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Multivalued mixed variational inequalities with locally Lipschitzian and locally cocoercive multivalued mappings

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ABSTRACT

In this paper, we study the problem of solving multivalued mixed variational inequalities. By using some sequential approximation techniques of fixed point theory, we solve the multivalued mixed variational inequalities involving locally Lipschitzian or locally cocoercive multivalued mappings. We establish that the convexity of the multivalued mapping values is not needed and construct by using the Banach contraction principle converging sequences to the solutions. Also, we show how to choose regularization parameters to compute these solutions.

1. Introduction

Let C be a nonempty closed convex subset of $\mathbb{R}^n$ and let $F : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ be a multivalued mapping such that $F(x)$ is nonempty closed subset, for every $x \in C$. Suppose further that $\varphi : C \to \mathbb{R}$ is a convex subdifferentiable function. We consider the following multivalued mixed variational inequality:

$$\begin{aligned}&\text{Find } x^* \in C \quad \text{such that}\\ &\exists w^* \in F\left(x^*\right), \quad \left\langle w^*, y - x^*\right\rangle + \varphi(y) - \varphi\left(x^*\right) \geq 0 \quad \forall y \in C.\end{aligned} \tag{1.1}$$

This problem has been considered by several authors (see [1–6]). One usually calls F the cost operator and C the set of constraints.

A large variety of problems arising in elasticity, fluid flow, economics, oceanography, transportation, optimization, pure and applied sciences can be seen as special cases of problem (1.1). See [4,7,8,5,9–11] and the references therein.

The methods based on fixed point theory for solving variational inequalities have been largely developed by several authors (see [1–4,12–16] and the references therein).

In this paper, we extend some results of [1] about solving the multivalued mixed variational inequality (1.1). By using the Banach contraction mapping principle, we introduce some sequential approximation techniques to construct converging sequences for solving the multivalued mixed variational inequality (1.1) when the involved multivalued mapping is locally Lipschitzian with respect to the Hausdorff metric or locally cocoercive. We prove that the multivalued mixed variational inequality (1.1) has a unique solution if F is locally Lipschitzian and strongly monotone or locally Lipschitzian monotone and φ is strongly convex. It has at least one solution if F is locally cocoercive. Also, we show how to choose regularization parameter α such that these solutions can be obtained by computing fixed points of a certain multivalued mapping.

Variational inequality problems as well as optimization, saddle points, Nash equilibrium, fixed points, complementarity problems and many other problems in nonlinear analysis are special cases of the more general concept of equilibrium

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problems (see, for example, [17–27,5,28–31] and the references therein). Recall that an equilibrium problem on C in the sense of Blum and Oettli [19] is a problem of the form

$$\text{Find } x^* \in C \quad \text{such that } \Phi\left(x^*, y\right) \geq 0 \; \forall y \in C \tag{1.2}$$

where $\Phi : C \times C \to \mathbb{R} \cup \{+\infty\}$ is a function such that $\Phi(x, x) = 0$, for every $x \in C$. The function Φ is called an equilibrium bifunction.

The last section of this paper is devoted to a discussion in order to compare the results obtained here for multivalued mixed variational inequalities with the old existing for equilibrium problems.

2. Preliminaries and fixed point formulation

Throughout the paper, $\langle \cdot, \cdot \rangle$ denotes the Euclidian inner product on $\mathbb{R}^n$ and $\|\cdot\|$ its associated norm.

Let $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a function. Recall that the domain of φ is

$$\operatorname{dom}(\varphi) \stackrel{\text{def}}{=} \left\{x \in \mathbb{R}^n \mid \varphi(x) < +\infty\right\}.$$

The function φ is said to be *proper* if $\operatorname{dom}(\varphi) \neq \emptyset$.

Let C be a nonempty convex subset of $\mathbb{R}^n$, that is:

$$\lambda x + (1 - \lambda) y \in C \quad \forall x, y \in C, \forall \lambda \in [0, 1].$$

Suppose that $C \subset \operatorname{dom}(\varphi)$. Recall that the function φ is said to be:

- *Convex* on C if
$$\forall x, y \in C, \qquad \forall \lambda \in [0, 1]$$
$$\varphi(\lambda x + (1 - \lambda) y) \leq \lambda \varphi(x) + (1 - \lambda) \varphi(y).$$
If the above inequality is strict whenever $x \neq y$, then the function φ is said to be *strictly convex*.
- *Strongly convex with modulus* $\eta > 0$ or briefly η*-strongly convex* on C if
$$\forall x, y \in C, \qquad \forall \lambda \in [0, 1]$$
$$\varphi(\lambda x + (1 - \lambda) y) \leq \lambda \varphi(x) + (1 - \lambda) \varphi(y) - \frac{1}{2} \lambda (1 - \lambda) \eta \|x - y\|^2.$$

Obviously, every η-strongly convex function is strictly convex and every strictly convex function is convex. The converse is false is general.

Let M be a nonempty subset of $\mathbb{R}^n$ and let $F : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ be a multivalued mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$ such that $M \subseteq \operatorname{dom}(F) \stackrel{\text{def}}{=} \{x \in \mathbb{R}^n \mid F(x) \neq \emptyset\}$. The multivalued mapping F is called:

- *Monotone* on M if
$$\forall x_1, x_2 \in M, \qquad \forall w_1 \in F(x_1), \forall w_2 \in F(x_2)$$
$$\langle w_1 - w_2, x_1 - x_2 \rangle \geq 0.$$
If the above inequality is strict whenever $x_1 \neq x_2$, then the multivalued mapping F is said to be *strictly monotone*.
- *Strongly monotone with modulus* $\beta > 0$ or briefly β*-strongly monotone* on M if
$$\forall x_1, x_2 \in M, \qquad \forall w_1 \in F(x_1), \forall w_2 \in F(x_2)$$
$$\langle w_1 - w_2, x_1 - x_2 \rangle \geq \beta \|x_1 - x_2\|^2.$$

Obviously, every β-strongly monotone multivalued mapping is strictly monotone and every strictly monotone multivalued mapping is monotone. The converse is false is general.

Recall that for a convex function $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$:

- *The subgradient* of φ at $x_0 \in \operatorname{dom}(\varphi)$ is the set
$$\partial \varphi(x_0) \stackrel{\text{def}}{=} \left\{z \in \mathbb{R}^n \mid \varphi(x) - \varphi(x_0) \geq \langle z, x - x_0 \rangle \; \forall x \in \mathbb{R}^n\right\}.$$
- The function φ is said to be *subdifferentiable* at x_0 if $\partial \varphi(x_0) \neq \emptyset$.

It is well known that the subdifferential operator $x \mapsto \partial \varphi(x)$ of a convex function φ is a monotone multivalued mapping and $\partial \varphi(x)$ is closed and convex, for every $x \in \operatorname{dom}(\varphi)$. One can consult [32] to see that a subdifferentiable function φ is strictly convex (*resp.*, β-strongly convex) if and only if $\partial \varphi$ is estimated strictly monotone (*resp.*, β-strongly monotone).

