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Maximal functions, Hardy spaces and Fourier multiplier theorems on unbounded Vilenkin groups

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ABSTRACT

We consider relations between various existing concepts of Hardy spaces on Vilenkin groups. The problem of an appropriate Fourier multiplier theorem for unbounded Vilenkin groups is revisited and the conditions made more precise under which such a theorem is established. Theorem 3.1 from Avdispahić and Memić (2010) [1] is rectified.

1. Introduction

Let $(p_n) = (p_n)_{n\in\mathbb{Z}}$ be a two-way infinite sequence of integers $p_n \geqslant 2$, and $\mathbb{Z}_{p_n} = \{0, 1, \dots, p_n - 1\}$, for every $n \in \mathbb{Z}$. We define the corresponding Vilenkin group G as $G = \{y = (y_n) \in \prod_{n\in\mathbb{Z}} \mathbb{Z}_{p_n},\ \lim_{n\to-\infty} y_n = 0\}$, where the group operation is the coordinate-wise addition modulo p_n.

Each element $y \in G$ is of the form $(\dots, 0, y_{-m}, \dots, y_{-1}, y_0, y_1, \dots, y_n, \dots)$, $y_n \in \mathbb{Z}_{p_{n+1}}$.

If every $\mathbb{Z}_{p_n}$ is endowed with the probability measure and the discrete topology, then the measure μ and the topology of G are obtained by taking the product measure and the product topology. A basis of neighborhoods of G is given by the family of open subgroups $G_n := \{(y_i) \in G\colon\ y_i = 0,\ \forall i < n\}$. For every $y = (y_n) \in G$ let $G_n(\dots, y_{n-i}, \dots, y_{n-1})$ denote the set $y + G_n$.

The dual group of G is denoted by Γ. It is the union of the increasing sequence of groups $\Gamma_n = \{\gamma \in \Gamma\colon\ \gamma(x) = 1,\ \forall x \in G_n\}$. Choose the Haar measures μ, λ on G and Γ respectively, such that $\mu(G_0) = \lambda(\Gamma_0) = 1$, and $\mu(G_n) = (\lambda(\Gamma_n))^{-1} = m_n^{-1}$, where

$$m_n = p_1 p_2 \dots p_n \quad \text{and} \quad m_{-n}^{-1} = p_0 p_{-1} p_{-2} \dots p_{-n+1} \quad (n \geqslant 1).$$

The group G is said to be bounded if $\sup_n p_n < \infty$. Otherwise, it is unbounded.

2. Maximal functions and Hardy spaces

We recall the definitions of p-atoms, maximal function f^{**} and Hardy spaces H^p_{**} used in [4–6].

P. Simon [4] introduced the following decomposition of the sets $\{0, 1, \dots, p_n - 1\}$, $n \in \mathbb{N}$:

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$$U_1 = \left\{0, 1, \ldots, \left[\frac{p_n}{2}\right] - 1\right\}, \qquad U_2 = \left\{\left[\frac{p_n}{2}\right], \left[\frac{p_n}{2}\right] + 1, \ldots, p_n - 1\right\},$$

$$U_3 = \left\{0, 1, \ldots, \left[\frac{[\frac{p_n}{2}] - 1}{2}\right] - 1\right\}, \qquad U_4 = \left\{\left[\frac{[\frac{p_n}{2}] - 1}{2}\right], \left[\frac{[\frac{p_n}{2}] - 1}{2}\right] + 1, \ldots, \left[\frac{p_n}{2}\right] - 1\right\}, \ldots.$$

Any restricted interval I_n contained in $y + G_n$ has the form $\biguplus_{i \in U_k} G_{n+1}(\ldots, y_{n-1}, i)$, for some U_k. The definition of restricted intervals is such that two intervals are either disjoint or one of them is a subset of the other.

