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Complete characterization of Hadamard powers preserving Loewner positivity, monotonicity, and convexity ☆

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A B S T R A C T

Entrywise powers of symmetric matrices preserving positivity, monotonicity or convexity with respect to the Loewner ordering arise in various applications, and have received much attention recently in the literature. Following FitzGerald and Horn (1977) [8], it is well-known that there exists a *critical exponent* beyond which all entrywise powers preserve positive definiteness. Similar phenomena have also recently been shown by Hiai (2009) to occur for monotonicity and convexity. In this paper, we complete the characterization of all the entrywise powers below and above the critical exponents that are positive, monotone, or convex on the cone of positive semidefinite matrices. We then extend the original problem by fully classifying the positive, monotone, or convex powers in a more general setting where additional rank constraints are imposed on the matrices. We also classify the entrywise powers that are super/sub-additive with respect to the Loewner ordering. Finally, we extend all the previous characterizations to matrices with negative entries. Our analysis consequently allows us to answer a question raised by Bhatia and Elsner (2007) regarding the smallest dimension for which even extensions of the power functions do not preserve Loewner positivity.

1. Introduction and main results

The study of positive definite matrices and of functions that preserve them arises naturally in many branches of mathematics and other disciplines. Given a function $f : \mathbb{R} \to \mathbb{R}$ and a matrix $A = (a_{ij})$, the matrix $f[A] := (f(a_{ij}))$ is obtained by applying f to the entries of A. Such mappings are called *Hadamard functions* (see [16, §6.3]) and appear naturally in many fields of pure and applied mathematics, probability, and statistics.

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Characterizing functions $f : \mathbb{R} \to \mathbb{R}$ such that $f[A]$ is positive semidefinite for every positive semidefinite matrix A is critical for many applications. For example, in modern high-dimensional probability and statistics, functions are often applied to the entries of covariance/correlation matrices in order to obtain regularized estimators with attractive properties (like sparsity, good condition number, etc.). Particular examples of functions used in practice include the so-called *hard* and *soft-thresholding* functions (see [5, 11–14]), and the power functions – see e.g. [20] and [24, §2.2]. The resulting matrices often serve as ingredients in other statistical procedures that require these matrices to be positive semidefinite. In order for such procedures to be widely applicable, it is therefore important to know whether a given Hadamard function preserves positivity.

Let A be a positive semidefinite matrix with nonnegative entries. In this paper, we study the properties of entrywise powers of A, i.e., the properties of $f[A]$ when $f(x) = x^\alpha$ is applied elementwise to A (for some $\alpha \geq 0$). This question of which entrywise (or Hadamard) powers x^α preserve Loewner positivity has been widely studied in the literature. One of the earliest works in this setting is by FitzGerald and Horn [8], who studied the set of entrywise powers preserving Loewner positivity among $n \times n$ matrices, in connection with the Bieberbach conjecture. They show that a certain phase transition occurs at $\alpha = n - 2$. More precisely, every $\alpha \geq n - 2$ as well as every positive integer preserve Loewner positivity, while no non-integers in $(0, n - 2)$ do so.

The phase transition at the integer $n - 2$ has been popularly referred to in the literature as the "critical exponent" (CE) for preserving Loewner positivity. (We remark that the notion of critical exponents in this paper differs from that in the physics literature, where it is used in the context of many-body systems.) Indeed, the study of critical exponents in the present context – and more generally of functions preserving a form of positivity – is an interesting and important endeavor in a wide variety of situations, and has been studied in many settings (see e.g. [21,17,10,22,18]).

While it is more common in the critical exponents literature to study matrices with nonnegative entries, positive semidefinite matrices containing negative entries also occur frequently in practice. In recent work, Hiai [15] extended previous work by FitzGerald and Horn by considering the odd and even extensions of the power functions to $\mathbb{R}$. Recall that for $\alpha \in \mathbb{R}$, the even and odd multiplicative extensions to $\mathbb{R}$ of the power function $f_\alpha(x) := x^\alpha$ are defined to be $\phi_\alpha(x) := |x|^\alpha$ and $\psi_\alpha(x) := \operatorname{sign}(x)|x|^\alpha$ at $x \neq 0$, and $f_\alpha(0) = \phi_\alpha(0) = \psi_\alpha(0) := 0$. In [15], Hiai studied the powers $\alpha > 0$ for which ϕ_α and ψ_α preserve Loewner positivity, and showed that the same phase transition also occurs at $n - 2$ for ϕ_α, ψ_α, as demonstrated in [8]. He also analyzed functions that are *monotone* and *convex* with respect to the Loewner ordering, and proved several deep results and connections between these classes of functions. These results are akin to the corresponding connections between positivity, monotonicity, and convexity for real functions of one variable. Before recalling these notions, we first introduce some notation. Given $n \in \mathbb{N}$ and $I \subset \mathbb{R}$, let $\mathbb{P}_n(I)$ denote the set of symmetric positive semidefinite $n \times n$ matrices with entries in I; denote $\mathbb{P}_n(\mathbb{R})$ by $\mathbb{P}_n$. We write $A \geq B$ when $A - B \in \mathbb{P}_n$. For a function $f : I \to \mathbb{R}$ and a matrix $A \in \mathbb{P}_n(I)$, we denote by $f[A]$ the matrix $f[A] := (f(a_{ij}))$. For a matrix A with nonnegative entries, the entrywise power $A^{\circ\alpha} := ((a_{ij}^\alpha))$ then equals $f_\alpha[A]$. Given a subset $V \subset \mathbb{P}_n(I)$, recall [15] that a function $f : I \to \mathbb{R}$ is

- *positive on* V with respect to the Loewner ordering if $f[A] \geq 0$ for all $0 \leq A \in V$;
- *monotone on* V with respect to the Loewner ordering if $f[A] \geq f[B]$ for all $A, B \in V$ such that $A \geq B \geq 0$;
- *convex on* V with respect to the Loewner ordering if $f[\lambda A + (1 - \lambda)B] \leq \lambda f[A] + (1 - \lambda)f[B]$ for all $0 \leq \lambda \leq 1$ and $A, B \in V$ such that $A \geq B \geq 0$;
- *super-additive on* V with respect to the Loewner ordering if $f[A + B] \geq f[A] + f[B]$ for all $A, B \in V$ for which $f[A + B]$ is defined;
- *sub-additive on* V with respect to the Loewner ordering if $f[A + B] \leq f[A] + f[B]$ for all $A, B \in V$ for which $f[A + B]$ is defined.

For convenience, functions satisfying these properties are henceforth termed Loewner positive, Loewner monotone, Loewner convex, and Loewner super/sub-additive respectively.

Note that many of the critical exponents for Hadamard powers preserving positivity, monotonicity, and convexity have already been determined in the literature [8,4,15]. Yet the sets of all powers preserving these properties have not been fully characterized. Specifically, the powers below the critical exponents have not been fully analyzed. However, when one uses power functions to regularize positive semidefinite matrices such as correlation matrices, the lower powers are crucially important, as they produce a lesser degree of perturbation from the original matrix. So it is also important to classify the powers below the critical exponent, which preserve positivity. There is thus a fundamental gap, which is addressed in this paper. Specifically, we completely characterize all powers $\alpha > 0$ for which the functions $f_\alpha(x)$, $\phi_\alpha(x)$, and $\psi_\alpha(x)$ are positive, monotone, convex, or super/sub-additive with respect to the Loewner ordering, thus completing the analysis.

An important refinement of the above problem is when an additional rank constraint is imposed. Specifically, we are interested in classifying the entrywise powers that are Loewner positive, monotone, or convex, when restricted to matrices in $\mathbb{P}_n$ of rank at most k, for fixed $1 \leq k \leq n$. Our motivation for imposing such constraints is twofold. First, for each non-integer power $\alpha < n-2$ below the critical exponent, one can find low rank $n \times n$ matrices whose positivity is not preserved by applying f_α entrywise. Preliminary results in this regard can be found in FitzGerald–Horn [8] and Bhatia–Elsner [4]; however, the role that rank plays in preserving positivity is not fully understood. It is thus natural to ask which entrywise powers are Loewner positive, monotone, or convex, when restricted to positive semidefinite matrices with low rank, or rank bounded above. Second, many applications in modern-day high-dimensional statistics require working with correlation matrices arising from small samples. Such matrices are very often rank-deficient in practice, and thus it is useful to characterize maps that preserve positivity when applied to matrices of a fixed rank.

Before stating our main result, we introduce some notation.

Definition 1.1. Fix integers $n \geq 2$ and $1 \leq k \leq n$, and subsets $I \subset \mathbb{R}$. Let $\mathbb{P}_n^k(I)$ denote the subset of matrices in $\mathbb{P}_n(I)$ that have rank at most k. Define:

$$\begin{aligned}
\mathcal{H}_{pos}(n,k) &:= \big\{\alpha \in \mathbb{R}\colon \text{ the function } x^\alpha \text{ is positive on } \mathbb{P}_n^k\big([0,\infty)\big)\big\},\\
\mathcal{H}_{pos}^{\phi}(n,k) &:= \big\{\alpha \in \mathbb{R}\colon \text{ the function } \phi_\alpha \text{ is positive on } \mathbb{P}_n^k(\mathbb{R})\big\},\\
\mathcal{H}_{pos}^{\psi}(n,k) &:= \big\{\alpha \in \mathbb{R}\colon \text{ the function } \psi_\alpha \text{ is positive on } \mathbb{P}_n^k(\mathbb{R})\big\}. \qquad (1.1)
\end{aligned}$$

Similarly, let $\mathcal{H}_J(n,k), \mathcal{H}_J^{\phi}(n,k), \mathcal{H}_J^{\psi}(n,k)$ denote the entrywise powers preserving Loewner properties on $\mathbb{P}_n^k([0,\infty))$ or $\mathbb{P}_n^k(\mathbb{R})$, with $J \in \{\text{positivity}, \text{monotonicity}, \text{convexity}, \text{super-additivity}, \text{sub-additivity}\}$.

We now state the main result of this paper in the form of Table 1.3.

Theorem 1.2 *(Main result). Fix an integer $n \geq 2$. The sets of entrywise real powers that are Loewner positive, monotone, convex, and super/sub-additive, are as listed in Table 1.3.*

As the present paper achieves a complete classification of the powers preserving various Loewner properties, previous contributions in the literature are also included in Table 1.3 for completeness. Note that there are many cases which had not been considered previously and which we settle completely in the paper. For sake of brevity, we will only briefly sketch proofs for the previously addressed cases (in order to mention how the rank constraint affects the problem). We instead focus our attention on the cases that remain open in the literature. Our original contributions in this paper are:

Table 1.3
Summary of real Hadamard powers preserving Loewner properties, with additional rank constraints (G–K–R refers to the current paper).

