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Exact controllability for a one-dimensional wave equation in non-cylindrical domains[☆]

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ABSTRACT

This paper is addressed to a study of the controllability for a one-dimensional wave equation in domains with moving boundary. This equation characterizes the motion of a string with a fixed endpoint and the other moving one. When the speed of the moving endpoint is less than the characteristic speed, by the Hilbert Uniqueness Method, the exact controllability of this equation is established. Also, an explicit dependence of the controllability time on the speed of the moving endpoint is given. Moreover, when the speed of the moving endpoint is equal to the characteristic speed, by a constructive method, we characterize a target set for the exact controllability with smooth controllers.

1. Introduction and main results

Let $T > 0$. Put $Q = (0, 1) \times (0, T)$ and for any given $k \in (0, 1]$, set $\alpha_k(t) = 1 + kt$ for $t \in [0, T]$. Denote by $\widehat{Q}_T^k$ the following non-cylindrical domain in $\mathbb{R}^2$:

$$\widehat{Q}_T^k = \left\{(y, t) \in \mathbb{R}^2; 0 < y < \alpha_k(t), t \in (0, T)\right\}.$$

It is easy to check that $\widehat{Q}_T^k = Q$ for $k = 0$.

Consider the following controlled wave equation in the non-cylindrical domain $\widehat{Q}_T^k$:

$$\begin{cases} u_{tt} - u_{yy} = 0 & & \text{in } \widehat{Q}_T^k, \\ u(0, t) = 0, & u(\alpha_k(t), t) = v(t) & \text{on } (0, T), \\ u(0) = u^0, & u_t(0) = u^1 & \text{in } (0, 1), \end{cases} \tag{1.1}$$

where u is the state variable, v is the control variable and $\left(u^0, u^1\right) \in L^2(0, 1) \times H^{-1}(0, 1)$ is any given initial value. (1.1) may describe the motion of a string with a fixed endpoint and a moving one. The constant k is said to be the speed of the moving endpoint. By Milla Miranda [7, pp. 451–452], for $0 < k < 1$, any $\left(u^0, u^1\right) \in L^2(0, 1) \times H^{-1}(0, 1)$ and $v \in L^2(0, T)$, (1.1) admits a unique solution in the sense of transposition.

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The goal of this paper is to study the exact controllability of (1.1) in the following sense.

Definition 1.1. (1.1) is said to be exactly controllable at the time T, if for any initial value $\left(u^0, u^1\right) \in L^2(0,1)\times H^{-1}(0,1)$ and any target $\left(u_d^0, u_d^1\right) \in L^2(0,\alpha_k(T)) \times H^{-1}(0,\alpha_k(T))$, one can always find a control $v \in L^2(0,T)$ such that the corresponding solution u of (1.1) in the sense of transposition satisfies

$$u(T) = u_d^0, \qquad u_t(T) = u_d^1.$$

For $k \in (0,1)$, write

$$T_k^* = \frac{e^{\frac{2k(1+k)}{1-k}} - 1}{k}. \tag{1.2}$$

Then, the first main result of this paper is stated as follows.

Theorem 1.1. *Suppose that $k \in (0,1)$. For any given $T > T_k^*$, (1.1) is exactly controllable at time T in the sense of Definition 1.1.*

Remark 1.1. It is easy to check that

$$T^* \triangleq \lim_{k\to 0} T_k^* = \lim_{k\to 0} \frac{e^{\frac{2k(1+k)}{1-k}} - 1}{k} = 2.$$

It is well known that the wave equation (1.1) in the cylindrical domain Q (or $\widehat{Q}_T^0$) is null controllable at any time $T > T^*$. However, we do not know whether the controllability time T_k^* is sharp.

Notice that the method used in the proof of Theorem 1.1 is not applicable to the case of $k = 1$ (see Remark 4.2 for more explanations). Therefore, when $k = 1$, we are interested in the following exact controllability for (1.1) with smooth controllers and initial data.

Definition 1.2. For any given initial value $(u^0, u^1) \in C^2([0,1]) \times C^1([0,1])$, a target $(u_d^0, u_d^1) \in C^2([0,1]) \times C^1([0,1])$ is exactly controllable with a smooth control at time T, if one can always find a control v in the class of

$$\mathbb{U} \triangleq \left\{ v \in C^2([0,T]);\ v(0) = u^0(1),\ v_t(0) = u_y^0(1) + u^1(1),\ v_{tt}(0) = 2u_{yy}^0(1) + 2u_y^1(1) \right\},$$

such that the corresponding classical solution u of (1.1) satisfies

$$u(T) = u_d^0, \qquad u_t(T) = u_d^1.$$

Remark 1.2. We impose some conditions on a smooth control v in Definition 1.2 (see the definition of $\mathbb{U}$). This is because a classical solution u of (1.1) has to satisfy certain compatibility conditions.

By a constructive method, we get the following exact controllability result for (1.1) in the case of $k = 1$.

Theorem 1.2. *Suppose that $k = 1$. Let $T > 1$. For any given initial value $(u^0, u^1) \in C^2([0,1]) \times C^1([0,1])$ satisfying $u^0(0) = u^1(0) = u_{yy}^0(0) = 0$, a target (u_d^0, u_d^1) is exactly controllable with a smooth control at time T in the sense of Definition 1.2 if and only if $(u_d^0, u_d^1) \in C^2([0,1]) \times C^1([0,1])$ satisfies the following assumptions:*

$$\begin{cases} u_d^0(0) = u_d^1(0) = u_{d,yy}^0(0) = 0, \\ u_{d,y}^0(y) - u_d^1(y) = u_y^0(T-y) + u^1(T-y) & y \in [T-1, T], \\ u_{d,y}^0(y) - u_d^1(y) = u_y^0(y-T) - u^1(y-T) & y \in [T, T+1], \\ u_{d,y}^1(T-1) - u_{d,yy}^0(T-1) = u_{yy}^0(1) + u_y^1(1). \end{cases}$$

Several further remarks are in order.

Remark 1.3. It would be quite interesting to study the exact controllability for multi-dimensional wave equations in a non-cylindrical domain without the additional conditions through a boundary controller or a locally distributed one. We will consider these problems in the forthcoming papers.

Remark 1.4. It seems natural to expect that the exact controllability for (1.1) in the sense of Definition 1.1 holds for the case of $k = 1$. However, we do not success to extend the approach developed in Theorem 1.1 to this case.

Remark 1.5. It would be quite interesting to study the controllability for (1.1) in the non-cylindrical domain $\widehat{Q}_T^k$ for the case of $k > 1$. However, it seems very difficult and remains to be done. This is because in order to guarantee the well-posedness of (1.1), one has to impose two boundary conditions on the moving boundary [4].

There are numerous works addressing the controllability problems of wave equations in a cylindrical domain (see e.g. [3,5,6,9–11] and the rich references cited therein). However, as far as we know, only a few papers have been published on the controllability of wave equations in non-cylindrical domains. In [2], the exact controllability of a multi-dimensional wave equation with constant coefficients in a non-cylindrical domain was established, while a control entered the system through the whole non-cylindrical domain. Also, [2] requires the boundary to be time like, and therefore in the case of $k = 1$, the non-cylindrical domain $\widehat{Q}_T^k$ is not considered in the one-dimensional case in [2]. In [1], the exact controllability of a semi-linear wave equation with variable coefficients in a non-cylindrical domain was investigated. But some additional conditions on the moving boundary were required, which enable the method used in [1] not to be applicable to the controllability problems of (1.1). In [7], the boundary controllability problem for a multi-dimensional wave equation with constant coefficients in a non-cylindrical domain was discussed. However, in the one-dimensional case, the following condition seems necessary:

$$\int_0^\infty |\alpha_k'(t)|dt < \infty.$$

It is easy to check that this condition is not satisfied for the moving boundary in (1.1). In order to overcome these difficulties and drop the additional conditions for the moving boundary, in the case of $0 < k < 1$, we transform (1.1) into an equivalent wave equation with variable coefficients in the cylindrical domain and establish the exact controllability of this equation by the Hilbert Uniqueness Method. The key point is to construct a suitable weighted energy for a wave equation with variable coefficients and characterize the polynomial decay rate for the energy explicitly (see (3.4)). On the other hand, in order to treat the case of $k = 1$, we introduce an auxiliary boundary value problem in a triangular domain (see (5.2)) and obtain the desired controllability result by a constructive method.