Definition 2.1. A complex function a is called a p-atom on G for $0 < p \leqslant 1$, if

(1) $\operatorname{supp}(a) \subset I_n$, for some restricted interval I_n,
(2) $\|a\|_\infty \leqslant (\mu(I_n))^{-\frac{1}{p}}$,
(3) $\int_G a(x)\,dx = 0$.

The maximal function f^{**} of any integrable function f is defined by

$$f^{**}(x) = \sup_I \left| \big(\mu(I)\big)^{-1} \int_I f(t)\,dt \right|,$$

where the supremum is taken over all restricted intervals containing the point x. This maximal function can be extended to martingales $f = (f_n,\ n \in \mathbb{Z})$ with respect to the sequence of σ-algebras $\mathcal{F}_n := \sigma\{x + G_n,\ x \in G\}$ (see [6]).

Then, H^p_{**} consists of martingales f for which $f^{**} \in L^p$. The norm in this space is given by $\|f\|_{H^p_{**}} := \|f^{**}\|_p$ $(0 < p < \infty)$. It is known for $0 < p \leqslant 1$ that $f \in H^p_{**}$ if and only if f allows a decomposition $f = \sum_i \lambda_i a_i$, where the functions $(a_i)_i$ are p-atoms and $\sum_i |\lambda_i|^p < +\infty$. Moreover, $\|f\|^p_{H^p_{**}} = \inf \sum_i |\lambda_i|^p$, where the infimum is taken over all possible atomic decompositions (see [6]).

The Hardy spaces $\mathcal{P}_p$ introduced in [6] have atomic decomposition with respect to the following family of atoms.

Definition 2.2. A complex function a is called a $\mathcal{P}_p$-atom on G for $0 < p \leqslant 1$, if

(1) $\operatorname{supp}(a) \subset x + G_n$, for some G_n and $x \in G$,
(2) $\|a\|_\infty \leqslant (\mu(G_n))^{-\frac{1}{p}}$,
(3) $\int_G a(x)\,dx = 0$.

The norm in $\mathcal{P}_p$ is defined through atomic decomposition as in the previous case when $0 < p \leqslant 1$. It can also be obtained by means of the families $(\lambda_n)_n$ of non-decreasing, non-negative functions with the property $m_n|\int_{x+G_n} f(t)\,dt| \leqslant \lambda_{n-1}(x)$, and such that λ_n is constant on the cosets of G_n. Then, $\|f\|_{\mathcal{P}_p} = \inf \|\sup_n \lambda_n\|_p$, where $0 < p < \infty$ and the infimum is taken over all such families of functions. The proof can be found in [6].

In [1] the following maximal functions were used:

$$f^*(x) = \sup_n \left| \int_{x+G_n} f(t)\,dt \right| \quad \text{and} \quad \tilde{M} f(x) = \sup_{n, I_n} \left| f * \big(\mu(I_n)\big)^{-1} 1_{I_n}(x) \right|,$$

where the sets I_n are of the form $I_n = \biguplus_{i=\alpha}^{\beta} G_{n+1}(\ldots, 0, i)$, $0 \leqslant \alpha \leqslant \beta < p_{n+1}$. They correspond respectively to the spaces H^p_* and $\tilde{H}^p$. Actually, we can express the function $\tilde{M} f$ using only intervals I_n for which $\alpha = \beta$, since

$$\big(\mu(I_n)\big)^{-1} \left| \int_{x-I_n} f(t)\,dt \right| \leqslant \max_i m_{n+1} \left| \int_{G_{n+1}(\ldots, x_{n-2}, x_{n-1}, i)} f(t)\,dt \right|,$$

where the maximum is taken over $i : G_{n+1}(\ldots, x_{n-2}, x_{n-1}, i) \subseteq x - I_n$. We recall that all Hardy spaces can be extended to martingales. It is known that $H^p_* \sim H^p_{**} \sim L^p$ for $1 < p < \infty$, where $\sim$ denotes the equivalence of norms and spaces (see [6]).

