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Boundedness in a quasilinear parabolic–parabolic Keller–Segel system with logistic source

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ABSTRACT

We consider a quasilinear parabolic–parabolic Keller–Segel system involving a source term of logistic type,

$$\begin{cases} u_t = \nabla\cdot\big(\phi(u)\nabla u\big) - \nabla\cdot\big(\psi(u)\nabla v\big) + g(u), & (x,t)\in\Omega\times(0,T),\\ v_t = \Delta v - v + u, & (x,t)\in\Omega\times(0,T),\end{cases} \tag{0.1}$$

with nonnegative initial data under Neumann boundary condition in a smooth bounded domain $\Omega\subset\mathbb{R}^n$, $n\geqslant 1$. Here, ϕ and ψ are supposed to be smooth positive functions satisfying $c_1s^p\leqslant\phi$ and $c_1s^q\leqslant\psi(s)\leqslant c_2s^q$ when $s\geqslant s_0$ with some $s_0>1$, and we assume that g is smooth on $[0,\infty)$ fulfilling $g(0)\geqslant 0$ and $g(s)\leqslant as-\mu s^2$ for all $s>0$ with constants $a\geqslant 0$ and $\mu>0$. Within this framework, it is proved that whenever $q<1$, for any sufficiently smooth initial data there exists a unique classical solution which is global in time and bounded. Our result is independent of p.

1. Introduction

This paper deals with the initial–boundary value problem for two coupled parabolic equations with logistic source,

$$\begin{cases} u_t = \nabla\cdot\big(\phi(u)\nabla u\big) - \nabla\cdot\big(\psi(u)\nabla v\big) + g(u), & (x,t)\in\Omega\times(0,T),\\ \tau v_t = \Delta v - v + u, & (x,t)\in\Omega\times(0,T),\\ \dfrac{\partial u}{\partial n} = \dfrac{\partial v}{\partial n} = 0, & (x,t)\in\partial\Omega\times(0,T),\\ u(x,0)=u_0(x),\quad v(x,0)=v_0(x), & x\in\Omega,\end{cases} \tag{1.1}$$

where $\Omega\in\mathbb{R}^n$ $(n\geqslant 1)$ is a bounded convex domain with smooth boundary and $\tau\in\{0,1\}$. The functions ϕ and ψ are assumed to satisfy

$$\phi,\psi\in C^2\big([0,\infty)\big),\qquad \phi(s)>0\quad\text{for all } s\geqslant 0, \tag{1.2}$$

$$c_1s^p\leqslant\phi(s),\quad s\geqslant s_0, \tag{1.3}$$

$$c_1s^q\leqslant\psi(s)\leqslant c_2s^q,\quad s\geqslant s_0 \tag{1.4}$$

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with some $c_2 > c_1 > 0$, $p, q \in \mathbb{R}$ and $s_0 > 1$. Initial data fulfill

$$u_0 \in C^\beta(\bar{\Omega}) \quad (0 < \beta < 1), \qquad v_0 \in W^{1,r}(\Omega) \quad (r > n). \tag{1.5}$$

Moreover, $g \in C^\infty([0, \infty))$ is supposed to be a smooth function which satisfies

$$g(0) \geqslant 0 \quad \text{as well as} \quad g(s) \leqslant as - \mu s^2 \quad \text{for all } s > 0 \tag{1.6}$$

with constants $a \geqslant 0$ and $\mu > 0$.

This is a version of the well-known Keller–Segel model which was initially introduced by Keller and Segel in 1970. The system is used in mathematical biology to describe chemotaxis processes, where certain bacteria move toward higher densities of a chemical substance emitted by themselves, and which diffuse at the same time. In this context, ϕ and ψ denote the diffusivity and chemotactic sensitivity, respectively.

When $g(u) \equiv 0$, the classical parabolic–elliptic chemotaxis model with $\tau = 0$, $\phi \equiv 1$ and $\psi \equiv u$ has been extensively studied through the past decades. There have been numerous results on criteria for existence of global bounded solutions, and on the detection of some solutions blowing up in either finite or infinite time. When $\tau = 1$, the analysis of fully parabolic system seems to be more involved (see [6,16,17]).

