\leqslant t\leqslant T. \tag{3.25}$$

Using the same method in the proof of Theorem 3.2, we have

$$\big\|f^\varepsilon(t)\big\|_p\leqslant\|f_0\|_p+\pi^3, \quad 0\leqslant t\leqslant T, \tag{3.26}$$

and

$$\big\|P_{u_{f^\varepsilon}}\big(f^\varepsilon\big)(t)\big\|_p\leqslant\|f_0\|_p+\pi^3, \quad 0\leqslant t\leqslant T. \tag{3.27}$$

Now, we show that $\int_{\mathbb{R}^3\times\mathbb{R}^3}|\xi|^{1+\delta}f^\varepsilon(t)\,dx\,d\xi$ is uniformly bounded for any $0\leqslant t\leqslant T$, if

$$0<\delta\leqslant\min\left\{\frac{q-3}{p'}-1,1\right\}. \tag{3.28}$$

Actually, we can use Hölder's inequality and Lemma A.4 to obtain that

$$\frac{d}{dt}\int_{\mathbb{R}^3\times\mathbb{R}^3}|\xi|^{1+\delta}f^\varepsilon\,dx\,d\xi=(1+\delta)\int_{\mathbb{R}^3\times\mathbb{R}^3}|\xi|^{\delta-1}\xi\cdot\left(E^\varepsilon+\xi\times B^\varepsilon\right)f^\varepsilon\,dx\,d\xi+\int_{\mathbb{R}^3\times\mathbb{R}^3}|\xi|^{1+\delta}Q_{u_{f^\varepsilon}}\left(f^\varepsilon\right)dx\,d\xi$$
$$\leqslant(1+\delta)\int_{\mathbb{R}^3\times\mathbb{R}^3}|\xi|^{\delta}\left|E^\varepsilon\right|f^\varepsilon\,dx\,d\xi+(C_p+1)\int_{\mathbb{R}^3\times\mathbb{R}^3}|\xi|^{1+\delta}f^\varepsilon\,dx\,d\xi.$$

According to Lemma 2.4, we have

$$\left\|\int_{\mathbb{R}^3}|\xi|^\delta f^\varepsilon(t)\,d\xi\right\|_{L^r(\mathbb{R}^3)}\leqslant C\left\|f^\varepsilon(t)\right\|_p^{\frac{1}{1+\delta+3/p'}}\left(\int_{\mathbb{R}^3\times\mathbb{R}^3}|\xi|^{1+\delta}f^\varepsilon(t)\,dx\,d\xi\right)^{\frac{\delta+3/p'}{1+\delta+3/p'}},$$

where $r=\frac{1+\delta+3/p'}{1+\delta+2/p'}$. From (3.28), we have $r'\leqslant q$, then

$$\frac{d}{dt}\int_{\mathbb{R}^3\times\mathbb{R}^3}|\xi|^{1+\delta}f^\varepsilon\,dx\,d\xi$$
$$\leqslant C\left\|f^\varepsilon(t)\right\|_p^{\frac{1}{1+\delta+3/p'}}\left(\int_{\mathbb{R}^3\times\mathbb{R}^3}|\xi|^{1+\delta}f^\varepsilon(t)\,dx\,d\xi\right)^{\frac{\delta+3/p'}{1+\delta+3/p'}}\left\|E^\varepsilon(t)\right\|_{L^{r'}(\mathbb{R}^3)}+(C_p+1)\int_{\mathbb{R}^3\times\mathbb{R}^3}|\xi|^{1+\delta}f^\varepsilon\,dx\,d\xi.$$

In view of (3.21), (3.26) and Gronwall's lemma, we can get that there exists a positive constant C independent of ε such that

$$\int_{\mathbb{R}^3\times\mathbb{R}^3}|\xi|^{1+\delta}f^\varepsilon(t)\,dx\,d\xi\leqslant C,\quad 0\leqslant t\leqslant T. \tag{3.29}$$

