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Simultaneously continuous retraction and Bishop–Phelps–Bollobás type theorem

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ABSTRACT

The dual space X^* of a Banach space X is said to admit a uniformly simultaneously continuous retraction if there is a retraction r from X^* onto its unit ball B_{X^*} which is uniformly continuous in norm topology and continuous in weak-$*$ topology. We prove that if a Banach space (resp. complex Banach space) X has a normalized unconditional Schauder basis with unconditional basis constant 1 and if X^* is uniformly monotone (resp. uniformly complex convex), then X^* admits a uniformly simultaneously continuous retraction. It is also shown that X^* admits such a retraction if $X = [\bigoplus X_i]_{c_0}$ or $X = [\bigoplus X_i]_{\ell_1}$, where $\{X_i\}$ is a family of separable Banach spaces whose duals are uniformly convex with moduli of convexity $\delta_i(\varepsilon)$ with $\inf_i \delta_i(\varepsilon) > 0$ for all $0 < \varepsilon < 1$. Let K be a locally compact Hausdorff space and let $C_0(K)$ be the real Banach space consisting of all real-valued continuous functions vanishing at infinity. As an application of simultaneously continuous retractions, we show that a pair $(X, C_0(K))$ has the Bishop–Phelps–Bollobás property for operators if X^* admits a uniformly simultaneously continuous retraction. As a corollary, $(C_0(S), C_0(K))$ has the Bishop–Phelps–Bollobás property for operators for every locally compact metric space S.

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1. Introduction

Let X be a real or complex Banach space and A be a subset of X. A continuous function $r : X \to A$ is said to be a *retraction* if r is the identity on A. Retractions have various applications in nonlinear geometric functional analysis [11,10,12]. Benyamini introduced the notion of simultaneously continuous retraction from the dual space X^* onto B_{X^*}. More precisely, the dual space X^* of a Banach space X is said to *admit a (resp. uniformly) simultaneously continuous retraction* if there is a retraction r from X^* onto B_{X^*} which is both weak-$*$ continuous and norm continuous (resp. uniformly norm-continuous). Benyamini [11] showed,

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in particular, that E^* admits uniformly simultaneously continuous retraction if E^* is a separable uniformly convex space, or E is the space $C(K)$ of all real-valued continuous functions on a compact metric space K.

As remarked in Proposition 4.22. [12], there is a connection between simultaneously continuous retractions and the denseness of norm attaining operators into $C(K)$. In this paper, we deal with the existence of uniformly simultaneous continuous retraction in a certain Banach space and its applications to Bishop–Phelps–Bollobás type theorem.

2. Uniformly simultaneously continuous retraction

Let $\{e_j\}$ be a normalized unconditional Schauder basis for X with unconditional basis constant 1. Its biorthogonal functionals will be denoted by $\{e_j^*\}$. In fact, it is easy to see that X and X^* are Banach lattices and, for every $x^* \in X^*$, we have

$$x^* = \text{weak} * \sum_{j=1}^{\infty} x^*(j) e_j^*,$$

where $x^*(j) = \langle x^*, e_j \rangle$. Recall that a Banach lattice X is uniformly monotone if, for all $\varepsilon > 0$,

$$M(\varepsilon) = \inf\bigl\{ \bigl\| |x| + |y| \bigr\| - 1 : \|x\| = 1, \|y\| \geq \varepsilon \bigr\} > 0.$$

It is easy to check that $\varepsilon \mapsto M(\varepsilon)$ is a monotone increasing function and $M(\varepsilon) \leq \varepsilon$ for all $\varepsilon > 0$. This M is called the *modulus of monotonicity* of X. It is easy to check that if X is uniformly monotone, then X is strictly monotone. That is, $\||x| + |y|\| > \|x\|$ for all $x \in X$ and for all nonzero element y in X. The uniform monotonicity of a Banach lattice is equivalent to the uniform complex convexity of its complexification [30,31]. The complex convexity has been used to study density of norm-attaining operators between Banach spaces [1,18].

Benyamini showed [11] that if X has a shrinking Schauder basis $\{e_j\}$ with $\{e_j^*\}$ being strictly monotone, then X^* admits a simultaneously continuous retraction. It is also shown that for $X = \ell_p$, $1 \leq p < \infty$ or $X = c_0$, X^* admits a uniformly simultaneously continuous retraction.

For $t \geq 0$, we define $M^{-1}(t) = \sup\{\varepsilon \geq 0 : M(\varepsilon) \leq t\}$ for a monotone increasing function M. The modulus of continuity for a function φ is defined by

$$\omega_\varphi(t) = \sup\bigl\{ \bigl\| \varphi(x^*) - \varphi(y^*) \bigr\| : \bigl\| x^* - y^* \bigr\| \leq t \bigr\}.$$

Let f be a nonnegative function on a deleted neighborhood of 0 with $\lim_{t \to 0+} f(t) = 0$. We say that X^* admits *an f-uniformly simultaneously continuous retraction* if there is a uniformly simultaneously continuous retraction φ with $\omega_\varphi(t) \leq f(t)$.

Theorem 2.1. *Suppose that a Banach space X has a normalized unconditional Schauder basis $\{e_j\}$ with unconditional basis constant 1. If X^* is uniformly monotone with modulus of monotonicity M, then X^* admits a uniformly simultaneously continuous retraction with modulus of continuity $2M^{-1}$.*

Proof. Notice that X^* is uniformly monotone and it is order-continuous (cf. [30]) and $\{e_j^*\}_{j=1}^\infty$ is a Schauder basis. Given $x^* = \sum_{j=1}^\infty a_j e_j^*$ with $x^* \notin B_{X^*}$, there is a unique n so that

$$\Biggl\| \sum_{j=1}^{n-1} a_j e_j^* \Biggr\| < 1, \quad \text{and} \quad \Biggl\| \sum_{j=1}^{n} a_j e_j^* \Biggr\| \geq 1.$$

By the strict monotonicity and convexity of norm, there is a unique $0 < t \leq 1$ so that

$$\left\| \sum_{j=1}^{n-1} a_j e_j^* + t a_n e_n \right\| = 1,$$

and we define $\varphi(x^*) = \sum_{j=1}^{n-1} a_j e_j^* + t a_n e_n$. Defining φ as an identity on B_{X^*}, we first show that φ is uniformly norm continuous.