J	$\mathcal{H}_J(n,k)$	$\mathcal{H}_J^{\phi}(n,k)$	$\mathcal{H}_J^{\psi}(n,k)$
Positivity			
$k=1$	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{R}$
	G–K–R	G–K–R	G–K–R
$2 \le k \le n$	$\mathbb{N} \cup [n-2,\infty)$	$2\mathbb{N} \cup [n-2,\infty)$	$(-1+2\mathbb{N}) \cup [n-2,\infty)$
	FitzGerald–Horn	FitzGerald–Horn, Hiai, Bhatia–Elsner, G–K–R	FitzGerald–Horn, Hiai, G–K–R
Monotonicity			
$k=1$	$[0,\infty)$	$[0,\infty)$	$[0,\infty)$
	G–K–R	G–K–R	G–K–R
$2 \le k \le n$	$\mathbb{N} \cup [n-1,\infty)$	$2\mathbb{N} \cup [n-1,\infty)$	$(-1+2\mathbb{N}) \cup [n-1,\infty)$
	FitzGerald–Horn	FitzGerald–Horn, Hiai, G–K–R	FitzGerald–Horn, Hiai, G–K–R
Convexity			
$k=1$	$[1,\infty)$	$[1,\infty)$	$[1,\infty)$
	G–K–R	G–K–R	G–K–R
$2 \le k \le n$	$\mathbb{N} \cup [n,\infty)$	$2\mathbb{N} \cup [n,\infty)$	$(-1+2\mathbb{N}) \cup [n,\infty)$
	Hiai, G–K–R	Hiai, G–K–R	Hiai, G–K–R
Super-additivity			
$1 \le k \le n$	$\mathbb{N} \cup [n,\infty)$	$2\mathbb{N} \cup [n,\infty)$	$(-1+2\mathbb{N}) \cup [n,\infty)$
	G–K–R	G–K–R	G–K–R
Sub-additivity			
$k=1$	$(-\infty,0] \cup \{1\}$ if $n=2$,		$\{0,1\}$ if $n=2$,
	$\{0,1\}$ if $n>2$	$\emptyset$	$\{1\}$ if $n>2$
	G–K–R	G–K–R	G–K–R
$2 \le k \le n$	$\{1\}$	$\emptyset$	$\{1\}$
	G–K–R	G–K–R	G–K–R

- We complete all of the previously unsolved cases involving powers preserving positivity, monotonicity, and convexity.
- We classify all powers preserving super-additivity and sub-additivity. These properties have not been explored in the literature in the entrywise setting.
- We also examine negative powers preserving Loewner properties, which were also previously unexplored.
- Finally, we extend all of the above results – as well as those in the literature – by introducing rank constraints. Once again, we are able to obtain a complete classification of all real powers preserving the five aforementioned Loewner properties.

Similar to many settings in the literature (see [18]), one can define Hadamard critical exponents for positivity, monotonicity, convexity, and super-additivity for $\mathbb{P}_n^k$ – these are the phase transition points akin to [8]. From Theorem 1.2, we immediately obtain the Hadamard critical exponents (CE) for the four Loewner properties for matrices with rank constraints:

Corollary 1.4. *Suppose $n \ge 2$ and $1 \le k \le n$. The Hadamard critical exponents for positivity, monotonicity, convexity, and super-additivity for $\mathbb{P}_n^k$ are $n-2$, $n-1$, n, n respectively if $2 \le k \le n$, and $0,0,1,n$ respectively if $k=1$. In particular, they are completely independent of the type of entrywise power used.*

An interesting consequence of Corollary 1.4 is that if $k \ge 2$, then the sets of fractional Hadamard powers f_α, ϕ_α, or ψ_α that are Loewner positive, monotone, convex, or super-additive on $\mathbb{P}_n^k$ do *not* depend on k. Thus, entrywise powers that preserve such properties on $\mathbb{P}_n^2$ automatically preserve them on all of $\mathbb{P}_n$. Corollary 1.4 also shows that the rank 1 case is different from that of other k, in that three of the critical exponents do not depend on n if $k=1$. This is not surprising for positivity because the functions $f_\alpha, \phi_\alpha, \psi_\alpha$ are all multiplicative. Furthermore, note that if $2 \le k \le n$, then entrywise maps are Loewner convex on $\mathbb{P}_n^k(I)$ if and only if they are Loewner super-additive. Finally, the structure of the $\mathcal{H}_J(n,k)$-sets is different for $J =$ sub-additivity, compared to the other Loewner properties.

1.1. Organization of the paper

We prove Theorem 1.2 by systematically studying entrywise powers that are (a) positive, (b) monotone, (c) convex, and (d) super/sub-additive with respect to the Loewner ordering. Thus in each of the next four sections, we gather previous results from the literature, and extend these in order to compute the sets $\mathcal{H}_J^I(n,k)$ for matrices with rank constraints. In doing so, as a special case we can complete the classification of powers $f_\alpha, \phi_\alpha, \psi_\alpha$ that are Loewner positive, monotone, or convex, for all matrices in $\mathbb{P}_n = \mathbb{P}_n^n$ (i.e., with no rank constraint). In Section 5, we then classify the entrywise real powers that are super/sub-additive and in the process demonstrate an interesting connection to Loewner convexity. We conclude the paper by discussing related questions and extensions to other power functions in Section 6.

2. Characterizing entrywise powers that are Loewner positive

The study of Hadamard powers originates in the work of FitzGerald and Horn [8]. We begin our analysis by stating one of their main results that characterizes the Hadamard powers preserving Loewner positivity.

Theorem 2.1. *(See FitzGerald and Horn [8, Theorem 2.2].) Suppose $A \in \mathbb{P}_n([0,\infty))$ for some $n \geq 2$. Then $A^{\circ\alpha} \in \mathbb{P}_n$ for all $\alpha \in \mathbb{N} \cup [n-2,\infty)$. If $\alpha \in (0,n-2)$ is not an integer, then there exists $A \in \mathbb{P}_n((0,\infty))$ such that $A^{\circ\alpha} \notin \mathbb{P}_n$. More precisely, Loewner positivity is not preserved for $A = ((1+\epsilon ij))_{i,j=1}^n$, for all sufficiently small $\epsilon > 0$ with $\alpha \in (0,n-2) \setminus \mathbb{N}$.*

Thus, $\mathcal{H}_{pos}(n,n) = \mathbb{N} \cup [n-2,\infty)$ for all $2 \leq n \in \mathbb{N}$. Additionally, Hiai [15] showed that the same results as above hold for the critical exponent for the even and odd extensions ϕ_α and ψ_α:

Theorem 2.2. *(See Hiai [15, Theorem 5.1].) If $n \geq 2$ and $\alpha \geq n-2$, then $\alpha \in \mathcal{H}_{pos}^\phi(n,n) \cap \mathcal{H}_{pos}^\psi(n,n)$.*

Remark 2.3. Fix $0 < R \leq \infty$. It is easy to see using a rescaling argument, that entrywise powers preserving positivity, monotonicity, or convexity on $\mathbb{P}_n(-R,R)$ also preserve the respective property on $\mathbb{P}_n(\mathbb{R})$, and vice versa. Thus in the present paper we only work with $\mathbb{P}_n(\mathbb{R})$ (or in the case of the usual powers $f_\alpha(x) = x^\alpha$, with $\mathbb{P}_n([0,\infty))$).

Theorem 2.2 shows that $[n-2,\infty)$ is contained in both $\mathcal{H}_{pos}^\phi(n,n)$ and $\mathcal{H}_{pos}^\psi(n,n)$. It is natural to ask for which $\alpha \in (0,n-2)$ do the Hadamard powers ϕ_α, ψ_α preserve positivity. This question was answered by Bhatia and Elsner for the powers ϕ_α:

Theorem 2.4. *(See Bhatia and Elsner [4, Theorem 2].) Suppose $n \geq 2$, and $r \in (0,n-2) \setminus 2\mathbb{N}$ is real. Then $r \notin \mathcal{H}_{pos}^\phi(n,n)$.*

Note that the above results correspond to the unconstrained-rank case: $\mathbb{P}_n(I) = \mathbb{P}_n^n(I)$. Our main result in this section refines Theorems 2.2 and 2.4, and completely characterizes the sets $\mathcal{H}_{pos}(n,k)$, $\mathcal{H}_{pos}^\phi(n,k)$, and $\mathcal{H}_{pos}^\psi(n,k)$ for all $1 \leq k \leq n$.

Theorem 2.5. *Suppose $2 \leq k \leq n$ are integers with $n \geq 3$. Then,*

$$\mathcal{H}_{pos}(n,k) = \mathbb{N} \cup [n-2,\infty), \qquad \mathcal{H}_{pos}^\phi(n,k) = 2\mathbb{N} \cup [n-2,\infty),$$
$$\mathcal{H}_{pos}^\psi(n,k) = (-1+2\mathbb{N}) \cup [n-2,\infty). \tag{2.1}$$

If instead $k = 1$ and/or $n = 2$, then

$$\mathcal{H}_{pos}(n,k) = \mathcal{H}_{pos}^\phi(n,k) = \mathcal{H}_{pos}^\psi(n,k) = (0,\infty). \tag{2.2}$$

To prove Theorem 2.5, recall the following classical result about generalized Dirichlet polynomials.

Lemma 2.6. *(See [19,4].) Suppose $\lambda_0 > \lambda_1 > \cdots > \lambda_m > 0$ are real, and $f : \mathbb{R} \to \mathbb{R}$ is of the form $f(x) := \sum_{i=0}^m a_i \lambda_i^x$ for some $a_i \in \mathbb{R}$ with $a_0 \neq 0$. Then f has at most m zeros on the real line.*

We now proceed to characterize all of the sets $\mathcal{H}_{pos}(n,k), \mathcal{H}^{\phi}_{pos}(n,k), \mathcal{H}^{\psi}_{pos}(n,k)$ for all $1 \leq k \leq n$.

Proof of Theorem 2.5. First suppose $k = 1$, $n \geq 2$, and $A = uu^T \in \mathbb{P}^1_n$ for some $u \in \mathbb{R}^n$. Since the functions $f_\alpha, \psi_\alpha, \phi_\alpha$ are multiplicative for all $\alpha \in \mathbb{R}$, we have $A^{\circ\alpha} = u^{\circ\alpha}(u^{\circ\alpha})^T \in \mathbb{P}^1_n$ for $u \in [0,\infty)^n$, and similarly for $\psi_\alpha[A], \phi_\alpha[A]$ for $u \in \mathbb{R}^n$. The result thus follows for $k = 1$. Furthermore, the result is obvious for $n = 2$ and all $\alpha \in \mathbb{R}$.

Now suppose that $2 \leq k \leq n$ and $n \geq 3$. We consider three cases corresponding to the three functions $f(x) = f_\alpha(x), \phi_\alpha(x)$, and $\psi_\alpha(x)$.

Case 1: $f(x) = f_\alpha(x)$. Consider the matrix

$$A := \begin{pmatrix} 1 & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1 & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1 \end{pmatrix} \oplus \mathbf{0}_{(n-3)\times(n-3)} \in \mathbb{P}^2_n\big([0,\infty)\big).$$

It is easily verified that $f_\alpha[A] \notin \mathbb{P}_n$ for all $\alpha \leq 0$. Thus using Theorem 2.1, we have

$$\mathbb{N} \cup [n-2,\infty) = \mathcal{H}_{pos}(n,n) \subset \mathcal{H}_{pos}(n,k) \subset (0,\infty).$$

Now note that the counterexample $((1+\epsilon ij)) \in \mathbb{P}^2_n([0,\infty))$ provided in Theorem 2.1 is a rank 2 matrix and hence $\alpha \notin \mathcal{H}_{pos}(n,2)$ for any $\alpha \in (0,n-2)\backslash\mathbb{N}$. Thus $\mathcal{H}_{pos}(n,2) = \mathbb{N} \cup [n-2,\infty)$. Finally, since $\mathcal{H}_{pos}(n,k) \subset \mathcal{H}_{pos}(n,2)$, it follows that $\mathcal{H}_{pos}(n,k) = \mathbb{N} \cup [n-2,\infty)$.