Using the same method in the proof of Theorem 3.2, we have that the sequences f^ε and $P_{u_{f^\varepsilon}}(f^\varepsilon)(t)$ are weakly compact in $L^1((0,T)\times B_R\times\mathbb{R}^3_\xi)$ for any $R>0$. According to Theorem 2.2,

$$\int_{\mathbb{R}^3}f^\varepsilon\varphi(\xi)\,d\xi\to\int_{\mathbb{R}^3}f\varphi(\xi)\,d\xi,\quad\text{in } L^1_{\mathrm{loc}}\left([0,T]\times\mathbb{R}^3\right),\quad\forall\varphi\in\mathcal{D}\left(\mathbb{R}^3\right), \tag{3.30}$$

where f is the weak limit of f^ε in $L^1((0,T)\times B_R\times\mathbb{R}^3_\xi)$. Then, we combine (3.29) and (3.30) to get that

$$\rho_{f^\varepsilon}\to\rho_f,\qquad\rho_{f^\varepsilon}u_{f^\varepsilon}\to\rho_f u_f,\quad\text{in } L^1_{\mathrm{loc}}\left([0,T]\times\mathbb{R}^3\right),$$

which yield that

$$Q_{u_{f^\varepsilon}}\left(f^\varepsilon\right)\to Q_{u_f}(f),\quad\text{in } \mathcal{D}'\left((0,T)\times\mathbb{R}^3\times\mathbb{R}^3\right).$$

And, it is obvious that

$$\left(E^\varepsilon+\xi\times B^\varepsilon\right)f^\varepsilon\to(E+\xi\times B)f\quad\text{in } \mathcal{D}'\left((0,T)\times\mathbb{R}^3\times\mathbb{R}^3\right).$$

Thus, $f(t,x,\xi)$ is a global weak solution to the Cauchy problem (1.4), and it is easy to verify the desired estimates (3.20). □

4. Global solutions for self-induced electrostatic fields

In this section, we give a reasonable short proof of a global existence result for the Cauchy problem (1.5)–(1.7) with L^2 initial datum f_0. The main theorem (Theorem 4.1) obtain by renormalization method extends the results in [34].

Theorem 4.1. *Let the initial datum $f_0(x,\xi)$ be a nonnegative function such that*

$$\left(1+|\xi|^2\right)f_0\in L^1\left(\mathbb{R}^3\times\mathbb{R}^3\right),\qquad f_0\in L^2\left(\mathbb{R}^3\times\mathbb{R}^3\right), \tag{4.1}$$

then there exists a global weak solution $f(t,x,\xi)$ to system (1.5)–(1.7) *such that*

$$\left\|f(t)\right\|_1=\|f_0\|_1,\qquad\left\|f(t)\right\|_2\leqslant\|f_0\|_2,\quad t\geqslant 0 \tag{4.2}$$

and

$$\mathcal{E}_k(f)(t)+\mathcal{E}_p(f)(t)\leqslant\mathcal{E}_k(f_0)+\mathcal{E}_p(f_0),\quad t\geqslant 0. \tag{4.3}$$

Consequently, there exists a positive constant $M = M(f_0)$ such that

$$\mathcal{E}_k(f)(t) = \int_{\mathbb{R}^3\times\mathbb{R}^3} |\xi|^2 f(t,x,\xi)\,dx\,d\xi \leqslant M, \quad t \geqslant 0. \tag{4.4}$$

Proof. Firstly, we regularize the initial datum. Choose $f_0^\varepsilon \in \mathcal{S}_+(\mathbb{R}^3\times\mathbb{R}^3)$, so that

$$\int_{\mathbb{R}^3\times\mathbb{R}^3} \left(1+|\xi|^2\right)\left|f_0 - f_0^\varepsilon\right| + \left|f_0 - f_0^\varepsilon\right|^2 dx\,d\xi \xrightarrow{\varepsilon} 0, \quad f_0^\varepsilon \geqslant \varepsilon \exp\left(-\left(|x|^2+|\xi|^2\right)\right). \tag{4.5}$$