Notice that if $\|x^*\| \geq 1$, then by the construction of φ and uniform monotonicity,

$$\|x^*\| = \big\| |\varphi(x^*)| + |x^* - \varphi(x^*)| \big\| \geq 1 + M\big(\|x^* - \varphi(x^*)\|\big)$$

and we have $M(\|x^* - \varphi(x^*)\|) \leq \|x^*\| - 1$. That is,

$$\|x^* - \varphi(x^*)\| < M^{-1}\big(\|x^*\| - 1\big).$$

We claim that for all x^*, y^* in X^*,

$$\|\varphi(x^*) - \varphi(y^*)\| \leq 2M^{-1}\big(\|x^* - y^*\|\big).$$

Because $M(\varepsilon) \leq \varepsilon$ for all $\varepsilon > 0$, we have $M^{-1}(t) \geq t$ for all $t > 0$. Hence this inequality is trivial if $\|x^*\| \leq 1$ and $\|y^*\| \leq 1$. If $\|x^*\| > 1$ and $\|y^*\| \leq 1$, then

$$\begin{aligned}
\|\varphi(x^*) - \varphi(y^*)\| &= \|\varphi(x^*) - y^*\| \leq \|\varphi(x^*) - x^*\| + \|x^* - y^*\| \\
&\leq M^{-1}\big(\|x^*\| - 1\big) + \|x^* - y^*\| \\
&\leq M^{-1}\big(\|x^*\| - \|y^*\|\big) + M^{-1}\big(\|x^* - y^*\|\big) \\
&\leq 2M^{-1}\big(\|x^* - y^*\|\big).
\end{aligned}$$

We assume that $\|x^*\| > 1$ and $\|y^*\| > 1$ and write

$$\varphi(x^*) = \sum_{j=1}^{n-1} x^*(j) e_j^* + t x^*(n) e_n^* \quad \text{and} \quad \varphi(y^*) = \sum_{j=1}^{m-1} y^*(j) e_j^* + s y^*(m) e_m^*$$

where $x^*(i) = x^*(e_i)$ and $y^*(i) = y^*(e_i)$ for every $i \in \mathbb{N}$.

For each $n \in \mathbb{N}$, let P_n be a projection on X defined by $P_n(\sum \alpha_i e_i) = \sum_{i=1}^{n} \alpha_i e_i$ and P_n^* be the adjoint operator. We may also assume that $m \geq n$ and then $\varphi(x^*) = \varphi(P_m^*(x^*))$, $\varphi(y^*) = \varphi(P_m^*(y^*))$ and $\|P_m^* x^* - P_m^* y^*\| \leq \|x^* - y^*\|$ shows that we can replace x^* and y^* by $P_m^*(x^*)$ and $P_m^*(y^*)$ respectively. That is,

$$x^* = \sum_{j=1}^{m} x^*(j) e_j^* \quad \text{and} \quad y^* = \sum_{j=1}^{m} y^*(j) e_j^*.$$

If $m > n$, $x^*(m)$ and $y^*(m)$ can be replaced by $s y^*(m)$ and $x^*(m) - y^*(m) + s y^*(m)$ without changing $\|x^* - y^*\|$, $\varphi(x^*)$ and $\varphi(y^*)$. On the other hand, if $n = m$ and $t \leq s$, then, letting

$$x_1^* = \sum_{j=1}^{n-1} x^*(j) e_j^* + s x^*(n) e_n^*,$$

we have $\|x_1\| \geq 1$, $\varphi(x^*) = \varphi(x_1^*)$ and $\|x_1^* - \varphi(y^*)\| \leq \|x^* - y^*\|$. Hence we get

$$\begin{aligned}\|\varphi(x^*)-\varphi(y^*)\| = \|\varphi(x_1^*)-\varphi(y^*)\| &\le \|\varphi(x_1^*)-x_1^*\| + \|x_1^*-\varphi(y^*)\| \\ &\le M^{-1}(\|x_1^*\|-1) + \|x^*-y^*\| \\ &\le 2M^{-1}(\|x^*-y^*\|),\end{aligned}$$

since $\|x_1^*\| \le \|\varphi(y^*)\| + \|\varphi(y^*)-x_1^*\| \le 1 + \|\varphi(y^*)-x_1^*\|$ and $t \le M^{-1}(t)$ for all $t \ge 0$.

Now, we will show that φ is weak-$*$ continuous. Suppose that a net $\{x_\alpha^*\}$ converges weak-$*$ to x^*. Since the range of φ is bounded and X has the Schauder basis $\{e_j\}$, it is enough to check that $\lim_\alpha \langle \varphi(x_\alpha^*), e_j\rangle = \langle \varphi(x^*), e_j\rangle$ for all j. Given $x^* \in X^*$, suppose first that there exists a unique n such that

$$\left\|\sum_{j=1}^{n-1} x^*(j)e_j^*\right\| < 1 \quad \text{and} \quad \left\|\sum_{j=1}^{n} x^*(j)e_j^*\right\| \ge 1$$

and $\varphi(x^*) = \sum_{j=1}^{n-1} x^*(j)e_j^* + tx^*(n)e_n^*$ for some $0 < t \le 1$. Since $P_{n-1}^*(x_\alpha^*)$ converges to $P_{n-1}^*(x^*)$ in norm, it is clear that $\lim_\alpha \langle \varphi(x_\alpha^*), e_j\rangle = \lim_\alpha \langle x_\alpha^*, e_j\rangle = x^*(j)$ for each $1 \le j \le n-1$. Hence, we may assume that $\|P_{n-1}^*(x_\alpha^*)\| < 1$ for all α. We claim that $\lim_\alpha \varphi(x_\alpha^*)(j) = 0 = \varphi(x^*)(j)$ for all $j \ge n+1$. Otherwise, there exist a $j_0 \ge n+1$, a subnet (x_β^*) and an $\varepsilon_0 > 0$ such that $|\varphi(x_\beta^*)(j_0)| \ge \varepsilon_0$ for all β. Then

$$\varepsilon_0 \le \left|\varphi(x_\beta^*)(j_0)\right| \le \left|x_\beta^*(j_0)\right| \to \left|x^*(j_0)\right|.$$

Hence $\|P_{j_0}^*(x^*)\| > 1$ and we may assume that $\|P_{j_0}^*(x_\alpha^*)\| > 1$. So there exist $n \le n_\beta \le j_0$ such that for some $0 < t_\beta \le 1$,

$$\varphi(x_\beta^*) = \sum_{j=1}^{n_\beta - 1} x_\beta^*(j)e_j^* + t_\beta x_\beta^*(n_\beta)e_{n_\beta}^*.$$

Since $\varphi(x_\beta^*)(j_0) \ne 0$, we have $j_0 \le n_\beta$. So $n_\beta = j_0$ for all β. We may assume that $\lim_\beta t_\beta = t_0$. Then

$$1 = \lim_\beta \|\varphi(x_\beta^*)\| = \lim_\beta \left\|\sum_{j=1}^{j_0-1} x_\beta^*(j)e_j^* + t_\beta x_\beta^*(j_0)e_{j_0}^*\right\| = \left\|\sum_{j=1}^{j_0-1} x^*(j)e_j^* + t_0 x^*(j_0)e_{j_0}^*\right\|.$$

Because $j_0 \ge n+1$, we get $t_0 = 0$, which is a contradiction to that $|t_\beta x_\beta^*(j_0)| = |\varphi(x_\beta^*)(j_0)| \ge \varepsilon_0$ for all β.

