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Singularities of improper affine maps and their Hessian equation☆

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ABSTRACT

We study improper affine spheres with some admissible singularities, called improper affine maps, and associated to the unimodular Hessian equation. In particular, we characterize when a curve of $\mathbb{R}^3$ is the singular curve of some improper affine map with prescribed cuspidal edges and swallowtails. Also, we consider improper affine maps with isolated singularities and show some similarities and differences between the Hessian $+1$ equation and the Hessian -1 equation. As a consequence, we construct global examples with the desired singularities.

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1. Introduction

A celebrated fact in geometric analysis is the correspondence between the solutions of the Monge–Ampère equation

$$f_{xx}f_{yy} - f_{xy}^2 = \varepsilon = \pm 1 \tag{1.1}$$

and the umbilical surfaces of the unimodular affine theory in $\mathbb{R}^3$, obtained locally as the graphs of $f(x, y)$ and called improper affine spheres; see [2,15,13,22].

Thus, in the definite case ($\varepsilon = +1$), the lack of global examples seems a natural consequence of the famous result by Jörgens [10], which states that all the solutions of the (elliptic) Hessian $+1$ equation on $\mathbb{R}^2$ are quadratic polynomials. Actually, up to unimodular affine transformations, the elliptic paraboloid is the unique improper affine sphere with complete definite affine metric; see [3,5,21].

This situation motivates the study of solutions and surfaces with some singularities. In particular, Jörgens proved in [11] that the revolution surfaces provide the only entire solutions with at most an isolated singularity.

Recently, thanks to the conformal representation of the definite improper affine spheres obtained in [6,7], the above theorem has been extended to the finitely punctured plane in [8]. Moreover, from a local viewpoint, we remark that around a non-removable isolated singularity the conformal structure is always that of an annulus and the solution is determined by a planar convex analytic curve; see [1].

Now, although the indefinite case ($\varepsilon = -1$) is different, we use similar methods. However, with the corresponding geometric model, we can also construct solutions of the (non-elliptic) Hessian -1 equation with isolated singularities and the conformal structure of a punctured disk.

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First, following [16,19], we extend the conformal representation given in [18] and introduce the improper affine maps as improper affine spheres with some admissible singularities. These are, mainly, isolated singularities and singular curves with cuspidal edges and swallowtails; see [9,12].

Then, we solve the associated Björling problem and prove that any indefinite improper affine map can be recovered in terms of its set of singularities. Moreover, we give a priori conditions for a curve of $\mathbb{R}^3$ to be the singular curve of some indefinite (or definite) improper affine map with prescribed cuspidal edges and swallowtails. Thus, one can obtain interesting examples with the desired singularities.

Finally, we construct indefinite improper affine maps with isolated singularities, some of them with the conformal structure of an annulus and determined by a planar convex analytic curve, and others with the conformal structure of a punctured disk. As consequence, we get entire solutions of the Hessian -1 equation in the punctured plane, but the corresponding improper affine maps are not revolution surfaces.

2. Improper affine maps

Consider $\psi : \Sigma \longrightarrow \mathbb{R}^3$ an improper affine sphere, that is, an immersion with constant affine normal ξ. Then, see [14,20], up to an unimodular affine transformation, one has $\xi = (0, 0, 1)$, and ψ can be locally seen as the graph of a solution $f(x, y)$ of the unimodular Hessian equation (1.1).

In such a case, the affine conormal N and the affine metric h of ψ are given by

$$\begin{aligned} &N = (-f_x, -f_y, 1),\\ &h = f_{xx}dx^2 + 2f_{xy}dxdy + f_{yy}dy^2, \end{aligned} \tag{2.1}$$

and (1.1) is equivalent to

$$(df_x)^2 + \varepsilon\, dy^2 = f_{xx}h, \qquad (df_y)^2 + \varepsilon\, dx^2 = f_{yy}h. \tag{2.2}$$

Hence, the coordinates of N and ψ provide conformal parameters for h. Note that the ruled solution $f(x, y) = xy$ seems special, because $f_{xx} = 0 = f_{yy}$; however, up the unimodular change (x, y) by $(x - y, x + y)/\sqrt{2}$, we can take $f(x, y) = (x^2 - y^2)/2$ and recover h from (2.2).