Then, we consider the regularized system

$$\partial_t f + \xi\cdot\nabla_x f + E\cdot\nabla_\xi f = Q_{u_f}(f), \qquad f(0,x,\xi) = f_0^\varepsilon(x,\xi), \tag{4.6}$$

$$-\triangle_x U(t,x) = \rho_f(t,x), \qquad \lim_{|x|\to\infty} U(t,x) = 0, \tag{4.7}$$

$$E(t,x) = -\nabla_x U(t,x). \tag{4.8}$$

Theorem 1.1 in [34] implies that there exists a global weak solution f^ε to the system (4.6)–(4.8), and

$$\left\|f^\varepsilon(t)\right\|_1 = \left\|f_0^\varepsilon\right\|_1, \qquad \left\|f^\varepsilon(t)\right\|_\infty \leqslant \left\|f_0^\varepsilon\right\|_\infty < \infty, \tag{4.9}$$

$$\mathcal{E}_k\left(f^\varepsilon\right)(t) + \mathcal{E}_p\left(f^\varepsilon\right)(t) \leqslant \mathcal{E}_k\left(f_0^\varepsilon\right) + \mathcal{E}_p\left(f_0^\varepsilon\right). \tag{4.10}$$

Analogously, we have

$$\left\|f^\varepsilon(t)\right\|_2 \leqslant \left\|f_0^\varepsilon\right\|_2. \tag{4.11}$$

In fact, due to (4.6), (4.9) and the Lipschitz continuity of the function $\beta(u) = u^2$ on the interval $[0, \|f_0^\varepsilon\|_\infty]$, we have

$$\partial_t\beta\left(f^\varepsilon\right) + \xi\cdot\nabla_x\beta\left(f^\varepsilon\right) + E^\varepsilon\cdot\nabla_\xi\beta\left(f^\varepsilon\right) = \beta'\left(f^\varepsilon\right)Q_{u_{f^\varepsilon}}\left(f^\varepsilon\right), \qquad f^\varepsilon(0,x,\xi) = f_0^\varepsilon(x,\xi),$$

where $E^\varepsilon(t,x)$ is the force field generated by $\rho_{f^\varepsilon}(t,x)$. Integrating it against (x,ξ) we obtain

$$\frac{d}{dt}\int_{\mathbb{R}^3\times\mathbb{R}^3} \left|f^\varepsilon\right|^2 dx\,d\xi = 2\int_{\mathbb{R}^3\times\mathbb{R}^3} Q_{u_{f^\varepsilon}}\left(f^\varepsilon\right)f^\varepsilon\,dx\,d\xi = -2\int_{\mathbb{R}^3\times\mathbb{R}^3} \left[Q_{u_{f^\varepsilon}}\left(f^\varepsilon\right)\right]^2 dx\,d\xi \leqslant 0,$$

which obviously implies (4.11).

Based on estimate (4.9)–(4.11), we obtain that

$$f^\varepsilon \to f \quad \text{weakly in } L^1_{\mathrm{loc}}\left([0,\infty)\times\mathbb{R}^3\times\mathbb{R}^3\right).$$

Let $\beta_\delta(u) = \frac{u}{1+\delta u}$, we know that $\beta_\delta(t)$ is Lipschitz continuous and

$$\beta_\delta\left(f^\varepsilon\right), \quad E^\varepsilon\cdot\beta_\delta\left(f^\varepsilon\right), \quad \beta'_\delta\left(f^\varepsilon\right)\cdot Q_{u_{f^\varepsilon}}\left(f^\varepsilon\right) \in L^1_{\mathrm{loc}}\left((0,\infty)\times\mathbb{R}^3\times\mathbb{R}^3\right),$$

so $\beta_\delta(f^\varepsilon)$ satisfies

$$\partial_t\beta_\delta\left(f^\varepsilon\right) + \xi\cdot\nabla_x\beta_\delta\left(f^\varepsilon\right) + E^\varepsilon\cdot\nabla_\xi\beta_\delta\left(f^\varepsilon\right) = \beta'_\delta\left(f^\varepsilon\right)Q_{u_{f^\varepsilon}}\left(f^\varepsilon\right)$$

in the sense of distributions. And it is easy to get that

$$\left\|E^\varepsilon\beta_\delta\left(f^\varepsilon\right)\right\|_{L^2((0,\infty)\times\mathbb{R}^3\times\mathbb{R}^3)} \leqslant \frac{1}{\delta}\|E\|_2, \qquad \left\|\beta'_\delta\left(f^\varepsilon\right)Q_{u_{f^\varepsilon}}\left(f^\varepsilon\right)\right\|_{L^2((0,T)\times\mathbb{R}^3\times\mathbb{R}^3)} \leqslant 2T\left\|f^\varepsilon(t)\right\|_2.$$

