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Existence and asymptotic properties of solutions of nonlinear multivalued differential inclusions with nonlocal conditions ☆

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ABSTRACT

This paper deals with a nonlocal problem governed by the nonlinear differential inclusions with multivalued perturbations in Banach spaces. First using the approach of geometry of Banach space, Hausdorff metric, the measure of noncompactness and fixed point, existence results are obtained. We have removed from previous papers the crucial restriction on the semigroup. Then we establish the asymptotic properties of integral solutions when $t \to +\infty$. Finally, for illustration, a partial differential equation is worked out.

1. Introduction

In this paper, we investigate the existence of integral solutions for the following nonlinear set-valued differential inclusion with nonlocal initial conditions

$$\begin{cases} u'(t) \in Au(t) + F\big(t, u(t)\big), & 0 < t \leqslant T, \\ u(0) = g(u), \end{cases} \tag{1.1}$$

where $A : D(A) \subseteq X \to X$ is a nonlinear m-dissipative operator which generates a contraction semigroup $S(t)$ and F is weakly upper semicontinuous multifunction with respect to its second variable in a real Banach space X. In the sequel, using the properties of almost nonexpansive curves, we discuss the asymptotic properties of solutions of the above problem when t converges to infinity on $[0, +\infty)$.

The nonlocal problem has been studied by many authors by using various frameworks and techniques. Currently, two directions are studied extensively. One is the semilinear case, i.e., the case in which A is the generator of a C_0-semigroup. In this respect, it should be noted the pioneering work of Byszewski and Lakshmikantham [12,14] who considered f and g satisfy Lipschitz conditions with the special type of g. In this context they studied and obtained the existence and uniqueness of mild solutions for nonlocal semilinear differential equations. Subsequently, as the nonlocal problem can be applied with better effect than the classical initial one, they have been studied extensively in recent years. In [32], Ntouyas and Tsamatos study the case with compactness conditions. Byszewski and Akca [13] establish the existence of solution to functional–differential equation when the semigroup is compact, and g is convex and compact on a given ball. In [21],

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Fu and Ezzinbi study neutral functional–differential equation with nonlocal conditions. Benchohra and Ntouyas [9] discuss second order differential equation under compact conditions.

Another direction is the nonlinear case, i.e., the case in which A is the generator of a nonlinear semigroup, with A nonlinear and m-dissipative. The nonlinear nonlocal problem has been considered for the first time by Aizicovici and Gao [1], where both F and g satisfy Lipschitz conditions. For subsequent developments, since allowing the item of perturbation F to be multi-valued, the nonlinear multi-valued case has been pursued in the literature through the leading work of Deimling [18], Lakshmikantham et al. [28], Bothe [11], Couchouron and Kamenskii [16], Aizicovici and McKibben [3], Benchohra and Ntouyas [9,10], Aizicovici and Lee [2], Xue [38], Paicu and Vrabie [34] and many others. Later, this study has been a subject of intensive research in many papers [4,22,29,40,41] because of its wide applicability in optimal control and feedback stabilizations and so on. For general or more advanced information we refer to [33,35,36,38–41] and references therein.

For the nonlocal nonlinear problem, many authors supposed that the semigroup $S(t)$ generated by A is compact and the multi-valued perturbation F is upper semicontinuous or lower semicontinuous, which can be found in many previous articles. However, the above assumptions are stronger restrictions on m-accretive operator A and the perturbed item F, which are not satisfied usually in many practical problems. Thus the problem arises whether or not there exist the integral solutions in the nonlinear case when $S(t)$ loses the compactness and F loses the strong semi-continuity.

Here we investigate the nonlinear differential inclusion described by system (1.1) without the assumptions of the compactness of the semigroup and the strong semi-continuity of the multi-valued perturbation. Actually, we need the equi-continuity of $S(t)$ and the weakly upper semi-continuity of F at the most. The main difficulties in this article are due to the nonlinearity of m-accretive operator A. In order to overcome them, we prove the integral inequality about the integral solution and the perturbed item for the fist time for the nonlinear nonlocal problem in nonseparable Banach space, which is very important for the nonlinear problem as well as for the semilinear problem. As a matter of fact, it should be mentioned that our main results essentially improve and extend some known ones in this area.

The paper is divided into six sections, the second one being merely concerned with some necessary background material. In Section 3, we give the existence result when $S(t)$ is equicontinuous, i.e. Theorem 3.1, while in Section 4, we give the existence result when $S(t)$ is not compact and not equicontinuous, i.e. Theorem 4.1. Section 5 studies the asymptotic properties of integral solutions, which extend some known results to the case that the given initial values are replaced by the nonlocal initial conditions. Finally, in Section 6, an example is given to illustrate our Theorem 3.1.

2. Preliminaries

Let X be a real Banach space with the norm $\|\cdot\|$ and X^* be its dual. We denote weak convergence in X by "$\rightharpoonup$". The duality multivalued mapping $J : X \to X^*$ is defined by

$$J(x) = \left\{x^* \in X^*\colon\ x^*(x) = \|x\|^2 = \left\|x^*\right\|^2\right\}, \quad \forall x \in X.$$

At first, we introduce the following concepts about the convexity of Banach space X.

Banach space X is said to be strictly convex whenever $S(X) = \{x \in X\colon\ \|x\| = 1\}$ contains no non-trivial line segments, i.e., each point of $S(X)$ is an extreme point of $B(X) = \{x \in X\colon\ \|x\| \leqslant 1\}$. Banach space X is said to be uniformly convex whenever given $\varepsilon > 0$ there exists $\delta > 0$ such that if $x, y \in S(X)$ and $\|x - y\| \geqslant \varepsilon$ then $\|\frac{x+y}{2}\| \leqslant 1 - \delta$. X^* is uniformly convex if and only if X is uniformly smooth.

Now we recall some basic facts about the geometry of Banach spaces.

i) If X is uniformly smooth, then J is single-valued and uniformly continuous on bounded subsets of X.
ii) If X is reflexive and strictly convex, every nonempty closed convex subset K of X is a Chebyshev set (see [23]). In the case, we denote by P_K the nearest point projection map from X onto K.
iii) The norm of X is Fréchet differentiable if for each $x \in S(X)$, $\lim_{t\to 0} \frac{\|x+ty\|-\|x\|}{t}$ exists uniformly for $y \in S$.
iv) The dual space X^* has Fréchet differentiable norm if and only if X is reflexive, strictly convex and satisfies the following property: If $x_n \rightharpoonup x$ as $n \to +\infty$ and $\|x_n\| \to \|x\|$ as $n \to +\infty$, then $\|x_n - x\| \to 0$ as $n \to +\infty$. That is, X has Kadec–Klee norm (see [20]).

