Two results on composition operators on the Dirichlet space

Daniel Li [a,*], Hervé Queffélec [b], Luis Rodríguez-Piazza [c,1]

[a] *Univ Lille Nord de France, U-Artois, Laboratoire de Mathématiques de Lens EA 2462 & Fédération CNRS Nord-Pas-de-Calais FR 2956, Faculté des Sciences Jean Perrin, Rue Jean Souvraz, S.P. 18, F-62300 Lens, France*
[b] *Univ Lille Nord de France, USTL, Laboratoire Paul Painlevé U.M.R. CNRS 8524 & Fédération CNRS Nord-Pas-de-Calais FR 2956, F-59655 Villeneuve d'Ascq Cedex, France*
[c] *Universidad de Sevilla, Facultad de Matemáticas, Departamento de Análisis Matemático & IMUS, Apartado de Correos 1160, 41080 Sevilla, Spain*

A B S T R A C T

We show that the decay of approximation numbers of compact composition operators on the Dirichlet space $\mathcal{D}$ can be as slow as we wish. We also prove the optimality of a result of O. El-Fallah, K. Kellay, M. Shabankhah and H. Youssfi on boundedness on $\mathcal{D}$ of self-maps of the disk all of whose powers are norm-bounded in $\mathcal{D}$.

© 2015 Elsevier Inc. All rights reserved.

1. Introduction

Recall that if φ is an analytic self-map of $\mathbb{D}$, a so-called *Schur function*, the composition operator C_φ associated to φ is formally defined by

$$C_\varphi(f) = f \circ \varphi.$$

The Littlewood subordination principle [3, p. 30] tells us that C_φ maps the Hardy space H^2 to itself for every Schur function φ. Also recall that if H is a Hilbert space and $T : H \to H$ a bounded linear operator, the n-th approximation number $a_n(T)$ of T is defined as

$$a_n(T) = \inf\{\|T - R\|; \text{ rank } R < n\}, \quad n = 1, 2, \dots. \tag{1.1}$$

Doctopic: Functional Analysis ARTICLE IN PRESS YJMAA:19189

2 D. Li et al. / J. Math. Anal. Appl. ••• (••••) •••–•••

In [9], working on that Hardy space H^2 (and also on some weighted Bergman spaces), we have undertaken the study of approximation numbers $a_n(C_\varphi)$ of composition operators C_φ, and proved among other facts the following:

Theorem 1.1. *Let $(\varepsilon_n)_{n\geq 1}$ be a non-increasing sequence of positive numbers tending to 0. Then, there exists a compact composition operator C_φ on H^2 such that*

$$\liminf_{n\to\infty} \frac{a_n(C_\varphi)}{\varepsilon_n} > 0.$$

As a consequence, there are composition operators on H^2 which are compact but in no Schatten class.

The last item had been previously proved by Carroll and Cowen [2], the above statement with approximation numbers being more precise.

For the Dirichlet space, the situation is more delicate because not every analytic self-map of $\mathbb{D}$ generates a bounded composition operator on $\mathcal{D}$. When this is the case, we will say that φ is a *symbol* (understanding "of $\mathcal{D}$"). Note that every symbol is necessarily in $\mathcal{D}$.

In [8], we have performed a similar study on that Dirichlet space $\mathcal{D}$, and established several results on approximation numbers in that new setting, in particular the existence of symbols φ for which C_φ is compact without being in any Schatten class S_p. But we have not been able in [8] to prove a full analogue of Theorem 1.1. Using a new approach, essentially based on Carleson embeddings and the Schur test, we are now able to prove that analogue.