Beyond this, one type of refined models was pursued by Hillen and Painter [5] on the basis of the assumption that in contrast to chemicals, the bacterial cells have a positive size which is not negligible. The associated system accounting for this so-called volume-filling effect is then quasilinear and involves more general functions ϕ and ψ as in (1.2) and (1.4), and in the case $g \equiv 0$ this has been widely studied as well [2,7,16]. For instance, in the corresponding parabolic–elliptic version obtained when $\phi(s) = s^p$ and $\psi(s) = s^q$ for large s, and when the second equation is replaced with $0 = \Delta v - M + u$, where M denotes the spatial mean of u, the results are essentially complete in the sense that a critical exponent on the interplay of ϕ and ψ has been found: Namely, if $q - p < \frac{2}{n}$, then all solutions are global and uniformly bounded; however, if $q - p > \frac{2}{n}$ and $q > 0$, then there exist radial solutions which become unbounded in finite time [20]. Similarly, also in the fully parabolic system (1.1) with $\tau = 1$, the exponent $\frac{2}{n}$ also plays an important role when $g \equiv 0$: It is known that if $q - p < \frac{2}{n}$, the system exclusively possesses global bounded solutions [13], whereas if $q - p > \frac{2}{n}$ with $n \geqslant 2$, unbounded solutions do exist [16], and even finite-time blow-up may occur under the additional conditions $n \geqslant 3$ and $q \geqslant 1$ [1].

It is our purpose in this paper to investigate the effect of a logistic source. Indeed, in related classical semilinear chemotaxis systems when $\phi(u) \equiv 1$ and $\psi(u) = \chi u$ with $\chi > 0$, such proliferation mechanisms in the style of (1.6) are known to prevent chemotactic collapse: In [14], for instance, it is proved that when $\mu > \frac{n-2}{n}\chi$, solutions of the parabolic–elliptic system with $\tau = 0$ are global and remain bounded. The same conclusion is true for the parabolic–parabolic system with $\tau > 0$ when either $n = 2$ [11], or when $n \geqslant 3$ and $\mu > \mu_0$ with some constant $\mu_0 > 0$ [15]. This is in sharp contrast to the possibility of blow-up which is known to occur in such systems when $g \equiv 0$ and $n \geqslant 2$ [3,8,10,19]. In presence of dampening sources of logistic type, only a partial result on the existence of explosions seems available [18].

In the present paper, we shall study (1.1) with $\tau = 1$ under the conditions (1.2)–(1.6). In this context, our main result says that for any choice of $q < 1$, the logistic dampening rules out the occurrence of blow-up:

Theorem 1. *Suppose that $\Omega \in \mathbb{R}^n$, $n \geqslant 1$, is a convex bounded domain with smooth boundary. Assume that ψ and ϕ satisfy (1.2)–(1.4) with some $q < 1$, g satisfies (1.6) with $\mu > 0$. Then for any nonnegative $u_0 \in C^\beta(\bar{\Omega})$ with $0 < \beta < 1$ and $v_0 \in W^{1,r}(\Omega)$ with $r > n$, there exists a pair $(u, v) \in C^0(\bar{\Omega} \times [0, \infty)) \cap C^{2,1}(\bar{\Omega} \times (0, \infty))$ which solves (1.1) in the classical sense. Moreover, both u and v are bounded in $\Omega \times (0, \infty)$.*

We underline that the above result is independent of the value of p in (1.3).

The plan of this paper is as follows.

In Section 2, we are going to prove local existence of classical solution to (1.1). Theorem 1 will be proved in Section 3, based on a series of lemmata providing appropriate a priori estimates.

2. Local existence

The question of local solvability to (1.1) for sufficiently smooth initial data can be addressed by methods involving standard parabolic regularity theory in a suitable fixed point framework.

Now let us assert that the system is locally well-posed under appropriate assumptions. Moreover, we are going to make sure a solution terminates in finite time if and only if it blows up in a certain norm.