Actually, when $\varepsilon = +1$, it is well known that $N + i\,\xi \times \psi : \Sigma \longrightarrow \mathbb{C}^3$ is a global holomorphic curve, with respect to the conformal structure induced by the Riemannian metric h, where the standard inner product $\langle \xi \times \psi, X\rangle$ is the determinant $[\xi, \psi, X]$, for any $X \in \mathbb{R}^3$; see [4,7].

Similarly, when $\varepsilon = -1$, we can change $\mathbb{C}$ by the split-complex numbers

$$\mathbb{C}' = \{z = s + jt : s, t \in \mathbb{R}, j^2 = 1,\ 1j = j1\}$$

and prove that $N + j\,\xi \times \psi : \Sigma \longrightarrow \mathbb{C}'^3$ is a global split-holomorphic curve, with respect to the conformal structure induced by the Lorentzian metric h; see [18].

In fact, from (1.1) and (2.1), we get

$$h = -\langle dN, d\psi\rangle, \qquad \langle N, \xi\rangle = 1, \qquad \langle N, d\psi\rangle = 0, \tag{2.3}$$

and

$$\sqrt{h(\psi_x, \psi_y)^2 - h(\psi_x, \psi_x)h(\psi_y, \psi_y)} = [\psi_x, \psi_y, \xi] = -[N_x, N_y, N].$$

Thus, for a local conformal parameter z, we have

$$h = 2\rho\, dz\, d\bar{z}, \quad \rho = \langle N, \psi_{z\bar{z}}\rangle = -j[\psi_z, \psi_{\bar{z}}, \xi] = j[N, N_z, N_{\bar{z}}] > 0 \tag{2.4}$$

and the split-holomorphic condition

$$N_z = j\xi \times \psi_z, \qquad N_{\bar{z}} = -j\xi \times \psi_{\bar{z}}, \tag{2.5}$$

with the usual notation $\bar{z} = s - jt$, $\mathrm{Re}(z) = s$, $\mathrm{Im}(z) = t$ and the partial derivatives

$$\psi_z = \frac{1}{2}(\psi_s + j\psi_t), \qquad \psi_{\bar{z}} = \frac{1}{2}(\psi_s - j\psi_t).$$

Moreover, from (2.3)–(2.5), we obtain

$$\psi_{z\bar{z}} = \rho\xi = jN_z \times N_{\bar{z}}, \qquad N_{z\bar{z}} = 0$$

and the Lelieuvre formula

$$\psi = 2\,\mathrm{Re}\int jN_z \times N dz = -2\,\mathrm{Re}\int j\left(\Phi + \overline{\Phi}\right) \times \Phi_z dz, \tag{2.6}$$

with the global split-holomorphic curve

$$\Phi = \frac{1}{2}N + \frac{j}{2}\,\xi \times \psi. \tag{2.7}$$

Conversely, we can introduce the class of indefinite improper affine spheres with admissible singularities, where the affine metric h is degenerated, but the affine conormal is well defined.

Definition 2.1. A map $\psi : \Sigma \longrightarrow \mathbb{R}^3$ is an indefinite improper affine map, with constant affine normal ξ, if it admits the representation (2.6) for a split-holomorphic curve Φ such that $[\Phi+\overline{\Phi}, \Phi_z, \overline{\Phi_z}]$ does not vanish identically and $2\langle\Phi, \xi\rangle = 1$.