In the following, $C([0,T]; X)$ is the space of continuous functions from $[0,T]$ to X endowed with the supremum norm $\|u\|_\infty = \sup\{\|u(t)\|\colon\ t \in [0,T]\}$ and $L^1([0,T]; X)$ is the space of X-valued Bochner integrable functions from $[0,T]$ to X endowed with the norm $\|u\|_1 = \int_0^T \|u(t)\|\,dt$, where $T > 0$. Denote by $P(X)$ the family of all nonempty subsets of X. Define $P_b(X) = \{B \subset X$ and B is nonempty and bounded$\}$ and $P_f(X) = \{B \subset X$ and B is nonempty and closed$\}$. By $P_{bf}(X)$ (resp., $P_{fc}(X)$) we mean that all nonempty, bounded and closed (resp., nonempty, closed and convex) subsets of X. Define Hausdorff measure of noncompactness $\beta : P_b(X) \to R^+$ by

$$\beta(B) = \inf\{r > 0\colon\ B \text{ can be covered by finite number of balls with radio } r\}.$$

Let $A, B \in P_{bf}(X)$ and let $x \in A$. Then by

$$d(x, B) = \inf\{d(x, y):\ y \in B\}$$

and

$$\rho(A, B) = \sup\{d(x, B):\ x \in A\}.$$

The function $H : P_{bf}(X) \times P_{bf}(X) \to R^+$ defined by

$$H(A, B) = \max\{\rho(A, B), \rho(B, A)\}$$

is a metric and is called the Hausdorff metric on X.

Definition 2.1. (See [10] or [15].) A multivalued map $\mathcal{F} : [0, T] \to P_f(X)$ is said to measurable, if $d(x, \mathcal{F}(.))$ is measurable for every $x \in X$.

It is clear that if $\mathcal{F}$ is measurable on separable Banach space X, then $\mathcal{F}$ admits a measurable selection (see [10]).

Definition 2.2. (See [11].) $\mathcal{F} : [0, T] \to P_f(X)$ is said to be upper semicontinuous on $[0, T]$, usc for short, if $\mathcal{F}^{-1}(V) := \{t \in [0, T]:\ \mathcal{F}(t) \cap V \neq \emptyset\}$ is closed on $[0, T]$ whenever $V \subset X$ is closed. $\mathcal{F}$ is called weakly usc, if $\mathcal{F}^{-1}(V)$ is closed for all weakly closed $V \subset X$.

Definition 2.3. (See [10].) A multivalued operator $\mathcal{F} : X \to P_f(X)$ is called

(1) γ-Lipschitz if and only if there exists $\gamma > 0$ such that

$$H(\mathcal{F}(x), \mathcal{F}(y)) \leqslant \gamma d(x, y), \quad \text{for each } x, y \in X,$$

(2) contraction if and only if it is γ-Lipschitz with $\gamma < 1$,
(3) $\mathcal{F}$ has a fixed point if there is $x \in X$ such that $x \in \mathcal{F}(x)$. The fixed point set of the multivalued operator $\mathcal{F}$ will be denoted by *Fix* $\mathcal{F}$.

Definition 2.4. (See [8].) A multivalued mapping A with domain $D(A)$ is said to be dissipative if $\|x_1 - x_2\| \leqslant \|x_1 - x_2 - \lambda(y_1 - y_2)\|$ for all $\lambda > 0$, $[x_i, y_i] \in A$, $i = 1, 2$. If also the range $R(I - A) = X$, then A is said to be m-dissipative.

By [17], if A is m-dissipative, then A generates a contraction semigroup $\{S(t):\ t \geqslant 0\}$ on $\overline{D(A)}$. The semigroup $\{S(t):\ t \geqslant 0\}$ is said to be equicontinuous if $\{S(t)x:\ x \in A\}$ is equicontinuous at any $t > 0$ for any bounded subset $A \subset X$.

Let A be m-dissipative. For given $x_0 \in \overline{D(A)}$ and $f \in L^1([0, T]; X)$, let us consider the initial value problem:

$$\begin{cases} u'(t) \in Au(t) + f(t), & t \in [0, T], \\ u(0) = x_0. \end{cases} \tag{2.1}$$

Definition 2.5. A function $u : [0, T] \to X$ is called an integral solution of (2.1) on $[0, T]$ if $u \in C([0, T]; X)$ with $u(0) = x_0$ and the inequality

$$\|u(t) - x\|^2 \leqslant \|u(s) - x\|^2 + 2\int_s^t \langle u(\tau) - x, f(\tau) + y\rangle_s \, d\tau$$

holding for all $[x, y] \in A$ and $0 \leqslant s \leqslant t \leqslant T$. Here the function $\langle\ ,\ \rangle_s : X \times X \to R$ is defined by $\langle x, y\rangle_s = \sup\{x^*(y):\ x^* \in J(x)\}$. Furthermore, we denote u by $u = K_{x_0} f$, where K_{x_0} is from $L^1([0, T]; X)$ to $C([0, T]; X)$.

Definition 2.6. A continuous function $u(t)$ is said to be an integral solution to (1.1) if there exists $f \in L^1([0, T]; X)$ with $f(t) \in F(t, u(t))$, a.e. on $[0, T]$ such that u is an integral solution of (2.1) in the sense of Definition 2.5 with $u_0 = g(u)$.

Lemma 2.1. *(See [28].) Let X be a Banach space and let $A : D(A) \subseteq X \to X$ be m-dissipative. Then for each $x_0 \in \overline{D(A)}$ and $f \in L^1([0, T]; X)$, there exists a unique integral solution of* (2.1) *on* $[0, T]$ *which satisfies* $u(0) = x_0$.

Lemma 2.2. *(See [28].) Let X be a Banach space and let $A : D(A) \subseteq X \to X$ be m-dissipative. If $f_1, f_2 \in L^1([0,T]; X)$ and u, v are the integral solutions of* (2.1) *corresponding to f_1 and f_2, respectively. Then the inequalities*

$$\|u(t) - v(t)\| \leqslant \|u(s) - v(s)\| + \int_s^t \|f_1(\tau) - f_2(\tau)\|\, d\tau, \tag{2.2}$$

$$\|u(t) - v(t)\|^2 \leqslant \|u(s) - v(s)\|^2 + 2\int_s^t \langle u(\tau) - v(\tau), f_1(\tau) - f_2(\tau)\rangle_s\, d\tau \tag{2.3}$$

hold for $0 \leqslant s \leqslant t \leqslant T$.