Lemma 2.1. *Suppose $\Omega \in \mathbb{R}^n$, $n \geqslant 1$, is a convex bounded domain with smooth boundary, ψ, ϕ satisfy (1.2)–(1.4), g fulfills (1.6) and $u_0 \in C^\beta(\bar{\Omega})$ with $\beta \in (0, 1)$, $v_0 \in W^{1,r}(\Omega)$ with $(r > n)$ both are nonnegative. Then there exist $T_{max} \in (0, \infty)$ and a pair nonnegative functions $(u, v) \in C^0(\bar{\Omega} \times [0, T_{max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{max}))$ classically solving (1.1) in $\Omega \times (0, T_{max})$. Moreover, $T_{max} < \infty$ if and only if*

$$\limsup_{t \nearrow T_{max}} \left(\left\| u(\cdot, t) \right\|_{L^\infty(\Omega)} + \left\| v(\cdot, t) \right\|_{W^{1,r}(\Omega)} \right) = \infty. \tag{2.1}$$

Proof. Let $\|u_0\|_{L^\infty(\Omega)} = K$. We can pick smooth functions ϕ_K, ψ_K on $[0,\infty)$ such that $\phi_K \equiv \phi$, $\psi_K \equiv \psi$ when $0 \leqslant s \leqslant 2K$ and $\phi_K = 2K$, $\psi_K = 2K$ when $s \geqslant 2K$. We define the following closed convex subset of the Banach Space $C(\bar{\Omega} \times (0,T))$

$$S := \left\{ \tilde{u} \in C\left(\bar{\Omega} \times [0,T]\right) \;\middle|\; \|\tilde{u}\|_{L^\infty(\Omega\times[0,T])} \leqslant 2K \right\}$$

(with $T < 1$ to be fixed below), and consider a fixed point problem, $F(\tilde{u}) = u$, where u is the first component of the solution (u,v) to the decoupled problem

$$\begin{cases} u_t = \nabla\cdot\left(\phi_K(u)\nabla u\right) - \nabla\cdot\left(\psi_K(u)\nabla v\right) + g(u), \\ v_t = \Delta v - v + \tilde{u}, \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial v}{\partial n} = 0, \\ u_0 \in C^0(\bar{\Omega}), \quad v_0 \in W^{1,r}(\Omega). \end{cases} \tag{2.2}$$

By linear parabolic theory, there exists a unique solution $v(x,t) \in C^{1+\bar{\beta},\frac{1+\bar{\beta}}{2}}(\bar{\Omega}\times[0,T])$ which satisfies

$$\|\nabla v\|_{L^\infty(\Omega\times(0,T))} \leqslant \|v\|_{C^{1+\bar{\beta},\frac{1+\bar{\beta}}{2}}(\bar{\Omega}\times[0,T])} \leqslant C\left(K,T,\|v_0\|_{W^{1,r}(\Omega)}\right),$$

and combined with $\inf\phi_K > 0$ and $u_0 \in C^{\beta}(\bar{\Omega})$, we may derive that u enjoys a uniform bound in $C^{\beta,\frac{\beta}{2}}(\bar{\Omega}\times[0,T])$ for some $\beta \in (0,1)$ [12]. So $u \in C^{\beta,\frac{\beta}{2}}(\bar{\Omega}\times[0,T]) \subset C(\bar{\Omega}\times[0,T])$. Next, since $\|u_0\|_{L^\infty} = K$, by continuity we can choose appropriate $T > 0$ such that $u \leqslant 2K$ in $\bar{\Omega}\times[0,T]$, then $F(S) \subset S$. Moreover, $F(S)$ is a compact subset in S, and evidently F is continuous since the solution of the first equation is unique. Applying the Schauder fixed point theorem, we obtain that there exists at least one fixed point $u \in S$. Standard parabolic regularity [9] then implies that (u,v) actually is classical solution of (2.2) in $C^0(\bar{\Omega}\times[0,T_{max}))\cap C^{2,1}(\bar{\Omega}\times(0,T_{max}))$. Note that $\phi_K = \phi$, $\psi_K = \psi$ when $u \leqslant 2K$, whence (u,v) in fact classically solves (1.1). Since the maximum existence time depends on $\|u_0\|_{L^\infty(\Omega)}$ and $\|v_0\|_{W^{1,r}(\Omega)}$, (2.1) holds. By applying the maximum principle to each scalar equation, we finally obtain that (u,v) is nonnegative. □

Before we proceed to show global existence, let us first weaken the above extensibility criterion in the following lemma.