From (2.3), (2.4), and the above definition, it is clear that

$$\psi_z = -j(\Phi + \overline{\Phi}) \times \Phi_z, \qquad N = \Phi + \overline{\Phi}, \tag{2.8}$$

$$\psi_{z\overline{z}} = j\Phi_z \times \overline{\Phi_z} = \rho\xi, \quad \rho = j\left[\, \Phi + \overline{\Phi}, \Phi_z, \overline{\Phi_z}\,\right], \tag{2.9}$$

and that $z_0 \in \Sigma$ is a non-degenerate singular point of the map ψ if and only if

$$\rho(z_0) = 0, \qquad d\rho|_{z_0} \neq 0. \tag{2.10}$$

In this case, either $\psi(z_0)$ is an isolated singularity or the singular set of ψ around z_0 locally becomes a regular curve $\gamma : I \subset \mathbb{R} \longrightarrow \Sigma$ and we have the KRSUY criterion for the singular curve $\alpha = \psi \circ \gamma$.

Theorem 2.2 (*[12]*)**.** *If η is a vector field along γ, with $\eta(s) \neq 0$ in the kernel of $d\psi_{\gamma(s)}$ for any s in the interval I, then the following hold.*

1. *$\gamma(0) = z_0$ is a cuspidal edge if and only if* $\det(\gamma'(0), \eta(0)) \neq 0$, *where* det *denotes the determinant of 2×2 matrices and prime indicates differentiation with respect to s.*
2. *$\gamma(0) = z_0$ is a swallowtail if and only if* $\det(\gamma'(0), \eta(0)) = 0$ *and*

$$\left.\frac{d}{ds}\right|_{s=0} \det(\gamma'(s), \eta(s)) \neq 0.$$

3. Singular curves

First, we solve the *affine Björling problem*, consisting in finding the indefinite improper affine map containing an analytic curve α with a prescribed affine conormal U along it. That is, we determine the corresponding split-holomorphic curve Φ with α and U.

Motivated by (2.3), we say that a pair of analytic curves $\alpha, U : I \longrightarrow \mathbb{R}^3$ is *admissible* for a non-zero vector $\xi \in \mathbb{R}^3$ if the equations

$$0 = \langle\alpha', U\rangle, \qquad 1 = \langle\xi, U\rangle, \tag{3.1}$$

hold on the interval I.

Thus, see (2.7) and (2.9), if we take the split-holomorphic curve $\Phi : \Omega \longrightarrow \mathbb{C}'^3$ given by

$$\Phi(z) = \frac{1}{2}\big(U(z) + j\xi \times \alpha(z)\big), \quad z = s + jt \in \Omega \subset \mathbb{C}',$$

in a domain Ω containing I, where the split-holomorphic extensions of U and α exist, then

$$2\langle\Phi, \xi\rangle = \langle U, \xi\rangle = 1$$

in Ω (by analyticity) and

$$\begin{aligned}\rho &= j\left[\, \Phi + \overline{\Phi}, \Phi_z, \overline{\Phi_z}\,\right] = \frac{j}{4}\left[U, U' + j\xi \times \alpha', U' - j\xi \times \alpha'\right] \\ &= -\frac{1}{2}\langle U \times U', \xi \times \alpha'\rangle = -\frac{1}{2}\langle U', \alpha'\rangle\end{aligned}$$

in I. Hence, if we denote by I_0 the zero set of the function $\lambda : I \longrightarrow \mathbb{R}$,

$$\lambda = \langle\alpha'', U\rangle = -\langle\alpha', U'\rangle, \tag{3.2}$$

we can obtain the following extension of Theorem 3.1 in [18].

Theorem 3.1. *Let $\alpha, U : I \longrightarrow \mathbb{R}^3$ be an admissible pair of curves for a non-zero vector ξ, with $I_0 \neq I$. Then there exists a unique indefinite improper affine map ψ, containing $\alpha(I)$, with affine normal ξ, affine conormal $U(s)$ at $\alpha(s)$ for all $s \in I$, and $\alpha(I_0)$ contained in its set of singularities.*

The case $I_0 = I$ is special, because $\rho = 0 = \lambda$ in I and the map ψ can be recovered in terms of its set of singularities.