Theorem 2.3. *We have*

$$\|f\|_{H^p_*} \leqslant \|f\|_{H^p_{**}} \leqslant \|f\|_{\tilde{H}^p} \sim \|f\|_{\mathcal{P}_p} \quad (0 < p < \infty).$$

If the sequence (p_n) *is bounded then all the spaces are equivalent. If* (p_n) *is unbounded, the converse of the first* (*resp. second*) *inequality is not valid when* $0 < p \leqslant 1$ (*resp.* $0 < p < \infty$).

Proof. The first two inequalities are straightforward. The proof that the converse of the first one does not hold for $0 < p \leqslant 1$ in the unbounded case can be found in [5]. For the converse of the second inequality, take the functions

$$a_n(t) = \begin{cases} m_{n+1}, & t \in G_{n+1}; \\ -m_{n+1}, & t \in G_{n+1}(\dots, 0, \dots, 0, 1); \\ 0, & \text{otherwise}. \end{cases}$$

Then, it is easily seen that

$$\|a_n\|_{H^p_{**}} = 2^{\frac{1}{p}} m_{n+1}^{1-\frac{1}{p}} \quad \text{and} \quad \|a_n\|_{\tilde{H}^p} = m_{n+1} m_n^{-\frac{1}{p}}.$$

In order to prove the equivalence $\|f\|_{\tilde{H}^p} \sim \|f\|_{\mathcal{P}_p}$, consider the sequence of operators M_N given by

$$M_N f(x) = \sup_{n \leqslant N} \max_{i=0,\dots,p_{n+1}-1} m_{n+1} \left| \int_{G_{n+1}(\dots,x_{n-2},x_{n-1},i)} f(t)\, dt \right|.$$

The sequence $(M_N f)_N$ possesses the properties cited above for $(\lambda_n)_n$. Therefore, $\|\sup_N M_N f\|_p \geqslant \|f\|_{\mathcal{P}_p}$. Notice that

$$m_{n+1} \int_{G_{n+1}(\dots,x_{n-2},x_{n-1},i)} f(t)\, dt = \big(\mu(I_n)\big)^{-1} f * 1_{I_n}(x),$$

where $I_n = G_{n+1}(\dots, x_{n-2}, x_{n-1}, p_{n+1} - i)$. Hence, $\tilde{M} f = \sup_N M_N f$, and

$$\|f\|_{\tilde{H}^p} = \|\tilde{M} f\|_p = \left\| \sup_N M_N f \right\|_p \geqslant \|f\|_{\mathcal{P}_p}.$$

It remains to prove the inequality $\|\sup_N M_N f\|_p \leqslant \|f\|_{\mathcal{P}_p}$. We only need to check that $M_N f \leqslant \lambda_N$ for any family $(\lambda_N)_N$ with the properties mentioned above. For every n it is clear that

$$\max_{i=0,\dots,p_{n+1}-1} m_{n+1} \left| \int_{G_{n+1}(\dots,x_{n-2},x_{n-1},i)} f(t)\, dt \right| \leqslant \lambda_n.$$

Consequently, $M_N f \leqslant \max_{n \leqslant N} \lambda_n = \lambda_N$. It follows that $\|\sup_N M_N f\|_p \leqslant \|f\|_{\mathcal{P}_p}$.

The equivalence of the spaces H^p_*, H^p_{**} and $\mathcal{P}_p$ when $(p_n)_n$ is bounded can be found in [6]. $\square$

3. Multiplier theorems

In this section we give a more precise form of Theorem 3.1 of [1] and prove some additional results about multipliers.

Theorem 3.1. *Let $\phi \in L^\infty(\Gamma)$. Suppose that*

$$\sup_N \int_{G_N^c} \big(\mu(G_N)\big)^{-1} \left(\int_{G_N} \left|(\phi - \phi_N)^\vee (x-u)\right| du \right)^p dx = O(1),$$

where $\phi_N = \phi 1_{\Gamma_N}$ and $\wedge, \vee$ denote respectively the Fourier transform and the inverse Fourier transform. Then ϕ is a multiplier from $\mathcal{P}_p$ to H^p_{**} and H^p_*.