Lemma 2.2. *Suppose $u_0 \in C^0(\bar{\Omega})$, $v_0 \in W^{1,r}(\Omega)$ both are nonnegative. Then the solution constructed in Lemma 2.1 has the property that if for some $C_1 > 0$ and $T \in (0,T_{max})$ we have,*

$$\left\|u(\cdot,t)\right\|_{L^\infty(\Omega)} \leqslant C_1, \quad t \in [0,T],$$

then there exists $C_2 > 0$ such that

$$\left\|v(\cdot,t)\right\|_{W^{1,r}(\Omega)} \leqslant C_2, \quad t \in [0,T].$$

Moreover, if $T_{max} < \infty$, then

$$\limsup_{t\nearrow T_{max}} \left\|u(\cdot,t)\right\|_{L^\infty(\Omega)} = \infty. \tag{2.3}$$

Proof. Assume $\|u(\cdot,t)\|_{L^\infty(\Omega)} \leqslant C_1$ for all $t \in [0,T]$. Hence by semigroup estimate

$$\left\|v(\cdot,t)\right\|_{W^{1,r}(\Omega)} \leqslant C_2, \quad t \in [0,T] \tag{2.4}$$

for some $C_2 > 0$, which depends on $\|u(\cdot,t)\|_{L^\infty(\Omega)}$, $\|v_0\|_{W^{1,r}(\Omega)}$ and T.

Suppose on contrary that $T_{max} < \infty$, but there exists $C_1 > 0$,

$$\left\|u(\cdot,t)\right\|_{L^\infty(\Omega)} \leqslant C_1 \quad \text{for all } t \in [0,T_{max}), \tag{2.5}$$

then $\|v(\cdot,t)\|_{W^{1,r}(\Omega)} \leqslant C_2$ in $[0,T_{max})$, which contract (2.3). □

3. A priori estimates

In this section, we are going to establish an iteration step to develop the main ingredient of our result. The iteration depends on a series of a priori estimate. In the course of our proof to the estimate, we shall refer to the following result which is a consequence of Lemma 3.1 in [4], which is on so-called maximal Sobolev regularity. Since it will turn out to be quite crucial to our approach, we shall formulate it here for completeness. On the other hand, the classical regularity requires boundary condition on initial data. We don't have such an assumption initial data in this context. We would like to use any positive time as the "initial" time in the regularity, on which the solution fulfills the boundary condition naturally.

Lemma 3.1. *(See [4].) Let $r \in (1,\infty)$. Consider the following evolution equation*

$$\begin{cases} v_t = \Delta v + f, \\ \dfrac{\partial v}{\partial n} = 0, \\ v(x,0) = v_0(x). \end{cases} \tag{3.1}$$

For each $v_0 \in W^{2,r}(\Omega)$ such that $\frac{\partial v_0}{\partial \nu} = 0$ on $\partial\Omega$ and any $f \in L^r((0,T);L^r(\Omega))$, there exists a unique solution

$$v \in W^{1,r}\big((0,T);L^r(\Omega)\big) \cap L^r\big((0,T);W^{2,r}(\Omega)\big).$$

Moreover, there exists $C > 0$ such that

$$\begin{aligned} &\int_0^T \|v(\cdot,t)\|^r_{L^r(\Omega)}\,dt + \int_0^T \|v_t(\cdot,t)\|^r_{L^r(\Omega)}\,dt + \int_0^T \|\Delta v(\cdot,t)\|^r_{L^r(\Omega)}\,dt \\ &\quad \leqslant C\int_0^T \|f(\cdot,t)\|^r_{L^r(\Omega)}\,dt + C\|v_0\|^r_{L^r(\Omega)} + C\|\Delta v_0\|^r_{L^r(\Omega)}. \end{aligned} \tag{3.2}$$