The rest of this paper is organized as follows. In Section 2, we reduce the controllability problem of (1.1) to that of a wave equation with variable coefficients in a cylindrical domain. Section 3 is devoted to proving two key inequalities for a wave equation with variable coefficients. In Section 4, we prove an equivalent controllability result to Theorem 1.1. Finally, in Section 5, we give a proof of Theorem 1.2 by a constructive method.

2. Reduction to controllability problems in a cylindrical domain

When $0 < k < 1$, in order to prove the first main result of this paper (Theorem 1.1), we first transform (1.1) into a wave equation with variable coefficients in a cylindrical domain in this section. To this aim, set

$$x = \frac{y}{\alpha_k(t)} \quad \text{and} \quad w(x, t) = u(y, t) = u(\alpha_k(t)x, t) \text{ for } (y, t) \in \widehat{Q}_T^k.$$

Then, it is easy to check that (x, t) varies in Q. Also, (1.1) is transformed into the following equivalent wave equation in the cylindrical domain Q:

$$\begin{cases} w_{tt} - \left[\dfrac{\beta_k(x, t)}{\alpha_k(t)} w_x\right]_x + \dfrac{\gamma_k(x)}{\alpha_k(t)} w_{tx} = 0 & \text{in } Q, \\ w(0, t) = 0, \quad w(1, t) = v(t) & \text{on } (0, T), \\ w(0) = w^0, \quad w_t(0) = w^1 & \text{in } (0, 1), \end{cases} \tag{2.1}$$

where

$$\beta_k(x, t) = \frac{1 - k^2x^2}{\alpha_k(t)}, \qquad \gamma_k(x) = -2kx, \qquad w^0 = u^0, \qquad w^1 = u^1 + kxu_x^0. \tag{2.2}$$

By a similar method used in the proof of Theorem 5.1 in [8], we get that for any given $(u^0, u^1) \in L^2(0, 1) \times H^{-1}(0, 1)$ and $v \in L^2(0, T)$, (2.1) admits a unique solution in the sense of transposition

$$w \in C([0, T]; L^2(0, 1)) \bigcap C^1([0, T]; H^{-1}(0, 1)).$$

Therefore, the exact controllability of (1.1) (Theorem 1.1) is reduced to the following controllability result for the wave equation (2.1).

Theorem 2.1. *Suppose that $k \in (0, 1)$. Let $T > T_k^*$ (recall (1.2)). Then, for any initial value $(w^0, w^1) \in L^2(0, 1) \times H^{-1}(0, 1)$ and target $(w_d^0, w_d^1) \in L^2(0, 1) \times H^{-1}(0, 1)$, there exists a control $v \in L^2(0, T)$ such that the corresponding solution w of (2.1) in the sense of transposition satisfies*

$$w(T) = w_d^0, \qquad w_t(T) = w_d^1.$$

The proof of Theorem 2.1 will be given in Section 4.

3. Two inequalities for wave equations with variable coefficients

In this section, we prove two key inequalities for the following homogeneous wave equation:

$$\begin{cases} \alpha_k(t)z_{tt} - [\beta_k(x,t)z_x]_x + \gamma_k(x)z_{tx} = 0 & \text{in } Q, \\ z(0,t) = 0, \qquad z(1,t) = 0 & \text{on } (0,T), \\ z(0) = z^0, \qquad z_t(0) = z^1 & \text{in } (0,1), \end{cases} \tag{3.1}$$

where $k \in (0,1)$, $(z^0, z^1) \in H_0^1(0,1) \times L^2(0,1)$ is any given initial value, and α_k, β_k and γ_k are the functions given in (2.1). Similar to Theorem 3.2 in [8], we have that (3.1) has a unique weak solution

$$z \in C([0,T]; H_0^1(0,1)) \bigcap C^1([0,T]; L^2(0,1)).$$

Define the following weighted energy for (3.1):

$$E(t) = \frac{1}{2}\int_0^1 \left[\alpha_k(t)|z_t(x,t)|^2 + \beta_k(x,t)|z_x(x,t)|^2\right] dx \quad \text{for } t \geq 0,$$

where z is the solution of (3.1). It follows that

$$E_0 \triangleq E(0) = \frac{1}{2}\int_0^1 \left[|z^1(x)|^2 + \beta_k(x,0)|z_x^0(x)|^2\right] dx.$$

Note that this weighted energy is different from the usual one, but they are equivalent. In the sequel, we denote by C a positive constant depending only on T and k, which may be different from one place to another.

In order to prove Theorem 2.1, we need the following two key inequalities.

Theorem 3.1. *Let $T > 0$. For any $(z^0, z^1) \in H_0^1(0,1) \times L^2(0,1)$, there exists a constant $C > 0$ such that the corresponding solution z of (3.1) satisfies*

$$\int_0^T \beta_k(1,t)|z_x(1,t)|^2 dt \leq C\left(|z^0|^2_{H_0^1(0,1)} + |z^1|^2_{L^2(0,1)}\right). \tag{3.2}$$

Theorem 3.2. *Let $T > T_k^*$ (recall (1.2)). For any $(z^0, z^1) \in H_0^1(0,1) \times L^2(0,1)$, there exists a constant $C > 0$ such that the corresponding solution z of (3.1) satisfies*

$$\int_0^T \beta_k(1,t)|z_x(1,t)|^2 dt \geq C\left(|z^0|^2_{H_0^1(0,1)} + |z^1|^2_{L^2(0,1)}\right). \tag{3.3}$$

First, we prove two lemmas, which will be used in the proofs of these inequalities. The first lemma is related to the decay rate for the energy $E(t)$.

Lemma 3.1. *For any $(z^0, z^1) \in H_0^1(0,1) \times L^2(0,1)$ and $t \in [0,T]$, we have*

$$E(t) = \frac{1}{\alpha_k(t)} E_0. \tag{3.4}$$

Proof. Multiplying the first equation of (3.1) by z_t and integrating on $(0,1) \times (0,t)$ for any $0 < t \leq T$, we get

$$\begin{aligned} 0 &= \int_0^t \int_0^1 \left\{\alpha_k(s)z_{tt}(x,s)z_t(x,s) - [\beta_k(x,s)z_x(x,s)]_x z_t(x,s) + \gamma_k(x)z_{tx}(x,s)z_t(x,s)\right\} dxds \\ &\triangleq J_1 + J_2 + J_3. \end{aligned}$$

In the following, we calculate the above three integrals J_i $(i = 1,2,3)$, respectively. By the definition of α_k, it is easy to check that

$$\begin{aligned} J_1 &= \int_0^t \int_0^1 \frac{1}{2}\alpha_k(s)\left[|z_t(x,s)|^2\right]_t dxds \\ &= \frac{1}{2}\int_0^1 \alpha_k(s)|z_t(x,s)|^2 dx\Big|_0^t - \frac{k}{2}\int_0^t \int_0^1 |z_t(x,s)|^2 dxds. \end{aligned} \tag{3.5}$$