We have only to show that $\lim_\alpha \varphi(x_\alpha^*)(n) = \varphi(x^*)(n) = tx^*(n)$. If $\|x_\alpha^*\| \le 1$ or $x_\alpha(n) = 0$, then set $t_\alpha = 1$. If $\|x_\alpha^*\| > 1$ and $x_\alpha^*(n) \ne 0$, then choose $0 \le t_\alpha \le 1$ so that $\varphi(x_\alpha^*) = t_\alpha x_\alpha^*(n)$. So we have for all α, $\varphi(x_\alpha^*)(n) = t_\alpha x_\alpha^*(n)$. Notice that if $t_\alpha < 1$, then

$$\varphi(x_\alpha^*) = \sum_{j=1}^{n-1} x_\alpha^*(j)e_j^* + t_\alpha x_\alpha^*(n)e_n^*.$$

For any subnet (x_γ), we can find a further subnet (x_β) such that $\lim_\beta t_\beta = t_0$. Suppose first that $t_0 < 1$. Then we may assume that $t_\beta < 1$ for all β. This means that

$$1 = \lim_\beta \|\varphi(x_\beta^*)\| = \left\|\sum_{j=1}^{n-1} x^*(j)e_j^* + t_0 x^*(n)e_n^*\right\|.$$

By the strict monotonicity, we get $t_0 = t$ and

$$\lim_\gamma \varphi(x_\gamma^*)(n) = \lim_\gamma t_\gamma x_\gamma^*(n) = tx^*(n) = \varphi(x^*)(n).$$

Secondly, suppose that $t_0 = 1$. Then we have

$$1 = \lim_{\beta} \|\varphi(x^*_\beta)\| \geq \lim_{\beta} \left\| \sum_{j=1}^{n-1} x^*_\beta(j)e^*_j + t_\beta x^*_\beta(n)e^*_n \right\| = \left\| \sum_{j=1}^{n-1} x^*(j)e^*_j + x^*(n)e^*_n \right\| \geq 1.$$

This shows that $t = 1$ and

$$\lim_{\beta} \varphi(x^*_\beta)(n) = \lim_{\beta} t_\beta x^*_\beta(n) = x^*(n) = \varphi(x^*)(n).$$

Hence we conclude that $\lim_\alpha \varphi(x^*_\alpha)(n) = \varphi(x^*)(n)$.

Finally, suppose that $\|P^*_n(x^*)\| < 1$ for all n. So, $\|x^*\| \leq 1$. Fix $n \in \mathbb{N}$. Then there exists α_n such that $\|P^*_n(x^*_\alpha)\| < 1$ for all $\alpha \geq \alpha_n$. Hence this shows that

$$\lim_{\alpha} \langle \varphi(x^*_\alpha), e_j \rangle = \lim_{\alpha} \langle x^*_\alpha, e_j \rangle = \langle x^*, e_j \rangle = \langle \varphi(x^*), e_j \rangle$$

for all $j \leq n$. Since the equality holds for arbitrary n, we get the desired result. □

Example 2.2. It is easy to check that every ℓ_p ($1 \leq p < \infty$) is uniformly monotone. There has been an extensive study about the uniform monotonicity of Orlicz–Lorentz spaces (cf. [24,25]).

Recall that the uniform complex convexity is equivalent to the uniform monotonicity on Banach lattices [30,31]. Hence we have the following.

Corollary 2.3. *Suppose that a complex Banach space X has a normalized unconditional Schauder basis $\{e_j\}$ with unconditional basis constant 1. If X^* is uniformly complex convex, then X^* admits a uniformly simultaneously continuous retraction.*

It is observed [11] that if Y^* admits a (f-uniformly) simultaneously continuous retraction and X is a norm-one complemented subspace of Y, so does X^*. Concerning the stability under the direct sum, it is shown that if we take $p_n = 1 - \frac{1}{n}$, and $X = [\bigoplus_n \ell_{p_n}]_1$, then X^* does not admit a simultaneously continuous retraction. However we get the following affirmative result.

Now, we see some stability results. The following is clear and we omit the proof.

Proposition 2.4. *Let $\{X_i\}_{i\in\mathbb{N}}$ be a family of Banach spaces and let $X = [\bigoplus X_i]_{c_0}$ or $X = [\bigoplus X_i]_{\ell_p}$ for $1 \leq p < \infty$. If X^* admits an f-uniformly simultaneously continuous retraction φ, then each X^*_i admits an f-uniformly simultaneously continuous retraction.*

Proposition 2.5. *Let $\{X_j\}_{j\in J}$ be a family of Banach spaces and let $X = [\bigoplus X_j]_1$. Suppose that each X^*_j admits a uniformly simultaneously continuous retraction φ_j. If*

$$\lim_{\varepsilon\to 0+} \sup_{j\in J} \omega_{\varphi_j}(\varepsilon) = 0,$$

then X^* admits a uniformly simultaneously continuous retraction. In particular, the dual of a finite ℓ_1 sum of Banach spaces whose duals admits a uniformly simultaneously continuous retractions also admits uniformly simultaneously continuous retraction.

Proof. For each $x^* \in X^*$, define $\varphi(x^*) = (\varphi_j(x^*))_{j\in J}$. Then it is easy to check that φ is uniformly norm-continuous and weak-$*$ continuous. □

We do not know if the similar result of Proposition 2.5 holds for c_0 or ℓ_p sums for $1 < p < \infty$. However, we provide a positive result for separable uniformly smooth spaces. Recall that a Banach space X is said to be uniformly convex if the modulus of convexity

$$\delta_X(\varepsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : x, y \in S_X, \text{ and } \|x-y\| \geq \varepsilon\right\}$$

is positive for all $0 < \varepsilon < 1$. A Banach space X is uniformly smooth if and only if X^* is uniformly convex. In the proof, we will use the following lemma.

Lemma 2.6. *(See [3, Lemma 3.3].) Let $\{c_n\}$ be a sequence of complex numbers with $|c_n| \leqslant 1$ for every n, and let $\eta > 0$ be such that for a convex series $\sum \alpha_n$, $\operatorname{Re} \sum_{n=1}^{\infty} \alpha_n c_n > 1 - \eta$. Then for every $0 < r < 1$, the set $A := \{i \in \mathbb{N} : \operatorname{Re} c_i > r\}$, satisfies the estimate*

$$\sum_{i \in A} \alpha_i \geqslant 1 - \frac{\eta}{1-r}.$$

Theorem 2.7. *Let $X = [\bigoplus X_i]_{c_0}$, where X_i's are Banach spaces and let $\delta_i(\varepsilon)$ be the modulus of convexity of X_i^*. Suppose that each space X_i is separable and $\inf_i \delta_i(\varepsilon) > 0$ for all $0 < \varepsilon < 1$. Then, X^* admits a uniformly simultaneous continuous retraction.*

Proof. For each $i \in \mathbb{N}$ there exists a sequence of finite-dimensional subspaces $E_i^1 \subset E_i^2 \subset E_i^3 \subset \ldots$ such that $\dim E_i^n = n$ and $\bigcup_{n=1}^{\infty} E_i^n$ is dense in X_i. Let e_i be the standard basis of c_0 which ensures that $\bigcup_{i=1}^{\infty} (X_i \otimes e_i)$ is dense in X, where

$$X_i \otimes e_i = \{x \otimes e_i : x \in X_i\}.$$

For each $i, j \in \mathbb{N}$, we define a sequence of spaces

$$E_k = \left(\bigcup_{p+q<i+j} E_q^p \otimes e_q\right) \cup \left(\bigcup_{q \leqslant j} E_{i+j-q}^q \otimes e_{i+j-q}\right)$$

where $k = \frac{(i+j-1)(i+j-2)}{2} + j$. We clearly see that $E_k \subset E_{k+1}$ for every $k \in \mathbb{N}$.