Case 2: $f(x) = \phi_\alpha(x)$. Note that $2\mathbb{N} \subset \mathcal{H}^{\phi}_{pos}(n,k)$ by the Schur product theorem. Using Theorem 2.2 and Case 1, it remains to show that no odd integer $\alpha \in (0,n-2)$ belongs to $\mathcal{H}^{\phi}_{pos}(n,k)$. To do so and for later use, first define the matrix A_r for $r \in \mathbb{N}$ as follows:

$$(A_r)_{ij} := \big((\cos(i-j)\pi/r)\big), \quad 1 \leq i,j \leq r. \tag{2.3}$$

Note that $A_r \in \mathbb{P}^2_r$ since $A_r = uu^T + vv^T$, where $u := (\cos(j\pi/r))_{j=1}^r$ and $\mathbf{v} := (\sin(j\pi/r))_{j=1}^r$. By [4, Theorem 2], the matrix $\phi_p[A_{\alpha+3}] \notin \mathbb{P}_{\alpha+3}$ for all $p \in (\alpha-1,\alpha+1)$. In particular, $\phi_\alpha[A_{\alpha+3}] \notin \mathbb{P}_{\alpha+3}$. Since we are considering integer powers α such that $\alpha < n-2$, we have $\alpha + 3 \leq n$, so

$$A_{\alpha+3} \oplus \mathbf{0}_{(n-\alpha-3)\times(n-\alpha-3)} \in \mathbb{P}^2_n, \qquad \phi_\alpha[A_{\alpha+3} \oplus \mathbf{0}_{(n-\alpha-3)\times(n-\alpha-3)}] \notin \mathbb{P}_n,$$

which proves that $\alpha \notin \mathcal{H}^{\phi}_{pos}(n,2)$ for any odd integer $\alpha \in (0,n-2)$, as desired.

Case 3: $f(x) = \psi_\alpha(x)$. Note that $-1 + 2\mathbb{N} \subset \mathcal{H}^{\psi}_{pos}(n,k)$ by the Schur product theorem. Using Theorem 2.2 and Case 1, it remains to show that no even integer $\alpha \in (0,n-2)$ belongs to $\mathcal{H}^{\psi}_{pos}(n,k)$. The approach in this part is to prove a result similar to the main result of [4], but for the function ψ_α. The proof is also similar to [4] and is therefore omitted for brevity. For future reference we point out that the key step in the proof uses the following assertion:

$$\psi_p[A_{\alpha+3}] \notin \mathbb{P}_{\alpha+3} \quad \forall p \in (\alpha-1,\alpha+1), \tag{2.4}$$

where $A_{\alpha+3}$ is defined as in (2.3). $\square$

Remark 2.7. We now come to an open question raised by Bhatia and Elsner in [4, §3] – namely,

Given $p \in (0,\infty) \setminus 2\mathbb{N}$, *find the smallest* $n \in \mathbb{N}$ *such that* $\phi_p[A] \notin \mathbb{P}_n$ *for at least one matrix* $A \in \mathbb{P}_n$.

Our result on the full characterization of the even extensions of entrywise powers that preserve Loewner positivity, as given by Theorem 2.5, allows us to answer this question. By Theorem 2.5, the smallest $n \in \mathbb{N}$ such that $\phi_p[A] \notin \mathbb{P}_n$ for at least one matrix $A \in \mathbb{P}_n$, is $n = \lfloor p \rfloor + 3$. Similarly, one can ask the analogous question for ψ: *given* $p \in (0,\infty) \setminus (-1 + 2\mathbb{N})$, *find the smallest* $n \in \mathbb{N}$ *such that* $\psi_p[A] \notin \mathbb{P}_n$ *for at least one* $A \in \mathbb{P}_n$. Once again by Theorem 2.5, the answer to this question is $n = \lfloor p \rfloor + 3$.

Remark 2.8. We note that an alternate approach to proving the claim in (2.4) is to conjugate $\psi_\alpha[A_{\alpha+3}]$ by the orthogonal matrix $D_{\alpha+3} = \operatorname{diag}(1,-1,1,\dots,1)$ to obtain a circulant matrix, and then follow the approach in [4].

3. Characterizing entrywise powers that are Loewner monotone

We now characterize the entrywise powers that are monotone with respect to the Loewner ordering. The following theorem by FitzGerald and Horn that is analogous to Theorem 2.1 but for monotonicity, answers the question for matrices with nonnegative entries. In what follows, we denote by $\mathbf{1}_{n\times n}$ the $n \times n$ matrix with all entries equal to 1.

Theorem 3.1. *(See FitzGerald and Horn [8, Theorem 2.4].) Suppose* $A, B \in \mathbb{P}_n([0,\infty))$ *for some* $n \geq 1$. *If* $A \geq B \geq 0$, *then* $A^{\circ\alpha} \geq B^{\circ\alpha} \geq 0$ *for* $\alpha \in \mathbb{N} \cup [n-1,\infty)$. *If* $\alpha \in (0, n-1)$ *is not an integer, then there exist* $A \geq B \geq 0$ *in* $\mathbb{P}_n([0,\infty))$ *such that* $A^{\circ\alpha} \not\geq B^{\circ\alpha}$. *More precisely, Loewner monotonicity is not preserved for* $A = ((1+\epsilon ij))_{i,j=1}^n$, $B = \mathbf{1}_{n\times n}$, *for all sufficiently small* $\epsilon > 0$ *with* $\alpha \in (0, n-1) \setminus \mathbb{N}$.

We now discuss a parallel result to Theorem 3.1 for ϕ_α and ψ_α that was proved by Hiai in [15]. The theorem extends to the cone of positive semidefinite matrices the standard real analysis result that a differentiable function is nondecreasing if and only if its derivative is nonnegative.

Theorem 3.2. *(See Hiai [15, Theorems 3.2 and 5.1].) Suppose* $0 < R \leq \infty$, $I = (-R, R)$, *and* $f : I \to \mathbb{R}$.

(1) *For each* $n \geq 3$, f *is monotone on* $\mathbb{P}_n(I)$ *if and only if* f *is differentiable on* I *and* f' *is Loewner positive on* $\mathbb{P}_n(I)$.
(2) *If* $n \geq 1$ *and* $\alpha \geq n-1$, *then* $\alpha \in \mathcal{H}^{\phi}_{mono}(n,n) \cap \mathcal{H}^{\psi}_{mono}(n,n)$.

Theorem 3.2 is a powerful result, but cannot be applied directly to study entrywise functions preserving matrices in the more restricted set $\mathbb{P}_n^k$. We thus refine the first part of the theorem to also include rank constraints.

Proposition 3.3. *Fix* $0 < R \leq \infty$, $I = (-R, R)$, *and* $2 \leq k \leq n$. *Suppose* $f : I \to \mathbb{R}$ *is differentiable on* I *and Loewner monotone on* $\mathbb{P}_n^k(I)$. *If* $A \in \mathbb{P}_n^k(I)$ *is irreducible, then* $f'[A] \in \mathbb{P}_n$.

Proof. We first make the following observation (which in fact holds over any infinite field):

Suppose $A_{n\times n}$ *is a symmetric irreducible matrix. Then there exists a vector* $\zeta \in \operatorname{Im} A$ *(the image of* A*) with no zero component.*

To see why the observation is true, first suppose that all vectors in $\operatorname{Im} A$ have the ith component zero for some $1 \leq i \leq n$ – i.e., $e_i^T A v = 0$ for all vectors v. Then the ith row (and hence column) of A is zero, which contradicts irreducibility. Now fix vectors $w_i \in \operatorname{Im} A$ for all $1 \leq i \leq n$, such that the ith entry of w_i is nonzero. Let $W := [w_1|w_2|\cdots|w_n]$; then for all tuples $\mathbf{c} := (c_1, \dots, c_n)^T \in \mathbb{R}^n$,

$$W\mathbf{c} = \sum_{i=1}^{n} c_i w_i \in \operatorname{Im} A.$$

Consider the set S of all $\mathbf{c} \in \mathbb{R}^n$ such that $W\mathbf{c}$ has a zero entry. Then $S = \bigcup_{i=1}^{n} S_i$, where $\mathbf{c} \in S_i$ if $e_i^T W\mathbf{c} = 0$. Note that S_i is a proper subspace of $\mathbb{R}^n$ since $e_i \notin S_i$ by assumption on w_i. Since $\mathbb{R}$ is an infinite field, S is a proper subset of $\mathbb{R}^n$, which proves the observation.

Now given an irreducible matrix $A \in \mathbb{P}_n^k(I)$, choose a vector $\zeta \in \operatorname{Im} A$ as in the above observation. Let $A_\epsilon := A + \epsilon\zeta\zeta^T$ for $\epsilon > 0$; then $A_\epsilon \in \mathbb{P}_n^k(I)$ since $\zeta \in \operatorname{Im} A$. Therefore by monotonicity, $\frac{f[A_\epsilon]-f[A]}{\epsilon} \geq 0$. Letting $\epsilon \to 0^+$, it follows that $f'[A] \circ (\zeta\zeta^T) \geq 0$. Now let $\zeta^{\circ(-1)} := (\zeta_1^{-1}, \dots, \zeta_n^{-1})^T$; then by the Schur Product Theorem, it follows that $f'[A] = f'[A] \circ (\zeta\zeta^T) \circ (\zeta^{\circ(-1)}(\zeta^{\circ(-1)})^T) \geq 0$, which concludes the proof. □

With the above results in hand, we now completely classify the powers preserving Loewner monotonicity, and also specify them when rank constraints are imposed.

Theorem 3.4. *Suppose $2 \leq k \leq n$ are integers. Then,*

$$\begin{aligned} \mathcal{H}_{mono}(n,k) = \mathbb{N} \cup [n-1,\infty), \qquad \mathcal{H}^{\phi}_{mono}(n,k) = 2\mathbb{N} \cup [n-1,\infty), \\ \mathcal{H}^{\psi}_{mono}(n,k) = (-1+2\mathbb{N}) \cup [n-1,\infty). \end{aligned} \tag{3.1}$$

If instead $k = 1$, then

$$\mathcal{H}_{mono}(n,1) = \mathcal{H}^{\phi}_{mono}(n,1) = \mathcal{H}^{\psi}_{mono}(n,1) = (0,\infty). \tag{3.2}$$

Proof. First suppose $k = 1 < n$ and $A = uu^T, B = vv^T \in \mathbb{P}_n^1$. If $A \geq B \geq 0$, then we claim that $v = cu$ for some $c \in [-1,1]$. To see the claim, assume to the contrary that u, v are linearly independent. We can then choose $w \in \mathbb{R}^n$ such that w is orthogonal to u but not to v. But then $w^T(A-B)w = -(w^T v)^2 < 0$, which contradicts the assumption $A \geq B$. Thus u, v are linearly dependent. Since $A \geq B \geq 0$, it follows that $v = cu$ with $|c| \leq 1$. Now for all $\alpha \geq 0$ and all $A, B \in \mathbb{P}_n^1([0,\infty))$ such that $A \geq B \geq 0$, we use the multiplicativity of f_α to compute:

$$f_\alpha[A] - f_\alpha[B] = f_\alpha[u]f_\alpha[u]^T - f_\alpha[cu]f_\alpha[cu]^T = \big(1 - \big(c^2\big)^\alpha\big) f_\alpha[u]f_\alpha[u]^T \geq 0.$$

Thus f_α is monotone on $\mathbb{P}_n^1([0,\infty))$. Similar computations show the monotonicity of ϕ_α and ψ_α on $\mathbb{P}_n^1(\mathbb{R})$ for all $\alpha \geq 0$. The same computations also show that $f_\alpha, \phi_\alpha, \psi_\alpha$ are not monotone on $\mathbb{P}_n^1(I)$, for any $\alpha < 0$.