Further, by $z(0,t)=z(1,t)=0$ on $(0,T)$, it holds that

$$\begin{aligned} J_2 &= -\int_0^t \beta_k(x,s)z_x(x,s)z_t(x,s)ds\Big|_0^1 + \int_0^t\int_0^1 \beta_k(x,s)z_x(x,s)z_{tx}(x,s)dxds \\ &= \int_0^t\int_0^1 \frac{1}{2}\beta_k(x,s)\left[|z_x(x,s)|^2\right]_t dxds \\ &= \frac{1}{2}\int_0^1 \beta_k(x,s)|z_x(x,s)|^2dx\Big|_0^t - \frac{1}{2}\int_0^t\int_0^1 \beta_{k,t}(x,s)|z_x(x,s)|^2dxds. \end{aligned}$$

By (2.2), we have

$$\beta_{k,t}(x,t) = -\frac{k(1-k^2x^2)}{(1+kt)^2} = -\frac{k}{(1+kt)}\beta_k(x,t).$$

Therefore, it follows that

$$J_2 = \frac{1}{2}\int_0^1 \beta_k(x,s)|z_x(x,s)|^2dx\Big|_0^t + \frac{1}{2}\int_0^t \frac{k}{(1+ks)}\int_0^1 \beta_k(x,s)|z_x(x,s)|^2dxds. \tag{3.6}$$

Further, from the definition of γ_k, we see

$$\begin{aligned} J_3 &= \int_0^t\int_0^1 \frac{1}{2}\gamma_k(x)\left[|z_t(x,s)|^2\right]_x dxds = -\frac{1}{2}\int_0^t\int_0^1 \gamma_{k,x}(x)|z_t(x,s)|^2dxds \\ &= k\int_0^t\int_0^1 |z_t(x,s)|^2dxds. \end{aligned} \tag{3.7}$$

Therefore, by (3.5)–(3.7) and the definition of $E(t)$, we obtain that

$$\begin{aligned} E(t) &= E_0 - \frac{1}{2}\int_0^t \frac{k}{(1+ks)}\int_0^1 \beta_k(x,s)|z_x(x,s)|^2dxds - \frac{k}{2}\int_0^t\int_0^1 |z_t(x,s)|^2dxds \\ &= E_0 - \frac{1}{2}\int_0^t \frac{k}{(1+ks)}\int_0^1 \beta_k(x,s)|z_x(x,s)|^2dxds - \frac{1}{2}\int_0^t \frac{k}{(1+ks)}\int_0^1 \alpha_k(x,s)|z_t(x,s)|^2dxds \\ &= E_0 - \int_0^t \frac{k}{(1+ks)}E(s)ds. \end{aligned}$$

This implies that

$$E_t(t) = -\frac{k}{1+kt}E(t), \quad 0\le t\le T.$$

It follows that

$$[(1+kt)E(t)]_t = 0, \quad 0\le t\le T,$$

which completes the proof of Lemma 3.1. □

Remark 3.1. Notice that Lemma 3.1 also holds for the case of $k=1$. Therefore, it is easy to check that for any $0<k\le 1$, $(w^0,w^1)\in L^2(0,1)\times H^{-1}(0,1)$ and $v\in L^2(0,T)$, the solution of (2.1) in the sense of transposition is unique. This implies that for any $(u^0,u^1)\in L^2(0,1)\times H^{-1}(0,1)$ and $v\in L^2(0,T)$, the solution of (1.1) in the sense of transposition for any given $0<k\le 1$ is unique.

By the multiplier method, we have the following estimate for every weak solution of (3.1).

Lemma 3.2. *Suppose that $q\in C^1([0,1])$ is any given function. Then any solution z of (3.1) satisfies the following estimate:*

$$\begin{aligned} \left[\frac{1}{2}\int_0^T \beta_k(x,t)q(x)|z_x(x,t)|^2dt\right]\Big|_0^1 &= \frac{1}{2}\int_0^T\int_0^1 q_x(x)\left[\alpha_k(t)|z_t(x,t)|^2 + \beta_k(x,t)|z_x(x,t)|^2\right]dxdt \\ &- \int_0^T\int_0^1 \alpha_{k,t}(t)q(x)z_t(x,t)z_x(x,t)dxdt - \frac{1}{2}\int_0^T\int_0^1 \beta_{k,x}(x,t)q(x)|z_x(x,t)|^2dxdt \\ &+ \int_0^1 \left[\alpha_k(t)q(x)z_t(x,t)z_x(x,t) + \frac{1}{2}\gamma_k(x)q(x)|z_x(x,t)|^2\right]dx\Big|_0^T. \end{aligned} \tag{3.8}$$

Proof. Multiplying the first equation of (3.1) by qz_x and integrating on Q, we have

$$\begin{aligned} 0 &= \int_0^T \int_0^1 \alpha_k(t) z_{tt}(x,t) q(x) z_x(x,t) dxdt - \int_0^T \int_0^1 [\beta_k(x,t) z_x(x,t)]_x \, q(x) z_x(x,t) dxdt \\ &\quad + \int_0^T \int_0^1 \gamma_k(x) z_{tx}(x,t) q(x) z_x(x,t) dxdt \\ &\triangleq L_1 + L_2 + L_3. \end{aligned}$$

In the following, we calculate the above three integrals $L_i (i = 1, 2, 3)$, respectively. It is easy to check that

$$\begin{aligned} L_1 &= \int_0^1 \alpha_k(t) q(x) z_t(x,t) z_x(x,t) dx \Big|_0^T - \int_0^T \int_0^1 \left[\alpha_{k,t}(t) q(x) z_t(x,t) z_x(x,t) + \alpha_k(t) q(x) z_t(x,t) z_{tx}(x,t)\right] dxdt \\ &= \int_0^1 \alpha_k(t) q(x) z_t(x,t) z_x(x,t) dx \Big|_0^T - \int_0^T \int_0^1 \alpha_{k,t}(t) q(x) z_t(x,t) z_x(x,t) dxdt \\ &\quad + \frac{1}{2} \int_0^T \int_0^1 \alpha_k(t) q_x(x) |z_t(x,t)|^2 dxdt. \end{aligned} \tag{3.9}$$

Further,

$$\begin{aligned} L_2 &= -\int_0^T \beta_k(x,t) q(x) |z_x(x,t)|^2 dt \Big|_0^1 + \int_0^T \int_0^1 \left[\beta_k(x,t) q_x(x) |z_x(x,t)|^2 + \beta_k(x,t) q(x) z_x(x,t) z_{xx}(x,t)\right] dxdt \\ &= -\int_0^T \beta_k(x,t) q(x) |z_x(x,t)|^2 dt \Big|_0^1 + \int_0^T \int_0^1 \beta_k(x,t) q_x(x) |z_x(x,t)|^2 dxdt \\ &\quad + \frac{1}{2} \int_0^T \beta_k(x,t) q(x) |z_x(x,t)|^2 dt \Big|_0^1 - \frac{1}{2} \int_0^T \int_0^1 [\beta_k(x,t) q(x)]_x |z_x(x,t)|^2 dxdt \\ &= -\frac{1}{2} \int_0^T \beta_k(x,t) q(x) |z_x(x,t)|^2 dt \Big|_0^1 + \frac{1}{2} \int_0^T \int_0^1 \left[\beta_k(x,t) q_x(x) |z_x(x,t)|^2 - \beta_{k,x}(x,t) q(x) |z_x(x,t)|^2\right] dxdt \end{aligned} \tag{3.10}$$

and

$$L_3 = \frac{1}{2} \int_0^1 \gamma_k(x) q(x) |z_x(x,t)|^2 dx \Big|_0^T. \tag{3.11}$$

By (3.9)–(3.11), we arrive at the desired estimate (3.8). $\square$

Now, we give a proof of Theorem 3.1.