Let $R_k : E_k \to X$ be a natural embedding (for the convenience, we set $E_0 = \{0\}$ and $R_0 : \{0\} \to X$). By the uniform convexity, it is easy to check that there is a unique Hahn–Banach extension of every element of E_k^* to X^*. So let $H_k : {E_k}^* \to X^*$ be the map defined by the Hahn–Banach extension theorem.

We also define a map $\psi_k : {E_k}^* \to {E_{k+1}}^*$ by $\psi_k = {R_{k+1}}^* \circ H_k$. For each x^*, let $n(x^*) = \inf\{k : \|{R_k}^* x^*\| \geq 1\}$, where we use the convention that $\inf \emptyset = \infty$.

We define a retraction $\phi : X^* \to B_{X^*}$. If $\|x^*\| \leqslant 1$, then $\phi(x^*) = x^*$. If $\|x^*\| > 1$ and $n(x^*) = 1$, then we put $\phi(x^*) = H_1({R_1}^* x^* / \|{R_1}^* x^*\|)$. We assume that $\|x^*\| > 1$ and $n(x^*) > 1$. For the convenience we write $n = n(x^*)$. Since ${R_n}^* x^*|_{E_{n-1}} = \psi_{n-1}({R_{n-1}}^* x^*)|_{E_{n-1}} = x^*|_{E_{n-1}}$, we have $\|{R_{n-1}}^* x^*\| = \|\psi_{n-1}({R_{n-1}}^* x^*)\| < 1$. Hence, there exists a unique $0 < \lambda \leqslant 1$ such that $\|\lambda {R_n}^* x^* + (1-\lambda)\psi_{n-1}({R_{n-1}}^* x^*)\| = 1$. We put $\phi(x^*) = H_n(\lambda {R_n}^* x^* + (1-\lambda)\psi_{n-1}({R_{n-1}}^* x^*))$.

We now show that a retraction ϕ is weak-$*$ continuous. Suppose that (x_α^*) converges to x^* in the weak-$*$ topology.

First assume that $n = n(x^*) < \infty$. Since ${R_n}^* x_\alpha^*$ converges to ${R_n}^* x^*$ in norm, we have ${R_n}^* \phi(x_\alpha^*)$ converges to ${R_n}^* \phi(x^*)$ in norm. This implies that every weak-$*$ limit point of a net $(\phi(x_\alpha^*))$ is an extension of ${R_n}^* \phi(x^*)$. Since $\|{R_n}^* \phi(x^*)\| = 1 = \|\phi(x^*)\|$ and the Hahn–Banach extension is unique, $\phi(x_\alpha^*)$ weak-$*$ converges to $\phi(x^*)$. On the other hand, assume $\|{R_n}^* x^*\| < 1$ for every $n \in \mathbb{N}$. Since the net $(\phi(x_\alpha^*))$ is bounded, we have only to show that $\phi(x_\alpha^*)(x)$ converges to $\phi(x^*)(x)$ for all $x \in E_n$ and for all $n \ge 1$. Fix N. Then ${R_N}^* x_\alpha^*$ converges to ${R_N}^* x^*$ in norm and there exists α_0 such that $\|R_N^* x_\alpha^*\| < 1$ for all $\alpha > \alpha_0$ and $\phi(x_\alpha^*)$ is an extension of $R_N^* x_\alpha^*$ for all $\alpha > \alpha_0$. That is, $\phi(x_\alpha^*)(x) = ({R_N}^* x_\alpha^*)(x)$ for each $\alpha > \alpha_0$ and $x \in E_N$. Hence $\phi(x_\alpha^*)(x)$ converges to $\phi(x^*)(x)$ for all $x \in E_N$. Because N is arbitrary, $\phi(x_\alpha^*)$ converges to $\phi(x^*)$ in the weak-$*$ topology.

We calculate the norm-modulus of continuity of ϕ. For $\epsilon > 0$, we fix $x^*, y^* \in X^*$ satisfying $\|x^* - y^*\| < \delta(\epsilon)^2$, and let $n = n(x^*) \leqslant n(y^*) = m$. If $n = \infty$, then it is clear. So assume first that $n \le m < \infty$.

Without loss of generality, we assume that $\phi(y^*)$ is an extension of ${R_n}^* y^*$. Indeed, if $n < m$, then this follows from the definition of ϕ. On the other hand, if $n = m$, then we choose $u^* \in X^*$ which annihilates E_{n-1}. Since ${R_n}^* y^* - \psi_n({R_{n-1}}^* y^*)$ and ${R_n}^* x^* - \psi_n({R_{n-1}}^* x^*)$ both annihilate E_{n-1}, we see that they are multiples of ${R_n}^* u^*$. This fact and the convexity of $\|\cdot\|$ imply that there exists α so that either

$$\big\|{R_n}^*\big(y^* + \alpha u^*\big)\big\| = 1 \quad \text{and} \quad \big\|{R_n}^*\big(x^* + \alpha u^*\big)\big\| \geqslant 1 \quad \text{or}$$
$$\big\|{R_n}^*\big(y^* + \alpha u^*\big)\big\| \geqslant 1 \quad \text{and} \quad \big\|{R_n}^*\big(x^* + \alpha u^*\big)\big\| = 1.$$

Hence, we assume $\|{R_n}^*(y^* + \alpha u^*)\| = 1$ and $\|{R_n}^*(x^* + \alpha u^*)\| \geqslant 1$ (otherwise, we change the role of x^* and y^*.) We now take $x^* + \alpha u^*$ and $y^* + \alpha u^*$ instead of x^* and y^*.

For any element z in a space of vector-valued sequence like X and X^*, we write $z = (z(1), z(2), \ldots)$. Choose $x \in S_{E_n}$ so that ${R_n}^* \phi(x^*)(x) = 1$, then we see that $1 = \frac{{R_n}^* \phi(x^*)(i)}{\|{R_n}^* \phi(x^*)(i)\|}(x(i)) = \frac{{R_n}^* \phi(x^*)(i)}{\|\phi(x^*)(i)\|}(x(i))$ for every $i \in C$, where $C = \{i : {R_n}^* \phi(x^*)(i) \neq 0\}$.