According to (4.9)–(4.11), we get that for any fixed T, R, δ, $\|\beta_\delta(f^\varepsilon)\|_{L^2((0,T)\times\mathbb{R}^3\times\mathbb{R}^3)}$ and $\|g_\delta^\varepsilon\|_{L^2((0,T)\times\mathbb{R}^3_x;H^{-1}(B_R))}$ are uniformly bounded, where

$$g_\delta^\varepsilon = \beta'_\delta\left(f^\varepsilon\right)Q_{u_{f^\varepsilon}}\left(f^\varepsilon\right) - E^\varepsilon\cdot\nabla_\xi\beta_\delta\left(f^\varepsilon\right).$$

By the velocity averaging lemma in [9], we have

$$\int_{\mathbb{R}^3} \beta_\delta\left(f^\varepsilon\right)\varphi(\xi)\,d\xi \in H^{\frac{1}{4}}\left((0,T)\times\mathbb{R}^3\right), \quad \forall\varphi\in\mathcal{D}\left(\mathbb{R}^3\right),$$

and

$$\left\| \int_{\mathbb{R}^3} \beta_\delta(f^\varepsilon)\varphi(\xi)\,d\xi \right\|_{H^{\frac{1}{4}}((0,T)\times\mathbb{R}^3)} \leqslant C\left[\left\|\beta_\delta(f^\varepsilon)\right\|_{L^2((0,T)\times\mathbb{R}^3\times\mathbb{R}^3)} + \left\|g_\delta^\varepsilon\right\|_{L^2((0,T)\times\mathbb{R}_x^3;H^{-1}(B_R))}\right],$$

where R is big enough such that $\operatorname{supp}(\varphi)\subset B_R$. Consequently,

$$\int_{\mathbb{R}^3} \beta_\delta(f^\varepsilon)\varphi(\xi)\,d\xi \xrightarrow{\varepsilon} \int_{\mathbb{R}^3} f_\delta\varphi(\xi)\,d\xi \quad \text{in } L^1_{\mathrm{loc}}([0,\infty)\times\mathbb{R}^3), \quad \forall\varphi\in\mathcal{D}(\mathbb{R}^3), \tag{4.12}$$

where f_δ is the weak limit of $\beta_\delta(f^\varepsilon)$ in $L^1_{\mathrm{loc}}([0,\infty)\times\mathbb{R}^3\times\mathbb{R}^3)$.

Next, we show

$$\int_{\mathbb{R}^3} f^\varepsilon\varphi(\xi)\,d\xi \xrightarrow{\varepsilon} \int_{\mathbb{R}^3} f\varphi(\xi)d\xi \quad \text{in } L^1_{\mathrm{loc}}([0,\infty)\times\mathbb{R}^3), \quad \forall\varphi\in\mathcal{D}(\mathbb{R}^3). \tag{4.13}$$

Notice that

$$0\leqslant f^\varepsilon-\beta_\delta(f^\varepsilon)\leqslant\delta(f^\varepsilon)^2,$$

we can get

$$\sup\int_0^T dt\int_{B_R\times B_R}\left|f^\varepsilon-\beta_\delta(f^\varepsilon)\right|dx\,d\xi\to 0, \quad \delta\to 0^+.$$

Besides,

$$\begin{aligned}
\int_0^T dt\int_{B_R\times B_R}|f-f_\delta|\,dx\,d\xi &= \int_0^T dt\int_{B_R\times B_R}\left(f-f^\varepsilon\right)\operatorname{sign}(f-f_\delta)\,dx\,d\xi\\
&\quad+\int_0^T dt\int_{B_R\times B_R}\left(f^\varepsilon-\beta_\delta(f^\varepsilon)\right)\operatorname{sign}(f-f_\delta)\,dx\,d\xi\\
&\quad+\int_0^T dt\int_{B_R\times B_R}\left(\beta_\delta(f^\varepsilon)-f_\delta\right)\operatorname{sign}(f-f_\delta)\,dx\,d\xi\\
&\to 0, \quad \delta\to 0^+.
\end{aligned}$$