Theorem 1.2. *For every sequence $(\varepsilon_n)_{n\geq 1}$ of positive numbers tending to 0, there exists a compact composition operator C_φ on the Dirichlet space $\mathcal{D}$ such that*

$$\liminf_{n\to\infty} \frac{a_n(C_\varphi)}{\varepsilon_n} > 0.$$

Turning now to the question of necessary or sufficient conditions for a Schur function φ to be a symbol, we can observe that, since $(z^n/\sqrt{n})_{n\geq 1}$ is an orthonormal sequence in $\mathcal{D}$ and since formally $C_\varphi(z^n) = \varphi^n$, a necessary condition is as follows:

$$\varphi \text{ is a symbol} \quad \Longrightarrow \quad \left\|\varphi^n\right\|_{\mathcal{D}} = O(\sqrt{n}). \tag{1.2}$$

It is worth noting that, for any Schur function, one has:

$$\varphi \in \mathcal{D} \quad \Longrightarrow \quad \left\|\varphi^n\right\|_{\mathcal{D}} = O(n)$$

(of course, this is an equivalence). Indeed, anticipating on the next section, we have for any integer $n \geq 1$:

$$\left\|\varphi^n\right\|_{\mathcal{D}}^2 = \left|\varphi(0)\right|^{2n} + \int_{\mathbb{D}} n^2 \left|\varphi(z)\right|^{2(n-1)} \left|\varphi'(z)\right|^2 dA(z)$$

$$\leq \left|\varphi(0)\right|^2 + \int_{\mathbb{D}} n^2 \left|\varphi'(z)\right|^2 dA(z) \leq n^2 \|\varphi\|_{\mathcal{D}}^2,$$

giving the result.

Now, the following sufficient condition was given in [4]:

$$\left\|\varphi^n\right\|_{\mathcal{D}} = O(1) \quad \Longrightarrow \quad \varphi \text{ is a symbol.} \tag{1.3}$$

Doctopic: Functional Analysis   ARTICLE IN PRESS   YJMAA:19189

D. Li et al. / J. Math. Anal. Appl. ••• (••••) •••–•••   3

In view of (1.2), one might think of improving this condition, but it turns out to be optimal, as says the second main result of that paper.

Theorem 1.3. *Let $(M_n)_{n \geq 1}$ be an arbitrary sequence of positive numbers tending to ∞. Then, there exists a Schur function $\varphi \in \mathcal{D}$ such that:*

1) $\|\varphi^n\|_{\mathcal{D}} = O(M_n)$ *as $n \to \infty$;*
2) φ *is not a symbol on $\mathcal{D}$.*

The organization of this paper will be as follows: in Section 2, we give the notation and background. In Section 3, we prove Theorem 1.2; in Section 3.1, we prove Theorem 1.3; and we end with a section of remarks and questions.

2. Notation and background

We denote by $\mathbb{D}$ the open unit disk of the complex plane and by A the normalized area measure $dx\,dy/\pi$ of $\mathbb{D}$. The unit circle is denoted by $\mathbb{T} = \partial\mathbb{D}$. The notation $A \lesssim B$ indicates that $A \leq cB$ for some positive constant c.

A Schur function is an analytic self-map of $\mathbb{D}$ and the associated composition operator is defined, formally, by $C_\varphi(f) = f \circ \varphi$. The operator C_φ maps the space $\mathcal{H}ol(\mathbb{D})$ of holomorphic functions on $\mathbb{D}$ into itself.

The Dirichlet space $\mathcal{D}$ is the space of analytic functions $f \colon \mathbb{D} \to \mathbb{C}$ such that

$$\|f\|_{\mathcal{D}}^2 := |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2\, dA(z) < +\infty. \tag{2.1}$$

If $f(z) = \sum_{n=0}^{\infty} c_n z^n$, one has:

$$\|f\|_{\mathcal{D}}^2 = |c_0|^2 + \sum_{n=1}^{\infty} n|c_n|^2. \tag{2.2}$$

Then $\| \,.\, \|_{\mathcal{D}}$ is a norm on $\mathcal{D}$, making $\mathcal{D}$ a Hilbert space, and $\| \,.\, \|_{H^2} \leq \| \,.\, \|_{\mathcal{D}}$. For further information on the Dirichlet space, the reader may see [1] or [11].