From the definition of ϕ, we have $\operatorname{Re} {R_n}^*(x^*)(x) \geqslant 1$, and so

$$\begin{aligned} 1 - \delta(\epsilon)^2 &< \operatorname{Re}\big({R_n}^*\big(x^*\big)\big)(x) - \big\|{R_n}^*\big(x^* - y^*\big)\big\| \\ &\leqslant \operatorname{Re} {R_n}^*\big(y^*\big)(x) = \sum \operatorname{Re} {R_n}^*\big(y^*\big)(i)\big(x(i)\big) \end{aligned}$$

Define a set $A = \{i : \operatorname{Re} \frac{{R_n}^*(y^*)(i)}{\|\phi(y^*)(i)\|}(x(i)) > 1 - \delta(\epsilon),\ \|\phi(y^*)(i)\| \neq 0\}$. Then, Lemma 2.6 shows that

$$\sum_A \big\|\phi\big(y^*\big)(i)\big\| > 1 - \delta(\epsilon), \quad \text{and} \quad \sum_{A^c} \big\|\phi\big(y^*\big)(i)\big\| < \delta(\epsilon).$$

Since $\phi(y^*)$ is an extension of ${R_n}^* y^*$, for each $i \in A \cap C$, we get

$$\begin{aligned} \left\| \frac{\phi(y^*)(i)}{\|\phi(y^*)(i)\|} + \frac{\phi(x^*)(i)}{\|\phi(x^*)(i)\|} \right\| &\geqslant \frac{{R_n}^*(y^*)(i)}{\|\phi(y^*)(i)\|}\big(x(i)\big) + \frac{{R_n}^* \phi(x^*)(i)}{\|\phi(x^*)(i)\|}\big(x(i)\big) \\ &> 2 - \delta(\epsilon) \end{aligned}$$

and so,

$$\left\| \frac{\phi(y^*)(i)}{\|\phi(y^*)(i)\|} - \frac{\phi(x^*)(i)}{\|\phi(x^*)(i)\|} \right\| < \epsilon.$$

Moreover, for each $i \in A \cap C$,

$$\begin{aligned} \big\|\phi\big(y^*\big)(i) - \phi\big(x^*\big)(i)\big\| &= \left\| \frac{\phi(y^*)(i)}{\|\phi(y^*)(i)\|} - \frac{\phi(x^*)(i)}{\|\phi(y^*)(i)\|} \right\| \big\|\phi\big(y^*\big)(i)\big\| \\ &< \left(\left\| \frac{\phi(y^*)(i)}{\|\phi(y^*)(i)\|} - \frac{\phi(x^*)(i)}{\|\phi(x^*)(i)\|} \right\| + \left\| \frac{\phi(x^*)(i)}{\|\phi(x^*)(i)\|} - \frac{\phi(x^*)(i)}{\|\phi(y^*)(i)\|} \right\| \right) \big\|\phi\big(y^*\big)(i)\big\| \\ &< \epsilon \big\|\phi\big(y^*\big)(i)\big\| + \big|\big\|\phi\big(y^*\big)(i)\big\| - \big\|\phi\big(x^*\big)(i)\big\|\big|. \end{aligned}$$

So we have for all $i \in A$,

$$\|\phi(y^*)(i) - \phi(x^*)(i)\| < \epsilon\|\phi(y^*)(i)\| + \big|\|\phi(y^*)(i)\| - \|\phi(x^*)(i)\|\big|.$$

On the other hand, the assumption $\|x^* - y^*\| < \delta(\epsilon)^2$ implies that $\|{R_n}^*x^* - {R_n}^*y^*\| < \delta(\epsilon)^2$, and so $\sum |\|{R_n}^*x^*(i)\| - \|{R_n}^*y^*(i)\|| \leqslant \sum \|{R_n}^*x^*(i) - {R_n}^*y^*(i)\| < \delta(\epsilon)^2$. Since $\|{R_n}^*y^*(i)\| \leqslant \|\phi(y^*)(i)\|$, $\|{R_n}^*x^*\| \geqslant 1$, and $\|\phi(y^*)\| = 1$, we have, setting $P = \{i : \|\phi(y^*)(i)\| \geq \|{R_n}^*x^*(i)\|\}$ and $Q = \{\|\phi(y^*)(i)\| < \|{R_n}^*x^*(i)\|\}$,

$$\begin{aligned}
&\sum \big|\|\phi(y^*)(i)\| - \|{R_n}^*x^*(i)\|\big| \\
&\quad = \sum_P \big(\|\phi(y^*)(i)\| - \|{R_n}^*x^*(i)\|\big) + \sum_Q \big(\|{R_n}^*x^*(i)\| - \|\phi(y^*)(i)\|\big) \\
&\quad = 1 - \sum_Q \|\phi(y^*)(i)\| - \sum_P \|{R_n}^*x^*(i)\| + \sum_Q \big(\|{R_n}^*x^*(i)\| - \|\phi(y^*)(i)\|\big) \\
&\quad \leq \sum_Q \|{R_n}^*x^*(i)\| - \sum_Q \|\phi(y^*)(i)\| + \sum_Q \big(\|{R_n}^*x^*(i)\| - \|\phi(y^*)(i)\|\big) \\
&\quad \leq 2\sum_Q \big(\|{R_n}^*x^*(i)\| - \|R_n^*y^*(i)\|\big) < 2\delta(\varepsilon)^2.
\end{aligned}$$

Notice also that $R_n^*x^*$ and $R_{n-1}^*x^*$ may have only one different term. Suppose that this different term is n_1th term of $R_n^*x^*$. Then $\|R_{n-1}^*x^*(i)\| = \|R_n^*x^*(i)\| = \|\phi(x^*)(i)\|$ for all $i \neq n_1$. Therefore we have $\sum_{i \neq n_1} |\|\phi(y^*)(i)\| - \|\phi(x^*)(i)\|| < 2\delta(\epsilon)^2$.

Since $\sum \|\phi(y^*)(i)\| = \sum \|\phi(x^*)(i)\| = 1$, we have $|\|\phi(y^*)(n_1)\| - \|\phi(x^*)(n_1)\|| < 2\delta(\epsilon)^2$. Moreover, the fact that $\sum |\|\phi(y^*)(i)\| - \|\phi(x^*)(i)\|| < 4\delta(\epsilon)^2$ shows

$$\begin{aligned}
\sum_{A^c} \|\phi(x^*)(i)\| &\leqslant \sum_{A^c} \|\phi(x^*)(i) - \phi(y^*)(i)\| + \sum_{A^c} \|\phi(y^*)\| \\
&< 4\delta(\epsilon)^2 + \delta(\epsilon).
\end{aligned}$$

Hence, we deduce that

$$\begin{aligned}
\|\phi(x^*) - \phi(y^*)\| &= \sum_A \|\phi(y^*)(i) - \phi(x^*)(i)\| + \sum_{A^c} \|\phi(y^*)(i) - \phi(x^*)(i)\| \\
&\leqslant \sum_A \epsilon\|\phi(y^*)(i)\| + \sum_A \big|\|\phi(y^*)(i)\| - \|\phi(x^*)(i)\|\big| + \sum_{A^c} \|\phi(y^*)(i)\| + \sum_{A^c} \|\phi(x^*)(i)\| \\
&< \epsilon + 4\delta(\epsilon)^2 + 4\delta(\epsilon)^2 + \delta(\epsilon) + \delta(\epsilon) \\
&= \epsilon + 8\delta(\epsilon)^2 + 2\delta(\epsilon).
\end{aligned}$$