The function φ is said to be:

- *Lower semicontinuous at* $x \in C$ if for every sequence $(x_n)_n$, we have
$$\lim_{n\to+\infty} x_n = x \Longrightarrow \liminf_{n\to+\infty} \varphi(x_n) \geq \varphi(x)$$
where $\liminf_{n\to+\infty} \varphi(x_n) = \sup_{n\in\mathbb{N}} \inf_{k\geq n} \varphi(x_k)$.
- *Lower semicontinuous on* C if it is lower semicontinuous at every point of C.

It is well known that if C is a nonempty closed convex subset of $\mathbb{R}^n$ and f is a strictly convex function on C to $\mathbb{R}$, then the optimization problem

$$\min_{x\in C} f(x)$$

has at most one solution. If f is lower semicontinuous and η-strongly convex on C, then the problem has a unique solution.

Existence and uniqueness of the solutions of variational inequalities have been already studied by several authors. (See [12,10] and the references therein). The following result concerns uniqueness of solutions of the multivalued mixed variational inequality (1.1).

Theorem 2.1. *Suppose that one of the following two conditions holds:*

1. *The multivalued mapping F is strictly monotone on C.*
2. *The multivalued mapping F is monotone and φ is strictly convex function on C.*

Then, the multivalued mixed variational inequality (1.1) has at most one solution.

To study the existence of solutions of the multivalued mixed variational inequality (1.1), we need the following well-known result (see [33,1,34,4,5,10]) which provides us with a construction of a multivalued mapping which is essential to solve the multivalued mixed variational inequality (1.1).

Theorem 2.2. *Let G be a symmetric positive definite matrix. Let x and w two points such that $F(x) \neq \emptyset$ and $w \in F(x)$.*

1. *The optimization problem*
$$\min_{y\in C}\left\{\frac{1}{2}\langle y-x, G(y-x)\rangle + \langle w, y-x\rangle + \varphi(y)\right\} \tag{2.1}$$
has a unique solution. We denote this solution by $h(x,w)$.
2. *A point $h \in C$ is a solution of the problem (2.1) if and only if there exists $z \in \partial\varphi(h)$ such that*
$$\langle w + G(h-x) + z, y-h\rangle \geq 0 \quad \forall y \in C. \tag{2.2}$$

Following Theorem 2.2, we can associate to each pair (x,w) with $x \in \operatorname{dom}(F)$ and $w \in F(x)$ a unique point $h(x,w)$ which is the unique solution of the optimization problem (2.1). Now, we consider the multivalued mapping $H: \mathbb{R}^n \to 2^{\mathbb{R}^n}$ defined by

$$H(x) = \begin{cases} \{h(x,w) \mid w \in F(x)\} & \text{if } x \in \operatorname{dom}(F), \\ \emptyset & \text{otherwise.} \end{cases}$$

We have $C \subseteq \operatorname{dom}(F) = \operatorname{dom}(H)$.

The following lemma gives a characterization of the solutions of the multivalued mixed variational inequality (1.1) by means of fixed points of the multivalued mapping H. See [33,1].

Lemma 2.3. *A point x^* is a solution of the multivalued mixed variational inequality (1.1) if and only if x^* is a fixed point of H. More precisely, we have $x^* = h(x^*, w^*) \in H(x^*)$ if and only if*

$$\langle w^*, y - x^*\rangle + \varphi(y) - \varphi(x^*) \geq 0 \quad \forall y \in C.$$

In the sequel we shall restrict our attention to the important case of $G = \alpha I$ where $\alpha > 0$ and I is the identity matrix. In this case, the optimization problem (2.1), for $x \in C$ and $w \in F(x)$, becomes

$$\min_{y\in C}\left\{\frac{\alpha}{2}\|y-x\|^2 + \langle w, y-x\rangle + \varphi(y)\right\}.$$

We need again some notions of continuous properties of multivalued mappings. The multivalued mapping F is said to be:

- *Closed at* x if for every sequences $(x_n)_n$ and $(y_n)_n$, we have

$$\left.\begin{array}{l}\lim\limits_{n\to+\infty} x_n = x,\\ \lim\limits_{n\to+\infty} y_n = y,\\ y_n \in F(x_n), \forall n\end{array}\right\} \Longrightarrow y \in F(x).$$

- *Closed on* M if it is closed at every point of M.
- *Upper semicontinuous at* x if for every open subset G containing $F(x)$, there exists an open subset U containing x such that $F(x') \subseteq G$, for every $x' \in U$.
- *Upper semicontinuous on* M if it is upper semicontinuous at every point of M.

Remark 2.1.
- A multivalued mapping F is upper semicontinuous at x if and only if it is continuous at x as a mapping from $\mathbb{R}^n$ to the set of subsets of $\mathbb{R}^n$ endowed with the upper Vietoris topology (see [35,36]).
- If F is closed at x, then $F(x)$ is closed.
- If F is upper semicontinuous at x and if $F(x)$ is closed, then F is closed at x.

3. Solutions with locally Lipschitzian multivalued mappings

Let A and B be two nonempty closed subsets of $\mathbb{R}^n$. Recall that *the Hausdorff metric* d_H between A and B is given by

$$d_H(A, B) \stackrel{\text{def}}{=} \max\left\{\sup_{a\in A}\inf_{b\in B}\|a-b\|, \sup_{b\in B}\inf_{a\in A}\|a-b\|\right\}.$$

This metric could take the value $+\infty$; see [37].

Let F be a multivalued mapping such that $F(x)$ is closed, for every $x \in M$. The multivalued mapping F is said to be *Lipschitzian with a constant* $L > 0$ or briefly L*-Lipschitzian on* M if

$$d_H\left(F(x), F(x')\right) \le L\|x - x'\| \quad \forall x, x' \in M.$$

In particular, F is said to be L*-contraction on* M if $L < 1$ and *nonexpansive on* M if $L = 1$.

The following lemma is useful for the sequel.

Lemma 3.1. *Let* $F : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ *be a multivalued mapping such that, for every* $x \in M$, $F(x)$ *is nonempty closed subset. Then,*

$$\begin{array}{l}\forall x, x' \in M, \qquad \forall w_x \in F(x), \quad \exists w_{x'} \in F(x')\\ \|w_x - w_{x'}\| \le d_H\left(F(x), F(x')\right).\end{array}$$

Proof. Let $x, x' \in M$ and let $w_x \in F(x)$. Assume that $d_H\left(F(x), F(x')\right)$ is finite, otherwise we are finished. For every $n \ge 1$, let $w(n) \in F(x')$ such that

$$\|w_x - w(n)\| < \frac{1}{n} + d_H\left(F(x), F(x')\right).$$

This is possible since $d_H\left(F(x), F(x')\right) < \frac{1}{n} + d_H\left(F(x), F(x')\right)$. Without loss of generality, we may assume that $(w(n))_n$ converges to some $w_{x'}$ and since $F(x')$ is closed, then $w_{x'} \in F(x')$. It is clear that

$$\|w_x - w_{x'}\| \le d_H\left(F(x), F(x')\right). \quad \square$$

The last lemma extends Lemma 2.2 of [1] to multivalued mappings with nonempty closed values. Recall that such a $w_{x'}$ is not necessarily unique and if $F(x')$ is nonempty closed convex subset, then $w_{x'}$ could be taken the orthogonal projection of w_x on $F(x')$.