Proof of Theorem 3.1. First, we choose $q(x) = x$ for $x \in [0, 1]$ in (3.8). Noting that $\alpha_k'(t) = k$, $\beta_{k,x}(x,t) = \frac{-2k^2 x}{1+kt}$ and $\gamma_k(x) = -2kx$, it follows that

$$\begin{aligned} \frac{1}{2} \int_0^T \beta_k(1,t) |z_x(1,t)|^2 dt &= \int_0^T E(t) dt - \int_0^T \int_0^1 kx z_t(x,t) z_x(x,t) dxdt \\ &\quad + \int_0^T \int_0^1 \frac{k^2 x^2}{1+kt} |z_x(x,t)|^2 dxdt + \int_0^1 \left[\alpha_k(t) x z_t(x,t) z_x(x,t) - kx^2 |z_x(x,t)|^2\right] dx \Big|_0^T. \end{aligned} \tag{3.12}$$

Next, we estimate every terms on the right side of (3.12). Notice that $1 \le \alpha_k(t) \le 1 + kT$ and $0 < \frac{1-k^2}{1+kT} \le \beta_k(x,t) \le 1$ for any $(x,t) \in Q$. By (3.4), we have

$$\begin{aligned} &\int_0^T E(t) dt - \int_0^T \int_0^1 kx z_t(x,t) z_x(x,t) dxdt + \int_0^T \int_0^1 \frac{k^2 x^2}{1+kt} |z_x(x,t)|^2 dxdt \\ &\le \int_0^T E(t) dt + C \int_0^T \int_0^1 \left[|z_t(x,t)|^2 + |z_x(x,t)|^2\right] dxdt \\ &\le \int_0^T E(t) dt + C \int_0^T \int_0^1 \left[\alpha_t(t) |z_t(x,t)|^2 + \beta_k(x,t) |z_x(x,t)|^2\right] dxdt \le CE_0. \end{aligned} \tag{3.13}$$

On the other hand, for each $t \in [0, T]$ and $\varepsilon > 0$, it holds that

$$\begin{aligned}
&\left|\int_0^1 \left[\alpha_k(t)xz_t(x,t)z_x(x,t) - kx^2|z_x(x,t)|^2\right]dx\right| \\
&\le \sqrt{1+kt}\left[\frac{1}{2\varepsilon}\int_0^1 \alpha_k(t)|z_t(x,t)|^2dx + \frac{\varepsilon}{2}\int_0^1 x^2|z_x(x,t)|^2dx\right] + k\int_0^1 x^2|z_x(x,t)|^2dx \\
&\le \frac{\sqrt{1+kt}}{2\varepsilon}\int_0^1 \alpha_k(t)|z_t(x,t)|^2dx + \left(\frac{\sqrt{1+kt}}{2}\varepsilon + k\right)\int_0^1 x^2|z_x(x,t)|^2dx \\
&\le \frac{\sqrt{1+kt}}{\varepsilon}\frac{1}{2}\int_0^1 \alpha_k(t)|z_t(x,t)|^2dx + \frac{2\left(\frac{\sqrt{1+kt}}{2}\varepsilon + k\right)(1+kt)}{1-k^2}\frac{1}{2}\int_0^1 \beta_k(x,t)|z_x(x,t)|^2dx.
\end{aligned}$$

Take $\varepsilon = \frac{1-k}{\sqrt{1+kt}}$, then it is easy to check that

$$\varepsilon > 0 \quad \text{and} \quad \frac{\sqrt{1+kt}}{\varepsilon} = \frac{2\left(\frac{\sqrt{1+kt}}{2}\varepsilon + k\right)(1+kt)}{1-k^2} = \frac{1+kt}{1-k}.$$

This implies that for any $t \in [0, T]$,

$$\left|\int_0^1 \left[\alpha_k(t)xz_t(x,t)z_x(x,t) - kx^2|z_x(x,t)|^2\right]dx\right| \le \frac{1+kt}{1-k}E(t) = \frac{1}{1-k}E_0.$$

It follows that

$$\left|\int_0^1 \left[\alpha_k(t)xz_t(x,t)z_x(x,t) - kx^2|z_x(x,t)|^2\right]dx\Big|_0^T\right| \le \frac{2}{1-k}E_0. \tag{3.14}$$

Therefore, by (3.12)–(3.14), we have

$$\frac{1}{2}\int_0^T \beta_k(1,t)|z_x(1,t)|^2dt \le CE_0 \le C\left(|z^0|^2_{H_0^1(0,1)} + |z^1|^2_{L^2(0,1)}\right). \quad \square$$

Remark 3.2. Theorem 3.1 implies that for any $(z_0, z_1) \in H_0^1(0,1) \times L^2(0,1)$, the corresponding solution z of (3.1) satisfies $z_x(1,\cdot) \in L^2(0,T)$.

In the following, we give a proof of Theorem 3.2.

Proof of Theorem 3.2. We give an estimate from below for the terms on the right side of (3.12). First, for any given $\varepsilon \in \left(0, \frac{1}{2}\right)$, we have

$$\begin{aligned}
&\int_0^T E(t)dt - \int_0^T\int_0^1 kxz_t(x,t)z_x(x,t)dxdt + \int_0^T\int_0^1 \frac{k^2x^2}{1+kt}|z_x(x,t)|^2dxdt \\
&\ge \int_0^T\int_0^1 \left\{\frac{1-\varepsilon}{2}\alpha_k(t)|z_t(x,t)|^2 + \left[\frac{1}{2}\beta_k(x,t) + \left(1-\frac{1}{2\varepsilon}\right)\frac{k^2x^2}{1+kt}\right]|z_x(x,t)|^2\right\}dxdt \\
&= \int_0^T\int_0^1 \left\{(1-\varepsilon)\frac{\alpha_k(t)}{2}|z_t(x,t)|^2 + \left[1 + \left(2-\frac{1}{\varepsilon}\right)\frac{k^2x^2}{1-k^2x^2}\right]\frac{\beta_k(x,t)}{2}|z_x(x,t)|^2\right\}dxdt \\
&\ge \int_0^T\int_0^1 \left\{(1-\varepsilon)\frac{\alpha_k(t)}{2}|z_t(x,t)|^2 + \left[1 + \left(2-\frac{1}{\varepsilon}\right)\frac{k^2}{1-k^2}\right]\frac{\beta_k(x,t)}{2}|z_x(x,t)|^2\right\}dxdt.
\end{aligned}$$

Take $\varepsilon = \frac{k}{1+k}$, then it is easy to check that

$$0 < \varepsilon < \frac{1}{2} \quad \text{and} \quad 1-\varepsilon = 1 + \left(2-\frac{1}{\varepsilon}\right)\frac{k^2}{1-k^2} = \frac{1}{1+k}.$$

It follows that

$$\begin{aligned}
&\int_0^T E(t)dt - \int_0^T\int_0^1 kxz_t(x,t)z_x(x,t)dxdt + \int_0^T\int_0^1 \frac{k^2x^2}{1+kt}|z_x(x,t)|^2dxdt \\
&\ge \frac{1}{1+k}\int_0^T E(t)dt = \frac{1}{1+k}\int_0^T \frac{1}{1+kt}dtE_0.
\end{aligned} \tag{3.15}$$

Hence, by (3.12), (3.15) and (3.14), we obtain

$$\begin{aligned}\frac{1}{2}\int_0^T \beta_k(1,t)|z_x(1,t)|^2dt &\geq \frac{1}{1+k}\int_0^T \frac{1}{1+kt}dtE_0 - \frac{2}{1-k}E_0\\ &= \frac{1}{k(1+k)}\ln(1+kT)E_0 - \frac{2}{1-k}E_0 = \left[\frac{1}{k(1+k)}\ln(1+kT) - \frac{2}{1-k}\right]E_0.\end{aligned}$$

If $T > T_k^*$ (recall (1.2)), it holds that $\frac{1}{k(1+k)}\ln(1+kT) - \frac{2}{1-k} > 0$. Also,

$$\frac{1}{2}\int_0^T \beta_k(1,t)|z_x(1,t)|^2dt \geq C\left[\frac{1}{k(1+k)}\ln(1+kT) - \frac{2}{1-k}\right](|z^0|^2_{H_0^1(0,1)} + |z^1|^2_{L^2(0,1)}).$$

This completes the proof of Theorem 3.2. □

4. Exact controllability in the case of $0 < k < 1$

In this section, we prove the exact controllability for the wave equation (2.1) in the cylindrical domain Q (Theorem 2.1) for $0 < k < 1$ by the Hilbert Uniqueness Method.