Proof. Using the first inequality in Theorem 2.3, we only need to prove the boundedness of $Tf = (\phi f^\wedge)^\vee$ from $\mathcal{P}_p$ to H^p_{**}.

In order to prove that ϕ is a multiplier it suffices to verify that the operator $Tf = (\phi f^\wedge)^\vee$ is bounded on the set of atoms of $\mathcal{P}_p$. Let a be an atom whose support is a subset of some G_N. We have

$$\int_G \left|\big(T(a)\big)^{**}(x)\right|^p dx = \int_{G_N} \left|\big(T(a)\big)^{**}(x)\right|^p dx + \int_{G_N^c} \left|\big(T(a)\big)^{**}(x)\right|^p dx.$$

Since $\|f^{**}\|_2$ is equivalent to the L^2 norm [6], then the standard L^2 argument in [3] can be used to estimate the first term. Namely,

$$\begin{aligned} \int_{G_N} \left|\big(T(a)\big)^{**}(x)\right|^p dx &= \int \left|\big(T(a)\big)^{**}(x)\right|^p 1_{G_N}(x)\, dx \\ &\leqslant \left\| \big(T(a)\big)^{**} \right\|_2^p \|1_{G_N}\|_2^{2-p} \leqslant C_p \left\| T(a) \right\|_2^p \|1_{G_N}\|_2^{2-p} \leqslant C_p \|\phi\|_\infty^p \|a\|_2^p \big(\mu(G_N)\big)^{1-\frac{p}{2}} \\ &\leqslant C_p \|\phi\|_\infty^p \big(\mu(G_N)\big)^{1-\frac{p}{2}} \big(\mu(G_N)\big)^{\frac{p}{2}-1} = C_p \|\phi\|_\infty^p. \end{aligned}$$

A similar estimate was applied in the proof of [1, Theorem 3.1] using [1, Proposition 2.2]. Here we mention that the assertion in [1, Proposition 2.2] is not valid for the maximal function $\tilde{M}f$. Namely, $\tilde{M}f$ is not bounded in L^2 by Theorem 2.3.

To estimate the second integral, we write $T(a)$ in the form

$$T(a)=\left(\phi a^{\wedge}\right)^{\vee}=\phi^{\vee}*a=\left(\sum_{j=-\infty}^{\infty}\Delta_j\phi\right)^{\vee}*a=\sum_{j=-\infty}^{\infty}(\Delta_j\phi)^{\vee}*a,$$

where the equality holds in the sense of distributions, and $\Delta_j\phi=\phi 1_{\Gamma_{j+1}}-\phi 1_{\Gamma_j}$. As $\int_{G_N}a=0$, it follows that $a^{\wedge}$ vanishes on Γ_N. This means that $(\Delta_j\phi)a^{\wedge}=(\Delta_j\phi)^{\vee}*a\equiv 0$ if $j\leqslant N-1$. Therefore, $T(a)=(\phi-\phi_N)^{\vee}*a$.

It is easily seen that if $x\in G_N^c$, $y\in G_N$, then $\int_{I_n}a(t-y)\,dt=0$, for every interval I_n containing x. This is clearly true when $n<N$, because I_n either contains G_N or does not intersect it. Now if $n\geqslant N$, then $I_n-y\subset x+G_N\subset G_N^c$.