The following theorem provides us with a tool for solving the multivalued mixed variational inequality (1.1).

Theorem 3.2. *Let* C *be a nonempty closed convex subset of* $\mathbb{R}^n$ *and let* M *be a nonempty subset of* C. *Let* φ *be a proper convex subdifferentiable function on* C *and let* $F : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ *be an* L*-Lipschitzian multivalued mapping on* M *such that, for every* $x \in M$, $F(x)$ *is a nonempty closed subset. Then,*

$$\begin{array}{l}\forall x, x' \in M, \qquad \forall w_x \in F(x), \exists w_{x'} \in F(x')\\ \left\|h(x, w_x) - h(x', w_{x'})\right\| \le \delta\|x - x'\|\end{array}$$

where

1. $\delta = \sqrt{1 - \frac{2\beta}{\alpha} + \frac{L^2}{\alpha^2}}$ *with* $\alpha > \frac{L^2}{2\beta}$ *if* F *is* β*-strongly monotone on* M *and*
2. $\delta = \frac{\sqrt{L^2+\alpha^2}}{\alpha+\eta}$ *with* $\alpha > \frac{L^2-\eta^2}{2\eta}$ *if* F *is monotone on* M *and* φ *is* η*-strongly convex on* C.

Proof. Let $x, x' \in M$, $w \in F(x)$ and $w' \in F(x')$. Let $h(x, w)$ and $h(x', w')$ the unique solutions of the optimization problem (2.1)

$$\min_{y\in C} \left\{ \frac{\alpha}{2} \|y - x\|^2 + \langle w, y - x\rangle + \varphi(y) \right\}$$

associated to x and w and to x' and w' respectively. From Theorem 2.2, let $z \in \partial\varphi(h(x, w))$ and $z' \in \partial\varphi(h(x', w'))$ be such that

$$\langle \alpha (h(x, w) - x) + w + z, y - h(x, w)\rangle \geq 0 \quad \forall y \in C$$

and

$$\langle \alpha (h(x', w') - x') + w' + z', y - h(x', w')\rangle \geq 0 \quad \forall y \in C.$$

Substituting $h(x', w')$ for y in the first inequality, $h(x, w)$ for y in the second, and by addition, we obtain

$$\left\langle x - x' - \frac{1}{\alpha}(w - w') - \frac{1}{\alpha}(z - z'), h(x, w) - h(x', w')\right\rangle - \langle h(x, w) - h(x', w'), h(x, w) - h(x', w')\rangle \geq 0.$$

It follows that

$$\begin{aligned} \|h(x, w) - h(x', w')\|^2 &\leq \left\langle x - x' - \frac{1}{\alpha}(w - w'), h(x, w) - h(x', w')\right\rangle - \frac{1}{\alpha}\langle z - z', h(x, w) - h(x', w')\rangle \\ &\leq \left\| x - x' - \frac{1}{\alpha}(w - w') \right\| \|h(x, w) - h(x', w')\| - \frac{1}{\alpha}\langle z - z', h(x, w) - h(x', w')\rangle. \end{aligned}$$

If F is L-Lipschitzian and β-strongly monotone on M, we use the monotonicity of $\partial\varphi$ and obtain

$$\|h(x, w) - h(x', w')\|^2 \leq \left\| x - x' - \frac{1}{\alpha}(w - w') \right\| \|h(x, w) - h(x', w')\|.$$

Then

$$\begin{aligned} \|h(x, w) - h(x', w')\|^2 &\leq \left\| x - x' - \frac{1}{\alpha}(w - w') \right\|^2 \\ &= \|x - x'\|^2 - \frac{2}{\alpha}\langle x - x', w - w'\rangle + \frac{1}{\alpha^2}\|w - w'\|^2. \end{aligned}$$

By Lemma 3.1, replace w by w_x and take $w_{x'}$ in place of w' such that $\|w_x - w_{x'}\| \leq d_H(F(x), F(x'))$. Since F is L-Lipschitzian and β-strongly monotone on M, we have then

$$\|h(x, w_x) - h(x', w_{x'})\|^2 \leq \left(1 - \frac{2\beta}{\alpha} + \frac{L^2}{\alpha^2}\right) \|x - x'\|^2$$

and then

$$\|h(x, w_x) - h(x', w_{x'})\| \leq \sqrt{1 - \frac{2\beta}{\alpha} + \frac{L^2}{\alpha^2}} \|x - x'\| \quad \forall x, x' \in M.$$

In the case where F is L-Lipschitzian and monotone on M and φ is η-strongly convex on C, we use the strong monotonicity of $\partial\varphi$ and obtain

$$\|h(x, w) - h(x', w')\|^2 \leq \left\| x - x' - \frac{1}{\alpha}(w - w') \right\| \|h(x, w) - h(x', w')\| - \frac{\eta}{\alpha} \|h(x, w) - h(x', w')\|^2.$$

Then

$$\begin{aligned} \left(1 + \frac{\eta}{\alpha}\right)^2 \|h(x, w) - h(x', w')\|^2 &\leq \left\| x - x' - \frac{1}{\alpha}(w - w') \right\|^2 \\ &= \|x - x'\|^2 - \frac{2}{\alpha}\langle x - x', w - w'\rangle + \frac{1}{\alpha^2}\|w - w'\|^2. \end{aligned}$$

By Lemma 3.1, replace w by w_x and take $w_{x'}$ in place of w' such that $\|w_x - w_{x'}\| \le d_H\left(F(x), F(x')\right)$. Since F is L-Lipschitzian and monotone on M, we have then

$$\left(1+\frac{\eta}{\alpha}\right)^2 \left\|h(x, w_x) - h\left(x', w_{x'}\right)\right\|^2 \le \left(1+\frac{L^2}{\alpha^2}\right)\|x-x'\|^2$$

and then

$$\left\|h(x, w_x) - h\left(x', w_{x'}\right)\right\| \le \frac{\sqrt{L^2+\alpha^2}}{\alpha+\eta}\|x-x'\| \quad \forall x, x' \in M. \quad \square$$

The following theorem allows us to construct by the Banach contraction principle, a convergent sequence to the unique fixed point of the multivalued mapping H and then to the unique solution of the multivalued variational inequality (1.1). This theorem is based on some techniques of fixed point theory (see [38,39]). Recall that, for $x_0 \in \mathbb{R}^n$ and $r > 0$,

$$B(x_0, r) \stackrel{\text{def}}{=} \left\{x \in \mathbb{R}^n \mid \|x - x_0\| < r\right\}$$

denotes the open ball around x_0 with radius r.