Proof of Theorem 2.1. We divide the proof of Theorem 2.1 into three parts.

Step 1. First, we define a linear operator $\Lambda : H_0^1(0,1) \times L^2(0,1) \to H^{-1}(0,1) \times L^2(0,1)$.

For any $(z^0, z^1) \in H_0^1(0,1) \times L^2(0,1)$, denote by z the corresponding solution of (3.1). Consider the following homogeneous wave equation:

$$\begin{cases}\eta_{tt} - \left[\dfrac{\beta_k(x,t)}{\alpha_k(t)}\eta_x\right]_x + \dfrac{\gamma_k(x)}{\alpha_k(t)}\eta_{tx} = 0 & \text{in } Q,\\ \eta(0,t) = 0, \qquad \eta(1,t) = z_x(1,t) & \text{on } (0,T),\\ \eta(T) = \eta_t(T) = 0 & \text{in } (0,1).\end{cases} \tag{4.1}$$

Then, it is well known that (4.1) admits a unique solution in the sense of transposition

$$\eta \in C([0,T]; L^2(0,1))\bigcap C^1([0,T]; H^{-1}(0,1)).$$

Moreover, by Theorem 3.1, there exists a constant C such that

$$\begin{aligned}|\eta|_{C([0,T];L^2(0,1))\cap C^1([0,T];H^{-1}(0,1))} &= \left(\sup_{t\in[0,T]}|\eta(\cdot,t)|^2_{L^2(0,1)} + \sup_{t\in[0,T]}|\eta_t(\cdot,t)|^2_{H^{-1}(0,1)}\right)^{\frac{1}{2}}\\ &\leq C|z_x(1,\cdot)|_{L^2(0,T)} \leq C\left(|z^0|_{H_0^1(0,1)} + |z^1|_{L^2(0,1)}\right).\end{aligned} \tag{4.2}$$

For any $(z^0, z^1) \in H_0^1(0,1) \times L^2(0,1)$, define a linear operator Λ:

$$\begin{aligned}&\Lambda : H_0^1(0,1) \times L^2(0,1) \to H^{-1}(0,1) \times L^2(0,1),\\ &(z^0, z^1) \mapsto (\eta_t(\cdot,0) + \gamma_k(\cdot)\eta_x(\cdot,0) - k\eta(\cdot,0), -\eta(\cdot,0)),\end{aligned}$$

where we use z to denote the solution of (3.1) associated to z_0 and z_1, and η denotes the solution of (4.1) associated to z. Write $F = H_0^1(0,1) \times L^2(0,1)$ and denote by F' its conjugate space. Also, define a bilinear form A on $\left(H_0^1(0,1) \times L^2(0,1)\right)^2$ as follows:

$$\begin{aligned}A\left((z^0,z^1),(y^0,y^1)\right) &\triangleq \langle \Lambda(z^0,z^1),(y^0,y^1)\rangle_{F',F}\\ &= \langle \eta_t(\cdot,0) + \gamma_k(\cdot)\eta_x(\cdot,0), y^0\rangle_{H^{-1},H_0^1} - \int_0^1 \left[k\eta(x,0)y^0(x) + \eta(x,0)y^1(x)\right]dx,\end{aligned}$$

for any $(z^0,z^1),(y^0,y^1) \in H_0^1(0,1) \times L^2(0,1)$, where η denotes the solution of (3.1) and (4.1) associated to (z^0,z^1).

Step 2. We prove that Λ is an isomorphism, when $T > T_k^*$.

Multiplying the first equation of (4.1) by $\alpha_k(t)z$ and integrating on Q, by (3.1), we obtain that

$$\int_0^T \beta_k(1,t)|z_x(1,t)|^2dt = \langle \eta_t(\cdot,0) + \gamma_k(\cdot)\eta_x(\cdot,0), z^0\rangle_{H^{-1},H_0^1} - \int_0^1 \left[k\eta(x,0)z^0(x) + \eta(x,0)z^1(x)\right]dx.$$

Combining the above equality with the definition of Λ, we have

$$\int_0^T \beta_k(1,t)|z_x(1,t)|^2 dt = \langle \Lambda(z^0,z^1),(z^0,z^1)\rangle_{F',F}. \tag{4.3}$$

By Theorems 3.1 and 3.2, it suffices to prove that Λ is surjective. Notice that Theorem 3.2 and (4.3) imply A is a coercive bilinear form. Moreover, by (4.2), it is easy to check that A is bounded. Therefore, by the Lax–Milgram Theorem, Λ is a surjection. It follows that Λ is an isomorphism.

Step 3. We prove the exact controllability of (2.1).

First, for any target $(w_d^0, w_d^1) \in L^2(0,1)\times H^{-1}(0,1)$, the following wave equation

$$\begin{cases} \xi_{tt} - \left[\dfrac{\beta_k(x,t)}{\alpha_k(t)}\xi_x\right]_x + \dfrac{\gamma_k(x)}{\alpha_k(t)}\xi_{tx} = 0 & \text{in } Q,\\ \xi(0,t)=0, \quad \xi(1,t)=0 & \text{on } (0,T),\\ \xi(T)=w_d^0, \quad \xi_t(T)=w_d^1 & \text{in } (0,1), \end{cases} \tag{4.4}$$

has a unique solution $\xi \in C([0,T];L^2(0,1))\bigcap C^1([0,T];H^{-1}(0,1))$.

Since Λ is an isomorphism, for any initial value $(w^0,w^1)\in L^2(0,1)\times H^{-1}(0,1)$, there exists $(z^0,z^1)\in H_0^1(0,1)\times L^2(0,1)$ such that

$$\Lambda(z^0,z^1) = \left([w^1-\xi_t(0)] - k[w^0-\xi(0)] + \gamma_k[w_x^0-\xi_x(0)], -[w^0-\xi(0)]\right). \tag{4.5}$$

Denote by η the solution of (3.1) and (4.1) associated to (z^0,z^1). Then, by the definition of Λ and (4.5), it is easy to check that $(\eta(0),\eta_t(0)) = \left(w^0-\xi(0), w^1-\xi_t(0)\right)$. Therefore, if we set $w=\xi+\eta$, then w is the solution of (2.1) associated to $v = z_x(1,\cdot)$. Furthermore, $(w(0),w_t(0)) = (w^0,w^1)$ and $(w(T),w_t(T)) = (w_d^0,w_d^1)$. This completes the proof of Theorem 2.1. □

Remark 4.1. By the equivalent transformation in Section 2, Theorem 2.1 implies the exact controllability for (1.1) in the non-cylindrical domain $\widehat{Q}_T^k$ for $0<k<1$ at the time $T>T_k^*$ (Theorem 1.1).

Remark 4.2. Notice that the method used in the proof of Theorem 2.1 seems not to be applicable to the case of $k=1$, since $\beta_k(1,t)=0$ on $(0,T)$ for $k=1$.