The Bergman space $\mathfrak{B}$ is the space of analytic functions $f \colon \mathbb{D} \to \mathbb{C}$ such that:

$$\|f\|_{\mathfrak{B}}^2 := \int_{\mathbb{D}} |f(z)|^2\, dA(z) < +\infty.$$

If $f(z) = \sum_{n=0}^{\infty} c_n z^n$, one has $\|f\|_{\mathfrak{B}}^2 = \sum_{n=0}^{\infty} \frac{|c_n|^2}{n+1}$. If $f \in \mathcal{D}$, one has by definition:

$$\|f\|_{\mathcal{D}}^2 = \|f'\|_{\mathfrak{B}}^2 + |f(0)|^2.$$

Recall that, whereas every Schur function φ generates a bounded composition operator C_φ on Hardy and Bergman spaces, it is no longer the case for the Dirichlet space (see [10, Proposition 3.12], for instance).

We denote by $b_n(T)$ the n-th *Bernstein number* of the operator $T \colon H \to H$, namely:

$$b_n(T) = \sup_{\dim E = n} \left(\inf_{f \in S_E} \|Tx\| \right) \tag{2.3}$$

where S_E denotes the unit sphere of E. It is easy to see [8] that

$$b_n(T) = a_n(T) \quad \text{for all } n \geq 1$$

(recall that the approximation numbers are defined in (1.1)).

If φ is a Schur function, let

$$n_\varphi(w) = \#\{z \in \mathbb{D}; \ \varphi(z) = w\} \geq 0 \tag{2.4}$$

be the associated *counting function*. If $f \in \mathcal{D}$ and $g = f \circ \varphi$, the change of variable formula provides us with the useful following equation [12,8]:

$$\int_{\mathbb{D}} |g'(z)|^2 \, dA(z) = \int_{\mathbb{D}} |f'(w)|^2 n_\varphi(w) \, dA(w) \tag{2.5}$$

(the integrals might be infinite). In those terms, a necessary and sufficient condition for φ to be a symbol is as follows [12, Theorem 1]. Let:

$$\rho_\varphi(h) = \sup_{\xi \in \mathbb{T}} \int_{S(\xi,h)} n_\varphi \, dA \tag{2.6}$$

where $S(\xi, h) = \mathbb{D} \cap D(\xi, h)$ is the Carleson window centered at ξ and of size h. Then φ is a symbol if and only if:

$$\sup_{0 < h < 1} \frac{1}{h^2} \rho_\varphi(h) < \infty. \tag{2.7}$$

This is not difficult to prove. In view of (2.5), the boundedness of C_φ amounts to the existence of a constant C such that:

$$\int_{\mathbb{D}} |f'(w)|^2 n_\varphi(w) \, dA(w) \leq C \int_{\mathbb{D}} |f'(z)|^2 \, dA(z), \quad \forall f \in \mathcal{D}.$$

Since $f' = h$ runs over $\mathfrak{B}$ as f runs over $\mathcal{D}$, and with equal norms, the above condition reads:

$$\int_{\mathbb{D}} |h(w)|^2 n_\varphi(w) \, dA(w) \leq C \int_{\mathbb{D}} |h(z)|^2 \, dA(z), \quad \forall h \in \mathfrak{B}.$$

This exactly means that the measure $n_\varphi \, dA$ is a Carleson measure for $\mathfrak{B}$. Such measures have been characterized in [6] and that characterization gives (2.7).

But this condition is very abstract and difficult to test, and sometimes more "concrete" sufficient conditions are desirable. In [8], we proved that, even if the Schur function extends continuously to $\overline{\mathbb{D}}$, no Lipschitz condition of order α, $0 < \alpha < 1$, on φ is sufficient for ensuring that φ is a symbol. It is worth noting that the limiting case $\alpha = 1$, so restrictive it is, guarantees the result.

Proposition 2.1. *Suppose that the Schur function φ is in the analytic Lipschitz class on the unit disk, i.e. satisfies:*

$$|\varphi(z) - \varphi(w)| \leq C|z - w|, \quad \forall z, w \in \mathbb{D}.$$

Then C_φ is bounded on $\mathcal{D}$.