Theorem 3.3. *Let $r > 0$, C a nonempty closed convex subset of $\mathbb{R}^n$ and let $x_0 \in C$. Let φ be a proper convex subdifferentiable function on C and let $F : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ be a closed multivalued mapping on C and L-Lipschitzian on $C \cap B(x_0, r)$ such that $F(x)$ is a nonempty subset, for every $x \in C \cap B(x_0, r)$. Suppose further that there exists $w_0 \in F(x_0)$ such that*

$$\|h(x_0, w_0) - x_0\| < (1-\delta)\, r$$

where $\delta = \sqrt{1 - \frac{2\beta}{\alpha} + \frac{L^2}{\alpha^2}}$ with $\alpha > \frac{L^2}{2\beta}$ if F is β-strongly monotone on C and $\delta = \frac{\sqrt{L^2+\alpha^2}}{\alpha+\eta}$ with $\alpha > \frac{L^2-\eta^2}{2\eta}$ if F is monotone and φ is η-strongly convex on C. Then the problem (1.1) has a unique solution $x^* \in C$.
More precisely, there exist a sequence $(x_n)_n$ in $C \cap B(x_0, r)$ converging to x^* and a sequence $(w_n)_n$ converging to $w^* \in F(x^*)$ such that $w_n \in F(x_n)$,

$$\|x_{n+2} - x_{n+1}\| \le \delta\|x_{n+1} - x_n\|, \qquad \|x_n - x^*\| \le \frac{1}{1-\delta}\|x_{n+1} - x_n\|,$$
$$\|w_{n+1} - w_n\| \le L\|x_{n+1} - x_n\| \quad \text{and} \quad \|w_n - w^*\| \le \frac{\delta^{n+1}}{1-\delta}L\|x_1 - x_0\|,$$

for every $n \in \mathbb{N}$.

Proof. Put $x_1 = h(x_0, w_0)$. We have then

$$\|x_1 - x_0\| < (1-\delta)\, r.$$

We will construct two sequences $(x_n)_n$ and $(w_n)_n$ satisfying the above conditions. The first step of the recurrence being similar to the step of order n, then suppose that $(x_k)_{k\le n+1}$ and $(w_k)_{k\le n}$ are constructed. By Lemma 3.1, let $w_{n+1} \in F(x_{n+1})$ be such that

$$\|w_n - w_{n+1}\| \le d_H(F(x_n), F(x_{n+1}))$$

and put $x_{n+2} = h(x_{n+1}, w_{n+1})$. By Theorem 3.2, we have then

$$\|x_{n+2} - x_{n+1}\| \le \delta\|x_{n+1} - x_n\|.$$

It follows that $x_{n+2} \in C \cap B(x_0, r)$, since

$$\|x_{n+2} - x_{n+1}\| \le \delta^{n+1}\|x_1 - x_0\| < (1-\delta)\,\delta^{n+1} r$$

and then

$$\begin{aligned}\|x_{n+2} - x_0\| &\le \sum_{i=0}^{n+1}\|x_{i+1} - x_i\| < \sum_{i=0}^{n+1}(1-\delta)\,\delta^i r \\ &= \frac{1-\delta^{n+2}}{1-\delta}(1-\delta)\, r = \left(1-\delta^{n+2}\right) r < r.\end{aligned}$$

Now, we will prove that the sequence $(x_n)_n$ is converging in C. For every $n \in \mathbb{N}$ and every $p \in \mathbb{N}^*$, we have

$$\begin{aligned}\|x_{n+p} - x_n\| &\le \sum_{i=0}^{p-1}\|x_{n+i+1} - x_{n+i}\| \le \sum_{i=0}^{p-1}\delta^i\|x_{n+1} - x_n\| \\ &= \frac{1-\delta^p}{1-\delta}\|x_{n+1} - x_n\| < \frac{1-\delta^p}{1-\delta}(1-\delta)\,\delta^n r = \left(1-\delta^p\right)\delta^n r.\end{aligned}$$

Then the sequence $(x_n)_n$ is a Cauchy sequence in C. It converges then to some $x^* \in C$. Also, by tending p to $+\infty$, we have

$$\|x_n - x^*\| \leq \frac{1}{1-\delta}\|x_{n+1} - x_n\|.$$

Now, we will prove that the sequence $(w_n)_n$ is converging in $\mathbb{R}^n$. By its construction, for every $n \in \mathbb{N}$, we have

$$\|w_{n+1} - w_n\| \leq L\|x_{n+1} - x_n\|.$$

It follows that, for every $n \in \mathbb{N}$ and every $p \in \mathbb{N}^*$, we have

$$\|w_{n+p} - w_n\| \leq L\|x_{n+p} - x_n\| < \left(1-\delta^p\right)\delta^n Lr.$$

This means that $(w_n)_n$ is a Cauchy sequence in $\mathbb{R}^n$ too and then converges to some $w^* \in \mathbb{R}^n$. Since F is a closed multivalued mappings on C and since $w_n \in F(x_n)$, for every $n \in \mathbb{N}$, then $w^* \in F(x^*)$.

It remains to prove that $x^* = h(x^*, w^*) \in H(x^*)$ and then x^* is the unique solution of the multivalued mixed variational inequality (1.1). Since $x_{n+1} = h(x_n, w_n)$, for every $n \in \mathbb{N}$, we have then

$$\frac{1}{2}\alpha\|y - x_n\|^2 + \langle w_n, y - x_n\rangle + \varphi(y) \geq \frac{1}{2}\alpha\|x_{n+1} - x_n\|^2 + \langle w_n, x_{n+1} - x_n\rangle + \varphi(x_{n+1}) \quad \forall y \in C.$$

By the continuity of the inner product and since φ is lower semicontinuous, we have

$$\frac{1}{2}\alpha\|y - x^*\|^2 + \left\langle w^*, y - x^*\right\rangle + \varphi(y) \geq \frac{1}{2}\alpha\|x^* - x^*\|^2 + \left\langle w^*, x^* - x^*\right\rangle + \varphi\left(x^*\right) \quad \forall y \in C.$$

Thus $x^* = h(x^*, w^*)$ and then by Theorem 2.1 and Lemma 2.3, x^* is the unique solution of the multivalued mixed variational inequality (1.1). □

As a corollary, we obtain Theorem 3.1 of [1].

Corollary 3.4. *Let C be a nonempty closed convex subset of $\mathbb{R}^n$. Let φ be a proper convex subdifferentiable function on C and let $F : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ be a closed L-Lipschitzian multivalued mapping on C such that $F(x)$ is a nonempty convex subset, for every $x \in C$. If F is β-strongly monotone on C or F is monotone and φ is η-strongly convex on C, then the problem (1.1) has a unique solution $x^* \in C$.*

Proof. Choose $x_0 \in C$ and $r > 0$ such that $\|x_1 - x_0\| < (1-\delta)r$ where $x_1 = h(x_0, w_0)$ is the unique solution of the optimization problem (2.1)

$$\min_{y\in C}\left\{\frac{\alpha}{2}\|y - x_0\|^2 + \langle w_0, y - x_0\rangle + \varphi(y)\right\}.$$

Now, apply Theorem 3.3. □

Here an example where the conditions of Theorem 3.3 are fulfilled.