We obtain

$$\begin{aligned}\left(\mu(I)\right)^{-1}\int_I T(a)(t)\,dt&=\left(\mu(I)\right)^{-1}\int_I\left((\phi-\phi_N)^{\vee}*a\right)(t)\,dt=\left(\mu(I)\right)^{-1}\int_I\int(\phi-\phi_N)^{\vee}(y)a(t-y)\,dy\,dt\\&=\int_{G_N^c}(\phi-\phi_N)^{\vee}(y)\left(\mu(I)\right)^{-1}\int_I a(t-y)\,dt\,dy,\end{aligned}$$

for every interval I that contains the point $x\in G_N^c$. Consequently,

$$\begin{aligned}\int_{G_N^c}\left|\left(T(a)\right)^{**}(x)\right|^p dx&=\int_{G_N^c}\sup_{I,x\in I}\left|\left(\mu(I)\right)^{-1}\int_I T(a)(t)\,dt\right|^p dx\\&=\int_{G_N^c}\sup_{I,x\in I}\left|\int_{G_N^c}(\phi-\phi_N)^{\vee}(y)\left(\mu(I)\right)^{-1}\int_I a(t-y)\,dt\,dy\right|^p dx\\&\leqslant\int_{G_N^c}\left(\int_{G_N^c}\left|(\phi-\phi_N)^{\vee}(y)\right|\sup_{I,x\in I}\left(\mu(I)\right)^{-1}\left|\int_I a(t-y)\,dt\right|dy\right)^p dx.\end{aligned}$$

Now, from $\int_{I_n}a(t-y)\,dt=0$, when $n<N$, and $(\mu(I_n))^{-1}|\int_{I_n}a(t-y)\,dt|\leqslant(\mu(G_N))^{-\frac{1}{p}}1_{G_N}(x-y)$, when $n\geqslant N$, we obtain

$$\begin{aligned}&\int_{G_N^c}\left(\int_{G_N^c}\left|(\phi-\phi_N)^{\vee}(y)\right|\sup_{I,x\in I}\left(\mu(I)\right)^{-1}\left|\int_I a(t-y)\,dt\right|dy\right)^p dx\\&\quad\leqslant\left(\mu(G_N)\right)^{-1}\int_{G_N^c}\left(\int_{G_N^c}\left|(\phi-\phi_N)^{\vee}(y)\right|1_{G_N}(x-y)\,dy\right)^p dx\\&\quad\leqslant\int_{G_N^c}\left(\mu(G_N)\right)^{-1}\left(\int_{G_N}\left|(\phi-\phi_N)^{\vee}(x-u)\right|du\right)^p dx=O(1).\qquad\square\end{aligned}$$

We derive two corollaries analogous to Corollaries 5 and 6 proved in [3].

Corollary 3.2. *Let $\phi\in L^{\infty}(\Gamma)$. If*

$$\sup_N\int_{G_N^c}\sum_{j=N+1}^{\infty}\left|(\Delta\phi_j)^{\vee}(x)\right|dx=O(1),$$

then ϕ is a multiplier from $\mathcal{P}_1$ to H^1_{**} and H^1_*.

This is obviously true as

$$\int_{G_N^c}\left(\mu(G_N)\right)^{-1}\int_{G_N}\left|(\phi-\phi_N)^{\vee}(x-u)\right|du\,dx=\int_{G_N^c}\left|(\phi-\phi_N)^{\vee}(x)\right|dx\leqslant\int_{G_N^c}\sum_{j=N+1}^{\infty}\left|(\Delta\phi_j)^{\vee}(x)\right|dx.$$

Corollary 3.3. *Let $\phi \in L^\infty(\Gamma)$ and $0 < p \leqslant 1$. If*

$$\sum_{N=-\infty}^{j} \big(\mu(G_N)\big)^{1-p} \bigg(\int\limits_{G_N \setminus G_{N+1}} \big|(\Delta\phi_j)^\vee(x)\big|\,dx \bigg)^p \leqslant C\big(\mu(G_j)\big)^{1-p},$$

then ϕ is a multiplier from $\mathcal{P}_p$ to H^p_{**} and H^p_*.