From now on, without loss of generality, we will fix $\xi = (0, 0, 1)$.

Theorem 3.2. *Let $\alpha : I \longrightarrow \mathbb{R}^3$ be an analytic curve satisfying*

$$[\alpha', \alpha'', \alpha''']^2 \neq [\alpha', \alpha'', \xi]^4 \neq 0, \quad \forall s \in I. \tag{3.3}$$

Then, there exists a unique indefinite improper affine map ψ containing $\alpha(I)$ in its set of singularities. Actually, α is a singular curve of ψ and $\alpha(s)$ is a cuspidal edge for all $s \in I$.

Proof. From (3.1) and (3.2), with $\lambda \equiv 0$, there is a unique $U : I \longrightarrow \mathbb{R}^3$,

$$U = \frac{\alpha' \times \alpha''}{[\alpha', \alpha'', \xi]}, \tag{3.4}$$

such that $\{\alpha, U\}$ is an admissible pair of curves for ξ and the map ψ is defined by (2.6) with

$$\Phi = \frac{\alpha_z \times \alpha_{zz}}{2[\alpha_z, \alpha_{zz}, \xi]} + \frac{j}{2}\, \xi \times \alpha,$$

in a neighborhood of I in $\mathbb{C}'$, where the split-holomorphic extension of α exists.

In fact, from (2.8) and (3.4), we have, along I,

$$\begin{aligned} \psi_z &= -j(\Phi + \overline{\Phi}) \times \Phi_z = \frac{-j}{2}\, U \times U' - \frac{1}{2}\, U \times \left(\xi \times \alpha'\right) \\ &= \frac{1}{2}\, \alpha' - \frac{j}{2}\, U \times U' = \frac{1}{2}\, \alpha' - \frac{j}{2}\, \frac{[\alpha', \alpha'', \alpha''']}{[\alpha', \alpha'', \xi]^2} \alpha', \end{aligned}$$

and ψ contains the curve α with

$$\psi_s = \alpha', \qquad \psi_t = -\frac{[\alpha', \alpha'', \alpha''']}{[\alpha', \alpha'', \xi]^2} \alpha'. \tag{3.5}$$

Thus, from (2.9), (2.10), (3.3) and (3.5), we get $[\psi_s, \psi_t, \xi](s, 0) = 0, \ \forall s \in I$,

$$\begin{aligned} \left.\frac{d}{dt}\right|_{(s,0)} [\psi_s, \psi_t, \xi] &= [\psi_{ts}, \psi_t, \xi](s, 0) + [\psi_s, \psi_{ss}, \xi](s, 0) \\ &= [\alpha', \alpha'', \xi] \left(1 - \frac{[\alpha', \alpha'', \alpha''']^2}{[\alpha', \alpha'', \xi]^4}\right) \neq 0, \end{aligned} \tag{3.6}$$

and α is a singular curve. Moreover, the kernel of $d\psi$ at $\gamma(s) = (s, 0)$ is spanned by

$$\eta = ([\alpha', \alpha'', \alpha'''], [\alpha', \alpha'', \xi]^2),$$

and we conclude from

$$\det(\gamma', \eta) = [\alpha', \alpha'', \xi]^2 \neq 0$$

and Theorem 2.2 that $\alpha(s)$ is a cuspidal edge for all $s \in I$. □

Example 3.3. If we take the curve $\alpha : \mathbb{R} \longrightarrow \mathbb{R}^3$ given by

$$\alpha(s) = (\cos(s), \sin(s), as),$$

then $[\alpha', \alpha'', \xi] = 1$ and $[\alpha', \alpha'', \alpha'''] = a$. So, from Theorem 3.2, when $a \in \mathbb{R} - \{\pm 1\}$,

$$U(s) = (a \sin(s), -a \cos(s), 1),$$

and the associated improper affine map $\psi : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ has coordinates

$$\begin{aligned} \psi_1(s, t) &= \cos(s) \cos(t) + a \sin(s) \sin(t), \\ \psi_2(s, t) &= \sin(s) \cos(t) - a \cos(s) \sin(t), \\ \psi_3(s, t) &= as - \frac{1}{2}(1 + a^2)t + \frac{1}{4}(1 - a^2) \sin(2t). \end{aligned}$$