Thus, (4.13) is proved. Notice that $\mathcal{E}_k(f^\varepsilon)$ is uniformly bounded, we can get

$$\rho_{f^\varepsilon}\to\rho_f, \qquad \rho_{f^\varepsilon}u_{f^\varepsilon}\to\rho_f u_f \quad \text{in } L^1_{\mathrm{loc}}([0,\infty)\times\mathbb{R}^3),$$

which yield that

$$Q_{u_{f^\varepsilon}}(f^\varepsilon)\to Q_{u_f}(f) \quad \text{in } \mathcal{D}'((0,\infty)\times\mathbb{R}^3\times\mathbb{R}^3).$$

Finally, we show

$$E^\varepsilon f^\varepsilon\to Ef \quad \text{in } \mathcal{D}'((0,\infty)\times\mathbb{R}^3\times\mathbb{R}^3), \quad \text{as } \varepsilon\to 0. \tag{4.14}$$

On one hand, we have (extracting a subsequence if necessary)

$$\int_{\mathbb{R}^3} f^\varepsilon\varphi(\xi)\,d\xi \xrightarrow{\varepsilon} \int_{\mathbb{R}^3} f\varphi(\xi)\,d\xi \quad \text{in } L^2_{\mathrm{loc}}([0,\infty)\times\mathbb{R}^3), \quad \forall\varphi\in\mathcal{D}(\mathbb{R}^3). \tag{4.15}$$

Actually, it follows from (4.11) and Refs. [9,16] that there exists a nonnegative super-quadratic function $\beta\in C^\infty(\mathbb{R})$ such that $\beta(0)=0$, $\lim_{u\to\infty}u^{-2}\beta(u)=\infty$ and

$$\int_{\mathbb{R}^3\times\mathbb{R}^3}\beta\left(f^\varepsilon(t)\right)dx\,d\xi\leqslant C\left\|f^\varepsilon(t)\right\|_2\leqslant C\left\|f_0^\varepsilon\right\|_2,$$

where $C>0$ is a constant independent of f^ε. At first, we claim that (extracting a subsequence if necessary)

$$\left(\int\limits_{\mathbb{R}^3} f^\varepsilon \varphi(\xi)\,d\xi\right)^2 \xrightarrow{\varepsilon} \left(\int\limits_{\mathbb{R}^3} f\varphi(\xi)\,d\xi\right)^2 \quad \text{in } L^1_{\mathrm{loc}}\big([0,\infty)\times\mathbb{R}^3\big), \quad \forall\varphi\in\mathcal{D}\big(\mathbb{R}^3\big). \tag{4.16}$$

In fact, it follows from (4.13) that $\int_{\mathbb{R}^3} f^\varepsilon\varphi(\xi)\,d\xi \xrightarrow{\varepsilon} \int_{\mathbb{R}^3} f\varphi(\xi)\,d\xi$ in measure and passing to another subsequence we can assume this convergence also hold a.e. on $[0,\infty)\times\mathbb{R}^3$. In consideration of Schur's theorem, it is sufficient to show that $(\int_{\mathbb{R}^3} f^\varepsilon\varphi(\xi)\,d\xi)^2$ is relatively compact in $L^1_{\mathrm{loc}}([0,\infty)\times\mathbb{R}^3)$. We use Dunford–Pettis theorem to prove it. (4.11) implies that $(\int_{\mathbb{R}^3} f^\varepsilon\varphi(\xi)\,d\xi)^2$ is uniformly bounded in $L^1_{\mathrm{loc}}([0,\infty)\times\mathbb{R}^3)$. Furthermore, for any given $K \Subset [0,\infty)\times\mathbb{R}^3$ we can get that

$$\sup_\varepsilon \int\limits_A \left(\int\limits_{\mathbb{R}^3} f^\varepsilon\varphi(\xi)\,d\xi\right)^2 dx\,dt \to 0, \quad |A|\to 0,$$