3. $S(t)$ is equicontinuous

In this section, X^* is supposed to be uniformly convex. We consider the nonlinear nonlocal multivalued problem (1.1) under the following assumptions:

(H_A) Let A is m-dissipative such that A generates an equicontinuous semigroup $\{S(t)\colon t > 0\}$ on $\overline{D(A)}$.
(H_g) $g : C([0,T]; X) \to X$ satisfies:
(1) g is continuous and compact;
(2) there exist constants a, b such that for all $u \in C([0,T]; X)$, $\|g(u)\| \leqslant a\|u\|_\infty + b$.
(H_F) $F : [0,T] \times X \to P_{fc}(X)$ satisfies:
(1) for every $x \in X$, the multifunction $F(\cdot, x)$ is measurable;
(2) for a.e. $t \in [0,T]$, the multifunction $F(t, \cdot)$ is weakly *usc*;
(3) there exist $\alpha, \eta \in L^1([0,T]; R^+)$ such that

$$|F(t,x)| := \sup\{\|y\|\colon\ y \in F(t,x)\} \leqslant \alpha(t)\|x\| + \eta(t)$$

for a.e. $t \in [0,T]$ and $x \in X$;
(4) there exists $\mu \in L^1([0,T]; R^+)$ such that $\beta(F(t,B)) \leqslant \mu(t)\beta(B)$ for a.e. $t \in [0,T]$ and every bounded subset $B \subset X$.

Remark 3.1. Under the conditions (H_F)(1)–(3), for each $u \in C([0,T]; X)$,

$$Sel(u) := \{f \in L^1([0,T]; X)\colon\ f(t) \in F(t, u(t)) \text{ a.e. on } [0,T]\}$$

is a nonempty, closed and convex subset of $L^1([0,T]; X)$ (see [4], Lemma 6).

In the sequel, we need some auxiliary statements.
Given $\emptyset \neq \Omega \subset X$, let $\beta_\Omega(B)$ be defined by

$$\beta_\Omega(B) = \inf\left\{r > 0\colon\ B \subset \bigcup_{i=1}^m B_r(x_i) \text{ for some } m \geqslant 1 \text{ and } x_1, \ldots, x_m \in \Omega\right\}$$

for bounded $B \subset \Omega$, i.e., the centers of the covering balls are chosen from Ω instead of X. Then $\beta(B) \leqslant \beta_\Omega(B) \leqslant 2\beta(B)$ for all bounded $B \subset \Omega$.

Moreover, we has the following representation about the Hausdorff measure of noncompactness:

Lemma 3.1. *(See [11] or [38].) Let Y be a separable Banach space and $\{Y_m\}_{m\geqslant 1}$ is an increasing sequence of finite dimensional subspaces such that $Y = \overline{\bigcup_{m=1}^\infty Y_m}$. Then*

$$\beta(A) = \lim_{m\to\infty} \limsup_{k\to\infty} d(x_k, Y_m),$$

for any bounded set $A = \{x_k\colon\ k \geqslant 1\} \subset Y$.

The following lemma is obtained in [37].

Lemma 3.2. *If X is uniformly smooth, $Y \subset X$ is a finite dimensional subspace, A satisfies* (H_A) *and $B \subset L^1([0,T]; Y)$ is uniformly integrable, then $KB \subset C([0,T]; X)$ is precompact.*

The forthcoming lemma plays a key role in the proof of our main theorem in this section.

Lemma 3.3. *If X^* is uniformly convex and A satisfies (H_A), then for any uniformly integrable sequence $\{w_k\}_{k=1}^{\infty} \subset L^1([0,T];X)$ and relatively compact subset $\{x_k\}_{k=1}^{\infty} \subset \overline{D(A)}$, we have*

$$\beta\big(\big\{(K_{x_k} w_k)(t)\colon k \geqslant 1\big\}\big) \leqslant \int_0^t \beta\big(\big\{w_k(s)\colon k \geqslant 1\big\}\big)\,ds, \quad t \in [0,T].$$

Proof. Since $w_k \in L^1([0,T];X)$ is strongly measurable for each $k \geqslant 1$, we may assume that $X_0 = \overline{span}(\bigcup_{k=1}^{\infty} w_k([0,T]))$ is separable. Since X is in particular reflexive, by Theorem V.2.3 in [19], there is a closed separable subspace Y of X, containing X_0, and a linear continuous projection $\mathcal{P}$ from X onto Y with $\|\mathcal{P}\| = 1$. For bounded $B \subset Y$, we therefore have $\beta(B) = \beta_Y(B)$. Since Y is separable, there exists an increasing sequence $\{Y_m\}_{m=1}^{\infty}$ of finite dimensional subspaces such that $Y = \overline{\bigcup_{m=1}^{\infty} Y_m}$. Define $\mathcal{P}_m : Y \to Y_m$ by $\mathcal{P}_m x = \{y \in Y_m\colon \|x-y\| = d(x, Y_m)\}$, which is compact and convex and $\|\mathcal{P}_m x\| \leqslant 2\|x\|$ for any $x \in Y$. For $w(t) \in L^1([0,T];Y)$,

$$\mathcal{P}_m w(t) = Y_m \cap \big(w(t) + d\big(w(t), Y_m\big) S(Y)\big). \tag{3.1}$$

It implies that

$$S^1_{\mathcal{P}_m w(.)} = \big\{v \in L^1\big([0,T];\ Y_m\big),\ v(t) \in \mathcal{P}_m w(t) \text{ a.e.}\big\} \neq \emptyset.$$

Define $\mathcal{Q}_m : L^1([0,T];Y) \to P(L^1([0,T];Y_m))$ by $\mathcal{Q}_m w = S^1_{\mathcal{P}_m w(.)}$.

From (3.1) and the inequality (2.2) in Lemma 2.2, we obtain

$$d\big((K_{x_k} w_k)(t), (K_{x_k} \mathcal{Q}_m w_k)(t)\big) \leqslant \int_0^t d\big(w_k(s), Y_m\big)\,ds$$

for any $k \geqslant 1$. Since $\{K_{x_k} w_k\colon k < n\} \subset C([0,T];X)$ is compact, we have

$$\beta\big(\big\{(K_{x_k} w_k)(t)\colon k \geqslant 1\big\}\big) = \beta\big(\{K_{x_k} w_k\colon k \geqslant n\}\big).$$

By Lemma 3.2 and Theorem 2.1 in [7], it follows that

$$\beta\big(\big\{(K_{x_k} w_k)(t)\colon k \geqslant 1\big\}\big) \leqslant \rho\big(\big\{K_{x_k} w_k(t)\colon\ k \geqslant n\big\}, \big\{\big(K_{x_k}(Q_m w_k)\big)(t)\colon k \geqslant 1\big\}\big) \leqslant \sup\left\{\int_0^t d\big(w_k(s), Y_m\big)\,ds,\ k \geqslant n\right\}.$$

Let $n, m \to \infty$, Lemma 3.1 applies and therefore

$$\beta\big(\big\{(K_{x_k} w_k)(t)\colon k \geqslant 1\big\}\big) \leqslant \int_0^t \beta_Y\big(\big\{w_k(s)\colon k \geqslant 1\big\}\big)\,ds = \int_0^t \beta\big(\big\{w_k(s)\colon k \geqslant 1\big\}\big)\,ds.$$

This completes the proof. □

The following multivalued fixed point theorem can be found in [11, Lemma 1].