Example 3.1. Let $C = \{(x, 0) \mid x \geq 0\}$ be a closed convex subset of $\mathbb{R}^2$. Define a multivalued mapping F on C by

$$F((x,0)) = \begin{cases} \{(2x, y) \mid 0 \leq y \leq x\} & \text{if } x < 4, \\ \left\{(2x, y) \mid 0 \leq y \leq x^2\right\} & \text{if } x \geq 4. \end{cases}$$

The multivalued mapping F is strongly monotone on C with modulus $\beta = 2$ and Lipschitzian on $\{(x, 0) \mid x \geq 0$ and $|x-1| < 3\}$ with constant $L = \sqrt{5}$. It is easily seen that F is a closed and locally Lipschitzian–non-Lipschitzian multivalued mapping on C. For the sake of simplicity, take $\varphi : C \to \mathbb{R}$ the constant function such that $\varphi(X) = 1$, for every $X \in C$. Note that φ is not strongly convex. Let $X_0 = (1, 0)$ and $W_0 = (2, 1) \in F(X_0)$. Let $r = 3$ and $\alpha = 2 > \frac{L^2}{2\beta} = \frac{5}{4}$. By a simple calculation, we have $\delta = \sqrt{1 - \frac{2\beta}{\alpha} + \frac{L^2}{\alpha^2}} = \frac{1}{2}$. The problem of optimization (2.1) associated to X_0 and $W_0 \in F(X_0)$ is

$$\min_{(y,0)\in C}\left\{\frac{\alpha}{2}\|\,(y,0) - (1,0)\,\|^2 + \langle (2,1), (y,0) - (1,0)\rangle + \varphi((y,0))\right\}.$$

The objective function f is defined by

$$\begin{aligned} f((y,0)) &= \frac{\alpha}{2}\,\|(y,0) - (1,0)\|^2 + \langle (2,1), (y,0) - (1,0)\rangle + \varphi((y,0)) \\ &= (y-1)^2 + 2(y-1) + 1 = y^2 - 2y + 1 + 2y - 2 + 1 = y^2. \end{aligned}$$

It is clear that $X_1 = (0, 0)$ is the unique solution of the above optimization problem and $\|X_1 - X_0\| = 1 < \frac{3}{2} = (1-\delta)r$. By Theorem 3.3, the multivalued mixed variational inequality (1.1) associated to C, φ and F has a unique solution.

In what follows by ε-solution of the multivalued mixed variational inequality (1.1) we mean a point $x \in C$ such that $\|x - x^*\| \leq \varepsilon$ where x^* is an exact solution of the multivalued mixed variational inequality (1.1).

Applying Theorem 3.3, we have the following algorithm for solving the multivalued mixed variational inequality (1.1).

Algorithm 3.1. Choose a tolerance $\varepsilon \geq 0$.

Choose $\alpha > \frac{L^2}{2\beta}$ if F is β-strongly monotone and $\alpha > \frac{L^2-\eta^2}{2\eta}$ if φ is η-strongly convex.

Fix $r > 0$, $x_0 \in C$ and $x_1 \in C \cap B(x_0, r)$.

Iteration n $(n = 1, 2 \cdots)$.

Find $w_n \in F(x_n)$ such that $\|w_n - w_{n-1}\| \leq d_H(F(x_n), F(x_{n-1}))$.

Solve the strongly convex program

$$\min_{y \in C} \left\{ \frac{\alpha}{2} \|y - x_n\|^2 + \langle w_n, y - x_n \rangle + \varphi(y) \right\}$$

to obtain its unique solution x_{n+1}.

If $\|x_{n+1} - x_n\| \leq (1 - \delta)\varepsilon$, then terminate: x_n is an ε-solution of the problem (1.1).

Otherwise, let $n = n + 1$ and go to iteration n.

4. Solutions with locally cocoercive multivalued mappings

Now we deal with the case when the multivalued mapping F is cocoercive. Note that in this case, the multivalued mixed variational inequality (1.1) is not necessarily uniquely solvable. Recall that a multivalued mapping F is said to be *cocoercive with a constant* γ or briefly (γ*-cocoercive) on* M if

$$\forall x, x' \in M, \qquad \forall w \in F(x), \qquad \forall w' \in F(x')$$
$$\gamma d_H^2(F(x), F(x')) \leq \langle w - w', x - x' \rangle.$$

As noted in [1], we will say that F is *projectively cocoercive with a constant* γ or briefly *(projectively* γ*-cocoercive) on* M if

$$\forall x, x' \in M, \qquad \forall w_x \in F(x), \quad \exists w_{x'} \in F(x')$$
$$\|w_x - w_{x'}\| \leq d_H(F(x), F(x')) \leq \sqrt{\frac{\langle w_x - w_{x'}, x - x' \rangle}{\gamma}}.$$

Every γ-cocoercive multivalued mapping on M is projectively γ-cocoercive on M.

Theorem 4.1. *Let C be a nonempty closed convex subset of $\mathbb{R}^n$ and let M be a nonempty subset of C. Let φ be a proper convex subdifferentiable function on C and let $F : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ be a projectively γ-cocoercive multivalued mapping on M such that $F(x)$ is a nonempty closed subset, for every $x \in M$. Then,*

$$\forall x, x' \in M, \qquad \forall w_x \in F(x), \quad \exists w_{x'} \in F(x')$$
$$\|h(x, w_x) - h(x', w_{x'})\| \leq \|x - x'\|.$$

Proof. By the same way as in the proof of Theorem 3.2, for every $x', x' \in M$, we have

$$\forall w \in F(x), \qquad \forall w' \in F(x')$$
$$\|h(x, w) - h(x', w')\|^2 \leq \|x - x'\|^2 - \frac{2}{\alpha}\langle x - x', w - w' \rangle + \frac{1}{\alpha^2}\|w - w'\|^2.$$

Since F is s projectively γ-cocoercive on M, take w_x in place of w and choose $w_{x'}$ satisfying the definition of cocoerciveness. We have then

$$\|h(x, w_x) - h(x', w_{x'})\|^2 \leq \|x - x'\|^2 - \frac{2\gamma}{\alpha} d_H^2(F(x), F(x')) + \frac{1}{\alpha^2}\|w_x - w_{x'}\|^2$$

and then

$$\|h(x, w_x) - h(x', w_{x'})\|^2 \leq \|x - x'\|^2 - \left(\frac{2\gamma}{\alpha} - \frac{1}{\alpha^2}\right)\|w_x - w_{x'}\|^2.$$

Choose $\alpha \geq \frac{1}{2\gamma}$. Then

$$\|h(x, w_x) - h(x', w_{x'})\| \leq \|x - x'\| \quad \forall x, x' \in M. \quad \square$$

Recall that, for $x_0 \in \mathbb{R}^n$ and $r > 0$,

$$\overline{B}(x_0, r) \stackrel{\text{def}}{=} \left\{x \in \mathbb{R}^n \mid \|x - x_0\| \le r\right\}$$

denotes the closed ball around x_0 with radius r.