Now suppose $2 \leq k \leq n$. Note that if $A \geq B \geq 0$, then one inductively shows using the Schur product theorem that

$$A^{\circ m} \geq B^{\circ m} \quad \forall m \leq N \quad \Rightarrow \quad A^{\circ(N+1)} - B^{\circ(N+1)} = \sum_{m=0}^{N} A^{\circ m} \circ (A-B) \circ B^{\circ(N-m)} \geq 0. \tag{3.3}$$

Therefore every positive integer Hadamard power is monotone on $\mathbb{P}_n$. We now consider three cases corresponding to the three functions $f(x) = f_\alpha(x), \phi_\alpha(x),$ and $\psi_\alpha(x)$.

Case 1: $f(x) = f_\alpha(x)$. First suppose $\alpha < 1$. By considering the matrices $B = \mathbf{1}_{2\times 2}$ and $A = B + uu^T$ with $u = (1,-1)^T$, we immediately obtain that f_α is not monotone on $\mathbb{P}_2^2([0,\infty))$, and hence not on $\mathbb{P}_n^k([0,\infty))$

for all $\alpha < 1$. Now Theorem 3.1 and the above analysis imply that $\mathcal{H}_{mono}(n,k) \subset \mathbb{N} \cup [n-1,\infty)$, since $((1+\epsilon ij)), \mathbf{1}_{n\times n} \in \mathbb{P}_n^2([0,\infty)) \subset \mathbb{P}_n^k([0,\infty))$ provide the necessary counterexample for $\alpha \in (0,n-1)\setminus\mathbb{N}$. Furthermore by Theorem 3.1, $\mathbb{N} \cup [n-1,\infty) = \mathcal{H}_{mono}(n,n) \subset \mathcal{H}_{mono}(n,k)$, and thus $\mathcal{H}_{mono}(n,k) = \mathbb{N} \cup [n-1,\infty)$.

Case 2: $f(x) = \phi_\alpha(x)$. By Eq. (3.3), $\phi_{2n}[A] = A^{\circ 2n}$ preserves monotonicity on $\mathbb{P}_n$. From this observation and Theorem 3.2, it follows that $2\mathbb{N} \cup [n-1,\infty) \subset \mathcal{H}^\phi_{mono}(n,n) \subset \mathcal{H}^\phi_{mono}(n,k)$. We now claim that

$$\mathcal{H}^\phi_{mono}(n,k) \subset \mathcal{H}^\phi_{pos}(n,k) \cap \mathcal{H}_{mono}(n,k) \subset \{n-2\} \cup 2\mathbb{N} \cup [n-1,\infty).$$

Indeed, the first inclusion above follows because every monotone function on $\mathbb{P}_n^k(\mathbb{R})$ is simultaneously monotone on $\mathbb{P}_n^k([0,\infty))$ and positive on $\mathbb{P}_n^k(\mathbb{R})$ by definition. The second inclusion above holds by Theorem 2.5 and Case 1.

It thus remains to consider if ϕ_{n-2} is monotone on $\mathbb{P}_n^k(\mathbb{R})$. We consider three sub-cases: if $n > 2$ is even, then $n-2 \in 2\mathbb{N} \cup [n-1,\infty)$. Hence $\mathcal{H}^\phi_{mono}(n,k) = 2\mathbb{N} \cup [n-1,\infty)$ by the analysis mentioned above in Case 2. Next if $n = 3$, we produce a three-parameter family of matrices $A \geq \mathbf{1}_{3\times 3} \geq 0$ in $\mathbb{P}_3(\mathbb{R})$ such that $\phi_1[A] \not\geq \phi_1[\mathbf{1}_{3\times 3}]$. Indeed, choose any $a > b > 0$ and $c \in (a^{-1}, b^{-1})$, and define

$$v := (a,b,-c)^T, \qquad B := \mathbf{1}_{3\times 3}, \qquad A := B + vv^T.$$

Then both A, B are in $\mathbb{P}_3^2(\mathbb{R})$, and $\phi_1[A], \phi_1[B] \in \mathbb{P}_3(\mathbb{R})$ by Theorem 2.5. However,

$$\det\big(\phi_1[A] - \phi_1[B]\big) = \det \begin{pmatrix} a^2 & ab & ac-2 \\ ab & b^2 & -bc \\ ac-2 & -bc & c^2 \end{pmatrix} = -4b^2(ac-1)^2 < 0.$$

Thus ϕ_1 is not monotone on $\mathbb{P}_3^2(\mathbb{R})$.

Finally, suppose $n > 3$ is odd and that ϕ_{n-2} is monotone on $\mathbb{P}_n^k(\mathbb{R})$. We then obtain a contradiction as follows: recall from Eq. (2.4) that the matrix A_n constructed in Eq. (2.3) satisfies: $\psi_{n-3}[A_n] \notin \mathbb{P}_n$. (Here, $\alpha = n-3 = p$ is an even integer in $(0, n-2)$, since $n > 3$ is odd.) Moreover, $A_n \in \mathbb{P}_n^2(\mathbb{R}) \subset \mathbb{P}_n^k(\mathbb{R})$ is irreducible. Hence if ϕ_{n-2} is monotone on $\mathbb{P}_n^k(\mathbb{R})$, then by Proposition 3.3, $\psi_{n-3}[A_n] = \frac{1}{n-2}(\phi_{n-2})'[A_n] \in \mathbb{P}_n$. This is a contradiction and so ϕ_{n-2} is not monotone for odd integers $n > 3$. This concludes the classification of the powers ϕ_α that preserve Loewner monotonicity.

Case 3: $f(x) = \psi_\alpha(x)$. This case follows similarly to Case 2 and is therefore omitted. □

4. Characterizing entrywise powers that are Loewner convex

We next characterize the entrywise powers that preserve Loewner convexity. Before proving the main result of this section, we need a few preliminary results. Recall that an $n\times n$ matrix A is said to be *completely positive* if $A = CC^T$ for some $n\times m$ matrix C with nonnegative entries. We denote the set of $n\times n$ completely positive matrices by CP_n.

Lemma 4.1. *Suppose $I \subset \mathbb{R}$ is convex, $n \geq 2$, and $f : I \to \mathbb{R}$ is continuously differentiable. Given two fixed matrices $A, B \in \mathbb{P}_n(I)$ such that*

(1) $A - B \in \mathrm{CP}_n$;
(2) $f[\lambda A + (1-\lambda)B] \leq \lambda f[A] + (1-\lambda) f[B]$ *for all* $0 \leq \lambda \leq 1$.

Then $f'[A] \geq f'[B]$.

Proof. Since $A - B \in \mathrm{CP}_n$, there exist vectors $v_1, \dots, v_m \in [0,\infty)^n$ such that

$$A - B = v_1 v_1^T + \dots + v_m v_m^T. \tag{4.1}$$

For $1 \le k \le m$, let $A_k = B + v_{k+1}v_{k+1}^T + \dots + v_m v_m^T$. Then $A =: A_0 \ge A_1 \ge \dots \ge A_{m-1} \ge A_m := B$. The rest of the proof is the same as the first part of the proof of [15, Theorem 3.2(1)]. $\square$

Just as Proposition 3.3 was used in proving Theorem 3.4, we need the following preliminary result to classify the powers that preserve convexity.

Proposition 4.2. *Fix $0 < R \le \infty$, $I = (-R, R)$, and $2 \le k \le n$. Suppose $f : I \to \mathbb{R}$ is twice differentiable on I and Loewner convex on $\mathbb{P}_n^k(I)$. If $A \in \mathbb{P}_n^k(I)$ is irreducible, then $f''[A] \in \mathbb{P}_n$.*

Proof. Given an irreducible matrix $A \in \mathbb{P}_n^k(I)$, choose a vector $\zeta \in \operatorname{Im} A$ as in the observation at the beginning of the proof of Proposition 3.3. We now adapt the proof of [15, Theorem 3.2(1)] for the $k = n$ case, to the $2 \le k < n$ case. Let $A_1 := A + \zeta\zeta^T$; then $A_1 \in \mathbb{P}_n^k(I)$ for $\|\zeta\|$ small enough since $\zeta \in \operatorname{Im} A$. More generally, it easily follows that $\lambda A_1 + (1-\lambda)A \in \mathbb{P}_n^k(I)$ for all $\lambda \in [0,1]$. Since

$$f\big[\lambda A_1 + (1-\lambda)A\big] \le \lambda f[A_1] + (1-\lambda) f[A] \quad \forall 0 \le \lambda \le 1$$

by convexity, it follows for $0 < \lambda < 1$ that

$$\frac{f[A + \lambda(A_1 - A)] - f[A]}{\lambda} \le f[A_1] - f[A],$$
$$\frac{f[A_1 + (1-\lambda)(A - A_1)] - f[A_1]}{1-\lambda} \le f[A] - f[A_1].$$

Letting $\lambda \to 0^+$ or $\lambda \to 1^-$, we obtain

$$(A_1 - A) \circ f'[A] \le f[A_1] - f[A], \qquad (A - A_1) \circ f'[A_1] \le f[A] - f[A_1].$$

Summing these two inequalities gives $(A_1 - A) \circ (f'[A_1] - f'[A]) \ge 0$. Note that $(A_1 - A)^{\circ -1} = (\zeta\zeta^T)^{\circ -1} \in \mathbb{P}_n^1$ and so $f'[A_1] - f'[A] \ge 0$.