If $s \in (0,T)$, and $v(s)$ satisfies $v(s) \in W^{2,r}(\Omega)$ with $\frac{\partial v}{\partial n}(s) = 0$ on $\partial\Omega$, then with the same constant $C > 0$ as above, we have the following,

$$\begin{aligned} &\int_s^T \|v(\cdot,t)\|^r_{L^r(\Omega)}\,dt + \int_s^T \|v_t(\cdot,t)\|^r_{L^r(\Omega)}\,dt + \int_s^T \|\Delta v(\cdot,t)\|^r_{L^r(\Omega)}\,dt \\ &\quad \leqslant C\int_s^T \|f(\cdot,t)\|^r_{L^r(\Omega)}\,dt + C\big\|v(\cdot,s)\big\|^r_{L^r(\Omega)} + C\big\|\Delta v(s)\big\|^r_{L^r(\Omega)}. \end{aligned} \tag{3.3}$$

Proof. If $v_0 = 0$, (3.2) is precisely proved in [4]. The general case can be easily derived by letting $\tilde{v} := v - \chi(t)v_0$, where $\chi \in C_0^\infty([0,\infty))$ is a cut-off function such that $\chi(t) \leqslant 1$ for any $t \leqslant \max\{\frac{d}{4},1\}$. Finally, (3.3) follows upon replacing $v(t)$ by $v(t+s)$. $\square$

In order to proceed, let us now pick any $s \in (0,T_{max})$ and $s \leqslant 1$. Then by the regularity principle asserted by Lemma 2.1, we have $(u(\cdot,s),v(\cdot,s)) \in C^2(\bar{\Omega})$ with $\frac{\partial v(\cdot,s)}{\partial n} = 0$ on $\partial\Omega$, so that in particular, we can pick $K > 0$ such that

$$\sup_{0\leqslant\tau\leqslant s}\big\|u(\tau)\big\|_{L^\infty(\Omega)} \leqslant K, \qquad \sup_{0\leqslant\tau\leqslant s}\big\|v(\tau)\big\|_{L^\infty(\Omega)} \leqslant K \quad \text{and} \quad \big\|\Delta v(s)\big\|_{L^\infty(\Omega)} \leqslant K. \tag{3.4}$$

Now we proceed to derive a priori estimate which will construct the main part of this work.

Lemma 3.2. *Assume that g satisfies (1.4). Then there exist $C > 0$ such that for any $T \in (0,T_{max})$ the solution of (1.1) satisfies*

$$\int_\Omega u \leqslant C \quad \text{for all } t \in (0,T), \qquad \int_s^T\int_\Omega u^2 \leqslant C(T+1). \tag{3.5}$$

Proof. Integrating the first equation in (1.1) and using Hölder's inequality gives

$$\begin{aligned} \frac{d}{dt}\int_\Omega u &\leqslant a\int_\Omega u - \mu\int_\Omega u^2 \\ &\leqslant a\int_\Omega u - \frac{\mu}{|\Omega|}\bigg(\int_\Omega u\bigg)^2 \quad \text{for all } t \in (s,T). \end{aligned} \tag{3.6}$$

This yields

$$\int_\Omega u \leqslant \max\bigg\{\frac{a|\Omega|}{\mu}, K|\Omega|\bigg\} \quad \text{for all } t \in (s,T). \tag{3.7}$$

Whereupon by integrating (3.6) on (s,T) with respect to t, we obtain

$$\int_s^T\int_\Omega u^2 \leqslant \frac{a}{\mu}\int_s^T\int_\Omega u + \frac{1}{\mu}\int_\Omega u(s)$$
$$\leqslant \left(\frac{a}{\mu}\int_\Omega u\right)T + \frac{K}{\mu}|\Omega| \quad \text{for all } t \in (s,T). \tag{3.8}$$

Therefore, by an evident choice of C we complete the proof. □

Having (3.5) as a rough a priori estimate, we are in a position to improve the regularity of u in a higher L^p space. The following lemma shows how this can be achieved. The technique again is based on maximal Sobolev regularity, that is, on Lemma 3.1.