Theorem 4.2. *Let $r > 0$, C a nonempty compact convex subset of $\mathbb{R}^n$ and $x_0 \in C$. Let φ be a proper convex subdifferentiable function on C and let $F : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ be a closed projectively γ-cocoercive multivalued mapping on $C \cap \overline{B}(x_0, r)$ such that $F\left(C \cap \overline{B}(x_0, r)\right)$ is bounded and $F(x)$ is a nonempty subset, for every $x \in C \cap \overline{B}(x_0, r)$. Let $\alpha \ge \frac{1}{2\gamma}$ and suppose further that $x - x_0 \ne \theta(h(x, w) - x_0)$ whenever $0 < \theta < 1$, $x \in C$ such that $\|x - x_0\| = r$ and $w \in F(x)$. Then the problem (1.1) has at least one solution $x^* \in C$.*

More precisely, there exist a sequence $(y_n)_n$ in C, two sequences $(x_n)_n$ and $(z_n)_n$ in $C \cap \overline{B}(x_0, r)$ and a sequence $(w_n)_n$ such that

1. *the sequence $(x_n)_n$ has at least a cluster point x^*,*
2. *every cluster point of $(x_n)_n$ is a cluster point of $(y_n)_n$ and $(z_n)_n$ and it is a solution of the problem (1.1),*
3. *the sequence $(w_n)_n$ has at least a cluster point $w^* \in F(x^*)$ and $w_n \in F(x_n)$, $\forall n \in \mathbb{N}$,*
4. $\|y_{n+1} - y_n\| \le \|x_{n+1} - x_n\|$, $\forall n \in \mathbb{N}$,
5. *the sequence $(\|x_n - z_n\|)_n$ is decreasing and*
6. $\lim_{n\to+\infty} \|x_n - z_n\| = 0$.

Proof. First, define a (radical retraction) mapping $f_r : C \to C \cap \overline{B}(x_0, r)$ by

$$f_r(x) = \begin{cases} x & \text{if } \|x - x_0\| \le r, \\ r\dfrac{x - x_0}{\|x - x_0\|} + x_0 & \text{if } \|x - x_0\| > r. \end{cases}$$

The mapping f_r is nonexpansive (see [40]). Fix $0 < \lambda < 1$ and define by induction the sequences $(x_n)_n$, $(w_n)_n$, $(z_n)_n$ and $(y_n)_n$ as follows: Choose $w_0 \in F(x_0)$, put $y_0 = h(x_0, w_0)$ the unique solution of the optimization problem (2.1)

$$\min_{y \in C} \left\{\frac{\alpha}{2}\|y - x_0\|^2 + \langle w_0, y - x_0\rangle + \varphi(y)\right\}$$

and put $z_0 = f_r(y_0)$. Suppose now x_n, w_n, y_n and z_n are constructed. Put

$$x_{n+1} = (1 - \lambda)x_n + z_n.$$

By assumption, choose $w_{n+1} \in F(x_{n+1})$ such that

$$\|w_n - w_{n+1}\| \le d_H(F(x_n), F(x_{n+1})) \le \sqrt{\frac{\langle w_n - w_{n+1}, x_n - x_{n+1}\rangle}{\gamma}}.$$

Put $y_{n+1} = h(x_{n+1}, w_{n+1})$ and finally, put $z_{n+1} = f_r(y_{n+1})$.

For every $n \in \mathbb{N}$, by Theorem 4.1, we have $\|y_{n+1} - y_n\| \le \|x_{n+1} - x_n\|$ and since f_r is a nonexpansive mapping, it follows that

$$\begin{aligned} \lambda\|x_n - z_n\| &= \|x_{n+1} - x_n\| = \|(1 - \lambda)x_n + \lambda z_n - ((1 - \lambda)x_{n-1} - \lambda z_{n-1})\| \\ &\le (1 - \lambda)\|x_n - x_{n-1}\| + \lambda\|z_n - z_{n-1}\| \\ &\le \|x_n - x_{n-1}\| = \lambda\|x_{n-1} - z_{n-1}\|. \end{aligned}$$

Thus the sequence $(\|z_n - x_n\|)_n$ is decreasing.

Before proving that $\lim_{n\to+\infty} \|x_n - z_n\| = 0$, we shall prove the following statement: for each $i, n \in \mathbb{N}$,

$$\|z_{i+n} - x_i\| \ge (1 - \lambda)^{-n}(\|z_{i+n} - x_{i+n}\| - \|z_i - x_i\|) + (1 + \lambda n)\|z_i - x_i\|.$$

We proceed with induction on n. Since it is obvious for $n = 0$, suppose that the statement is true for a given n and all i. Replacing i with $i + 1$, we obtain

$$\|z_{i+n+1} - x_{i+1}\| \ge (1 - \lambda)^{-n}(\|z_{i+n+1} - x_{i+n+1}\| - \|z_{i+1} - x_{i+1}\|) + (1 + \lambda n)\|z_{i+1} - x_{i+1}\|.$$

On the other hand, we have

$$\begin{aligned} \|z_{i+n+1} - x_{i+1}\| &= \|z_{i+n+1} - ((1 - \lambda)x_i + \lambda z_i)\| \\ &\le \lambda\|z_{i+n+1} - z_i\| + (1 - \lambda)\|z_{i+n+1} - x_i\| \\ &\le (1 - \lambda)\|z_{i+n+1} - x_i\| + \lambda\sum_{k=0}^{n}\|z_{i+k+1} - z_{i+k}\| \\ &\le (1 - \lambda)\|z_{i+n+1} - x_i\| + \lambda\sum_{k=0}^{n}\|x_{i+k+1} - x_{i+k}\|. \end{aligned}$$

It follows that

$$\begin{aligned}\|z_{i+n+1}-x_i\| \geq\ & (1-\lambda)^{-(n+1)}\left(\|z_{i+n+1}-x_{i+n+1}\|-\|z_{i+1}-x_{i+1}\|\right)+(1-\lambda)^{-1}(1+\lambda n)\|z_{i+1}-x_{i+1}\| \\ & -\lambda(1-\lambda)^{-1}\sum_{k=0}^{n}\|x_{i+k+1}-x_{i+k}\|.\end{aligned}$$

Since $\|x_{i+k+1}-x_{i+k}\| = \lambda\|z_{i+k}-x_{i+k}\|$, then

$$\begin{aligned}\|z_{i+n+1}-x_i\| \geq\ & (1-\lambda)^{-(n+1)}\left(\|z_{i+n+1}-x_{i+n+1}\|-\|z_{i+1}-x_{i+1}\|\right)+(1-\lambda)^{-1}(1+\lambda n)\|z_{i+1}-x_{i+1}\| \\ & -\lambda^2(1-\lambda)^{-1}(n+1)\|z_i-x_i\| \\ =\ & (1-\lambda)^{-(n+1)}\left(\|z_{i+n+1}-x_{i+n+1}\|-\|z_i-x_i\|\right) \\ & +\left((1-\lambda)^{-1}(1+\lambda n)-(1-\lambda)^{-(n+1)}\right)\|z_{i+1}-x_{i+1}\| \\ & +\left((1-\lambda)^{-(n+1)}-\lambda^2(1-\lambda)^{-1}(n+1)\right)\|z_i-x_i\|.\end{aligned}$$

Since the sequence $(\|z_n-x_n\|)_n$ is decreasing and $1+\lambda n \leq (1-\lambda)^{-n}$, we have then

$$\begin{aligned}\|z_{i+n+1}-x_i\| \geq\ & (1-\lambda)^{-(n+1)}\left(\|z_{i+n+1}-x_{i+n+1}\|-\|z_i-x_i\|\right)+\left((1-\lambda)^{-1}(1+\lambda n)-(1-\lambda)^{-(n+1)}\right)\|z_i-x_i\| \\ & +\left((1-\lambda)^{-(n+1)}-\lambda^2(1-\lambda)^{-1}(n+1)\right)\|z_i-x_i\| \\ =\ & (1-\lambda)^{-(n+1)}\left(\|z_{i+n+1}-x_{i+n+1}\|-\|z_i-x_i\|\right)+(1+\lambda(n+1))\|z_i-x_i\|.\end{aligned}$$

Thus the statement holds for $n+1$.