Finally, given $\epsilon > 0$, define $A_\epsilon := A + \epsilon\zeta\zeta^T$. Then $A_\epsilon \in \mathbb{P}_n^k(I)$ and $f'[A_\epsilon] \ge f'[A]$ by the previous paragraph for $\sqrt{\epsilon}\zeta$. Therefore, for all $\epsilon > 0$, $\frac{f'[A_\epsilon] - f'[A]}{\epsilon} \ge 0$. Letting $\epsilon \to 0^+$, it follows that $f''[A] \circ (\zeta\zeta^T) \ge 0$. Now let $\zeta^{\circ(-1)} := (\zeta_1^{-1}, \dots, \zeta_n^{-1})^T$; then by the Schur Product Theorem,

$$0 \le f''[A] \circ \big(\zeta\zeta^T\big) \circ \big(\zeta^{\circ(-1)}\big(\zeta^{\circ(-1)}\big)^T\big) = f''[A],$$

which concludes the proof. $\square$

Note that Lemma 4.1 and Proposition 4.2 generalize to the cone $\mathbb{P}_n$ of matrices with the Loewner ordering, the elementary results from real analysis that the first and second derivatives of a convex (twice) differentiable function are nondecreasing and nonnegative, respectively. These parallels have been explored by Hiai in detail for $I = (-R, R)$; see [15, Theorems 3.2 and 5.1]. We now state some assertions from [15] that concern Loewner convexity.

Theorem 4.3. *(See Hiai [15, Theorems 3.2 and 5.1].) Suppose $0 < R \le \infty$, $I = (-R, R)$, and $f : I \to \mathbb{R}$.*

(1) *For each $n \ge 2$, f is convex on $\mathbb{P}_n(I)$ if and only if f is differentiable on I and f' is monotone on $\mathbb{P}_n(I)$.*
(2) *If $n \ge 1$ and $\alpha \ge n$, then $\alpha \in \mathcal{H}^{\phi}_{conv}(n,n) \cap \mathcal{H}^{\psi}_{conv}(n,n)$.*

With the above results in hand, we now extend them in order to completely classify the powers preserving Loewner convexity, and also specify them when rank constraints are imposed.

Theorem 4.4. *Suppose $2 \leq k \leq n$ are integers. Then,*

$$\mathcal{H}_{conv}(n,k) = \mathbb{N} \cup [n,\infty), \qquad \mathcal{H}^{\phi}_{conv}(n,k) = 2\mathbb{N} \cup [n,\infty), \qquad \mathcal{H}^{\psi}_{conv}(n,k) = (-1+2\mathbb{N}) \cup [n,\infty). \tag{4.2}$$

If instead $k = 1$, then

$$\mathcal{H}_{conv}(n,1) = \mathcal{H}^{\phi}_{conv}(n,1) = \mathcal{H}^{\psi}_{conv}(n,1) = [1,\infty). \tag{4.3}$$

Proof. Suppose that $k = 1 < n$ and $A = uu^T, B = vv^T \in \mathbb{P}^1_n$. If $A \geq B \geq 0$, then by the proof of Theorem 3.4, $v = cu$ for some $c \in [-1,1]$. Now for any $\alpha > 0$ and $\lambda \in [0,1]$,

$$\lambda f_\alpha[A] + (1-\lambda) f_\alpha[B] - f_\alpha\big[\lambda A + (1-\lambda)B\big] = \big(\lambda + (1-\lambda)c^{2\alpha} - \big(\lambda + c^2(1-\lambda)\big)^\alpha\big) f_\alpha[A].$$

So f_α is convex on $\mathbb{P}^1_n(\mathbb{R})$ if and only if (using $b = c^2$)

$$\lambda + (1-\lambda)b^\alpha \geq \big(\lambda + b(1-\lambda)\big)^\alpha, \quad \forall \lambda, b \in [0,1].$$

This condition is equivalent to the function $x \mapsto x^\alpha$ being convex on $[0,1]$ and hence on $[0,\infty)$ – in other words, if and only if $\alpha \geq 1$. A similar argument can be applied to analyze ϕ_α, ψ_α. If on the other hand $\alpha < 1$, then set $A := \mathbf{1}_{2\times 2} \oplus \mathbf{0}_{(n-2)\times(n-2)} \in \mathbb{P}^1_n(I)$ and $B := \mathbf{0}_{n\times n}$, and compute:

$$\frac{1}{2} f[A] + \frac{1}{2} f[B] - f\left[\frac{1}{2}(A+B)\right] = \frac{1}{2} f[A] - f\left[\frac{1}{2} A\right] = \big(2^{-1} - 2^{-\alpha}\big) f[A], \tag{4.4}$$

which is clearly not in $\mathbb{P}_n(\mathbb{R})$ if $\alpha < 1$. It follows that none of the functions $f = f_\alpha, \phi_\alpha, \psi_\alpha$ is convex on $\mathbb{P}^k_n(I)$ for $\alpha < 1$, $n \geq 2$, and $1 \leq k \leq n$.

We now assume that $2 \leq k \leq n$, and show that $\mathbb{N} \cup [n,\infty) \subset \mathcal{H}_{conv}(n,k)$. We first assert that for any differentiable function $f : [0,\infty) \to \mathbb{R}$ such that $f'(x)$ is monotone on $\mathbb{P}_n([0,\infty))$, then f is convex on $\mathbb{P}_n([0,\infty))$. This assertion parallels one implication in Theorem 4.3(1) for $I = [0,\infty)$ instead of $I = (-R,R)$. As the proof is similar to the proof of [15, Proposition 2.4], it is omitted.

Next, letting $f(x) = x^\alpha$ for $\alpha \in [n,\infty) \cup \mathbb{N}$, it follows immediately from Theorem 3.4 that f is convex on $\mathbb{P}_n([0,\infty))$. Thus $\mathbb{N} \cup [n,\infty) \subset \mathcal{H}_{conv}(n,n) \subset \mathcal{H}_{conv}(n,k)$. Now note that for any $\alpha \geq 1$, $\phi'_\alpha(x) = \alpha\psi_{\alpha-1}(x)$ and $\psi'_\alpha(x) = \alpha\phi_{\alpha-1}(x)$. Thus using Theorem 4.3, it follows that $2\mathbb{N} \cup [n,\infty) \subset \mathcal{H}^{\phi}_{conv}(n,k)$ and $(-1+2\mathbb{N}) \cup [n,\infty) \subset \mathcal{H}^{\psi}_{conv}(n,k)$.

Note also from Eq. (4.4) that $\mathcal{H}_{conv}(n,k) \subset [1,\infty)$, and similarly for $\mathcal{H}^{\phi}_{conv}(n,k)$ and $\mathcal{H}^{\psi}_{conv}(n,k)$. Thus to show the reverse inclusions, i.e., that $\mathcal{H}_{conv}(n,k) \subset \mathbb{N} \cup [n,\infty)$ (and analogously for ϕ_α, ψ_α), we consider three cases corresponding to the three functions $f = f_\alpha, \phi_\alpha, \psi_\alpha$.

Case 1: $f(x) = f_\alpha(x)$. Let $\alpha \in \mathcal{H}_{conv}(n,k)$ and consider the matrices $A = A_\epsilon = ((1+\epsilon ij))^n_{i,j=1}$ and $B = \mathbf{1}_{n\times n}$ for $\epsilon > 0$. Since $A - B \in \mathrm{CP}_n$, by Lemma 4.1 for $I = [1, 1+\epsilon n^2]$, we have $f'_\alpha[A] \geq f'_\alpha[B]$. Thus by Theorem 3.1, it follows that $\alpha - 1 \geq n-1$ or $\alpha \in \mathbb{N}$. Therefore $\mathcal{H}_{conv}(n,k) \subset \mathbb{N} \cup [n,\infty)$.

Case 2: $f(x) = \phi_\alpha(x)$. Given $\alpha \in \mathcal{H}^{\phi}_{conv}(n,k)$ for $k \geq 2$, first note by Case 1 that

$$\mathcal{H}^{\phi}_{conv}(n,k) \subset \mathcal{H}^{\phi}_{conv}(n,2) \subset \mathcal{H}_{conv}(n,2) = \mathbb{N} \cup [n,\infty). \tag{4.5}$$

Thus it suffices to show that there is no odd integer in $S := (0, n) \cap \mathcal{H}^{\phi}_{conv}(n, 2)$. First note that for every odd integer $\alpha \in S$, the function ϕ_α is convex on $\mathbb{P}^2_{\alpha+1}(\mathbb{R})$. There are now two cases: first if $\alpha > 1$, then define $A_{\alpha+1} \in \mathbb{P}^2_{\alpha+1}(\mathbb{R})$ as in (2.3). Now $A_{\alpha+1}$ is irreducible since $\alpha \geq 3$. Applying Proposition 4.2 to $A_{\alpha+1}$, we obtain $\phi''_\alpha[A_{\alpha+1}] \geq 0$. Now if $2 \leq \alpha < n$, then this contradicts Case 2 of the proof of Theorem 2.5 since α is an odd integer. Therefore $\alpha \notin \mathcal{H}^{\phi}_{conv}(n, 2)$ for all odd integers $\alpha \in (1, n)$. The second case is when $\alpha = 1$. Recall from [15, Proposition 2.4] that if $\alpha = 1$ and $\phi_1 : \mathbb{R} \to \mathbb{R}$ is convex on $\mathbb{P}^2_{\alpha+1}(\mathbb{R}) = \mathbb{P}_2(\mathbb{R})$, then $\phi_1(x) = |x|$ would be differentiable on $\mathbb{R}$, which is false. We conclude that $\mathcal{H}^{\phi}_{conv}(n, 2) \subset 2\mathbb{N} \cup [n, \infty)$.

Case 3: $f(x) = \psi_\alpha(x)$. We now prove that $\mathcal{H}^{\psi}_{conv}(n, 2) \subset (-1 + 2\mathbb{N}) \cup [n, \infty)$. Once again it suffices to show that no even integer $\alpha \in (0, n)$ lies in $\mathcal{H}^{\psi}_{conv}(n, 2)$. First assume that $\alpha > 2$. Then an argument similar to that for ϕ_α above (together with the analogous example in Theorem 2.5 for ψ_α) shows that $\alpha \notin \mathcal{H}^{\psi}_{conv}(n, 2)$. Finally, if $\alpha = 2$, we provide a three-parameter family of counterexamples to show that ψ_2 is not convex on $\mathbb{P}^2_3(\mathbb{R})$ (and hence not convex on $\mathbb{P}^2_n(\mathbb{R})$ by adding blocks of zeros). To do so, choose $0 < b < a < \infty$ and $c \in (a^{-1}, \min(b^{-1}, 2a^{-1}))$, and define:

$$v := (a, b, -c)^T, \qquad B := \mathbf{1}_{3\times 3}, \qquad A := B + vv^T.$$

Clearly, $A, B \in \mathbb{P}^2_3(\mathbb{R})$ and $A \geq B$. Moreover,

$$C := \frac{1}{2}\big(\psi_2[A] + \psi_2[B]\big) - \psi_2\big[(A + B)/2\big] = \frac{1}{4}\begin{pmatrix} a^4 & a^2b^2 & (3ac-2)(2-ac) \\ a^2b^2 & b^4 & b^2c^2 \\ (3ac-2)(2-ac) & b^2c^2 & c^4 \end{pmatrix}.$$

Now verify that $\det C = -4^{-3}[2b(ac - 1)]^4 < 0$. Thus $A \geq B \geq 0$ provide a family of counterexamples to the convexity of ψ_2 on $\mathbb{P}^2_3(\mathbb{R})$ (with $\lambda = 1/2$). $\square$

Remark 4.5. In [15, Lemma 5.2], Hiai generalizes the arguments of FitzGerald and Horn in [8, Theorems 2.2, 2.4] and proves that for a given non-integer $0 < \alpha < n$, if $A_\epsilon := ((1 + \epsilon ij))^n_{i,j=1}$, then

$$f_\alpha\left[\frac{A_\epsilon + A_{\epsilon'}}{2}\right] \not\leq \frac{f_\alpha[A_\epsilon] + f_\alpha[A_{\epsilon'}]}{2}$$

for some $\epsilon, \epsilon' \geq 0$ small enough. Hiai's result can be used to deduce that $\mathcal{H}_{conv}(n, k) \subset \mathbb{N} \cup [n, \infty)$ for $2 \leq k < n$. The proof of Theorem 4.4 above is different in that it builds directly on the results obtained for monotonicity as compared to constructing specific matrices. The proof also has the additional advantage that it classifies the integer powers x^m for $m < n$, which are Loewner convex.