5. Exact controllability in the case of $k = 1$

In this section, we study the exact controllability of (1.1) with a smooth control in the non-cylindrical domain $\widehat{Q}_T^1$. To this aim, consider the following wave equation:

$$\begin{cases} u_{tt}-u_{yy}=0 & \text{in } \widehat{Q}_T^1,\\ u(0,t)=0, \quad u(t+1,t)=v(t) & \text{on } (0,T),\\ u(0)=u^0, \quad u_t(0)=u^1 & \text{in } (0,1), \end{cases} \tag{5.1}$$

where $\widehat{Q}_T^1 = \left\{(y,t)\in\mathbb{R}^2; 0<y<t+1, t\in(0,T)\right\}$, $(u^0,u^1)\in C^2([0,1])\times C^1([0,1])$ satisfying $u^0(0)=u^1(0)=u_{yy}^0(0)=0$ and $v\in\mathbb{U}$ (see Definition 1.2).

In order to establish the existence of classical solutions for (5.1), we introduce the following auxiliary boundary value problem in a triangular domain:

$$\begin{cases} U_{tt}-U_{yy}=0 & \text{in } \widehat{\Omega}_T,\\ U(y,-(y+1))=\Psi(y) & \text{on } (-(T+1),0),\\ U(y,y-1)=\Phi(y) & \text{on } (0,T+1), \end{cases} \tag{5.2}$$

where $\widehat{\Omega}_T \triangleq \{(y,t)\in\mathbb{R}^2; -(t+1)<y<t+1, t\in(-1,T)\}$ and $(\Psi,\Phi)\in C^2([-(T+1),0])\times C^2([0,T+1])$ satisfying $\Psi(0)=\Phi(0)$. It is easy to check that for any $T>-1$, one can always find a classical solution U of (5.2) as follows:

$$U(y,t) = \Phi\left(\frac{y+t+1}{2}\right) + \Psi\left(\frac{y-t-1}{2}\right) - \Phi(0), \quad (y,t)\in\overline{\widehat{\Omega}_T}. \tag{5.3}$$

Then, we have the following well-posedness result for classical solutions for (5.1).

Lemma 5.1. *Suppose that $T>1$. For any $(u^0,u^1)\in C^2([0,1])\times C^1([0,1])$ satisfying $u^0(0)=u^1(0)=u_{yy}^0(0)=0$ and $v\in\mathbb{U}$, there exists a unique classical solution u of (5.1).*

Proof. First, we extend (5.1) to the domain $\widehat{\Omega}_T$ in a suitable way. Then, we can find a solution U (defined by (5.3)) of (5.2) in $\overline{\widehat{\Omega}_T}$. Furthermore, we show that $u \triangleq U|_{\overline{Q_T^1}}$ is a unique classical solution of (5.1).

Step 1. For any $(u^0, u^1) \in C^2([0,1]) \times C^1([0,1])$ satisfying $u^0(0) = u^1(0) = u^0_{yy}(0) = 0$ and $v \in \mathbb{U}$, set

$$\Psi(y) = \begin{cases} \widehat{\psi_1}(y) & y \in \left[-\frac{1}{2}, 0\right], \\ \widehat{\psi_2}(y) & y \in \left[-1, -\frac{1}{2}\right], \\ \widehat{\psi_3}(y) & y \in \left[-\frac{T+1}{2}, -1\right], \\ \widehat{\psi_4}(y) & y \in \left[-(T+1), -\frac{T+1}{2}\right], \end{cases} \quad \text{and} \quad \Phi(y) = \begin{cases} \widehat{\varphi_1}(y) & y \in \left[0, \frac{1}{2}\right], \\ \widehat{\varphi_2}(y) & y \in \left[\frac{1}{2}, 1\right], \\ v(y-1) & y \in [1, T+1], \end{cases}$$

where $\widehat{\psi_i}$ $(i = 1,2,3,4)$ and $\widehat{\varphi_i}$ $(i = 1,2)$ are suitable functions to be specified later, such that $\Psi \in C^2([-(T+1), 0])$, $\Phi \in C^2([0, T+1])$ and $\Psi(0) = \Phi(0) = 0$.

Then, there exists a classical solution U (defined by (5.3)) of (5.2) in $\overline{\widehat{\Omega}_T}$. Notice that for $y \in (0,1)$, $\frac{y+1}{2} \in (\frac{1}{2}, 1)$ and $\frac{y-1}{2} \in (-\frac{1}{2}, 0)$. Therefore, by (5.3), if $\Phi(0) = 0$, it holds that $U(y,0) = u^0(y)$ and $U_t(y,0) = u^1(y)$ in $(0,1)$ if and only if for any $y \in [0,1]$, we have that

$$\begin{cases} \Phi\left(\frac{y+1}{2}\right) + \Psi\left(\frac{y-1}{2}\right) - \Phi(0) = \widehat{\varphi_2}\left(\frac{y+1}{2}\right) + \widehat{\psi_1}\left(\frac{y-1}{2}\right) = u^0(y), \\ \frac{1}{2}\Phi'\left(\frac{y+1}{2}\right) - \frac{1}{2}\Psi'\left(\frac{y-1}{2}\right) = \frac{1}{2}\widehat{\varphi_2}'\left(\frac{y+1}{2}\right) - \frac{1}{2}\widehat{\psi_1}'\left(\frac{y-1}{2}\right) = u^1(y). \end{cases}$$

This implies that

$$\begin{cases} \widehat{\psi_1}(y) = \frac{1}{2}\left[-u^0(1) + \int_0^1 u^1(y)dy + u^0(2y+1) - \int_0^{2y+1} u^1(\tau)d\tau\right] & y \in \left[-\frac{1}{2}, 0\right], \\ \widehat{\varphi_2}(y) = \frac{1}{2}\left[u^0(1) - \int_0^1 u^1(y)dy + u^0(2y-1) + \int_0^{2y-1} u^1(\tau)d\tau\right] & y \in \left[\frac{1}{2}, 1\right]. \end{cases} \tag{5.4}$$

On the other hand, notice that for $t \in (0,1)$, $\frac{t+1}{2} \in (\frac{1}{2}, 1)$ and for $t \in (1,T)$, $\frac{t+1}{2} \in (1, \frac{T+1}{2})$. Therefore, if $\Phi(0) = 0$, it holds that $U(0,t) = 0$ on $(0,T)$ if and only if the following conditions hold:

$$\begin{cases} \Phi\left(\frac{t+1}{2}\right) + \Psi\left(-\frac{t+1}{2}\right) - \Phi(0) = \widehat{\varphi_2}\left(\frac{t+1}{2}\right) + \widehat{\psi_2}\left(\frac{-t-1}{2}\right) = 0 & t \in [0,1], \\ \Phi\left(\frac{t+1}{2}\right) + \Psi\left(-\frac{t+1}{2}\right) - \Phi(0) = v\left(\frac{t+1}{2} - 1\right) + \widehat{\psi_3}\left(\frac{-t-1}{2}\right) = 0 & t \in [1,T]. \end{cases}$$

By the change of variable, we conclude that

$$\begin{cases} \widehat{\psi_2}(y) = -\widehat{\varphi_2}(-y) & y \in \left[-1, -\frac{1}{2}\right], \\ \widehat{\psi_3}(y) = -v(-y-1) & y \in \left[-\frac{T+1}{2}, -1\right]. \end{cases} \tag{5.5}$$

Moreover, we take $\widehat{\psi_4} \in C^2(\left[-(T+1), -\frac{T+1}{2}\right])$ and $\widehat{\varphi_1} \in C^2(\left[0, \frac{1}{2}\right])$ such that

$$\begin{aligned} &\widehat{\varphi_1}(0) = 0, \qquad \widehat{\varphi_1}\left(\frac{1}{2}\right) = \frac{1}{2}\left[u^0(1) - \int_0^1 u^1(y)dy\right], \\ &\widehat{\varphi_1}'\left(\frac{1}{2}\right) = u^0_y(0), \qquad \widehat{\varphi_1}''\left(\frac{1}{2}\right) = 2u^1_y(0), \\ &\widehat{\psi_4}\left(-\frac{T+1}{2}\right) = -v\left(\frac{T-1}{2}\right), \qquad \widehat{\psi_4}'\left(-\frac{T+1}{2}\right) = v_t\left(\frac{T-1}{2}\right), \\ &\widehat{\psi_4}''\left(-\frac{T+1}{2}\right) = -v_{tt}\left(\frac{T-1}{2}\right). \end{aligned} \tag{5.6}$$

Then, by $u^0(0)=u^1(0)=u^0_{yy}(0)=0$, $v\in\mathbb{U}$ and (5.6), it is easy to check that Ψ is C^2 at $y=-\frac{T+1}{2},-1,-\frac{1}{2}$ and Φ is C^2 at $y=\frac{1}{2},1$. Therefore, $\Psi\in C^2([-(T+1),0])$, $\Phi\in C^2([0,T+1])$ and $\Psi(0)=\Phi(0)=0$.