Lemma 3.3. *Let $\alpha \geqslant 1$. Then there exist $C_2 > 0$ and $M > 0$, depending on μ, q, a, K and $|\Omega|$ only, such that if for some $T \in (0, T_{max})$ and some $C_1 > 0$, we have*

$$\int_\Omega u^\alpha \leqslant C_1(T+1) \quad \text{for any } t \in (s,T), \qquad \int_s^T\int_\Omega u^{\alpha+1} \leqslant C_1(T+1), \tag{3.9}$$

then

$$\int_\Omega u^\gamma \leqslant C_2M^\gamma C_1(T+1) \quad \text{for any } t \in (s,T), \qquad \int_s^T\int_\Omega u^{\gamma+1} \leqslant C_2M^\gamma C_1(T+1), \tag{3.10}$$

where $\gamma = (2-q)\alpha + 1 - q$.

Proof. We multiply the first equation by $\gamma u^{\gamma-1}$, then integrate by parts and use (1.3), (1.4) to obtain

$$\frac{d}{dt}\int_\Omega u^\gamma = -\gamma(\gamma-1)\int_\Omega \phi(u)u^{\gamma-2}|\nabla u|^2 + \gamma(\gamma-1)\int_\Omega \psi(u)u^{\gamma-2}\nabla u\cdot\nabla v + \gamma\mu\int_\Omega u^{\gamma-1}g(u)$$
$$\leqslant c_2\gamma(\gamma-1)\int_\Omega u^{q+\gamma-2}\nabla u\cdot\nabla v + a\gamma\int_\Omega u^\gamma - \mu\gamma\int_\Omega u^{\gamma+1} \quad \text{for all } t \in (s,T), \tag{3.11}$$

where c_2 is provided by (1.4). Since $q+\gamma-1>0$, and with the help of the second equation, we see that

$$c_2\gamma(\gamma-1)\int_\Omega u^{q+\gamma-2}\nabla u\cdot\nabla v = \frac{c_2\gamma(\gamma-1)}{q+\gamma-1}\int_\Omega \nabla u^{q+\gamma-1}\cdot\nabla v$$
$$= -\frac{c_2\gamma(\gamma-1)}{q+\gamma-1}\int_\Omega u^{\gamma+q-1}(v_t+v-u)$$
$$\leqslant -\frac{c_2\gamma(\gamma-1)}{q+\gamma-1}\int_\Omega u^{q+\gamma-1}v_t + \frac{c_2\gamma(\gamma-1)}{q+\gamma-1}\int_\Omega u^{q+\gamma}$$
$$\leqslant -c_3\gamma\int_\Omega u^{q+\gamma-1}v_t + c_4\gamma\int_\Omega u^{q+\gamma} \tag{3.12}$$

for all $t \in (s,T)$, where $c_3 = \inf_{\gamma\geqslant 3-2q}\frac{c_2(\gamma-1)}{\gamma+q-1} > 0$, $c_4 = \sup_{\gamma\geqslant 3-2q}\frac{c_2(\gamma-1)}{\gamma+q-1} > 0$. Furthermore, Young's inequality entails

$$-\int_\Omega u^{\gamma+q-1}v_t \leqslant \frac{\mu}{4c_3}\int_\Omega u^{(\gamma+q-1)s_1} + C(s_1,\mu)\int_\Omega |v_t|^{\frac{s_1}{s_1-1}}, \tag{3.13}$$

where

$$\begin{cases} s_1 = \dfrac{\gamma+1}{\gamma+q-1}, \\ C(s_1,\mu) = (s_1-1)s_1^{-\frac{s_1}{s_1-1}}\left(\dfrac{\mu}{4c_3}\right)^{-\frac{1}{s_1-1}} = \dfrac{2-q}{\gamma+q-1}\left(1+\dfrac{2-q}{\gamma+q-1}\right)^{-\frac{\gamma+1}{2-q}}\left(\dfrac{\mu}{4c_3}\right)^{-\frac{\gamma+q-1}{2-q}}. \end{cases}$$