where $A\subset K$ is measurable. In fact, for $\sigma>0$ large enough we have

$$\begin{aligned}
\sup_\varepsilon \int\limits_A \left(\int\limits_{\mathbb{R}^3} f^\varepsilon\varphi(\xi)\,d\xi\right)^2 dx\,dt &\leqslant \sup_\varepsilon \int\limits_A \bigg(\int\limits_{f^\varepsilon\leqslant\sigma} + \int\limits_{f^\varepsilon>\sigma}\bigg)\big(f^\varepsilon\big)^2 \chi_{\mathrm{supp}(\varphi)}(\xi)\,d\xi\,dt\,dx \cdot \int\limits_{\mathbb{R}^3} |\varphi|^2\,d\xi \\
&\leqslant \left[\sigma^2|A|\big|\mathrm{supp}(\varphi)\big| + \sup_{u>\sigma}\frac{u^2}{\beta(u)}\int\limits_A\int\limits_{\mathbb{R}^3}\beta\big(f^\varepsilon\big)\,d\xi\,dx\,dt\right]\int\limits_{\mathbb{R}^3}|\varphi|^2\,d\xi \\
&\leqslant \left[\sigma^2|A|\big|\mathrm{supp}(\varphi)\big| + \sup_{u>\sigma}\frac{Cu^2}{\beta(u)}\int\limits_K\int\limits_{\mathbb{R}^3}\big|f_0^\varepsilon\big|^2\,d\xi\,dx\,dt\right]\int\limits_{\mathbb{R}^3}|\varphi|^2\,d\xi,
\end{aligned}$$

letting $|A|\to 0$ and $\sigma\to\infty$ in succession, we obtain the desired limit. So $(\int_{\mathbb{R}^3} f^\varepsilon\varphi(\xi)\,d\xi)^2$ is relatively compact in $L^1_{\mathrm{loc}}([0,\infty)\times\mathbb{R}^3)$. Secondly, we have proven that $\int_{\mathbb{R}^3} f^\varepsilon\varphi(\xi)\,d\xi$ is bounded in $L^2_{\mathrm{loc}}([0,\infty)\times\mathbb{R}^3)$, without loss of generality, we may assume that

$$\int\limits_{\mathbb{R}^3} f^\varepsilon\varphi(\xi)\,d\xi \xrightarrow{\varepsilon} \int\limits_{\mathbb{R}^3} f\varphi(\xi)\,d\xi \quad \text{weakly in } L^2_{\mathrm{loc}}\big([0,\infty)\times\mathbb{R}^3\big), \quad \forall\varphi\in\mathcal{D}\big(\mathbb{R}^3\big). \tag{4.17}$$

Combining (4.16) with (4.17), we get (4.15).

On the other hand, from (4.10) we have that there exist $\tilde{E}\in L^2_{\mathrm{loc}}([0,\infty)\times\mathbb{R}^3)$ such that (extracting a subsequence if necessary)

$$E^\varepsilon \xrightarrow{\varepsilon} \tilde{E} \quad \text{weakly in } L^2_{\mathrm{loc}}\big([0,\infty)\times\mathbb{R}^3\big),$$

and from (4.7), (4.8) we have

$$E^\varepsilon \xrightarrow{\varepsilon} E \quad \text{in } \mathcal{D}'\big((0,\infty)\times\mathbb{R}^3\big),$$

so $E^\varepsilon \xrightarrow{\varepsilon} E$ weakly in $L^2_{\mathrm{loc}}([0,\infty)\times\mathbb{R}^3)$. Combining this result with (4.15), we get (4.14). Thus, $f(t,x,\xi)$ is a global weak solution to the system (1.5)–(1.7), and we can take the limits in (4.9)–(4.11) to get the desired estimates. □

Remark 4.2. Obviously, Theorem 4.1 can be extended to any initial datum f_0 such that $f_0\in L^p(\mathbb{R}^3\times\mathbb{R}^3)$ for some $p>2$. In this case, all results are not altered except for the second estimate in (4.2), which should be redescribed by $\|f(t)\|_p \leqslant \|f_0\|_p$, $t\geqslant 0$.

Acknowledgments

The authors would like to thank anonymous referees for several helpful comments and valuable suggestions concerning the presentation of this paper.

Appendix A. Properties of wave-particle collision integral

In this appendix, we summarize some basic properties of the collision operator $Q_{u_f}(f)$, which we have used in the last section. For details, see Refs. [5–7,13].