In conclusion, (5.4) and (5.5) imply that

$$\Psi(y)=\begin{cases}\frac{1}{2}\left[-u^0(1)+\int_0^1 u^1(y)dy+u^0(2y+1)-\int_0^{2y+1}u^1(\tau)d\tau\right] & y\in\left[-\frac{1}{2},0\right],\\ \frac{1}{2}\left[-u^0(1)+\int_0^1 u^1(y)dy-u^0(-2y-1)-\int_0^{-2y-1}u^1(\tau)d\tau\right] & y\in\left[-1,-\frac{1}{2}\right],\\ -v(-y-1) & y\in\left[-\frac{T+1}{2},-1\right],\\ \widehat{\psi_4}(y) & y\in\left[-(T+1),-\frac{T+1}{2}\right],\end{cases}$$

and

$$\Phi(y)=\begin{cases}\widehat{\varphi_1}(y) & y\in\left[0,\frac{1}{2}\right],\\ \frac{1}{2}\left[u^0(1)-\int_0^1 u^1(y)dy+u^0(2y-1)+\int_0^{2y-1}u^1(\tau)d\tau\right] & y\in\left[\frac{1}{2},1\right],\\ v(y-1) & y\in[1,T+1],\end{cases}$$

where $\widehat{\psi_4}$ and $\widehat{\varphi_1}$ are two functions satisfying (5.6).

Step 2. We give an expression of the classical solution U in $\overline{\widehat{Q_T^1}}$.

To this aim, write $\overline{\widehat{Q_T^1}}=A\bigcup B\bigcup C_1\bigcup C_2\bigcup D$, where

$$\begin{cases}A=\left\{(y,t)\in\mathbb{R}^2;\ t\le y\le -t+1 \text{ and } 0\le t\le\frac{1}{2}\right\},\\ B=\left\{(y,t)\in\mathbb{R}^2;\ -t+1\le y\le t+1 \text{ and } 0\le t\le\frac{1}{2},\ \text{or } t\le y\le t+1 \text{ and } \frac{1}{2}\le t\le T\right\},\\ C_1=\left\{(y,t)\in\mathbb{R}^2;\ -t+1\le y\le t \text{ and } \frac{1}{2}\le t\le 1,\ \text{or } t-1\le y\le t \text{ and } 1\le t\le T\right\},\\ C_2=\left\{(y,t)\in\mathbb{R}^2;\ y\le t\le 1-y \text{ and } 0\le y\le\frac{1}{2}\right\},\\ D=\left\{(y,t)\in\mathbb{R}^2;\ 0\le y\le t-1 \text{ and } 1\le t\le T\right\}.\end{cases}$$

For any $(y,t)\in A$, $\frac{1}{2}\le\frac{y+t+1}{2}\le 1$ and $-\frac{1}{2}\le\frac{y-t-1}{2}\le 0$. Then, by (5.3) and noting the definitions of Φ and Ψ, we have that for any $(y,t)\in A$,

$$\begin{aligned}U(y,t)&=\frac{1}{2}\left[u^0(1)-\int_0^1 u^1(y)dy+u^0(y+t)+\int_0^{y+t}u^1(\tau)d\tau\right]\\&\quad+\frac{1}{2}\left[-u^0(1)+\int_0^1 u^1(y)dy+u^0(y-t)-\int_0^{y-t}u^1(\tau)d\tau\right]\\&=\frac{1}{2}\left[u^0(y+t)+u^0(y-t)+\int_{y-t}^{y+t}u^1(\tau)d\tau\right].\end{aligned}\tag{5.7}$$

Further, for any $(y,t)\in B$, $1\le\frac{y+t+1}{2}\le 1+T$ and $-\frac{1}{2}\le\frac{y-t-1}{2}\le 0$. Then, similarly, we get that for any $(y,t)\in B$,

$$U(y,t)=v\left(\frac{y+t-1}{2}\right)+\frac{1}{2}\left[-u^0(1)+\int_0^1 u^1(y)dy+u^0(y-t)-\int_0^{y-t}u^1(\tau)d\tau\right].\tag{5.8}$$

Further, for any $(y,t)\in C_1$, $1\le\frac{y+t+1}{2}\le T+\frac{1}{2}$ and $-1\le\frac{y-t-1}{2}\le-\frac{1}{2}$. Therefore, it is easy to check that for any $(y,t)\in C_1$,

$$U(y,t)=v\left(\frac{y+t-1}{2}\right)+\frac{1}{2}\left[-u^0(1)+\int_0^1 u^1(y)dy-u^0(-y+t)-\int_0^{-y+t}u^1(\tau)d\tau\right].\tag{5.9}$$

Further, for any $(y,t)\in C_2$, $\frac{1}{2}\le\frac{y+t+1}{2}\le 1$ and $-1\le\frac{y-t-1}{2}\le-\frac{1}{2}$. Then, we have that for any $(y,t)\in C_2$,

$$\begin{aligned}U(y,t)&=\frac{1}{2}\left[u^0(1)-\int_0^1 u^1(y)dy+u^0(y+t)+\int_0^{y+t}u^1(\tau)d\tau\right]\\&\quad+\frac{1}{2}\left[-u^0(1)+\int_0^1 u^1(y)dy-u^0(-y+t)-\int_0^{-y+t}u^1(\tau)d\tau\right]\\&=\frac{1}{2}\left[u^0(y+t)-u^0(-y+t)+\int_{-y+t}^{y+t}u^1(\tau)d\tau\right].\end{aligned}\tag{5.10}$$

Further, for any $(y,t)\in D$, $1\le\frac{y+t+1}{2}\le T$ and $-\frac{T+1}{2}\le\frac{y-t-1}{2}\le-1$. Therefore, we obtain that for any $(y,t)\in D$,

$$U(y,t)=v\left(\frac{y+t-1}{2}\right)-v\left(\frac{-y+t-1}{2}\right).\tag{5.11}$$

Step 3. Set $u=U|_{\overline{Q_T^1}}$. Then, (5.8) implies that $u(t+1,t)=v(t)$ on $(0,T)$. Therefore, it is easy to check that u is a classical solution of (5.1). By (5.7)–(5.11), u is given as follows:

$$u(y,t)=\begin{cases}\frac{1}{2}\left[u^0(y+t)+u^0(y-t)+\int_{y-t}^{y+t}u^1(\tau)d\tau\right] & \text{in } A,\\ v\left(\frac{y+t-1}{2}\right)-\frac{1}{2}\left[u^0(1)-\int_0^1 u^1(y)dy-u^0(y-t)+\int_0^{y-t}u^1(\tau)d\tau\right] & \text{in } B,\\ v\left(\frac{y+t-1}{2}\right)-\frac{1}{2}\left[u^0(1)-\int_0^1 u^1(y)dy+u^0(-y+t)+\int_0^{-y+t}u^1(\tau)d\tau\right] & \text{in } C_1,\\ \frac{1}{2}\left[u^0(y+t)-u^0(-y+t)+\int_{-y+t}^{y+t}u^1(\tau)d\tau\right] & \text{in } C_2,\\ v\left(\frac{y+t-1}{2}\right)-v\left(\frac{-y+t-1}{2}\right) & \text{in } D.\end{cases}\tag{5.12}$$

Since classical solutions of (5.1) must be solutions of it in the sense of transposition, by Remark 3.1, the classical solution u of (5.1) is unique. This finishes the proof of Lemma 5.1. $\square$

Now, we give a proof of Theorem 1.2 by a constructive method.