Since $\frac{2-q}{\gamma+q-1} \leqslant \frac{3-2q}{\gamma}$, we may choose

$$\begin{cases} M_1 > \max\left\{\left(\dfrac{4c_3}{\mu}\right)^{\frac{1}{2-q}}, 1\right\}, \\ c_5 > \sup\limits_{\gamma > 3-2q} (3-2q)\left(1 + \dfrac{2-q}{\gamma+q-1}\right)^{-\frac{\gamma+1}{2-q}} \left(\dfrac{\mu}{4c_3}\right)^{-\frac{q-1}{2-q}}. \end{cases}$$

We rewrite (3.13) as

$$-\int_\Omega u^{\gamma+q-1} v_t \leqslant \frac{\mu}{4c_3} \int_\Omega u^{\gamma+1} + \frac{c_5}{\gamma} {M_1}^\gamma \int_\Omega |v_t|^{\alpha+1}, \tag{3.14}$$

here we use $\frac{s_1}{s_1-1} = \alpha + 1$.

By supposing that c_5 and $M_1 > 1$ are large enough, a similar computation gives

$$\int_\Omega u^{q+\gamma} \leqslant \frac{\mu}{4c_4} \int_\Omega u^{\gamma+1} + \frac{c_5}{\gamma} {M_1}^\gamma |\Omega|, \tag{3.15}$$

and

$$\int_\Omega u^\gamma \leqslant \frac{\mu}{4a} \int_\Omega u^{\gamma+1} + \frac{c_5}{\gamma} {M_1}^\gamma |\Omega|. \tag{3.16}$$

By (3.14)–(3.16) we conclude that

$$\begin{aligned} \frac{d}{dt}\int_\Omega u^\gamma &\leqslant -\frac{\gamma\mu}{4}\int_\Omega u^{\gamma+1} + c_3c_5{M_1}^\gamma \int_\Omega |v_t|^{\alpha+1} + c_4c_5{M_1}^\gamma|\Omega| + ac_5{M_1}^\gamma|\Omega| \\ &\leqslant -\frac{\mu\gamma}{4}\int_\Omega u^{\gamma+1} + c_6{M_1}^\gamma \int_\Omega |v_t|^{\alpha+1} + c_6{M_1}^\gamma, \end{aligned} \tag{3.17}$$

for all $t \in (s, T)$, where we choose $c_6 := \max\{ac_5|\Omega| + c_4c_5|\Omega|, c_3c_5\}$. Integrating (3.17) on (s, T), we have

$$\int_\Omega u^\gamma + \frac{\mu\gamma}{4}\int_s^T\int_\Omega u^{\gamma+1} \leqslant \int_\Omega u^\gamma(s) + c_6{M_1}^\gamma \int_s^T\int_\Omega |v_t|^{\alpha+1} + c_6{M_1}^\gamma(T+1) \tag{3.18}$$

for all $t \in (s, T)$. Now, we multiply the second equation by $(\alpha+1)v^\alpha$, integrate by parts and apply Young's inequality again to see

$$\begin{aligned} \frac{d}{dt}\int_\Omega v^{\alpha+1} &= -\alpha(\alpha+1)\int_\Omega v^{\alpha-1}|\nabla v|^2 - (\alpha+1)\int_\Omega v^{\alpha+1} + (\alpha+1)\int_\Omega uv^\alpha \\ &\leqslant -\int_\Omega v^{\alpha+1} + \int_\Omega u^{\alpha+1} \end{aligned} \tag{3.19}$$

for all $t \in (s, T)$. This yields

$$\begin{aligned} \int_s^t\int_\Omega v^{\alpha+1} &\leqslant \int_\Omega v^{\alpha+1}(s) + \int_s^T\int_\Omega u^{\alpha+1} \\ &\leqslant K^{\alpha+1}|\Omega| + \int_s^T\int_\Omega u^{\alpha+1} \\ &\leqslant K^{\alpha+1}|\Omega| + C_1(T+1) \\ &\leqslant c_7{M_2}^\alpha C_1(T+1), \end{aligned} \tag{3.20}$$