Lemma 3.4. *Let X be a Banach space, $\emptyset \neq D \subset X$ compact convex and $F : D \to P(D)$ usc with closed contractible values. Then F has a fixed point.*

The next statement follows from Theorem 2.1 of [39].

Lemma 3.5. *If A generates an equicontinuous semigroup $S(t)$, $B \in L^1([0,T];X)$ is uniformly integrable and $C \subset \overline{D(A)}$ is compact, then the set $\Pi = \{u\colon u$ is the integral solutions of (2.1) for some $f \in B$ and some $u_0 \in C\}$ is bounded and equicontinuous in $C([0,T];X)$.*

To discuss the existence we also need the following lemmas.

Lemma 3.6. *(See [11], Proposition 2.) Let X be a Banach space, $\Omega \neq \emptyset$ a subset of another Banach space and $F : \Omega \to P(X)$ has weakly compact values. Then the following conclusion holds: If the values of F are also convex, then F is weakly usc if and only if $\{x_n\} \subset \Omega$ with $x_n \to x_0 \in \Omega$ and $y_n \in F(x_n)$ implies $y_{n_k} \rightharpoonup y_0 \in F(x_0)$ for some subsequence $\{y_{n_k}\}$ of $\{y_n\}$.*

Lemma 3.7. *(See [11], Lemma 2.) Let X be a Banach space and let $W \subset L^1([0,T];X)$ be uniformly integrable. Suppose that there exist weakly relatively compact sets $C(t) \subset X$ such that $f(t) \in C(t)$ a.e. on $[0,T]$, for all $f \in W$. Then W is weakly relatively compact in $L^1([0,T];X)$.*

Lemma 3.8. *(See [37], Lemma 2.3.2.) Let X be a real Banach space whose topological dual is uniformly convex, and let $\{u_n\}$, $\{v_k\}$ be two sequences in $C([a,b];X)$, and $\{f_n\}$, $\{f'_k\}$ two sequences in $L^1([a,b];X)$. If $\lim_{n\to+\infty} u_n = u$, $\lim_{k\to+\infty} v_k = v$ strongly in $C([a,b];X)$, and $\lim_{n\to+\infty} f_n = f$, $\lim_{k\to+\infty} f'_k = f'$ weakly in $L^1([a,b];X)$, then*

$$\lim_{n,k\to+\infty} \int_a^b \big\langle u_n(s) - v_k(s), f_n(s) - f'_k(s)\big\rangle_s \, ds = \int_a^b \big\langle u(s) - v(s), f(s) - f'(s)\big\rangle_s \, ds.$$

Our main existence result is the following:

Theorem 3.1. *If the hypotheses (H_A), (H_g)(1)–(2) and (H_F)(1)–(4) are true. Then the nonlocal multivalued problem (1.1) has at least one integral solution provided that $a + \|\alpha\|_1 < 1$, where $\|\alpha\|_1 = \int_0^T \alpha(t)\,dt$.*

Proof of Theorem 3.1. Let $G : C([0,T];X) \to C([0,T];X)$ be defined by

$$G(v) = \big\{u \in C\big([0,T];X\big):\ u \text{ is the unique integral solution of (2.1) for some } f \in Sel(v) \text{ and } u(0) = g(v)\big\}.$$

That is, $G(v) = \{K_{g(v)} f:\ f \in Sel(v)\}$.

For any fixed $[x,y] \in A$ and any $u \in G(v)$,

$$\big\|u(t)\big\| \leqslant a\|v\| + b + 2\|x\| + T\|y\| + \|\eta\|_1 + \int_0^T \alpha(s)\big\|v(s)\big\|\,ds$$

for $t \in [0,T]$. As $a + \|\alpha\|_1 < 1$, put

$$r = \frac{2\|x\| + b + T\|y\| + \|\eta\|_1}{1 - a - \|\alpha\|_1},$$

where $\|\eta\|_1 = \int_0^T \eta(s)\,ds$. Then $\|u\|_\infty \leqslant r$ when $\|v\|_\infty \leqslant r$.

Define $W_0 = \{u \in C([0,T];X):\ \|u\|_\infty \leqslant r\}$ and $W_1 = \overline{conv}(GW_0)$, where $\overline{conv}$ means the closure of the convex hull in $C([0,T];X)$, then $W_1 \subset W_0$. As g is compact, $\{S(t):\ t > 0\}$ is equicontinuous and (H_F) is true, we know that W_1 is equicontinuous by Lemma 3.5.

Moreover, we define by $W_{n+1} = \overline{conv}(GW_n)$, for $n = 1, 2, \ldots,$ then we obtain that $W_{n+1} \subset W_n$ for $n = 1, 2, \ldots$ as $W_1 \subset W_0$. It is obvious that $\{W_n\}_{n=1}^{\infty}$ is a decreasing sequence of bounded closed convex equicontinuous subsets of $W_0 \subset C([0,T];X)$. By the inequality (2), p. 673 of [6], then for any $\varepsilon > 0$, there exist sequences $\{u_k\}_{k=1}^{+\infty} \subset W_n$ and $\{f_k\}_{k=1}^{+\infty} \subset L^1([0,T];X)$ such that $f_k \in Sel(u_k)$ for all $k \geqslant 1$ and

$$\beta\big(W_{n+1}(t)\big) = \beta\big(\big\{K_{g(u)} f:\ f \in Sel(u),\ u \in W_n\big\}\big) \leqslant 2\beta\big(\{K_{g(u_k)} f_k:\ k \geqslant 1\}\big) + \varepsilon.$$

As g is a compact mapping, it implies by Lemma 3.3 that

$$\beta\big(W_{n+1}(t)\big) \leqslant 2\int_0^t \beta\big(\big\{f_k(s)\,ds:\ k \geqslant 1\big\}\big)\,ds + \varepsilon \leqslant 2\int_0^t \beta\big(F\big(s, W_n(s)\big)\big)\,ds + \varepsilon \leqslant 2\int_0^t \mu(s)\beta\big(W_n(s)\big)\,ds + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we have

$$\beta\big(W_{n+1}(t)\big) \leqslant 2\int_0^t \mu(s)\beta\big(W_n(s)\big)\,ds.$$

Let $n \to \infty$, it follows that

$$\lim_{n\to\infty} \beta\big(W_n(t)\big) \leqslant 2\int_0^t \mu(s)\Big(\lim_{n\to\infty} \beta\big(W_n(s)\big)\Big)\,ds, \quad \text{for } t \in [0,T].$$

From the Gronwall's inequality, this yields that

$$\lim_{n\to\infty} \beta\big(W_n(t)\big) \equiv 0 \quad \text{for all } t \in [0,T].$$

Moreover, we know that $\{W_n\}_{n\geqslant 0}$ is a decreasing sequence of bounded and equicontinuous subsets of $C([0,T];X)$. This means that $\lim_{n\to\infty} \beta_c(W_n) = 0$, where β_c means the Hausdorff measure of $C([0,T];X)$. By Chapter 2 in [7], we obtain that $W = \bigcap_{n\geqslant 0} W_n$ is nonempty, convex and compact in $C([0,T];X)$ and $G(W) \subseteq W$.