Doctopic: Functional Analysis ARTICLE IN PRESS YJMAA:19189

D. Li et al. / J. Math. Anal. Appl. • • • (• • • •) • • • – • • • 5

Proof. Let $f \in \mathcal{D}$; one has:

$$\left\| C_\varphi(f) \right\|_{\mathcal{D}}^2 = \left| f(\varphi(0)) \right|^2 + \int_{\mathbb{D}} \left| f'(\varphi(z)) \right|^2 \left| \varphi'(z) \right|^2 dA(z)$$

$$\leq \left| f(\varphi(0)) \right|^2 + \|\varphi'\|_\infty^2 \int_{\mathbb{D}} \left| f'(\varphi(z)) \right|^2 dA(z).$$

This integral is nothing but $\|C_\varphi(f')\|_{\mathfrak{B}}^2$ and hence, since C_φ is bounded on the Bergman space $\mathfrak{B}$, we have, for some constant K_1:

$$\int_{\mathbb{D}} \left| f'(\varphi(z)) \right|^2 dA(z) \leq K_1^2 \|f'\|_{\mathfrak{B}}^2 \leq K_1^2 \|f\|_{\mathcal{D}}^2.$$

On the other hand,

$$\left| f(\varphi(0)) \right| \leq \left(1 - \left| \varphi(0) \right|^2 \right)^{-1/2} \|f\|_{H^2} \leq \left(1 - \left| \varphi(0) \right|^2 \right)^{-1/2} \|f\|_{\mathcal{D}},$$

and we get

$$\left\| C_\varphi(f) \right\|_{\mathcal{D}}^2 \leq K^2 \|f\|_{\mathcal{D}}^2,$$

with $K^2 = K_1^2 + (1 - |\varphi(0)|^2)^{-1}$. $\square$

3. Proof of Theorem 1.2

We are going to prove Theorem 1.2 mentioned in the Introduction, which we recall here.

Theorem 3.1. *For every sequence (ε_n) of positive numbers with limit 0, there exists a compact composition operator C_φ on $\mathcal{D}$ such that*

$$\liminf_{n \to \infty} \frac{a_n(C_\varphi)}{\varepsilon_n} > 0.$$

Before entering really in the proof, we may remark that, without loss of generality, by replacing ε_n with $\inf(2^{-8}, \sup_{k \geq n} \varepsilon_k)$, we can, and do, assume that $(\varepsilon_n)_n$ decreases and $\varepsilon_1 \leq 2^{-8}$.

Moreover, we can assume that $(\varepsilon_n)_n$ decreases "slowly", as said in the following lemma.

Lemma 3.2. *Let (ε_i) be a decreasing sequence with limit zero and let $0 < \rho < 1$. Then, there exists another sequence $(\widehat{\varepsilon_i})$, decreasing with limit zero, such that $\widehat{\varepsilon_i} \geq \varepsilon_i$ and $\widehat{\varepsilon_{i+1}} \geq \rho \widehat{\varepsilon_i}$, for every $i \geq 1$.*

Proof. We define inductively $\widehat{\varepsilon_i}$ by $\widehat{\varepsilon_1} = \varepsilon_1$ and

$$\widehat{\varepsilon_{i+1}} = \max(\rho \widehat{\varepsilon_i}, \varepsilon_{i+1}).$$

It is seen by induction that $\widehat{\varepsilon_i} \geq \varepsilon_i$ and that $\widehat{\varepsilon_i}$ decreases to a limit $a \geq 0$. If $\widehat{\varepsilon_i} = \varepsilon_i$ for infinitely many indices i, we have $a = 0$. In the opposite case, $\widehat{\varepsilon_{i+1}} = \rho \widehat{\varepsilon_i}$ from some index i_0 onwards, and again $a = 0$ since $\rho < 1$. $\square$

We will take $\rho = 1/2$ and assume for the sequel that $\varepsilon_{i+1} \geq \varepsilon_i/2$.

Doctopic: Functional Analysis

ARTICLE IN PRESS

YJMAA:19189

6

D. Li et al. / J. Math. Anal. Appl. ••• *(••••)* •••–•••

Proof of Theorem 3.1. We first construct a subdomain $\Omega = \Omega_\theta$ of $\mathbb{D}$ defined by a cuspidal inequality:

$$\Omega = \big\{ z = x + iy \in \mathbb{D};\ |y| < \theta(1-x),\ 0 < x < 1 \big\}, \tag{3.1}$$

where $\theta \colon [0,1] \to [0,1[$ is a continuous increasing function such that

$$\theta(0) = 0 \quad \text{and} \quad \theta(1-x) \leq 1-x. \tag{3.2}$$

Note that since $1 - x \leq \sqrt{1 - x^2}$, the condition $|y| < \theta(1-x)$ implies that $z = x + iy \in \mathbb{D}$. Note also that $1 \in \overline{\Omega}$ and that Ω is a Jordan domain.