where $M_2 := \max\{1, K\}$ and $c_7 := K + 1$. Lemma 3.1 thus entails

$$\int_s^t \int_\Omega |v_t|^{\alpha+1} \leqslant C_3 \int_s^t \int_\Omega u^{\alpha+1} + C_3 \int_s^t \int_\Omega v^{\alpha+1} + C_3 \int_\Omega \left|\Delta v(s)\right|^{\alpha+1} + C_3 \int_\Omega v^{\alpha+1}(s)$$
$$\leqslant C_1 C_3 (T+1) + c_7 {M_2}^{\alpha} C_1 C_3 (T+1) + 2K^{\alpha+1}|\Omega| C_3 (T+1)$$
$$\leqslant c_8 {M_2}^{\alpha} C_1 (T+1) \tag{3.21}$$

where C_3 is constant from Lemma 3.1 in (3.3), $c_8 := C_3(1 + c_7 + 2M_2|\Omega|)$. Combined with (3.18), this gives

$$\int_\Omega u^\gamma \leqslant M_2^\gamma |\Omega| + c_6 M_1^\gamma c_8 M_2^\alpha C_1 (T+1) + c_6 M_1 (T+1)$$
$$\leqslant c_{10} M^\gamma C_1 (T+1) \quad \text{for all } t \in (s, T), \tag{3.22}$$

and

$$\int_s^T \int_\Omega u^{\gamma+1} \leqslant \frac{4}{\mu} c_9 M^\gamma C_1 (T+1), \tag{3.23}$$

with $c_9 := |\Omega| + c_6 c_8 + c_6$, $M := M_1 M_2$ and $C_2 := \max\{c_9, \frac{4}{\mu} c_9\}$. $\square$

Now, we can set up the iteration procedure to derive the main result in this section.

Lemma 3.4. *Let $q < 1$, there exists $C = C(|\Omega|, q) > 0$ such that for any $T \in (0, T_{max})$, $\|u(\cdot,t)\|_\infty \leqslant C$ for all $t \in (0, T)$, where C is independent of T.*

Proof. Let $\gamma_0 = 1$, $\gamma_k = (2-q)\gamma_{k-1} + 1 - q$ $(k \geqslant 1)$. Then Lemma 3.2 and Lemma 3.3 give us

$$\int_\Omega u^{\gamma_k} \leqslant C_2^k M^{\sum_{i=1}^k \gamma_i} C(T+1) \quad \text{for all } t \in (s, T) \text{ and } k \geqslant 0. \tag{3.24}$$

Notice that by the definition of γ_k, there exist $a_1, a_2 > 0$ such that $a_1(2-q)^k < \gamma_k < a_2(2-q)^k$. Therefore

$$\left\|u(\cdot,t)\right\|_{L^{\gamma_k}(\Omega)} \leqslant C_2^{\frac{k}{a_1(2-q)^k}} C^{\frac{1}{a_1(2-q)^k}} (T+1)^{\frac{1}{a_1(2-q)^k}} M^{\frac{a_2 \sum_{i=1}^k (2-q)^k}{a_1(2-q)^k}} \quad \text{for all } t \in (s, T),\ k \geqslant 0. \tag{3.25}$$

Since $q < 1$, we have $\gamma_k \to \infty$ as $k \to \infty$. Thus letting $k \to \infty$ on both sides of (3.25), we find that

$$\left\|u(\cdot,t)\right\|_{L^\infty(\Omega)} \leqslant M^{\frac{a_2(2-q)}{a_1}} \quad \text{for all } t \in (s, T). \tag{3.26}$$

(3.4) gives

$$\left\|u(\cdot,t)\right\|_{L^\infty(\Omega)} \leqslant K \quad \text{for } t \in [0, s]. \tag{3.27}$$

By choosing $C := \max\{K, M^{\frac{a_2(2-q)}{a_1}}\}$, we complete the proof. $\square$

The assertion of Theorem 1 is an immediate consequence of the above lemmata.

Proof of Theorem 1. Suppose on contrary that $T_{max} < \infty$. By Lemma 3.4, we have $\|u(\cdot,t)\|_{L^\infty(\Omega)} \leqslant C$ for all $t \in (0, T_{max})$, where C is independent on T_{max}. This contracts Lemma 2.2, thus we derive that $T_{max} = \infty$. Thanks to (2.4) and embedding theorem, (u, v) is global and bounded. $\square$

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