Proof of Theorem 1.2. For any given initial value $(u^0,u^1)\in C^2([0,1])\times C^1([0,1])$ satisfying $u^0(0)=u^1(0)=u^0_{yy}(0)=0$ and a target (u_d^0,u_d^1), there exists a control $v\in\mathbb{U}$ such that the corresponding classical solution u (see (5.12)) of (5.1) satisfies $u(T)=u_d^0$ and $u_t(T)=u_d^1$, if and only if the following conditions (1)–(3) hold:

(1) By the expression of u in B, for $t=T$ and $y\in[T,T+1]$,

$$\begin{cases}v\left(\frac{y+T-1}{2}\right)-\frac{1}{2}\left[u^0(1)-\int_0^1 u^1(y)dy-u^0(y-T)+\int_0^{y-T}u^1(\tau)d\tau\right]=u_d^0(y),\\ \frac{1}{2}v_t\left(\frac{y+T-1}{2}\right)-\frac{1}{2}u_y^0(y-T)+\frac{1}{2}u^1(y-T)=u_d^1(y);\end{cases}\tag{5.13}$$

(2) By the expression of u in C_1, for $t=T$ and $y\in[T-1,T]$,

$$\begin{cases}v\left(\frac{y+T-1}{2}\right)-\frac{1}{2}\left[u^0(1)-\int_0^1 u^1(y)dy+u^0(-y+T)+\int_0^{-y+T}u^1(\tau)d\tau\right]=u_d^0(y),\\ \frac{1}{2}v_t\left(\frac{y+T-1}{2}\right)-\frac{1}{2}u_y^0(-y+T)-\frac{1}{2}u^1(-y+T)=u_d^1(y);\end{cases}\tag{5.14}$$

(3) By the expression of u in D, for $t=T$ and $y\in[0,T-1]$,

$$\begin{cases}v\left(\frac{y+T-1}{2}\right)-v\left(\frac{-y+T-1}{2}\right)=u_d^0(y),\\ \frac{1}{2}v_t\left(\frac{y+T-1}{2}\right)-\frac{1}{2}v_t\left(\frac{-y+T-1}{2}\right)=u_d^1(y).\end{cases}\tag{5.15}$$

Therefore, by (5.13), it follows that for any $y \in [T, T+1]$,

$$\begin{cases} v\left(\dfrac{y+T-1}{2}\right) = \dfrac{1}{2}\left[u^0(1) - \displaystyle\int_0^1 u^1(y)dy - u^0(y-T) + \int_0^{y-T} u^1(\tau)d\tau\right] + u_d^0(y), \\ u_{d,y}^0(y) - u_d^1(y) = u_y^0(y-T) - u^1(y-T). \end{cases} \tag{5.16}$$

By the change of variable, we obtain

$$\begin{aligned} v_1(t) &\triangleq v(t) \\ &= \frac{1}{2}\left[u^0(1) - \int_0^1 u^1(y)dy - u^0(2t-2T+1) + \int_0^{2t-2T+1} u^1(\tau)d\tau\right] \\ &\quad + u_d^0(2t-T+1) \quad \text{for } t \in \left[T - \frac{1}{2}, T\right]. \end{aligned} \tag{5.17}$$

Further, by (5.14), it holds that for any $y \in [T-1, T]$,

$$\begin{cases} v\left(\dfrac{y+T-1}{2}\right) = \dfrac{1}{2}\left[u^0(1) - \displaystyle\int_0^1 u^1(y)dy + u^0(-y+T) + \int_0^{-y+T} u^1(\tau)d\tau\right] + u_d^0(y), \\ u_{d,y}^0(y) - u_d^1(y) = u_y^0(-y+T) + u^1(-y+T). \end{cases} \tag{5.18}$$

By the change of variable, we have

$$\begin{aligned} v_2(t) &\triangleq v(t) \\ &= \frac{1}{2}\left[u^0(1) - \int_0^1 u^1(y)dy + u^0(2T-1-2t) + \int_0^{2T-1-2t} u^1(\tau)d\tau\right] \\ &\quad + u_d^0(2t-T+1) \quad \text{for } t \in \left[T-1, T - \frac{1}{2}\right]. \end{aligned} \tag{5.19}$$

Further, by (5.15), we see that for any constant C_0 and $y \in [0, T-1]$,

$$\begin{cases} v\left(\dfrac{y+T-1}{2}\right) = C_0 + \dfrac{1}{2}u_d^0(y) + \dfrac{1}{2}\displaystyle\int_0^y u_d^1(\tau)d\tau, \\ v\left(\dfrac{-y+T-1}{2}\right) = C_0 - \dfrac{1}{2}u_d^0(y) + \dfrac{1}{2}\displaystyle\int_0^y u_d^1(\tau)d\tau, \\ u_d^0(0) = u_d^1(0) = 0. \end{cases} \tag{5.20}$$

By the change of variable, we conclude that

$$\begin{cases} v_3(t) \triangleq v(t) = C_0 + \dfrac{1}{2}u_d^0(2t-T+1) + \dfrac{1}{2}\displaystyle\int_0^{2t-T+1} u_d^1(\tau)d\tau & \text{for } t \in \left[\dfrac{T-1}{2}, T-1\right], \\ v_4(t) \triangleq v(t) = C_0 - \dfrac{1}{2}u_d^0(-2t+T-1) + \dfrac{1}{2}\displaystyle\int_0^{-2t+T-1} u_d^1(\tau)d\tau & \text{for } t \in \left[0, \dfrac{T-1}{2}\right]. \end{cases} \tag{5.21}$$

Choose the following control function in (5.1):

$$v(t) = \begin{cases} v_1(t) & t \in \left[T - \dfrac{1}{2}, T\right], \\ v_2(t) & t \in \left[T-1, T - \dfrac{1}{2}\right], \\ v_3(t) & t \in \left[\dfrac{T-1}{2}, T-1\right], \\ v_4(t) & t \in \left[0, \dfrac{T-1}{2}\right], \end{cases}$$

where v_i ($i = 1, 2, 3, 4$) are the functions given in (5.17), (5.19) and (5.21). Then, by conditions (1)–(3), (u_d^0, u_d^1) is exactly controllable with a smooth control if and only if

$$\begin{cases} u_{d,y}^0(y) - u_d^1(y) = u_y^0(T-y) + u^1(T-y) & y \in [T-1, T], \\ u_{d,y}^0(y) - u_d^1(y) = u_y^0(y-T) - u^1(y-T) & y \in [T, T+1], \\ v \in \mathbb{U} \text{ (see Definition 1.2)}. \end{cases} \tag{5.22}$$

It is easy to check that $v \in \mathbb{U}$ if and only if the following conditions hold:

$$\begin{cases} C_0 = u^0(1) + \dfrac{1}{2}u_d^0(T-1) - \dfrac{1}{2}\displaystyle\int_0^{T-1} u_d^1(y)dy, \\ u_d^0(0) = u_d^1(0) = u_{d,yy}^0(0) = 0, \\ u_{d,y}^0(T-1) - u_d^1(T-1) = u_y^0(1) + u^1(1), \\ u_{d,y}^1(T-1) - u_{d,yy}^0(T-1) = u_{yy}^0(1) + u_y^1(1). \end{cases} \tag{5.23}$$

By (5.22) and (5.23), we arrive at the conclusion of Theorem 1.2. □

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