It is clear that the affine metric

$$h = [\psi_s, \psi_t, \xi](ds^2 - dt^2) = (1 - a^2) \cos(t) \sin(t)(ds^2 - dt^2)$$

does not vanish identically, because $a \neq \pm 1$.

Moreover, around $t = 0, \alpha = \psi(\ , 0)$ is the only singular curve with cuspidal edges (see Fig. 1, with $a = 0$ and $a = 0.1$).

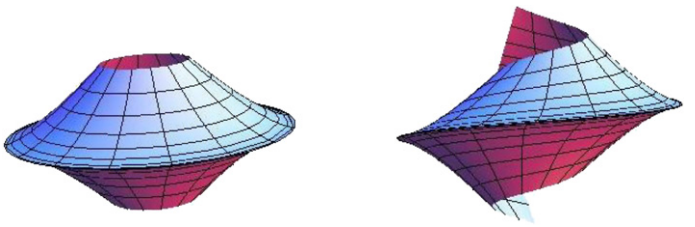

Fig. 1. Improper affine maps with cuspidal edges.

Theorem 3.4. *Let $\alpha : I \longrightarrow \mathbb{R}^3$ be an analytic curve such that*

$$[\alpha', \alpha'', \alpha''']^2 \neq [\alpha', \alpha'', \xi]^4 \neq 0, \quad \forall s \in I - \{0\},$$

and $0 \in I$ is a zero of α', $\alpha' \times \alpha''$, $[\alpha', \alpha'', \xi]$ and $[\alpha', \alpha'', \alpha''']$ of order 1, 2, 2 and 3 respectively.

Then, α is a singular curve of an unique indefinite improper affine map ψ, and $\alpha(0)$ is a swallowtail.

Proof. We follow the same arguments from (3.4) to (3.6). Note that U and ψ_t are well defined by the hypothesis and

$$\left.\frac{d}{dt}\right|_{(s,0)} [\psi_s, \psi_t, \xi] = [\alpha', \alpha'', \xi] - \frac{[\alpha', \alpha'', \alpha''']^2}{[\alpha', \alpha'', \xi]^3} \neq 0.$$

Now, the kernel of $d\psi$ at $\gamma(s) = (s, 0)$ is spanned by

$$\eta = \left(1, \frac{[\alpha', \alpha'', \xi]^2}{[\alpha', \alpha'', \alpha''']}\right),$$

and, from Theorem 2.2, $\alpha(0)$ is a swallowtail, because 0 is a zero of order 1 of

$$\det(\gamma', \eta) = \frac{[\alpha', \alpha'', \xi]^2}{[\alpha', \alpha'', \alpha''']}. \quad \square$$

Example 3.5. The curve $\alpha : \mathbb{R} \longrightarrow \mathbb{R}^3$ defined by

$$\alpha(s) = \left(\cos(s) + \frac{1}{2}\cos(2s), -\sin(s) + \frac{1}{2}\sin(2s), \frac{1}{6}\cos(3s)\right)$$

has

$$[\alpha', \alpha'', \xi] = 1 - \cos(3s) \quad \text{and} \quad [\alpha', \alpha'', \alpha'''] = \sin(3s) - \frac{1}{2}\sin(6s),$$

with the same 2π-periodic zeros $\frac{2}{3}\pi$, $\frac{4}{3}\pi$, and 2π. Thus, one can check the conditions of Theorem 3.4 and obtain an improper affine map with α as a singular curve with three swallowtails connected by three arcs with cuspidal edges. In fact,