Now let us check that $graph(G)$ is closed. The proof is divided into three steps.

First, let $v_n \subset W$ with $v_n \to v$ in $C([0,T];X)$ and $u_n \in G(v_n)$ with $u_n \to u$ in $C([0,T];X)$, and let $\{f_n\}_{n=1}^{\infty} \subset L^1([0,T];X)$ be a sequence satisfying $f_n \in Sel(v_n)$ for $n \geqslant 1$ and $u_n = K_{g(v_n)} f_n$. By $(\mathrm{H}_F)(3)$, $\{f_n\}_{n=1}^{\infty} \subset L^1([0,T];X)$ is uniformly integrable. Moreover, f_n satisfy $f_n(t) \in C(t) := F(t, \overline{\{v_n(t)\colon n \geqslant 1\}})$. Since X^* is uniformly convex, we know that X is reflexive, hence F has weakly compact values and the set $C(t)$ is weakly compact for every $t \in [0,T]$ due to Lemma 3.6. Therefore, we may assume that $f_n \rightharpoonup f$ in $L^1([0,T];X)$ by Lemma 3.7.

Subsequently, we prove that $f \in Sel(v)$. Indeed, there are $\overline{f}_n \in conv\{f_k\colon k \geqslant n\}$ such that $\overline{f}_n \to f$ in $L^1([0,T];X)$ by Mazur's theorem. Hence, for some subsequence $\{\overline{f}_{n_k}\}$, $\overline{f}_{n_k}(t) \to f(t)$ a.e. on $[0,T]$. Let $t \in [0,T]$ satisfy that $f_n(t) \in F(t, v_n(t))$ for all $n \geqslant 1$ and $\overline{f}_{n_k}(t) \to f(t)$. For given $x^* \in X^*$, $x^* \circ F(t,\cdot)$ is usc with compact convex values. Hence for any $\varepsilon > 0$, we have $x^*(f_n(t)) \in x^*(F(t, v(t))) + (-\varepsilon, \varepsilon)$ for all large n. Furthermore, the same inclusion holds for $x^*(\overline{f}_{n_k}(t))$ for all large k. Since F has closed convex values, we have $x^*(f(t)) \in x^*(F(t, v(t)))$ for all $x^* \in X^*$, which implies $f(t) \in F(t, v(t))$ a.e. on $[0,T]$.

Thirdly, as g is continuous, it follows that $g(v_n) \to g(v)$. By (2.3) in Lemma 2.2, we have

$$\begin{aligned}\big\|u_n(t) - (K_{g(v)} f)(t)\big\|^2 &= \big\|(K_{g(v_n)} f_n)(t) - (K_{g(v)} f)(t)\big\|^2 \\ &\leqslant \big\|g(v_n) - g(v)\big\|^2 + 2\int_0^t \big\langle u_n(\tau) - (K_{g(v)} f)(\tau), f_n(\tau) - f(\tau)\big\rangle_s \, d\tau.\end{aligned}$$

Therefore, let $n \to \infty$, the above inequality in conjunction with Lemma 3.8 shows that $u = K_{g(v)} f$ with $f \in Sel(v)$. That is, $u \in G(v)$. Hence, the multioperator G is usc on W.

Next we shall show that G has contractible values. Let $C = G(v)$ for some $v \in M$, fix $\overline{f} \in Sel(v)$ and define $h : [0,1] \times C \to C$ by

$$h(s,u)(t) = \begin{cases} u(t), & \text{if } t \in [0, sT], \\ \bar{u}(t; sT, u(sT)), & \text{if } t \in [sT, T], \end{cases}$$

where $\bar{u}(t; t_0, x_0)$ is the solution of $w'(t) \in Aw(t) + \overline{f}(t)$ on $[t_0, T]$, $w(t_0) = x_0$. Since $u = K_{g(v)} f$ for some $f \in Sel(v)$, we have $h(s,u) = K_{g(v)} \widehat{f}$ with $\widehat{f} := f\chi_{[0,sT]} + \overline{f}\chi_{[sT,T]} \in Sel(v)$, hence h maps into C. Moreover, h is continuous due to the continuous dependence of $\bar{u}(t; t_0, x_0)$ on the initial condition $(t_0, x_0) \in [0,T] \times \overline{D(A)}$, and $h(0,u) = K_{g(v)} \overline{f}$, $h(1,u) = K_{g(v)} f = u$.

Finally, an appeal to Lemma 3.4 shows that the multioperator G has a fixed point $u \in C([0,T];X)$. Obviously, the fixed point u is also an integral solution for the problem (1.1). This completes the proof. □

Remark 3.2. For the nonlinear nonlocal case, Lemma 3.3 is new in the nonseparable Banach space, which is also very crucial to our proof of Theorem 3.1. Since the valid of the conclusion of Lemma 3.3, we do not assume the compactness of the semigroup and the separability of Banach space X in Theorem 3.1. Therefore our result essentially extends and improves some known results in [4,11,38–40]. We emphasize that the condition that X is uniformly convex is essential in order to conclude the existence of integral solution. But nevertheless, in the case when A is densely defined and linear, the condition of uniform convexity on the topological dual of X may be discarded.

4. $S(t)$ is not compact and not equicontinuous

In this section we always suppose X is separable, we focus our attention on another case that $S(t)$ is not compact and not equicontinuous, which is of great interest in the theory of the nonlocal differential inclusions. We are now prepared to formulate the main result.

Theorem 4.1. *Let X^* be uniformly convex. Assume the conditions $(\mathrm{H}_F)(1)$–(2) and the following conditions (H'_g) and (H_1) are satisfied:*

(H'_g) *$g : C([0,T];X) \to \overline{D(A)}$ is Lipschitz continuous with constant k;*
(H_1) *there exist $p(t), q(t) \in L^1([0,T];R^+)$ such that $H(F(t,x), F(t,y)) \leqslant p(t)\|x-y\|$ and $\|F(t,x)\| \leqslant q(t)(1+\|x\|)$ for each $t \in [0,T]$ and $x, y \in X$.*

Then the nonlocal multivalued problem (1.1) has at least one integral solution on $[0,T]$ provided that $k + Q < 1$, where $Q = \int_0^T p(s)\,ds$.