Proof. The result is easily established by following the proof of Corollary 6 in [3], since the number U_N mentioned there is of the form $U_N = \int_{G_N^c} (\mu(G_N))^{-1} (\int_{G_N} |(\phi - \phi_N)^\vee(x-u)|\,du)^p\,dx$. $\square$

Theorem 3.4. *Let $\phi \in L^\infty(\Gamma)$ and $0 < p \leqslant 1$. Suppose that $(\phi - \phi_N)^\vee$ is supported on $\{y \in G,\ y_N = 0\}$ and*

$$\sup_N \int\limits_{I_N^c} \big(\mu(I_N)\big)^{-1} \bigg(\int\limits_{I_N} \big|(\phi - \phi_N)^\vee(x-u)\big|\,du \bigg)^p dx = O(1).$$

Then ϕ is a multiplier on H^p_{**}.

Proof. The first steps of this proof are the same as in Theorem 3.1. We easily obtain the estimation

$$\int\limits_{I_N} \big|\big(T(a)\big)^{**}(x)\big|^p\,dx = \int \big|\big(T(a)\big)^{**}(x)\big|^p 1_{I_N}(x)\,dx \leqslant C_p\|\phi\|_\infty^p \big(\mu(I_N)\big)^{1-\frac{p}{2}} \big(\mu(I_N)\big)^{\frac{p}{2}-1} = C_p\|\phi\|_\infty^p.$$

Similarly, $\int_{I_n} a(t-y)\,dt = 0$ if $x \in I_N^c$, $y \in G_{N+1}$ and I_n is an interval containing x. One has

$$\big(\mu(I)\big)^{-1} \int\limits_I T(a)(t)\,dt = \big(\mu(I)\big)^{-1} \int\limits_I \big((\phi - \phi_N)^\vee * a\big)(t)\,dt = \int\limits_{G_{N+1}^c} (\phi - \phi_N)^\vee(y)\big(\mu(I)\big)^{-1} \int\limits_I a(t-y)\,dt\,dy.$$

Now,

$$\begin{aligned}
\int\limits_{I_N^c} \big|\big(T(a)\big)^{**}(x)\big|^p\,dx &= \int\limits_{I_N^c} \sup_{I,x\in I} \bigg|\big(\mu(I)\big)^{-1} \int\limits_I T(a)(t)\,dt\bigg|^p dx \\
&= \int\limits_{I_N^c} \sup_{I,x\in I} \bigg| \int\limits_{G_{N+1}^c} (\phi - \phi_N)^\vee(y)\big(\mu(I)\big)^{-1} \int\limits_I a(t-y)\,dt\,dy\bigg|^p dx \\
&\leqslant \int\limits_{I_N^c} \bigg(\int\limits_{G_{N+1}^c} \big|(\phi - \phi_N)^\vee(y)\big| \sup_{I,x\in I} \big(\mu(I)\big)^{-1} \bigg| \int\limits_I a(t-y)\,dt\bigg|\,dy\bigg)^p dx.
\end{aligned}$$

Consider the intervals I_n that contain the point x. For $n < N$, the term $\int_{I_n} a(t-y)\,dt$ clearly vanishes. If $n > N$ then $(\mu(I_n))^{-1}|\int_{I_n} a(t-y)\,dt|$ is bounded by $(\mu(I_N))^{-\frac{1}{p}}$ and vanishes when $x - y \in I_N^c$. Now if $n = N$, then for $y_N = 0$, $I_n - y$ remains an interval that contains the point $x - y$. Therefore, $(\mu(I_n))^{-1}|\int_{I_n} a(t-y)\,dt|$ is similarly bounded by $(\mu(I_N))^{-\frac{1}{p}} 1_{I_N}(x-y)$. We get