5. Characterizing entrywise powers that are Loewner super/sub-additive

Powers that are Loewner super/sub-additive have been studied for matrix functions in parallel settings, where functions of matrices are evaluated through the Hermitian functional calculus instead of entrywise (see e.g. [1,2,23,7,6,3]). We now characterize the powers that are Loewner super/sub-additive when applied entrywise.

Theorem 5.1. *Suppose $1 \leq k \leq n$ are integers with $n \geq 2$. Then,*

(1) $\mathcal{H}_{super}(n, k) = \mathbb{N} \cup [n, \infty)$, $\mathcal{H}^{\phi}_{super}(n, k) = 2\mathbb{N} \cup [n, \infty)$, $\mathcal{H}^{\psi}_{super}(n, k) = (-1 + 2\mathbb{N}) \cup [n, \infty)$.

(2) (a) $\mathcal{H}_{sub}(n,k) = \begin{cases} \{1\}, & \text{if } 2\leq k\leq n, \\ \{0,1\}, & \text{if } k=1,\ n>2, \\ (-\infty,0]\cup\{1\}, & \text{if } (n,k)=(2,1). \end{cases}$

(b) $\mathcal{H}^{\phi}_{sub}(n,k) = \emptyset$ *for all* $1 \leq k \leq n$.

(c) $\mathcal{H}^{\psi}_{sub}(n,k) = \{0,1\}$ *if* $(n,k) = (2,1)$, *and* $\{1\}$ *otherwise.*

Before we prove the result, note that it yields a hitherto unknown connection between super-additivity and convexity with respect to the Loewner ordering.

Corollary 5.2. *Fix* $\alpha > 0$ *and integers* $2 \leq k \leq n$. *A fractional power function* $f = f_\alpha, \phi_\alpha, \psi_\alpha$ *is Loewner convex on* $\mathbb{P}_n^k(I)$ *if and only if* f *is Loewner super-additive. Here* $I = [0,\infty)$ *if* $f = f_\alpha$ *and* $I = \mathbb{R}$ *otherwise.*

Proof. The result follows from Theorems 4.4 and 5.1. $\square$

Remark 5.3. Note *a priori* that the defining inequalities for convexity and super-additivity go in opposite ways. More precisely, for $\alpha > 0$ Loewner convexity is equivalent to Loewner midpoint convexity by continuity. Thus, $f = f_\alpha, \phi_\alpha, \psi_\alpha$ is convex if and only if $f[A+B] \leq 2^{\alpha-1}(f[A]+f[B])$ for $A \geq B \geq 0$. On the other hand, super-additivity asserts that $f[A+B] \geq f[A] + f[B]$ for $A, B \geq 0$. However, recall that a convex function $f : [0,\infty) \to [0,\infty)$ is super-additive if and only if $f(0) = 0$. The characterization of entrywise Loewner convex powers in Corollary 5.2 in terms of the super-additive ones is thus akin to the aforementioned fact for $n = 1$.

In order to prove Theorem 5.1, we extend classical results about generalized Vandermonde determinants to the odd and even extensions of the power functions.

Proposition 5.4. *Let* $0 < R \leq \infty$. *Then,*

(1) *the functions* $\{f_\alpha\colon\ \alpha \in \mathbb{R}\} \cup \{f \equiv 1\}$ *are linearly independent on* $I = [0,R)$;
(2) *the functions* $\{\phi_\alpha, \psi_\alpha\colon\ \alpha \in \mathbb{R}\} \cup \{f \equiv 1\}$ *are linearly independent on* $I = (-R,R)$.

Proof. Fix $\alpha_1 < \cdots < \alpha_n$ and define $\alpha := (\alpha_1, \ldots, \alpha_n)$. We first show that the set of functions $\{x^{\alpha_i}\colon\ i = 1,\ldots,n\} \cup \{f \equiv 1\}$ is linearly independent on $[0,R)$. Indeed, fix $\mathbf{x} := (0, x_1, \ldots, x_n) \in \mathbb{R}^n$ for any $0 < x_1 < \cdots < x_n < R$; then by [9, Chapter XIII, §8, Example 1], the vectors

$$\mathbf{x}^{\circ\alpha_j} := \big(0, x_1^{\alpha_j}, \ldots, x_n^{\alpha_j}\big), \quad j = 1,\ldots,n,$$

and $(1,1,\ldots,1)$ are linearly independent.

We next show that the set of functions $\{\phi_{\alpha_i}, \psi_{\alpha_i}\colon\ i = 1,\ldots,n\} \cup \{f \equiv 1\}$ is linearly independent on $(-R,R)$. Indeed, fix $\mathbf{x}' := (x_1, \ldots, x_n)$ with $x_i \in (0,R)$ as above; then by the above analysis,

$$\Psi\big(\mathbf{x}', \boldsymbol{\alpha}\big) := \begin{pmatrix} (\phi_{\alpha_i}(x_j))_{i,j=1}^n & (\phi_{\alpha_i}(-x_j))_{i,j=1}^n \\ (\psi_{\alpha_i}(x_j))_{i,j=1}^n & (\psi_{\alpha_i}(-x_j))_{i,j=1}^n \end{pmatrix} \tag{5.1}$$

is a nonsingular matrix, since it is of the form $\begin{pmatrix} M & M \\ M & -M \end{pmatrix}$ for a nonsingular matrix M. But then $\begin{pmatrix} \Psi(\mathbf{x}',\alpha) & \mathbf{0}_{2n\times 1} \\ \mathbf{1}_{1\times 2n} & 1 \end{pmatrix}$ is also nonsingular, whence the points $\pm x_1, \ldots, \pm x_n, 0$ provide the required nonsingular matrix. This proves the second assertion. $\square$

Proposition 5.4 has the following consequence that is repeatedly used in proving Theorem 5.1.

Corollary 5.5. *Let* $0 < R \leq \infty$ *and* $I = (-R,R)$ *or* $I = [0,R)$. *Fix integers* $1 \leq m \leq n$ *and scalars* $c_1, \ldots, c_m$ *and* $\alpha_1 < \alpha_2 < \cdots < \alpha_m$. *Suppose* $\{g_1, \ldots, g_m\} \subset \{\phi_{\alpha_1}, \ldots, \phi_{\alpha_m}, \psi_{\alpha_1}, \ldots, \psi_{\alpha_m}\}$, *and define*

$F(x) := \sum_{i=1}^m c_i g_i(x)$. Then there exist vectors $u \in (I \cap (-1,1))^n$ and $v_i \in \mathbb{R}^n$ that do not depend on c_i, such that $v_i^T F[uu^T]v_i = c_i$ for all $i = 1, \dots, m$.

Proof. Suppose first $I = (-R, R)$. Choose scalars $\alpha_n > \alpha_{n-1} > \cdots > \alpha_{m+1} > \alpha_m$. By Proposition 5.4, the functions $\phi_{\alpha_1}, \dots, \phi_{\alpha_n}, \psi_{\alpha_1}, \dots, \psi_{\alpha_n}$ are linearly independent. Thus, as in the proof of Proposition 5.4, for all pairwise distinct $x_1, \dots, x_n \in (0,1) \cap I$, the matrix $\Psi(\mathbf{x}, \alpha)$ as in (5.1) is nonsingular, where $\mathbf{x} := (x_1, \dots, x_n)$ and $\alpha := (\alpha_1, \dots, \alpha_n)$. Now consider the submatrix $C_{m\times 2n}$ of $\Psi(\mathbf{x}, \alpha)$ whose rows correspond to the functions g_i for $1 \leq i \leq m$. Since C has full rank, choose elements $u_1, u_2, \dots, u_n$ from among the $\pm x_i$ such that the matrix $(g_i(u_j))_{i,j=1}^m$ is nonsingular. Now set $u := (u_1, \dots, u_n)^T$; then the vectors $g_i[u]$ are linearly independent. Choose v_i to be orthogonal to $g_j[u]$ for $j \neq i$, and such that $v_i^T g_i[u] = 1$. It follows that $v_i^T F[uu^T]v_i = c_i$ for all i. The proof is similar for $I = [0, R)$. $\square$

We now classify the entrywise powers that are Loewner super/sub-additive.

Proof of Theorem 5.1. *(1) Super-additivity.* Fix an integer $1 \leq k \leq n$. First apply the definition of super-additivity to $A = B = \mathbf{1}_{n\times n} \in \mathbb{P}_n^k([0,\infty))$ to conclude that if $\alpha \in \mathbb{R}$ and one of $f_\alpha, \phi_\alpha, \psi_\alpha$ is Loewner super-additive, then $\alpha \geq 1$. We now consider three cases corresponding to the three functions $f(x) = f_\alpha(x), \phi_\alpha(x)$, and $\psi_\alpha(x)$ for $\alpha \geq 1$.

Case 1: $f(x) = f_\alpha(x)$. That f_α is super-additive on $\mathbb{P}_n([0,\infty))$ for $\alpha \in \mathbb{N}$ follows by applying the binomial theorem. Now, suppose $\alpha \in (1,\infty) \setminus \mathbb{N}$. We adapt the argument in [8, Theorem 2.4] to our situation. First assume that $\alpha \geq n$; then for $A, B \in \mathbb{P}_n([0,\infty))$,

$$f_\alpha[A+B] = f_\alpha[A] + \alpha \int_0^1 B \circ f_{\alpha-1}\big[t(A+B) + (1-t)A\big]\,dt.$$

Note that $t(A+B)+(1-t)A = A+tB \geq tB$ for all $0 \leq t \leq 1$. Since $\alpha - 1 \geq n-1$, it follows by Theorem 3.1 that $f_{\alpha-1}[t(A+B) + (1-t)A] \geq t^{\alpha-1} f_{\alpha-1}[B]$. Therefore,

$$f_\alpha[A+B] \geq f_\alpha[A] + \alpha f_\alpha[B] \int_0^1 t^{\alpha-1}\,dt = f_\alpha[A] + f_\alpha[B].$$