Finally, assume that $n < m = \infty$. In this case, $\|y^*\| \leq 1$. If $\|x^*\| \leq 1$, then the desired result clearly holds. So assume that $\|x^*\| > 1$. Let $y_t^* = tx^* + (1-t)x^*$ and let

$$t_0 = \sup\{0 < t < 1 : \|y_t^*\| = 1\}.$$

It is clear that $0 \leq t_0 < 1$. For each $t_0 < s < 1$, $\|y_s^*\| > 1$ and $\|x^* - y_s^*\| \leq \|x^* - y^*\| < \delta(\varepsilon)^2$. From the previous result, we have

$$\|\phi(x^*) - \phi(y_s)\| < \varepsilon + 8\delta(\varepsilon)^2 + 2\delta(\varepsilon).$$

Since y_s^* converges to y_{t_0} as s tends to t_0, the weak-$*$ continuity of ϕ shows that

$$\left\|\phi(x^*) - \phi(y_{t_0})\right\| \le \varepsilon + 8\delta(\varepsilon)^2 + 2\delta(\varepsilon).$$

Since $\|y_{t_0}\| \le 1$, we have

$$\left\|\phi(x^*) - \phi(y)\right\| \le \left\|\phi(x^*) - \phi(y_{t_0})\right\| + \left\|y_{t_0}^* - y^*\right\| \le \varepsilon + 9\delta(\varepsilon)^2 + 2\delta(\varepsilon).$$

This completes the proof. □

For the ℓ_p sum of a countable family of Banach spaces, we get the following.

Proposition 2.8. *Let $\{X_i\}_{i\in\mathbb{N}}$ be a family of Banach spaces whose dual spaces are separable uniformly convex with moduli of convexity $\delta_i(\varepsilon)$ such that $\inf_i \delta_i(\varepsilon) > 0$ for all $0 < \varepsilon < 1$ and let $X = [\bigoplus X_i]_{\ell_p}$ for $1 \le p < \infty$. Then, X^* admits a uniformly simultaneous continuous retraction.*

Proof. Benyamini [11] showed that if X is a separable Banach space whose dual space is uniformly convex with modulus of convexity δ, then X^* admits a δ^{-1}-uniformly simultaneously continuous retraction. For $1 < p < \infty$, the ℓ_p sum of a countable family of separable uniformly convex spaces with uniformly lower bounded moduli of convexity is separable uniformly convex [21] and we get the desired result.

Finally, suppose that $p = 1$. By the assumption, for each $i \in \mathbb{N}$, we get a δ_i^{-1}-uniformly simultaneously continuous retraction φ_i on X_i^*. By Proposition 2.5, it is enough to show that

$$\lim_{\varepsilon\to 0+} \sup_i \delta_i^{-1}(\varepsilon) = 0.$$

Otherwise, there exists $t_0 \in (0,1)$ such that, for each $n \in \mathbb{N}$, there is $i_n \in \mathbb{N}$ satisfying $\delta_{i_n}^{-1}(\frac{1}{n}) > t_0$. Hence we have, for all n,

$$\frac{1}{n} \ge \delta_{i_n}(t_0) \ge \inf_i \delta_i(t_0).$$

It is a contradiction to $\inf_i \delta_i(t_0) > 0$ and this completes the proof. □

Proposition 2.9. *Let L be a locally compact Hausdorff space, K be the one-point compactification of L and let $M(L)$ and $M(K)$ be the Banach spaces of all scalar-valued Borel regular measures on L and K with the total variational norms, respectively. Suppose that $M(K)$ admits a uniformly simultaneously continuous retraction as a dual of $C(K)$. Then $M(L)$ admits a uniformly simultaneously continuous retraction as a dual of $C_0(L)$.*

Proof. Let $K = L \cup \{\infty\}$ and let ϕ be an f-uniformly simultaneously continuous retraction from $C(K)^*$ onto $B_{C(K)^*}$. Then for each $\mu \in M(L) = C(L)^*$ and for each Borel subset E of K, define $\tilde{\mu}(E) = \mu(E \setminus \{\infty\})$. Then it is clear that $\tilde{\mu} \in M(K)$. Define the map $\psi : M(L) \to B_{M(L)}$ by, for each $f \in C_0(L)$,

$$\langle f, \psi(\mu)\rangle = \int_L f\, d\phi(\tilde{\mu}).$$

Then it is easy to check that ψ is weak-$*$ continuous on $M(L) = C_0(L)^*$ and it is f-uniformly continuous with respect to the norm. □

Corollary 2.10. *Let L be a locally compact metrizable Hausdorff space. Then the real space $C_0(L)^*$ admits a uniformly simultaneously continuous retraction.*

Proof. It is shown that if K is compact metrizable space, then the real space $C(K)^*$ admits a uniformly simultaneously continuous retraction. Since L is metrizable, its one-point compactification $\hat{L}$ is compact metrizable. Hence the result follows from Proposition 2.9. $\square$

3. Retraction and Bishop–Phelps–Bollobás property

The Bishop–Phelps theorem [13] states that for a Banach space X, every element in its dual space X^* can be approximated by ones that attain their norms. Since then, there has been an extensive research to extend this result to bounded linear operators between Banach spaces [15,26,32,34,36,37] and non-linear mappings [2,5,9,17,19,29]. On the other hand, Bollobás [14] sharpened the Bishop–Phelps theorem which is called the Bishop–Phelps–Bollobás theorem.

Theorem 3.1 *(Bishop–Phelps–Bollobás theorem). Let X be a Banach space. If $x \in S_X$ and $x^* \in S_{X^*}$ satisfy $|x^*(x)-1| < \varepsilon^2/4$, then there exist $y \in S_X$ and $y^* \in S_{X^*}$ such that $y^*(y) = 1$, $\|x^*-y^*\| < \varepsilon$ and $\|x-y\| < \varepsilon$.*

Acosta, Aron, García and Maestre [3] introduced the Bishop–Phelps–Bollobás property to study extensions of the theorem above to operators between Banach spaces.

Definition 3.2. (See [3, Definition 1.1].) A pair of Banach spaces (X, Y) is said to have the *Bishop–Phelps–Bollobás property* (*BPBp* in short) for operators if, for every $\varepsilon \in (0, 1)$, there is $\eta(\varepsilon) > 0$ such that for every $T_0 \in L(X, Y)$ with $\|T_0\| = 1$ and every $x_0 \in S_X$ satisfying

$$\big\|T_0(x_0)\big\| > 1 - \eta(\varepsilon),$$

there exist $S \in L(X, Y)$ and $x \in S_X$ such that

$$1 = \|S\| = \|Sx\|, \qquad \|x_0 - x\| < \varepsilon \quad \text{and} \quad \|T_0 - T\| < \varepsilon.$$

In this case, we will say that (X, Y) has the BPBp with function $\varepsilon \mapsto \eta(\varepsilon)$. The pair (X, Y) is said to have the *Bishop–Phelps Property* (BPp) if the set of all norm-attaining operators is dense in $L(X, Y)$.