$$\begin{aligned}
&\int\limits_{I_N^c} \bigg(\int\limits_{G_{N+1}^c} \big|(\phi - \phi_N)^\vee(y)\big| \sup_{I,x\in I} \big(\mu(I)\big)^{-1} \bigg| \int\limits_I a(t-y)\,dt\bigg|\,dy\bigg)^p dx \\
&\quad \leqslant \big(\mu(I_N)\big)^{-1} \int\limits_{I_N^c} \bigg(\int\limits_{G_{N+1}^c} \big|(\phi - \phi_N)^\vee(y)\big| 1_{I_N}(x-y)\,dy\bigg)^p dx \\
&\quad \leqslant \int\limits_{I_N^c} \big(\mu(I_N)\big)^{-1} \bigg(\int\limits_{I_N} \big|(\phi - \phi_N)^\vee(x-u)\big|\,du\bigg)^p dx = O(1). \qquad \square
\end{aligned}$$

The next multiplier theorem bears on the form of conditions in related problems of integrability and summability on unbounded Vilenkin groups [2].

Theorem 3.5. *Let G be a compact Vilenkin group. Assume that $\phi \in L^\infty(\Gamma)$ satisfies*

$$\left(\sum_{s=0}^{N-1} m_{s+1}^{\frac{1}{p'}} \log p_{s+1}\right)\left(\sum_{k=m_{N+1}}^{\infty} \left|\Delta\phi(k)\right|^p\right)^{\frac{1}{p}} = O(1),$$

for some $p \in (1,2]$, where $\frac{1}{p'} + \frac{1}{p} = 1$, $\Delta\phi(k) = \phi(k) - \phi(k+1)$, and $(\phi - \phi_N)^\vee$ is supported on $\{y \in G,\ y_N = 0\}$, for each $N \in \mathbb{N}$. Then ϕ is a multiplier on H^1_{**}.

Proof. Let χ_{m_k} denote the element of Γ_{k+1} for which $\chi_{m_k}(x_k) = e^{\frac{2\pi i}{p_{k+1}}}$, and $\chi_n := \prod_{k=0}^s \chi_{m_k}^{a_k}$ if $n = \sum_{k=0}^s a_k m_k$ and $0 \leqslant a_k < p_{k+1}$. Writing explicitly the inverse Fourier transform and using the formula of partial summation, we get

$$\begin{aligned}
\int_{I_N^c} \left|(\phi - \phi_N)^\vee(x)\right| dx &= \int_{I_N^c} \left|\sum_{j=N}^{\infty} \sum_{k=m_j}^{m_{j+1}-1} \phi(k)\chi_k(x)\right| dx \\
&\leqslant \int_{G_N^c} \left|\sum_{j=N}^{\infty} \sum_{k=m_j}^{m_{j+1}-1} \phi(k)\chi_k(x)\right| dx + \int_{G_N \setminus I_N} \left|\sum_{j=N+1}^{\infty} \sum_{k=m_j}^{m_{j+1}-1} \phi(k)\chi_k(x)\right| dx \\
&\quad + \int_{G_N \setminus I_N} \left|\sum_{k=m_N}^{m_{N+1}-1} \phi(k)\chi_k(x)\right| dx \\
&\leqslant \int_{G_N^c} \left|\sum_{j=N}^{\infty} \sum_{k=m_j}^{m_{j+1}-2} \big(\phi(k) - \phi(k+1)\big) D_{k+1}(x)\right| dx \\
&\quad + \int_{G_N \setminus I_N} \left|\sum_{j=N+1}^{\infty} \sum_{k=m_j}^{m_{j+1}-2} \big(\phi(k) - \phi(k+1)\big) D_{k+1}(x)\right| dx \\
&\quad + \int_{G_N \setminus I_N} \left|\sum_{k=m_N}^{m_{N+1}-2} \big(\phi(k) - \phi(k+1)\big) D_{k+1}(x)\right| + \left|\phi(m_N) D_{m_N}(x)\right| dx,
\end{aligned}$$

since $D_{m_j}(x) = D_{m_{j+1}}(x) = 0$ for $j \geqslant N$ and $x \in G_N^c$.

Dealing with the expression above as done in [1], we easily obtain that each term is bounded independently on N. □

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