$$U(s) = \left(-\cos(s) + \frac{1}{2}\cos(2s), \sin(s) + \frac{1}{2}\sin(2s), 1\right)$$

and $\psi : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ has coordinates

$$\psi_1(s,t) = \cos(s)(\cos(t) + \sin(t)) + \frac{1}{2}\cos(2s)(\cos(2t) + \sin(2t)),$$

$$\psi_2(s,t) = -\sin(s)(\cos(t) + \sin(t)) + \frac{1}{2}\sin(2s)(\cos(2t) + \sin(2t)),$$

$$\psi_3(s,t) = \frac{1}{24}(12t + 12\cos(2t) - 3\cos(4t) + 4\cos(3s))(\cos(3t) + 3\sin(t)).$$

Now, the affine metric is

$$h = 2\sin(t)\Big(\sin(3t) - \cos(3s)\Big)(ds^2 - dt^2),$$

and $t = 0$ gives $\alpha = \psi(\ , 0)$ with the expected properties (see Fig. 2).

Theorem 3.6. *There are no indefinite improper affine maps containing a singular curve $\alpha : I \longrightarrow \mathbb{R}^3$ satisfying*

$$[\alpha', \alpha'', \xi] = 0, \quad \forall s \in I. \tag{3.7}$$

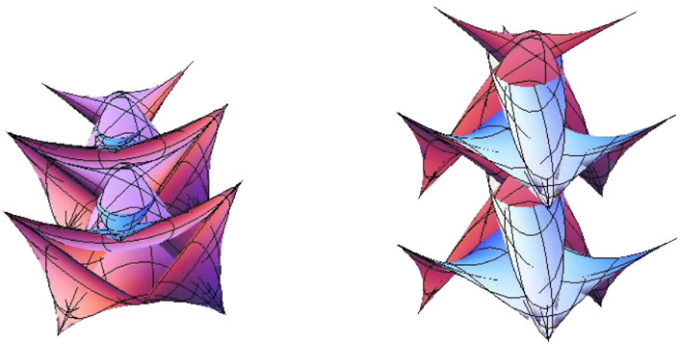

Fig. 2. Improper affine map with swallowtails.

Proof. If we assume that α is contained in an indefinite improper affine map ψ, with affine conormal U along α, then (3.1), (3.2) and (3.7), with $\lambda \equiv 0$, give

$$\alpha' \times \alpha'' = \langle \alpha' \times \alpha'', \xi \rangle U = 0,$$

and α is a line with direction vector v, such that $\langle v, U \rangle = 0$.

As a consequence, $\langle v, N \rangle = 0$, and $[N, N_z, N_{\bar{z}}]$ vanishes in a neighborhood of α, which is a contradiction. □

Remark 1. We can change $\mathbb{C}'$ by $\mathbb{C}$ and prove in a similar way the above theorems for definite improper affine maps, with the holomorphic curves $N + i\ \xi \times \psi$ and the conformal representation used in [1] for the classification of the isolated singularities of the Hessian $+1$ equation. Note that the 1 in (3.6) becomes -1 in the definite case and we can simplify the hypotheses; see [17].

4. Isolated singularities

Conversely, we can apply the ideas of [1] for the isolated singularities of the Hessian -1 equation, when the conformal structure of the affine metric around the singularity is that of an annulus $\mathcal{A}$.

Theorem 4.1. *Let $U : \mathbb{R} \longrightarrow \mathbb{R}^2 \times \{1\}$ be a 2π-periodic regular analytic parameterization of a convex curve.*

Then, there exists a unique indefinite improper affine map $\psi : \mathcal{A} \longrightarrow \mathbb{R}^3$, with an isolated singularity at the origin, where the affine conormal tends to U.