Suppose now that $\lim_{n\to+\infty}\|x_n-z_n\| = l > 0$ and choose a positive integer $k \geq \frac{2r}{l\lambda}$ and $\varepsilon > 0$ such that $\varepsilon < l(1-\lambda)^k$. Since $(\|y_n-x_n\|)_n$ is decreasing, there exists an integer i such that

$$0 \leq \|z_i-x_i\|-\|z_{k+i}-x_{k+i}\| \leq \varepsilon.$$

Therefore, by using the above statement, we arrive at the contradiction

$$\begin{aligned}2r+l \leq (1+k\lambda)\,l \leq\ & (1+k\lambda)\|z_i-x_i\| \\ \leq\ & \|z_{k+i}-x_i\|-(1-\lambda)^{-k}\left(\|z_{k+i}-x_{k+i}\|-\|z_i-x_i\|\right) \\ \leq\ & \|z_{k+i}-x_i\|+(1-\lambda)^{-k}\varepsilon < 2r+l.\end{aligned}$$

Then $\lim_{n\to+\infty}\|x_n-z_n\| = 0$.

By the construction and the convexity of $C\cap\overline{B}(x_0,r)$, the sequence $(x_n)_n$ lies in $C\cap\overline{B}(x_0,r)$. Without lost of generality, we may assume that $(x_n)_n$ converges to some $x^*\in C\cap\overline{B}(x_0,r)$.

Since $F(C\cap B(x_0,r))$ is bounded, then without lost of generality too we may assume that $(w_n)_n$ converges to w^* and since F is closed on $C\cap\overline{B}(x_0,r)$, we have $w^*\in F(x^*)$.

Also the sequences $(y_n)_n$ and $(z_n)_n$ lie in C and $C\cap\overline{B}(x_0,r)$ respectively and, once again, we may assume without lost of generality that $(y_n)_n$ converges to $y^*\in C$ and $(z_n)_n$ to $z^*\in C\cap\overline{B}(x_0,r)$.

It follows from the definition of the sequence $(x_n)_n$ that $z^*=x^*$.

We shall prove now that $y^*=x^*=h(x^*,w^*)\in H(x^*)$ and then x^* is a solution of the multivalued mixed variational inequality (1.1). Since $y_n=h(x_n,w_n)$, for every $n\in\mathbb{N}$, then

$$\frac{1}{2}\alpha\|y-x_n\|^2+\langle w_n,y-x_n\rangle+\varphi(y) \geq \frac{1}{2}\alpha\|y_n-x_n\|^2+\langle w_n,y_n-x_n\rangle+\varphi(y_n) \quad \forall y\in C.$$

By the continuity of the inner product and the lower semicontinuity of φ, we have

$$\frac{1}{2}\alpha\|y-x^*\|^2+\left\langle w^*,y-x^*\right\rangle+\varphi(y) \geq \frac{1}{2}\alpha\|y^*-x^*\|^2+\left\langle w^*,y^*-x^*\right\rangle+\varphi\left(y^*\right) \quad \forall y\in C.$$

Thus $y^*=h(x^*,w^*)$ and since $f_r(y^*)=z^*=x^*$, it follows that $y^*\in C\cap\overline{B}(x_0,r)$. Otherwise $r\frac{y^*-x_0}{\|y^*-x_0\|}+x_0=x^*$ and then $x^*-x_0=\frac{r}{\|y^*-x_0\|}(y^*-x_0)$ with $\|x^*-x_0\|=r$ and $0<\theta=\frac{r}{\|y^*-x_0\|}<1$. Contradiction.

Then $x^*=z^*=y^*=h(x^*,w^*)$ and then x^* is a solution of the multivalued mixed variational inequality (1.1). □

As a corollary, we obtain Theorem 3.4 of [1].

Corollary 4.3. *Let C be a nonempty compact convex subset of $\mathbb{R}^n$. Let φ be a proper convex subdifferentiable function on C and let $F:\mathbb{R}^n\to 2^{\mathbb{R}^n}$ be an upper semicontinuous γ-cocoercive multivalued mapping on C such that $F(x)$ is a nonempty compact convex subset, for every $x\in C$. Then the problem (1.1) has at least one solution $x^*\in C$.*

Proof. Choose $x_0\in C$, $r>\operatorname{diam}(C)$ and apply Theorem 4.2. □

Here is an example where the conditions of Theorem 4.2 are fulfilled.

Example 4.1. Let $C = \{(x, 0) \mid 0 \leq x \leq 10\}$ be a compact convex subset of $\mathbb{R}^2$. Define a multivalued mapping F on C by

$$F((x,0)) = \begin{cases} \{(2x, y) \mid 0 \leq y \leq x\} & \text{if } x \leq 4, \\ \left\{\left(2x, \dfrac{1}{x-4}\right)\right\} & \text{if } 4 < x < 5, \\ \{(10, 2)\} & \text{if } x \geq 5. \end{cases}$$

Clearly the multivalued mapping F is neither closed nor cocoercive on C. Also, it is neither Lipschitzian nor strongly monotone on C too. However, the multivalued mapping F is closed, bounded and cocoercive on the subset $\{(x, 0) \mid x \geq 0$ and $|x-1| \leq 3\}$ with modulus $\gamma = \frac{2}{5}$. For the sake of simplicity, take $\varphi : C \to \mathbb{R}$ the constant function such that $\varphi(X) = 1$, for every $X \in C$. Let $X_0 = (1, 0)$. Let $r = 3$ and $\alpha = 2 > \frac{1}{2\gamma} = \frac{5}{4}$. Let $X = (4, 0)$ be the unique point of C such that $\|X - X_0\| = 3$. Let $W = (8, a) \in F(X_0)$ where $0 \leq a \leq x$. The problem of optimization (2.1) associated to X and W is

$$\min_{(y,0)\in C} \left\{ \frac{\alpha}{2} \|(y, 0) - (4, 0)\|^2 + \langle (8, a), (y, 0) - (4, 0)\rangle + \varphi((y, 0)) \right\}.$$

The objective function f is defined by

$$\begin{aligned} f((y,0)) &= \frac{\alpha}{2} \|(y, 0) - (4, 0)\|^2 + \langle (8, a), (y, 0) - (4, 0)\rangle + \varphi((y, 0)) \\ &= (y-4)^2 + 8(y-4) + 1 = y^2 - 8y + 16 + 8y - 32 + 1 = y^2 - 15. \end{aligned}$$

It is clear that $h(X, W) = (0, 0)$ is the unique solution of the above optimization problem and, for every $0 < \theta < 1$, $X - X_0 = (3, 0) \neq \theta(-1, 0) = (h(X, W) - X_0)$. By Theorem 4.2, the multivalued mixed variational inequality (1.1) associated to C, φ and F has at least one solution.