It is clear that BPBp implies BPp. Recall that Bourgain [15] showed that (X, Y) has the BPp for every Banach space Y if X has the Radon–Nikodým property. However, it is shown [3] there exists a Banach space Y such that (ℓ_1, Y) does not have BPBp even though ℓ_1 has the Radon–Nikodým property.

In the study of the operators from a Banach space into $C(K)$, the following representation theorem is useful. We are stating a version of this representation theorem for operators into $C_0(S)$ space, which is a slight modification of [22, Theorem 1, p. 490] and we omit the proof.

Lemma 3.3. *Let X be a Banach space and let L be a locally compact Hausdorff topological space. Given an operator $T : X \to C_0(L)$, define $\mu : L \to X^*$ by $\mu(s) = T^*(\delta_s)$ for every $s \in L$. Then the relationship*

$$[Tx](s) = \mu(s)(x), \quad \forall x \in X,\ s \in L$$

defines an isometric isomorphism between $\mathcal{L}(X, C_0(L))$ and the space of w^*-continuous functions from L to X^* which vanishes at infinity, endowed with the supremum norm, i.e. $\|\mu\| = \sup\{\|\mu(s)\| : s \in L\}$. The subspace of compact operators corresponds to norm continuous functions which vanishes at infinity.

If $C(K)$ is the space of all continuous functions on a compact Hausdorff space K and X is a Banach space whose dual X^* admits a uniform simultaneously continuous retraction, then the norm-attaining operators are dense in the space $\mathcal{L}(X, C(K))$ of bounded linear operators from X into $C(K)$ [12, Proposition 4.22.]. So $L_\infty[0,1]$ does not admit a uniformly simultaneously continuous retraction because the pair $(L_1[0,1], C(S))$ does not have the BPp for a certain compact metric space S [37,27]. It is worth-while to note that $(L_1(\mu), L_\infty(\nu))$ has BPp if μ is any measure and ν is a localizable measure [23,35]. These results are refined to show that $(L_1(\mu), L_\infty(\nu))$ has BPBp if μ is any measure and ν is a localizable measure [7,20].

Let f be a nonnegative nondecreasing function such that $\lim_{t\to 0+} f(t) = 0 = f(0)$. A map $\varphi : X^* \to B_{X^*}$ is called an *f-approximate nearest point map* if $\|\varphi(x^*) - x^*\| \le d(x^*, B_{X^*}) + f(d(x^*, B_{X^*}))$ for all $x^* \in X^*$. This notion is introduced by Benyamini [11]. A dual space X^* is said to *admit weak-$*$ approximate nearest point map* if there exists a weak-$*$ continuous f-approximate nearest point map $\varphi : X^* \to B_{X^*}$. Notice that the weak-$*$ continuous approximate nearest point map is a weak-$*$ continuous retraction. It is easy to check that if X^* admits a uniformly simultaneously continuous retraction $\varphi : X^* \to B_{X^*}$, then φ is a weak-$*$ ω_φ-approximate nearest point map [11].

Theorem 3.4. *Let K be a locally compact Hausdorff space and let X be a Banach space. If X^* admits a weak-$*$ approximate nearest map, then the pair $(X, C_0(K))$ has the BPBp.*

Proof. Let $r : X^* \to B_{X^*}$ be a weak-$*$ f-approximate nearest point map. Given $\varepsilon > 0$, suppose that $\|T(x_0)\| > 1 - \varepsilon^2/4$ for some $T \in S_{L(X,C(K))}$ and $x_0 \in S_X$. Let $\varphi : K \to X^*$ be the function $\varphi(s) = T^*(\delta_s)$ for all $s \in K$. Choose $t_0 \in K$ such that $|T(x_0)(t_0)| = |\langle x_0, T^*(\delta_{t_0})\rangle| = |\varphi(t_0)(x_0)| > 1 - \varepsilon^2/4$. By the Bishop–Phelps–Bollobás Theorem 3.1, there exist a norm-attaining functional $x_1^* \in S_{X^*}$ and $x_1 \in S_X$ such that

$$\|x_0 - x_1\| < \varepsilon, \qquad \left\| x_1^* - \frac{\varphi(t_0)}{\|\varphi(t_0)\|} \right\| < \varepsilon.$$

Since $\|\varphi(t_0) - \frac{\varphi(t_0)}{\|\varphi(t_0)\|}\| = 1 - \|\varphi(t_0)\| < \varepsilon^2/4 < \varepsilon$, we have $\|x_1^* - \varphi(t_0)\| < 2\varepsilon$. Choose a function $f_0 \in C_0(K)$ such that $f_0(t_0) = 1$ and $0 \le f \le 1$. Define $\psi : K \to X^*$ by

$$\psi(t) = r\big(\varphi(t) + f_0(t)\big(x_1^* - \varphi(t_0)\big)\big) \quad (t \in K).$$

Then $\psi(t_0) = r(x_1^*) = x_1^*$. Let S be the corresponding operator and

$$1 \ge \|S\| \ge \|Sx_1\| \ge \big|\langle Sx_1, \delta_{t_0}\rangle\big| = \big|\big\langle \psi(t_0), x_1\big\rangle\big| = \big|\big\langle x_1^*, x_1\big\rangle\big| = 1.$$

Then we have

$$\begin{aligned}
\|S - T\| &= \sup_{t\in K}\big\|\varphi(t) - \psi(t)\big\| = \sup_{t\in K}\big\|\varphi(t) - r\big(\varphi(t) + f_0(t)\big(x_1^* - \varphi(t_0)\big)\big)\big\| \\
&\le \sup_{t\in K}\big\|\big(\varphi(t) + f_0(t)\big(x_1^* - \varphi(t_0)\big)\big) - r\big(\varphi(t) + f_0(t)\big(x_1^* - \varphi(t_0)\big)\big)\big\| + \big\|x_1^* - \varphi(t_0)\big\| \\
&\le d\big(\varphi(t) + f_0(t)\big(x_1^* - \varphi(t_0)\big), B_{X^*}\big) + f\big(d\big(\varphi(t) + f_0(t)\big(x_1^* - \varphi(t_0)\big), B_{X^*}\big)\big) + 2\varepsilon \\
&\le \big\|x_1^* - \varphi(t_0)\big\| + f\big(\big\|x_1^* - \varphi(t_0)\big\|\big) + 2\varepsilon \\
&\le 4\varepsilon + f(2\varepsilon).
\end{aligned}$$

This completes the proof. $\square$

Cascales, Guirao and Kadets [16] (cf. [6]) showed that every Asplund operator T from a Banach space X into a uniform algebra A can be approximated by norm-attaining Asplund operators. In particular,

$(X, C(K))$ has the BPBp if X is an Asplund space. Since $C[0,1]$ is not an Asplund space, the Banach space whose dual admits the uniformly simultaneously continuous retraction need not be an Asplund space. Benyamini also constructed an example which shows that there is a (Asplund) Banach space which is isomorphic to ℓ_2 whose dual does not admit a uniformly simultaneously continuous retraction [11].