Proof. Here, we take the constant curve $\alpha_0 : \mathbb{R} \longrightarrow \mathbb{R}^3$, $\alpha_0 \equiv 0$, and so

$$\Phi(z) = \frac{1}{2}\big(U(z) + j\xi \times 0\big) = \frac{1}{2}U(z),$$

in a neighborhood of $\mathbb{R} \times \{0\}$ in $\mathbb{C}'$, where the split-holomorphic extension of U exists.

We observe that Φ is well defined on the annulus

$$\mathcal{A} = \{z = s + jt \in \mathbb{C}' : 0 < t < r\}/(2\pi\mathbb{Z}),$$

and $2\langle \Phi, \xi \rangle = 1$, by the hypothesis.

Moreover, from (2.8) and (2.9), we get along the circle $\mathbb{S} \equiv \mathbb{R} \times \{0\}/(2\pi\mathbb{Z})$

$$\psi_s = 0, \qquad \psi_t = -U \times U', \qquad \rho = 0,$$

and

$$\left.\frac{d}{dt}\right|_{(s,0)} [\psi_s, \psi_t, \xi] = [\psi_{ts}, \psi_t, \xi](s, 0) = -[U, U', U''](s) \neq 0.$$

Thus, from (2.10), we conclude that $0 = \psi(\mathbb{S})$ is an isolated singularity, where the affine conormal of ψ tends to U. □

Example 4.2. Similarly to Example 3.3, if we take $U : \mathbb{R} \longrightarrow \mathbb{R}^2 \times \{1\}$ with

$$U(s) = (\cos(s), \sin(s), 1),$$

then Theorem 4.1 gives the revolution improper affine map $\psi : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ with

$$\psi(s, t) = \left(\cos(s)\sin(t),\ \sin(s)\sin(t),\ \frac{t}{2} + \frac{1}{4}\sin(2t)\right),$$

and it is clear that $\psi(\mathbb{R} \times \{0\}) = (0, 0, 0)$ is an isolated singularity (see Fig. 3).

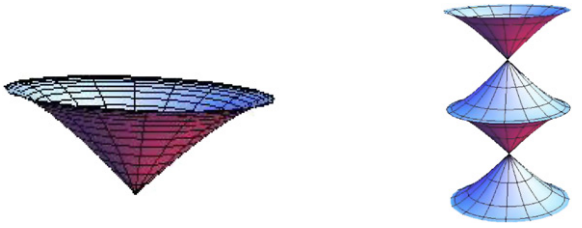

Fig. 3. Improper affine map with isolated singularities.

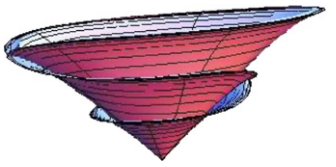

Fig. 4. Non-embedded isolated singularity.

We also observe that $\psi(\mathbb{R}\times[0,\frac{\pi}{4}])$ and $\psi(\mathbb{R}^2)$ provide a solution of the Hessian -1 equation on the same punctured disk.

Remark 2. If $U(\mathbb{R})$ is not a simple curve in $\mathbb{R}^2\times\{1\}$, then the isolated singularity is not embedded (see Fig. 4).

Unlike what happens in the definite case, where an isolated singularity is non-removable if and only if its underlying conformal structure is that of an annulus, see [11,8], we can construct indefinite improper affine maps with isolated singularities and the conformal structure of a punctured disk $\mathcal{D}^*$.

In general, if the split-holomorphic curve Φ of an indefinite improper affine map $\psi:\Sigma\longrightarrow\mathbb{R}^3$ has coordinates $(\Phi_1,\Phi_2,1/2)$, then

$$\rho=j\left[\Phi+\overline{\Phi},\Phi_z,\overline{\Phi_z}\right]=j(\Phi_{1z}\overline{\Phi_{2z}}-\Phi_{2z}\overline{\Phi_{1z}}). \tag{4.1}$$