We shall carry out the proof of Theorem 4.1 with the help of the following fixed point theorem for contraction multivalued operators.

Lemma 4.1. *(See [18].) Let (X,d) be a complete metric space. If $N : X \to P_f(X)$ is a contraction, then $\mathit{Fix}\, N \neq \emptyset$.*

Proof of Theorem 4.1. Let us define $N : C([0,T];X) \to C([0,T];X)$ by

$$Nv = \big\{ y \in C\big([0,T];X\big) \colon\ y \text{ is the integral solution of (2.1) with } u(0) = g(v) \text{ and } f \in \mathit{Sel}(v) \big\}.$$

It is evident that the fixed points of N are the integral solutions of the nonlocal multivalued problem (1.1). Consequently, it suffices to show that there exists a point $u \in C([0,T];X)$ such that $u \in \mathit{Fix}\, N$. To this aim, let us observe first that, by $(\mathrm{H}_F)(1)$ and the second part of (H_1), for each $v \in C([0,T];X)$, $\mathit{Sel}(v) \neq \emptyset$ since F has a measurable selection (see [15], Theorem III.6). In addition, using similar arguments with the proof of Theorem 3.1, we can easily deduce that the multioperator N defined as above has closed values.

Next, we prove that N is a contraction.

Let $v_1, v_2 \in C([0,T];X)$ and $u_1 \in N(v_1)$. Then there exists $f_1(t) \in F(t, v_1(t))$ such that u_1 is the integral solution of (2.1) on $[0,T]$ with $u_1(0) = g(v_1)$ and $f = f_1$.

It follows from (H_1) that

$$H\big(F\big(t, v_1(t)\big), F\big(t, v_2(t)\big)\big) \leqslant p(t)\big\|v_1(t) - v_2(t)\big\|.$$

Hence there is $z \in F(t, v_2(t))$ such that

$$\big\|f_1(t) - z\big\| \leqslant p(t)\big\|v_1(t) - v_2(t)\big\|, \quad t \in [0,T].$$

At this point, let us now define the operator $\phi : [0,T] \to P(X)$ by

$$\phi(t) = \big\{ z \in X \colon\ \big\|f_1(t) - z\big\| \leqslant p(t)\big\|v_1(t) - v_2(t)\big\| \big\}.$$

Bearing in mind the above, we can write

$$\phi(t) = f_1(t) + p(t)\big\|v_1(t) - v_2(t)\big\| S(X).$$

Obviously ϕ is measurable with closed values. In addition, $F(t, v_2(t))$ is also measurable with closed values (see [4]). Thus, the multivalued operator $\psi(t) = \phi(t) \bigcap F(t, v_2(t))$ is measurable with closed values, and hence there exists $f_2(t)$ a measurable selection for ψ (see [30]) such that

$$\big\|f_1(t) - f_2(t)\big\| \leqslant p(t)\big\|v_1(t) - v_2(t)\big\|, \quad \text{for each } t \in [0,T]. \tag{4.1}$$

Let $u_2 \in C([0,T];X)$ be the unique integral solution of (2.1) with $u_2(0) = g(v_2)$ and $f = f_2$. Then, by virtue of the inequality (2.2) in Lemma 2.2, we have

$$\big\|u_1(t) - u_2(t)\big\| \leqslant \big\|g(v_1) - g(v_2)\big\| + \int_0^t \big\|f_1(s) - f_2(s)\big\|\, ds.$$

Now, (4.1) in conjunction with (H_g') yields

$$\begin{aligned}
\big\|u_1(t) - u_2(t)\big\| &\leqslant k\|v_1 - v_2\|_\infty + \int_0^t p(s)\big\|v_1(s) - v_2(s)\big\|\, ds \\
&\leqslant \left(k + \int_0^t p(s)\, ds\right)\|v_1 - v_2\|_\infty \\
&\leqslant (k + Q)\|v_1 - v_2\|_\infty.
\end{aligned}$$

Then

$$\|u_1 - u_2\|_\infty \leqslant (k + Q)\|v_1 - v_2\|_\infty.$$

By the analogous relation, obtained by interchanging roles of v_1, v_2, it follows that

$$H\big(N(v_1), N(v_2)\big) \leqslant (k + Q)\|v_1 - v_2\|_\infty.$$

Hence, if $k + Q < 1$, then N is a contraction. From Lemma 4.1, we conclude that N has a fixed point $u \in C([0,T];X)$ which is an integral solution of (1.1), thereby completing the proof. □

Remark 4.1. In many previous papers, such as [3,4,11,32,38–40], the authors obtain the existence results of mild or integral solutions for corresponding differential equations under the assumption of compactness condition or the equicontinuous condition on the semigroup. However, we apply completely different method to deal with the differential equation, which is inspired by Benchohra and Ntouyas. This enables us to obtain the integral solutions for differential system (1.1) without the strong restriction on the semigroup.

Remark 4.2. The above result obtained on the compact interval $[0, T]$ can be extended to the infinite interval $[0, +\infty)$ by using the same fixed point theorem, if the Banach space of continuous functions $v : [0, +\infty) \to X$ such that $v(t)$ is bounded on $[0, +\infty)$, the norm on the space is given by $\|v\| = \sup_{t\in[0,+\infty)} \|v(t)\|$.

5. Asymptotic properties of integral solutions

The main objective of the section is to prove some new results concerning the following nonlinear set-valued perturbed differential inclusions:

$$\begin{cases} u'(t) \in Au(t) + F\big(t, u(t)\big), & t \in [0, +\infty), \\ u(0) = g(u), \end{cases} \tag{5.1}$$

where A is a dissipative operator in X. We study the asymptotic properties of not necessarily bounded integral solution $\{u(t)\}_{t\geqslant 0}$ of (5.1) (if any) under suitable conditions. Our results improve and extend some known results (see [26,35,36]).

In order to obtain some results concerning asymptotic behavior, we first introduce the following definitions (see [35]).

Definition 5.1. A curve $\{u(t)\}_{t\geqslant 0}$ in X is a continuous function u from $[0, +\infty)$ into X, i.e. $u \in C([0, +\infty); X)$.