Let $u\in\mathbb{R}^3$ be fixed, we define the linearized collision operator $Q_u(f)$ as follows: for any function $f(\xi)\in L^1(\mathbb{R}^3)$,

$$Q_u(f)(\xi) = P_u(f)(\xi) - f(\xi), \quad P_u(f)(\xi) = \frac{1}{4\pi}\int\limits_{\mathbb{S}^2} f\big(u+|\xi-u|\omega\big)\,d\omega. \tag{A.1}$$

Then, we have

Lemma A.1. *Let $f(\xi)$, $g(\xi) \in C_c^\infty(\mathbb{R}^3)$ be nonnegative functions and let $\psi \in C^\infty[0,\infty)$. Then*

(1) *$P_u(f)$ is a projector, i.e.,*

$$P_u^2(f) = P_u(f).$$

(2) *$Q_u(f)$ is symmetric, i.e.,*

$$\int\limits_{\mathbb{R}^3} Q_u(f) g \, d\xi = \int\limits_{\mathbb{R}^3} Q_u(g) f \, d\xi = -\int\limits_{\mathbb{R}^3} Q_u(f) Q_u(g) \, d\xi.$$

(3) *Infinite many collision invariants*:

$$\int\limits_{\mathbb{R}^3} \xi \, Q_u(f)(\xi) \, d\xi = \int\limits_{\mathbb{R}^3} \psi\big(|\xi - u|\big) Q_u(f)(\xi) \, d\xi = 0.$$

(4) *$Q_u(f) = 0$ if and only if there exist $u \in \mathbb{R}^3$ and $F \in C_c^\infty[0,\infty)$ such that $f(\xi) = F(|\xi - u|^2)$.*

(5) *H-theorem*:

$$\int\limits_{\mathbb{R}^3} Q_u(f) f \, d\xi = -\int\limits_{\mathbb{R}^3} Q_u(f) Q_u(f) \, d\xi \leqslant 0.$$

Lemma A.2. *Let $u(t,x), u_n(t,x) : (0,\infty) \times \mathbb{R}^3 \to \mathbb{R}^3$ be locally integrable functions such that $\lim_{n\to\infty} u_n = u$ in $L^1_{\mathrm{loc}}((0,\infty) \times \mathbb{R}^3)$, and let $f(t,x,\xi) \in C_c^\infty((0,\infty) \times \mathbb{R}^3 \times \mathbb{R}^3)$. Then*

(1) *For any $p, q \in [1,\infty]$ and $T > 0$, we have*

$$\big\| P_u(f) \big\|_{L^q((0,T);L^p(\mathbb{R}^3\times\mathbb{R}^3))} \leqslant \|f\|_{L^q((0,T);L^p(\mathbb{R}^3\times\mathbb{R}^3))}.$$

(2) *For any $p, q \in [1,\infty)$ and $T > 0$, we have*

$$\lim_{n\to\infty} P_{u_n}(f) = P_u(f) \quad \text{in } L^q\big((0,T);\ L^p\big(\mathbb{R}^3 \times \mathbb{R}^3\big)\big).$$

Remark A.3. Since $C_c^\infty((0,T) \times \mathbb{R}^3 \times \mathbb{R}^3)$ is dense in $L^q((0,T); L^p(\mathbb{R}^3 \times \mathbb{R}^3))$ for any $p, q \in [1,\infty)$, Lemma A.2(1) implies that the operator $Q_u(f)$ has a unique bounded extension on the whole space $L^q((0,T); L^p(\mathbb{R}^3 \times \mathbb{R}^3))$. It is in this manner that the operator $P_u(f)$ is defined. Consequently, the result of Lemma A.2(2) is valid for any $f \in L^q((0,T); L^p(\mathbb{R}^3 \times \mathbb{R}^3))$. On the other hand, it is obvious that the results in Lemma A.1 can be naturally extended to functions whenever the involved integrals are well defined.

In the proofs of the main theorems, we shall also need the following technical result which is a direct consequence of Hölder's inequality (see, e.g., [7]).

Lemma A.4. *For any $p \geqslant 1$ and any nonnegative function f with $(1 + |\xi|^p) f \in L^1(\mathbb{R}^3)$, we have*

$$\rho_f |u_f|^p \leqslant \int\limits_{\mathbb{R}^3} |\xi|^p f \, d\xi,$$

and

$$\int\limits_{\mathbb{R}^3} |\xi|^p P_{u_f}(f) \, d\xi \leqslant C_p \int\limits_{\mathbb{R}^3} |\xi|^p f \, d\xi.$$

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