Theorem 4.3. *Let $\Phi:\mathcal{D}\longrightarrow\mathbb{C}'^2\times\{1/2\}$ be a split-holomorphic curve with*

$$\Phi_{1z}\equiv j\quad\text{and}\quad\Phi_{2z}=F^2, \tag{4.2}$$

such that the split-holomorphic function $F:\mathcal{D}\longrightarrow\mathbb{C}'$ only vanishes at $z_0\in\mathcal{D}$. Then, the corresponding indefinite improper affine map $\psi:\mathcal{D}\longrightarrow\mathbb{R}^3$ is regular on $\mathcal{D}^*=\mathcal{D}-\{z_0\}$ and has an isolated singularity.

Proof. It is clear from (4.1) and (4.2), because $\rho=F^2+\overline{F^2}>0$ on $\mathcal{D}^*$. □

Remark 3. Similarly, we get a global indefinite improper affine map $\psi:\mathbb{C}'\longrightarrow\mathbb{R}^3$ with a finite number of singularities, associated to the zeros of a split-holomorphic function $F:\mathbb{C}'\longrightarrow\mathbb{C}'$.

In this way, in contrast to [11,8], we also find solutions of the Hessian -1 equation in the finitely punctured plane.

Example 4.4. Of course, the simplest choice in Theorem 4.3 is $F(z)=z$ and

$$\Phi(z)=\left(jz,\frac{1}{3}z^3,\frac{1}{2}\right),$$

with $z=s+jt\in\mathbb{C}'$. Thus, the affine conormal is given by

$$N(s,t)=\left(2t,\ \frac{2}{3}s^3+2st^2,\ 1\right),$$

and the corresponding improper affine map $\psi:\mathbb{R}^2\longrightarrow\mathbb{R}^3$ has

$$\psi(s,t)=\left(2s^2t+\frac{2}{3}t^3,-2s,\ \frac{1}{3}s^4-2s^2t^2-t^4\right).$$

Now, the affine metric is $h=4(s^2+t^2)(ds^2-dt^2)$, and $\psi(0,0)=(0,0,0)$ is the only singularity (see Fig. 5).

Moreover, we see that $\psi(\mathbb{R}^2-\{(0,0)\})$ is the graph of a global solution of the Hessian -1 equation on the punctured plane.

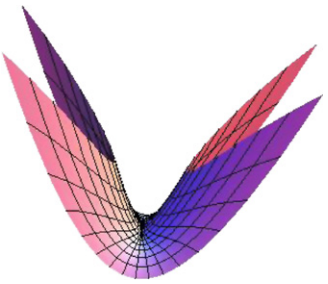

Fig. 5. Entire solution on the punctured plane.

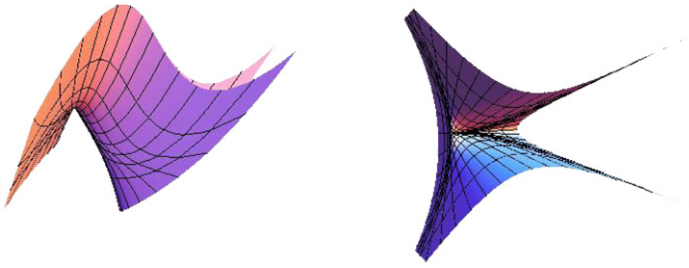

Fig. 6. Improper affine map with two isolated singularities.

Remark 4. We can follow with $F(z) = z(z-1)$ and obtain a solution on the twice-punctured plane. Alternately, we can distribute the zeros between the coordinates of Φ_z and choose, for instance,

$$\Phi(z) = \frac{1}{2}\Big(jz^3, z^3 - 3z^2 + 3z, 1\Big).$$

In this case, from (4.1), the associated improper affine map $\psi : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ has affine metric

$$h = 9\Big(t^2 + (s - s^2 + t^2)^2\Big)(ds^2 - dt^2),$$

and $\psi(0,0)$, $\psi(1,0)$ are the only singularities (see Fig. 6).

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