This shows that f_α is super-additive on $\mathbb{P}_n([0,\infty))$, and hence on $\mathbb{P}_n^k([0,\infty))$ if $\alpha \geq n$. The last remaining case is when $\alpha \in (1,n) \setminus \mathbb{N}$. Define $g_\alpha(x) := (1+x)^\alpha$. Given $\epsilon > 0$ and $v \in (0,1)^n$, apply Taylor's theorem entrywise to $g_\alpha[\epsilon vv^T]$ to obtain:

$$g_\alpha\big[\epsilon vv^T\big] = \mathbf{1}_{n\times n} + \sum_{i=1}^{\lfloor\alpha\rfloor} \epsilon^i \binom{\alpha}{i} f_i[v] f_i[v]^T + O\big(\epsilon^{1+\lfloor\alpha\rfloor}\big)C, \tag{5.2}$$

where $C = C(v)$ is an $n \times n$ matrix that is independent of ϵ. By Corollary 5.5 applied to $F(x) = \sum_{i=1}^{\lfloor\alpha\rfloor} \epsilon^i \binom{\alpha}{i} x^i - \epsilon^\alpha x^\alpha$ and $m = 1 + \lfloor\alpha\rfloor \leq n$, there exist $u \in (0,1)^n$ and $x_\alpha \in \mathbb{R}^n$ such that $x_\alpha^T F[uu^T]x_\alpha = -\epsilon^\alpha$. It follows that

$$x_\alpha^T\big(f_\alpha\big[\mathbf{1}_{n\times n} + \epsilon uu^T\big] - \mathbf{1}_{n\times n} - \epsilon^\alpha f_\alpha\big[uu^T\big]\big)x_\alpha = O\big(\epsilon^{1+\lfloor\alpha\rfloor}\big)x_\alpha^T C x_\alpha - \epsilon^\alpha,$$

and the last expression is negative for sufficiently small $\epsilon = \epsilon_0 > 0$. Hence $f_\alpha[\mathbf{1}_{n\times n} + \epsilon_0 uu^T] \not\geq f_\alpha[\mathbf{1}_{n\times n}] + f_\alpha[\epsilon_0 uu^T]$. This shows that f_α is not super-additive on $\mathbb{P}_n^1([0,\infty))$ and hence on $\mathbb{P}_n^k([0,\infty))$, for $\alpha \in (1,n)\setminus\mathbb{N}$.

Case 2: $f(x) = \phi_\alpha(x)$. Clearly, the assertion holds if $\alpha \in 2\mathbb{N}$ and $1 \leq k \leq n$, since in that case $\phi_\alpha \equiv x^\alpha$. Next if $\alpha \geq n \geq 2$, then as in Case 1, for $A, B \in \mathbb{P}_n(\mathbb{R})$,

$$\phi_\alpha[A+B] = \phi_\alpha[A] + \alpha \int_0^1 B \circ \psi_{\alpha-1}\big[t(A+B) + (1-t)A\big]\, dt.$$

Since $\alpha - 1 \geq n-1$, by Theorem 3.4 the function $\psi_{\alpha-1}$ is monotone on $\mathbb{P}_n(\mathbb{R})$. Thus,

$$\phi_\alpha[A+B] \geq \phi_\alpha[A] + \alpha B \circ \psi_{\alpha-1}[B] \int_0^1 t^{\alpha-1}\, dt = \phi_\alpha[A] + \phi_\alpha[B].$$

It follows that ϕ_α is super-additive on $\mathbb{P}_n^k(\mathbb{R})$ for $\alpha \in 2\mathbb{N} \cup [n, \infty)$. Next note by Case 1 that ϕ_α is not super-additive on $\mathbb{P}_n^k(\mathbb{R})$ for $\alpha \in (1, n) \setminus \mathbb{N}$. It thus remains to prove that ϕ_α is not super-additive on $\mathbb{P}_n^k(\mathbb{R})$ for $\alpha \in (-1+2\mathbb{N}) \cap [1, n)$. Note that for all $u, v \in \mathbb{R}^n$, if ϕ_α is super-additive, then

$$\phi_\alpha\big[uu^T + vv^T\big] \geq \phi_\alpha\big[uu^T\big] + \phi_\alpha\big[vv^T\big] = \phi_\alpha[u]\phi_\alpha[u]^T + \phi_\alpha[v]\phi_\alpha[v]^T \in \mathbb{P}_n(\mathbb{R}).$$

Thus, if ϕ_α is super-additive, then it is also positive on $\mathbb{P}_n^2(\mathbb{R})$. We conclude by Theorem 2.5 that ϕ_α is not super-additive for $\alpha \in (-1+2\mathbb{N}) \cap [1, n-2)$.

The only two powers left to consider are $\alpha = n-2$ with n odd, and $\alpha = n-1$ with n even. In other words, n is of the form $n = 2l$ or $n = 2l+1$ with $l \geq 1$. Thus, $\alpha = 2l-1 \geq 1$ in both cases. We claim that ϕ_{2l-1} is not super-additive on $\mathbb{P}_n^1(\mathbb{R})$. To show the claim, first observe that if $v \in (-1,1)^n$, then $1 + v_i v_j > 0$ for all i, j, and so by the binomial theorem,

$$\phi_{2l-1}\big[\mathbf{1}_{n\times n} + vv^T\big] - \mathbf{1}_{n\times n} - \phi_{2l-1}\big[vv^T\big] = \sum_{i=1}^{2l-1} \binom{2l-1}{i} g_i\big[vv^T\big] - \phi_{2l-1}\big[vv^T\big],$$

where $g_i(x) = x^i$ for $x \in \mathbb{R}$ and $i = 1, \dots, 2l-1$. By Corollary 5.5 applied to the functions $g_1, g_2, \dots, g_{2l-1} = \psi_{2l-1}, \phi_{2l-1}$ and $m = 2l \leq n$, there exist $u \in (-1,1)^n$ and $x \in \mathbb{R}^n$ such that

$$x^T\big(\phi_{2l-1}\big[\mathbf{1}_{n\times n} + uu^T\big] - \mathbf{1}_{n\times n} - \phi_{2l-1}\big[uu^T\big]\big)x = -1.$$

This shows that ϕ_{2l-1} is not super-additive on $\mathbb{P}_n^1(\mathbb{R})$, hence not on $\mathbb{P}_n^k(\mathbb{R})$.

Case 3: $f(x) = \psi_\alpha(x)$. The proof is similar to that of Case 2 and is thus omitted.

(2)(a) Sub-additivity for f_α. First note that if $\alpha \in \mathbb{R}$, applying the definition of sub-additivity to $A = B = \mathbf{1}_{n\times n} \in \mathbb{P}_n^1([0,\infty))$ shows that f_α is not Loewner sub-additive for $\alpha > 1$. Clearly f_1 is sub-additive on $\mathbb{P}_n(I)$, so it remains to study f_α for $\alpha < 1$. Now suppose $2 \leq k \leq n$ and $\alpha < 1$. By Theorem 2.5, there exists $A \in \mathbb{P}_n^2(I)$ such that $f_\alpha[A] \notin \mathbb{P}_n$. Setting $B = A$, we obtain: $f_\alpha[A] + f_\alpha[B] - f_\alpha[A+B] = (2 - 2^\alpha) f[A] \notin \mathbb{P}_n$. It follows that f_α is not sub-additive on $\mathbb{P}_n^k(I)$ for $\alpha < 1$. This settles the assertion for $2 \leq k \leq n$.

The last case is if $k = 1$ and $\alpha < 1$. For ease of exposition, the analysis in this case is divided into several sub-cases:

Sub-case 1: $\alpha \in (0,1)$. Given $0 < \epsilon < 1$ and $v \in (0,1)^n$, apply Taylor's theorem entrywise to $g_\alpha[\epsilon vv^T]$, where $g_\alpha(x) = (1+x)^\alpha$ as above. We obtain:

$$f_\alpha\big[\mathbf{1}_{n\times n} + \epsilon vv^T\big] - \mathbf{1}_{n\times n} - f_\alpha\big[\epsilon vv^T\big] = \epsilon\alpha vv^T - \epsilon^\alpha f_\alpha\big[vv^T\big] + O\big(\epsilon^2\big)C,$$

where $C = C(v)$ is an $n \times n$ matrix that is independent of ϵ. By Corollary 5.5 with $F(x) = \epsilon\alpha x - \epsilon^\alpha x^\alpha$ and $m = 2 \leq n$, there exist $u \in (0,1)^n$ and $x_\alpha \in \mathbb{R}^n$ such that

$$x_\alpha^T\big(f_\alpha\big[\mathbf{1}_{n\times n} + \epsilon uu^T\big] - \mathbf{1}_{n\times n} - f_\alpha\big[\epsilon uu^T\big]\big)x_\alpha = \epsilon\alpha + O\big(\epsilon^2\big)x_\alpha^T C x_\alpha,$$

which is positive for $\epsilon > 0$ small enough. Therefore f_α is not sub-additive on $\mathbb{P}_n^1([0,\infty))$ for $\alpha \in (0,1)$.

Sub-case 2: $\alpha = 0$. To see why f_0 is indeed sub-additive on $\mathbb{P}_n^1([0,\infty))$, given a subset $S \subset \{1,\dots,n\}$ we define $\mathbf{1}_S$ to be the matrix with (i,j) entry 1 if $i,j \in S$ and 0 otherwise. Now given $A = (a_{ij}) \in \mathbb{P}_n^1([0,\infty))$, define $S(A) := \{i\colon\ a_{ii} \neq 0\}$. Then $f_0[A] = \mathbf{1}_{S(A)}$, $f_0[B] = \mathbf{1}_{S(B)}$ for $A, B \in \mathbb{P}_n^1([0,\infty))$, and hence by inclusion–exclusion, $f_0[A] + f_0[B] - f_0[A+B] = \mathbf{1}_{S(A)\cap S(B)} \in \mathbb{P}_n^1([0,\infty))$. Thus f_0 is sub-additive on $\mathbb{P}_n^1([0,\infty))$ as claimed.

Sub-case 3: $\alpha < 0, n \geq 3$. The fact that f_α is not subadditive on $\mathbb{P}_n^1([0,\infty))$ for $\alpha < 0$ follows by an argument similar to Sub-case 1. The argument is omitted for the sake of brevity.