Definition 5.2. A curve $\{u(t)\}_{t\geqslant 0}$ in X is said to be almost nonexpansive if there exists $\delta > 0$ such that

$$\big\|u(t+h) - u(s+h)\big\| \leqslant \big\|u(t) - u(s)\big\| + \varepsilon(s,t)$$

for each $s, t \geqslant 0$ and $h \in [0, \delta]$, where $\{\varepsilon(s,t)\}_{s,t\geqslant 0}$ is bounded and $\lim_{s,t\to+\infty} \varepsilon(s,t) = 0$. (Note that in this definition $h \in [0, \delta]$ may be replaced by h belonging to any bounded interval $[0, N]$.)

An integral solution for (2.1) on $[0, T]$ is a curve $u(t)$ on $[0, T]$ satisfying $u(0) = g(u)$ and the inequality:

$$\frac{1}{2}\big\|u(t) - x\big\|^2 - \frac{1}{2}\big\|u(s) - x\big\|^2 \leqslant \int_s^t \big\langle u(\tau) - x, f(\tau) - y\big\rangle_s \, d\tau$$

for every $[x, y] \in A$ and every $0 \leqslant s \leqslant t \leqslant T$ (see [8]).

The following lemma is an easy consequence of the inequality (2.2) in Lemma 2.2 with $f_2(\tau) = f_1(\tau + (t-s))$ and $v(\tau) = u(\tau + (t-s))$.

Lemma 5.1. *If u is an integral solution for* (5.1)*, then u is the integral solution for* (2.1) *on $[0, +\infty)$ with $u(0) = g(u)$ and $f \in Sel(u)$. Moreover, if $f \in L^1_{loc}((0, +\infty); X)$, then we have*

$$\big\|u(t+h) - u(s+h)\big\| \leqslant \big\|u(t) - u(s)\big\| + \int_s^{s+h} \big\|f\big(\tau + (t-s)\big) - f(\tau)\big\| \, d\tau.$$

Finally we are able to state the main results of this section. As a direct consequence of Lemma 5.1, we first have

Theorem 5.1. *Let u be an integral solution to system* (5.1) *on $[0, T]$ for every $T > 0$, then there exists $f \in Sel(u)$ and u is the integral solution of* (2.1) *on $[0, T]$ for every $T > 0$. In addition, if there exist $f_\infty \in X$ and $\delta > 0$ such that*

$$\lim_{s\to+\infty} \int_s^{s+\delta} \big\|f(\tau) - f_\infty\big\| \, d\tau = 0, \tag{5.2}$$

then $\{u(t)\}_{t\geqslant 0}$ is an almost nonexpansive curve in X. In particular, if

$$f - f_\infty \in L^p\big((0, +\infty); X\big) \quad \text{for some } f_\infty \in X \text{ and } 1 \leqslant p < \infty, \tag{5.3}$$

then $\{u(t)\}_{t\geqslant 0}$ is also an almost nonexpansive curve in X.

Proof. We take

$$\varepsilon(t,s)=\begin{cases}\int_s^{s+\delta}\|f(\theta+(t-s))-f(\theta)\|\,d\theta, & \text{if } t\geqslant s,\\ \int_t^{t+\delta}\|f(\theta+(s-t))-f(\theta)\|\,d\theta, & \text{if } s\geqslant t.\end{cases}$$

It is not difficult to obtain the conclusion of Theorem 5.1 (similarly, see [36]). This completes the proof. □

In order to apply our results to set-valued dissipative system (5.1), we now study the asymptotic properties of almost nonexpansive curves in X, which appear in [35,36].

Lemma 5.2. *Let $\{u(t)\}_{t\geqslant 0}$ be an almost nonexpansive curve in a Banach space X. Then $\lim_{t\to+\infty}\|\frac{u(t)}{t}\|$ exists.*

Lemma 5.3. *Let $\{u(t)\}_{t\geqslant 0}$ be an almost nonexpansive curve in X and $C_h=\bigcap_{n=0}^{\infty}\overline{co}\{\{u(t+h)-u(t)\}_{t\geqslant n}\}$ for $h>0$. Then*

(1) *If X is reflexive, then $C_h\neq\emptyset$ and $\|\frac{u(t)}{t}\|\to\frac{d(0,C_h)}{h}$ as $t\to+\infty$ for each $h>0$.*
(2) *If X is reflexive and strictly convex, then $\frac{u(t)}{t}\rightharpoonup\frac{1}{h}P_{C_h}0$ as $t\to+\infty$ and $\|\frac{u(t)}{t}\|\to\frac{1}{h}\|P_{C_h}0\|$ as $t\to+\infty$ for each $h>0$.*
(3) *If X^* has Fréchet differentiable norm, then $\frac{u(t)}{t}\to\frac{1}{h}P_{C_h}0$ as $t\to+\infty$ for each $h>0$.*

Subsequently, having in mind the above results, it is easy to obtain

Theorem 5.2. *If u is an integral solution to system (5.1) on $[0,+\infty)$, then u is the integral solution for (2.1) on $[0,+\infty)$ with $u(0)=g(u)$ and $f\in Sel(u)$. Moreover, let $f-f_\infty\in L^p((0,+\infty);X)$ for some $f_\infty\in X$ and $1\leqslant p<\infty$ (or more generally f satisfies (5.2)). Then*

(1) *If X is reflexive, then $C_h\neq\emptyset$ and $\|\frac{u(t)}{t}\|\to\frac{d(0,C_h)}{h}$ as $t\to+\infty$ for each $h>0$.*
(2) *If X is reflexive and strictly convex, then $\frac{u(t)}{t}\rightharpoonup\frac{1}{h}P_{C_h}0$ as $t\to+\infty$ and $\|\frac{u(t)}{t}\|\to\frac{1}{h}\|P_{C_h}0\|$ as $t\to+\infty$ for each $h>0$.*
(3) *If X^* has Fréchet differentiable norm, then $\frac{u(t)}{t}\to\frac{1}{h}P_{C_h}0$ as $t\to+\infty$ for each $h>0$.*

Proof. By Definition 2.6, if u is an integral solution of (5.1), then there exists some $f\in Sel(u)$ such that u is the integral solution for (2.1) with $u(0)=g(u)$. Moreover, since f satisfies the (5.2) or (5.3), we know that $\{u(t)\}_{t\geqslant 0}$ is an almost nonexpansive curve in X by Theorem 5.1. Finally, applying Lemmas 5.2 and 5.3, we get the conclusion of Theorem 5.2. □

6. An example

In this section, we shall discuss an example to illustrate the applicability of our abstract theory.

$$\frac{\partial}{\partial t}v(t,x)=\Delta v(t,x)+F\big(t,v(t,x)\big),\quad t\in[0,T],\ x\in\Omega, \tag{6.1}$$

$$-\frac{\partial}{\partial\nu}v(t,x)\in\partial j\big(v(t,x)\big),\quad t\in[0,T],\ x\in\Gamma, \tag{6.2}$$

$$v(0,x)=\int_\Omega\int_0^T G\big(t,x,\xi,v(t,\xi)\big)\,dt\,d\xi,\quad x\in\Omega. \tag{6.3}$$

Here Ω is a bounded domain in R^n ($n\geqslant 1$) with smooth boundary Γ, $\frac{\partial}{\partial\nu}$ denotes the outward normal derivative to Γ, $j:R\to R\cup\{+\infty\}$ is proper, convex and lower semicontinuous function with $j(0)=0$, $F:R\times R\to R$ and $G:[0,T]\times\Omega\times\Omega\times R\to R$ are given functions.