Proposition 3.5. *Let $\{X_j\}_{j\in J}$ be a family of Banach spaces and let $X = [\bigoplus X_n]_1$. Suppose that each X_j^* admits a weak-$*$ f-approximate nearest point map φ_j with a common function f. Then X^* admits a weak-$*$ f-approximate nearest point map.*

Proposition 2.5 shows the following.

Corollary 3.6. *Let $\{X_j\}_{j\in J}$ be a family of Banach spaces and let $X = [\bigoplus X_j]_1$. Suppose that each X_j^* admits a uniformly simultaneously continuous retraction φ_j. If*

$$\lim_{\varepsilon\to 0+} \sup_{j\in J} \omega_{\varphi_j}(\varepsilon) = 0,$$

then $(X, C_0(L))$ has the BPBp for all locally compact Hausdorff spaces L.

For the range spaces, the stability of the BPBp under various direct sums of Banach spaces is studied in [8]. We get here some stability results for the domain spaces when the range is $C(K)$.

Example 3.7. Let X be a Banch space whose dual X^* admits a uniformly simultaneously continuous retraction like ℓ_p or $C(S)$ spaces for all compact Hausdorff space S. Then $(\ell_1(X), C(K))$ has the BPBp for all compact Hausdorff space K. Moreover, we also have the same result for the finite ℓ_1 sums of different Banach spaces whose dual admits a uniformly simultaneously continuous retraction. For example, we see that $(\ell_p \oplus_1 C(S), C(K))$ has the BPBp.

Recently it is shown [4] that the pair $(C(S), C(K))$ has the BPBp if $C(S)$ and $C(K)$ are spaces of real-valued continuous functions on a compact Hausdorff spaces S and K respectively. However it is still open for the spaces of complex-valued continuous functions. It is shown [10] that $C(S)^*$ admits weak-$*$ approximate nearest point map if S is a compact metric space.

Corollary 3.8. *Let S be a locally compact metrizable space and L a locally compact Hausdorff space. Then for real-spaces $C_0(S)$ and $C_0(L)$, the pair $(C_0(S), C_0(L))$ has the BPBp.*

It is worth-while to remark that the first-named author shows that (c_0, X) has the BPBp for all uniformly convex spaces X [28].

Let $K(X, Y)$ be a subspace of $L(X, Y)$ which consists of all compact operators from a Banach space X into a Banach space Y. Recently the notion of B^k was introduced by Martín [33]. A Banach space Y is said to have property B^k if for any Banach space X, the norm-attaining compact operators are dense in $K(X, Y)$. Johnson and Wolfe [26] showed that $C(K)$ space has property B^k. The following result is due to Aron, Cascales and Kozhushkina [6]. However, we give another proof using retraction.

Theorem 3.9. *Let K be a compact Hausdorff space and let E be a Banach space. Then for each $0 < \varepsilon < 1$, there is $\eta(\varepsilon) > 0$ such that if $T \in S_{K(E,C(K))}$ and $\|T(x_0)\| > 1 - \eta(\varepsilon)$, there exist $S \in S_{K(E,C(K))}$ and $x_1 \in S_E$ such that $\|S(x_1)\| = 1$, $\|x_0 - x_1\| < \varepsilon$ and $\|S - T\| < \varepsilon$. In fact we can take $\eta(\varepsilon) = \frac{\varepsilon^2}{64}$.*

Proof. Given $\varepsilon > 0$, suppose that $\|T(x_0)\| > 1 - \varepsilon^2/4$ for some $T \in S_{L(E,C(K))}$ and $x_0 \in S_E$. Let $\varphi : K \to E^*$ be the function $\varphi(s) = T^*(\delta_s)$ for all $s \in K$. Since T is compact, φ is norm-continuous. Choose $t_0 \in K$ such that $|T(x_0)(t_0)| = |\langle x_0, T^*(\delta_{t_0})\rangle| = |\varphi(t_0)(x_0)| > 1 - \varepsilon^2/4$. By the Bishop–Phelps–Bollobás Theorem 3.1, there exists a norm-attaining functional $x_1^* \in S_{E^*}$ and $x_1 \in S_E$ such that

$$\|x_0 - x_1\| < \varepsilon, \qquad \left\| x_1^* - \frac{\varphi(t_0)}{\|\varphi(t_0)\|} \right\| < \varepsilon.$$

Since $\|\varphi(t_0) - \frac{\varphi(t_0)}{\|\varphi(t_0)\|}\| = 1 - \|\varphi(t_0)\| < \varepsilon^2/4 < \varepsilon$, we have $\|x_1^* - \varphi(t_0)\| < 2\varepsilon$. Let $r : E^* \to B_{E^*}$ be the retraction defined by $r(x) = x$ if $\|x\| \le 1$ and $r(x) = \frac{1}{\|x\|}x$ if $\|x\| \ge 1$. Define the norm-continuous map $\psi : K \to E^*$ by

$$\psi(t) = r\big(\varphi(t) + x_1^* - \varphi(t_0)\big) \quad (t \in K).$$

Then $\psi(t_0) = r(x_1^*) = x_1^*$. Let S be the corresponding compact operator and

$$1 \ge \|S\| \ge \|Sx_1\| \ge \big|\langle Sx_1, \delta_{t_0}\rangle\big| = \big|\big\langle \psi(t_0), x_1\big\rangle\big| = \big|\big\langle x_1^*, x_1\big\rangle\big| = 1.$$

Hence we have $\|S\| = 1 = \|Sx_1\|$. Since $1 \le \|y\| \le 1 + \varepsilon$ implies that

$$\big\|r\big(y^*\big) - y^*\big\| \le \left\| r\big(y^*\big) - r\left(\frac{y^*}{\|y^*\|}\right) \right\| + \left\| \frac{y^*}{\|y^*\|} - y^* \right\| \le 2\varepsilon,$$

we have

$$\begin{aligned} \|S - T\| &= \sup_{t \in K} \big\|\varphi(t) - \psi(t)\big\| = \sup_{t \in K} \big\| r\big(\varphi(t) + x_1^* - \varphi(t_0)\big) - \varphi(t)\big\| \\ &\le 2\varepsilon + \big\|x_1^* - \varphi(t_0)\big\| \le 4\varepsilon. \end{aligned}$$

Therefore, by letting $\eta(\varepsilon) = \frac{\varepsilon^2}{64}$, we get the desired result. □

Because $C(K)$ space is a predual of an L_1 space, the above theorem is equivalent to the following which is proved in [4] and we omit the proof.

Theorem 3.10. *(See [4].) For each $0 < \varepsilon < 1$, there is $\eta(\varepsilon) > 0$ such that if E is any Banach space, Y is any predual of an L_1-space, $T \in S_{K(E,Y)}$ and $\|T(x_0)\| > 1 - \eta(\varepsilon)$, there exist $S \in S_{K(E,Y)}$ and $x_1 \in S_E$ such that $\|S(x_1)\| = 1$, $\|x_0 - x_1\| < \varepsilon$ and $\|S - T\| < \varepsilon$.*

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