Sub-case 4: $\alpha < 0, (n,k) = (2,1)$. The bulk of the work in classifying the sub-additive entrywise powers f_α lies in the remaining case of $\mathbb{P}_2^1$ with $\alpha < 0$. We first show that f_α is sub-additive on $\mathbb{P}_2^1([0,\infty))$ for all $\alpha < 0$. Setting $A := (a,b)^T(a,b)$ and $B := (c,d)^T(c,d)$, the problem translates to showing that

$$\big(f_\alpha\big(a^2\big) + f_\alpha\big(c^2\big) - f_\alpha\big(a^2+c^2\big)\big)\cdot\big(f_\alpha\big(b^2\big) + f_\alpha\big(d^2\big) - f_\alpha\big(b^2+d^2\big)\big) \geq \big(f_\alpha(ab) + f_\alpha(cd) - f_\alpha(ab+cd)\big)^2.$$

Note that if any of $a,b,c,d = 0$ then the inequality is clear. Thus we may assume $a,b,c,d > 0$. Now define

$$f(x,y) := x^\alpha + y^\alpha - (x+y)^\alpha, \qquad g(x,y) := \log f\big(e^x, e^y\big).$$

Then proving the above inequality is equivalent to showing that $(g(x_1,y_1)+g(x_2,y_2))/2 \geq g((x_1+x_2)/2, (y_1+y_2)/2)$, i.e., that g is midpoint-convex on $\mathbb{R}^2$. Since g is smooth, it suffices to show that its Hessian $H_g(x,y)$ is positive semidefinite at all points in $\mathbb{R}^2$. A straightforward but longwinded computation demonstrates that $\det H_g(x,y) = 0$ for all $x, y \in \mathbb{R}^2$. Thus it suffices to show that g_{xx} is nonnegative on $\mathbb{R}$ (and the result for g_{yy} follows by symmetry). We now compute, setting $E := e^x + e^y$ for notational convenience:

$$\begin{aligned} f\big(e^x,e^y\big)^2 g_{xx}(x,y) &= \big(e^{\alpha x} + e^{\alpha y} - E^\alpha\big)\big(\alpha^2 e^{\alpha x} - \alpha e^x E^{\alpha-1} - \alpha(\alpha-1)e^{2x}E^{\alpha-2}\big) - \alpha^2\big(e^x E^{\alpha-1} - e^{\alpha x}\big)^2 \\ &= \alpha^2 e^{\alpha(x+y)}E^{\alpha-2}\big(E^{2-\alpha} - \big(e^{(2-\alpha)x} + e^{(2-\alpha)y}\big)\big) + \alpha e^{x+y}E^{\alpha-2}\big(E^\alpha - \big(e^{\alpha x} + e^{\alpha y}\big)\big). \end{aligned}$$

Note that the difference in the first term is nonnegative because $x^{2-\alpha}$ is super-additive, while the difference in the second term is nonpositive because x^α is sub-additive. Thus both terms are nonnegative, which concludes the proof for f_α.

(b) Sub-additivity for ϕ_α. By considering the matrices $A = uu^T, B = vv^T$ with $u = (1,1)^T$ and $v = (1,-1)^T$, it immediately follows that ϕ_α is not Loewner sub-additive on $\mathbb{P}_2^1(\mathbb{R})$, and hence not sub-additive on $\mathbb{P}_n^k(\mathbb{R})$ for all (n,k).

(c) Sub-additivity for ψ_α. First note that $\psi_1(x) = x$ is sub-additive on $\mathbb{P}_n^k(\mathbb{R})$ for all (n,k). It is also not difficult to show that ψ_0 is sub-additive on $\mathbb{P}_2^1(\mathbb{R})$. Using part (a), it remains to prove that ψ_0 is not sub-additive on $\mathbb{P}_n^1(\mathbb{R})$ for $n > 2$, and ψ_α is not sub-additive on $\mathbb{P}_2^1(\mathbb{R})$ for $\alpha < 0$.

To see why ψ_0 is not sub-additive on $\mathbb{P}_n^1(\mathbb{R})$ for $n \geq 3$, use the following three-parameter family of rank one counterexamples:

$$\big(A(a,b,c) := (-a,c,c)^T(-a,c,c),\ B(a,b,c) := (c,-b,c)^T(c,-b,c)\big), \quad 0 < a < b < c.$$

It remains to show that ψ_α is not sub-additive on $\mathbb{P}_2^1(\mathbb{R})$ for any $\alpha < 0$. Let $A := (1,-1)^T(1,-1)$ and $B := (1, 1/2)^T(1, 1/2)$. Then,

$$\psi_\alpha[A] + \psi_\alpha[B] - \psi_\alpha[A+B] = \begin{pmatrix} 2-2^\alpha & -1+2^{1-\alpha} \\ -1+2^{1-\alpha} & 1+(1/4)^\alpha-(5/4)^\alpha \end{pmatrix} =: C_\alpha.$$

We claim that $\det C_\alpha < 0$ for all $\alpha < 0$, which shows that ψ_α is not sub-additive on $\mathbb{P}_2^1(\mathbb{R})$ and completes the proof. To see why the claim holds, compute for $\alpha < 0$:

$$4^\alpha\big(2-2^\alpha\big)\det C_\alpha = 4^\alpha + 2^\alpha - 1 - 5^\alpha.$$

Note that the function $f_\alpha(x) = x^\alpha$ is convex on $(0,\infty)$ for $\alpha < 0$, so Jensen's inequality yields:

$$2^\alpha = f_\alpha\left(\frac{3}{4}\cdot 1 + \frac{1}{4}\cdot 5\right) < \frac{3}{4} + \frac{1}{4}\cdot 5^a, \qquad 4^\alpha = f_\alpha\left(\frac{1}{4}\cdot 1 + \frac{3}{4}\cdot 5\right) < \frac{1}{4} + \frac{3}{4}\cdot 5^a.$$

Adding the two inequalities shows that $\det C_\alpha < 0$ for $\alpha < 0$, and the proof is complete. □

6. Concluding remarks

We conclude by discussing the following questions that naturally arise from the above analysis:

(1) Is it possible to find matrices of rank exactly k (for some $1 \le k \le n$) for which a non-integer power less than $n-2$ is not Loewner positive? Similar questions can also be asked for monotonicity, convexity, and super/sub-additivity.
(2) Can we compute the Hadamard critical exponents for convex combinations of the two-sided power functions ϕ_α, ψ_α?

We answer these questions in the remainder of this section.

6.1. Examples of fixed rank positive semidefinite matrices

In the previous sections, several matrices in $\mathbb{P}_n^k(I)$ were constructed, for which specific non-integer Hadamard powers were not Loewner positive (or monotone, convex, or super/sub-additive). Recall that $\mathbb{P}_n^k(I)$ consists of matrices of rank k *or less*; however, all of the examples above were in fact of rank 2 or rank 1. A natural question is thus if there are higher rank examples for which the Loewner properties are violated. This is important since in applications, when entrywise powers are applied to regularize covariance/correlation matrices, the rank corresponds to the sample size and is thus known. Hence, failure to maintain positive definiteness only for rank 2 matrices might not have serious consequences in practice. The following result strengthens Theorems 2.5, 3.4, 4.4, and 5.1 by constructing higher rank counterexamples from lower rank matrices.

Proposition 6.1. *Let $0 < R \le \infty$, $I = [0,R)$ or $(-R,R)$, and $f : I \to \mathbb{R}$ be continuous. Suppose $1 \le l < k \le n$ are integers, and $A, B \ge 0$ are matrices in $\mathbb{P}_n(I)$ such that* $\operatorname{rank} A = l$ *and one of the following properties is satisfied:*

(a) *$f[A] \notin \mathbb{P}_n$;*
(b) *$A \ge B \ge 0$ and $f[A] \not\ge f[B]$;*
(c) *$A \ge B \ge 0$ and $f[\lambda A + (1-\lambda)B] \not\le \lambda f[A] + (1-\lambda) f[B]$ for some $0 < \lambda < 1$;*

(d) $f[A+B] \not\geq f[A]+f[B]$;
(e) $f[A+B] \not\leq f[A]+f[B]$.

Then there exist $A', B' \geq 0$ such that $\operatorname{rank} A' = k$ *and the same property holds when A, B are replaced by A', B' respectively.*

Note that the special cases of $l = 1, 2$ answer question (1) above.

Proof of Proposition 6.1. We show the result for property (b) monotonicity; the analogous results for (a) positivity, (c) convexity, (d) super-additivity, and (e) sub-additivity are shown similarly. Suppose $A \geq B \geq 0$ and $\operatorname{rank} A = l$, but $f[A] \not\geq f[B]$. Then there exists a nonzero vector $v \in \mathbb{R}^n$ such that $v^T f[A]v < v^T f[B]v$. Now write $A = \sum_{i=1}^{l} \lambda_i u_i u_i^T$ where $\lambda_i \neq 0$ and u_i are the nonzero eigenvalues and eigenvectors respectively. Extend the u_i to an orthonormal set $\{u_1, \dots, u_k\}$, and define $C := \sum_{i=l+1}^{k} u_i u_i^T$. Clearly, $A + \epsilon C \geq B + \epsilon C \geq 0$ and $A + \epsilon C, B + \epsilon C \in \mathbb{P}_n(I)$ for small $\epsilon > 0$. Since

$$0 > v^T f[A]v - v^T f[B]v = \lim_{\epsilon \to 0^+} v^T \big(f[A+\epsilon C] - f[B+\epsilon C]\big)v$$

and f is continuous, there exists small $\epsilon_0 > 0$ such that $f[A+\epsilon_0 C] \not\geq f[B+\epsilon_0 C]$. Now setting $A' := A + \epsilon_0 C, B' := B + \epsilon_0 C$ completes the proof, since $A' \in \mathbb{P}_n^k(I)$. $\square$

6.2. Convex combinations of power functions

Another related question to Theorem 1.2 would be to compute Hadamard critical exponents for the function $(\phi_\alpha + \psi_\alpha)/2$, which equals x^α on $[0, \infty)$ and 0 on $(-\infty, 0)$. More generally, consider $f_\alpha^\lambda := \lambda\phi_\alpha + (1-\lambda)\psi_\alpha$ for $\lambda \in [0,1]$. Theorem 1.2 already yields much information about the sets $\mathcal{H}_J^{f^\lambda}(n,k)$ for all $n \geq 2$, $1 \leq k \leq n$, and $J \in \{\text{positivity}, \text{monotonicity}, \text{convexity}, \text{super/sub-additivity}\}$. In particular, it follows from Theorem 1.2 that given J, n, k, the Hadamard critical exponents for f^λ are the same as in Corollary 1.4. Namely, the critical exponents are:

$$CE_{pos}^{f^\lambda}(n,k) = n-2, \qquad CE_{mono}^{f^\lambda}(n,k) = n-1, \qquad CE_{conv}^{f^\lambda}(n,k) = n, \quad \forall 2 \leq k \leq n, \tag{6.1}$$

and are $0, 0, 1$ respectively, if $k = 1 < n$. To see why, first note that for $2 \leq k \leq n$, the sets of Loewner positive, monotone, or convex maps are closed under taking convex combinations. Thus the critical exponents for positivity, monotonicity, and convexity are at most $n-2, n-1, n$ respectively. Next, the function f_α^λ reduces to f_α on $[0,\infty)$ for all $\lambda \in [0,1]$. Thus using the specific matrices described in the proofs of Theorems 2.1 and 3.1 and [15, Lemma 5.2], we observe that for all $\lambda \in [0,1]$ and non-integer powers $0 < \alpha < n-2, n-1$, or n, the function f_α^λ is not Loewner positive, monotone, or convex respectively. The $k = 1$ case is handled similarly.

We now consider the super/sub-additivity of f_α^λ. By Theorem 5.1 and similar arguments as above, one verifies that $CE_{super}^{f^\lambda}(n,k) = n$ for $1 \leq k \leq n$. We next claim that f_α^λ is never sub-additive for $0 < \lambda < 1$, $\alpha > 0$, and $1 \leq k \leq n$. To see why, note that f_α^λ is not sub-additive on $\mathbb{P}_n^k(\mathbb{R})$ for $\alpha \in (0,\infty) \setminus \{1\}$ since f_α is not sub-additive on $\mathbb{P}_n([0,\infty))$. If $\alpha = 1$, using $A = \mathbf{1}_{2\times 2} \oplus \mathbf{0}_{(n-2)\times(n-2)}$ and $B = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \oplus \mathbf{0}_{(n-2)\times(n-2)}$ shows that f_1^λ is not sub-additive for $0 < \lambda < 1$.

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