Now, applying Theorem 4.2, we have the following algorithm for solving the multivalued mixed variational inequality (1.1).

Algorithm 4.1. *Step* 0. Choose a tolerance $\varepsilon > 0$, $\lambda \in\ ]0, 1[$, $\alpha \geq \frac{1}{2\gamma}$, $x_0 \in C$ and seek $w_0 \in F(x_0)$. Let $k = 0$.

Step 1. Solve the strongly convex program

$$\min_{y\in C} \left\{ \frac{\alpha}{2} \|y - x_n\|^2 + \langle w_n, y - x_n\rangle + \varphi(y) \right\}$$

to obtain its unique solution y_n. Take $z_n := f_r(y_n)$.

If $\|z_n - x_n\| < \varepsilon$, then the algorithm terminates. x_n is an ε-solution. Otherwise go to Step 2.

Step 2. Take $x_{n+1} := (1-\lambda)x_n + \lambda z_n$. Find $w_{n+1} \in F(x_{n+1})$ such that

$$\|w_{n+1} - w_n\| \leq d_H(F(x_{n+1}), F(x_n)) \leq \sqrt{\frac{\langle w_{n+1} - w_n, x_{n+1} - x_n\rangle}{\gamma}}.$$

Let $n = n + 1$ and return to Step 1.

Remark 4.1. Clearly, if $x_{n_0} = z_{n_0}$ for some n_0, then $x_{n_0} = f_r\left(h\left(x_{n_0}, w_{n_0}\right)\right) = h\left(x_{n_0}, w_{n_0}\right)$. It follows that x_{n_0} is a fixed point of H and therefore, it is a solution of the multivalued mixed variational inequality (1.1).

5. Equilibrium problems: discussion and conclusion

Many results concerning existence of solutions of equilibrium problems are known in the literature and they are generally based on two techniques. The first technique is related to the separation of convex sets, the second, to fixed point theory.

Any solution of the multivalued mixed variational inequality (1.1) is a solution of the equilibrium problem (1.2) where Φ is defined on $C \times C$ by

$$\Phi(x, y) = \sup_{z\in F(x)} \Psi_x(z, y) \quad \forall x, y \in C \tag{5.1}$$

and Ψ_x is a bifunction defined on $F(x) \times C$ by

$$\Psi_x(z, y) = \langle z, y - x\rangle + \varphi(y) - \varphi(x).$$

The converse is true under additional conditions (see, for example, [19,21,25,26,5]).

If φ is a convex function, then the bifunction Ψ_x defined above is convex in its second variable, for every $x \in C$. It follows from [26, Proposition 2.3] that if $x^* \in C$ is such that Ψ_{x^*} is weakly concavelike in its first variable, then the two following conditions are equivalent:

1. If x^* is a solution of the equilibrium problem (1.2) where Φ is defined by (5.1), then x^* is a solution of the multivalued mixed variational inequality (1.1),
2. $F(x^*)$ satisfies the following strong closedness condition: if

$$\left\{\left\{z \in F\left(x^*\right) \mid \Psi_{x^*}(z, y) < -\delta\right\} \mid y \in C, \delta > 0\right\}$$

covers $F(x^*)$, then it contains a finite subcover. We say in this case that $F(x^*)$ is *compactly adapted to* Ψ_{x^*}.

Obviously, if $F(x)$ is a compact convex subset, for every $x \in C$, then by the upper semicontinuity of Ψ_x in its first variable, the set $F(x)$ is compactly adapted to Ψ_x, for every $x \in C$ and then any solution of the equilibrium problem (1.2) where Φ is defined by (5.1) is a solution of the multivalued mixed variational inequality (1.1).

Remark 5.1. Observe that under assumptions of Theorems 3.3 and 4.2, the sets $(F(x))_{x\in C}$ are not supposed convex and neither closed are when $x \in C \setminus \overline{B}(x_0, r)$ under assumptions of Theorem 4.2. In other words, no Ψ_x is supposed weakly concavelike in its first variable and no $F(x)$ is supposed compactly adapted to Ψ_x except under assumptions of Theorem 4.2 when $x \in C \cap \overline{B}(x_0, r)$.

Generally in the literature, most results on existence of solutions of equilibrium problems, except Proposition 2.3 of [26], provide sufficient conditions. Standard assumptions on the equilibrium bifunction Φ are

1. monotonicity of Φ on C, that is, $\Phi(x, y) + \Phi(y, x) \leq 0, \forall x, y \in C$;
2. hemicontinuity of Φ in its first variable on C;
3. convexity and lower semicontinuity of Φ in its second variable on C.

The following coercivity condition on unbounded sets: there exist a compact subset L and a point $y_0 \in L \cap C$ such that

$$\Psi(x, y_0) < 0 \quad \forall x \in C \setminus L,$$

as well as some general convexity concepts are also usually assumed. Proofs of the results concerning existence of solutions of equilibrium problems are generally based on KKM Lemma and Ky Fan Lemma in infinite dimensional spaces (see [20,23,27]). See also [25] for an exposition of some classical and recent results concerning existence of solutions of equilibrium problems.

Remark 5.2. If Φ is defined by (5.1), then the monotonicity of Φ on the whole C is only assumed in Theorem 3.3 and no other property of continuity on Φ on the whole C is assumed in both Theorems 3.3 and 4.2. In comparison with [21, Lemma 2.15], the maximal monotonicity is nowhere assumed and therefore the sets $(F(x))_{x\in C}$ are not assumed to be compacts. The above coercivity condition becomes: there exist a compact subset L and a point $y_0 \in L \cap C$ such that

$$\varphi(x) - \varphi(y_0) > \sup_{z\in F(x)} \langle z, y_0 - x\rangle \quad \forall x \in C \setminus L.$$

Thus, the coercivity condition is in this case very close to the subdifferentiable function φ. Finally, we point out that under assumptions of Theorems 3.3 and 4.2, both the solutions obtained and their approximation sequences are constructed in the compact convex set $C \cap \overline{B}(x_0, r)$.

To our knowledge, there does not seem to be any result in the literature concerning existence of solutions of multivalued mixed variational inequalities and equilibrium problems when the subdifferentiable function φ is not constant and the multivalued mapping values are not necessarily convex. Also, there does not seem to be any consideration of local conditions on the multivalued mapping and on the equilibrium bifunction. In conclusion, the approach used in this paper provides again sufficient conditions for solving multivalued mixed variational inequalities and equilibrium problems and offers, at least, another attempt in order to reach optimal conditions.

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