We suppose that:

(H_1) There exists $h:[0,T]\times\Omega\times R\to R^+$ such that $F(t,v(t,x)):=\{u(t)\in L^1(0,T;L^2(\Omega));\ 0\leqslant u(t,x)\leqslant h(t,x,v(t,x)),\ t\in[0,T],\ x\in\Omega\}$, where h is a function satisfying the following conditions:
  (1) for almost every $(t,x)\in[0,T]\times\Omega$, $h(t,x,r)$ is a continuous function of r;
  (2) for each fixed $r\in R$, $h(t,x,r)$ is a measurable function of (t,x);
  (3) there exist two functions $h_1\in L^1([0,T];R^+)$ and $h_2\in L^1([0,T];L^2(\Omega)), h_2(t,x)\geqslant 0$ such that $h(t,x,r)\leqslant h_1(t)|r|+h_2(t,x)$.

(H_2) There exists $l\in L^1([0,T];R^+)$ such that $\beta(F(t,B))\leqslant l(t)\beta(B)$ for almost all $t\in[0,T]$ and every bounded subset $B\subset X$.

(H_3) The function $G:[0,T]\times\Omega\times\Omega\times R\to R$ satisfies the following conditions:
  (1) the Carathéodory condition, that is, $G(t,x,\xi,r)$ is a continuous function about r for a.e. $(t,x,\xi)\in[0,T]\times\Omega\times\Omega$; $G(t,x,\xi,r)$ is measurable about (t,x,ξ) for each fixed $r\in R$;

(2) $|G(t,x,\xi,r) - G(t,x',\xi,r)| \leqslant m_k(t,x,x',\xi)$ for all $(t,x,\xi,r),(t,x',\xi,r) \in [0,T] \times \Omega \times \Omega \times R$ with $|r| \leqslant k$, where $m_k \in L^1([0,T] \times \Omega \times \Omega \times R; R^+)$ satisfies $\lim_{x \to x'} \int_\Omega \int_0^T m_k(t,x,x',\xi)\,dt\,d\xi = 0$, uniformly in $x' \in \Omega$;
(3) $|G(t,x,\xi,r)| \leqslant \frac{\delta}{Tm(\Omega)}|r| + \eta(t,x,\xi)$ for all $r \in R$, where $\eta \in L^2([0,T] \times \Omega \times \Omega; R^+)$ and $\delta > 0$.

Let $X = L^2(\Omega)$ equipped with the norm $\|v\| = (\int_\Omega v^2(x)\,dx)^{\frac{1}{2}}$ for every $v \in L^2(\Omega)$, then X is a Hilbert space. For any $v \in D(A)$, define

$$Av = -\Delta v,$$

where $D(A) = \{v \in X\colon -\frac{\partial v}{\partial \nu} \in \partial j(v),\ \text{a.e. on } \Gamma\}$. Then A is the subdifferential of a proper, convex and lower semicontinuous function $\phi : X \to R \cup \{+\infty\}$, i.e., $A = \partial\phi$ (see [37]), and thus A is m-dissipative, as is proved in [8]. Furthermore, if j is not a positive function, then A generates a semigroup $S(t)$ on X, which may be only equicontinuous.

Next we define $F : [0,T] \times X \to P(X)$ and $g : C([0,T;X]) \to X$ by

$$F(t,v) = \{u \in X\colon 0 \leqslant u(x) \leqslant h(t,x,v(x)),\ x \in \Omega\},$$

and

$$g(v)(x) = \int_\Omega \int_0^T G(t,x,\xi,v(t,\xi))\,dt\,d\xi, \quad x \in \Omega.$$

In view of hypothesis (H_1), it is implied from p. 191 in [25] and Theorem 8.14 in [5] that F has nonempty, closed and convex values, $F(\cdot,v)$ is measurable for all $v \in X$ and $|F(t,v)| \leqslant h_1(t)\|v\| + \|h_2(t,\cdot)\|$. Moreover, by the methods and techniques of [24,25,27] we can prove that $F(t,\cdot)$ is weakly usc for a.e. $t \in [0,T]$. Subsequently, on account of (H_3), it follows directly from Theorem 4.2 in [31] that g is well defined and it is a completely continuous operator. Finally, we show that g satisfies the growth condition in (H_g)(2).

To this end, let $v \in C([0,T];X)$, from (H_3), we obtain

$$\begin{aligned}
|g(v)(x)| &\leqslant \int_\Omega \int_0^T G(t,x,\xi,v(t,\xi))\,dt\,d\xi \\
&\leqslant \frac{\delta}{Tm(\Omega)} \int_\Omega \int_0^T |v(t,\xi)|\,dt\,d\xi + \int_\Omega \int_0^T \eta(t,x,\xi)\,dt\,d\xi \\
&\leqslant \frac{\delta}{Tm(\Omega)} \int_0^T \left(\int_\Omega v^2(t,\xi)\,d\xi \right)^{\frac{1}{2}} dt + \int_\Omega \int_0^T \eta(t,x,\xi)\,dt\,d\xi \\
&= \frac{\delta}{\sqrt{m(\Omega)}} \|v\|_{C([0,T];L^2(\Omega))} + \int_\Omega \int_0^T \eta(t,x,\xi)\,dt\,d\xi,
\end{aligned}$$

where $\eta \in L^2([0,T] \times \Omega \times \Omega; R^+)$, the above inequality implies that g satisfies the condition (H_g)(2) with $a = \frac{\delta}{\sqrt{m(\Omega)}}$ and $b = (Tm(\Omega))^{\frac{1}{2}} \|\eta\|_{L^2([0,T] \times \Omega \times \Omega; R^+)}$.

Let us return now to Eqs. (6.1)–(6.3), and observe that it may be rewritten as the abstract nonlocal problem (1.1). On account of the above discussion, we infer that hypotheses (H_A), (H_g) and (H_F) are satisfied. Therefore, if the following inequality $\frac{\delta}{\sqrt{m(\Omega)}} + \int_0^T h_1(t)\,dt < 1$ holds, the nonlocal problem (6.1)–(6.3) has at least one integral solution $v \in C([0,T];L^2(\Omega))$ due to Theorem 3.1.

Further discussions about application of our results will be in our consequent papers.

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