We introduce a parameter δ with $\varepsilon_1 \leq \delta \leq 1 - \varepsilon_1$. We put:

$$\theta\big(\delta^j\big) = \varepsilon_j \delta^j \tag{3.3}$$

and we extend θ to an increasing continuous function from $(0,1)$ into itself (piecewise linearly, or more smoothly, as one wishes). We claim that:

$$\theta(h) \leq h \quad \text{and} \quad \theta(h) = o(h) \quad \text{as } h \to 0. \tag{3.4}$$

Indeed, if $\delta^{j+1} \leq h < \delta^j$, we have $\theta(h)/h \leq \theta(\delta^j)/\delta^{j+1} = \varepsilon_j/\delta$, which is $\leq \varepsilon_1/\delta \leq 1$ and which tends to 0 with h.

We define now $\varphi = \varphi_\theta \colon \overline{\mathbb{D}} \to \overline{\Omega}$ as a continuous map which is a Riemann map from $\mathbb{D}$ onto Ω, and with $\varphi(1) = 1$ (a cusp-type map). Since φ is univalent, one has $n_\varphi = \mathbb{1}_\Omega$, and since Ω is bounded, φ defines a symbol on $\mathcal{D}$, by (2.7). Moreover, (3.4) implies that $A[S(\xi,h) \cap \Omega] \leq h\theta(h)$ for every $\xi \in \mathbb{T}$; hence, ρ_φ being defined in (2.6), one has $\rho_\varphi(h) = o(h^2)$ as $h \to 0^+$. In view of [12], this little-oh condition guarantees the compactness of $C_\varphi \colon \mathcal{D} \to \mathcal{D}$.

It remains to minorate its approximation numbers.

The measure $\mu = n_\varphi \, dA$ is a Carleson measure for the Bergman space $\mathfrak{B}$, and it was proved in [7] that $C_\varphi^* C_\varphi$ is unitarily equivalent to the Toeplitz operator $T_\mu = I_\mu^* I_\mu \colon \mathfrak{B} \to \mathfrak{B}$ defined by:

$$T_\mu f(z) = \int_{\mathbb{D}} \frac{f(w)}{(1 - \overline{w}z)^2} \, dA(w) = \int_{\mathbb{D}} f(w) K_w(z) \, dA(w), \tag{3.5}$$

where $I_\mu \colon \mathfrak{B} \to L^2(\mu)$ is the canonical inclusion and K_w is the reproducing kernel of $\mathfrak{B}$ at w, i.e. $K_w(z) = \frac{1}{(1 - \overline{w}z)^2}$.

Actually, we can get rid of the analyticity constraint by considering, instead of T_μ, the operator $S_\mu = I_\mu I_\mu^* \colon L^2(\mu) \to L^2(\mu)$, which corresponds to the arrows:

$$L^2(\mu) \xrightarrow{I_\mu^*} \mathfrak{B} \xrightarrow{I_\mu} L^2(\mu).$$

We use the relation (3.5) which implies:

$$a_n(C_\varphi) = a_n(I_\mu) = a_n\big(I_\mu^*\big) = \sqrt{a_n(S_\mu)}. \tag{3.6}$$

We set:

$$c_j = 1 - 2\delta^j \quad \text{and} \quad r_j = \varepsilon_j \delta^j \tag{3.7}$$

One has $r_j = \varepsilon_j(1 - c_j)/2$.

Doctopic: Functional Analysis | ARTICLE IN PRESS | YJMAA:19189

D. Li et al. / J. Math. Anal. Appl. • • • *(* • • • • *)* • • • – • • •   7

Lemma 3.3. *The disks $\Delta_j = D(c_j, r_j)$, $j \geq 1$, are disjoint and contained in Ω.*

Proof. If $z = x + iy \in \Delta_j$, then $1 - x > 1 - c_j - r_j = (1 - c_j)(1 - \varepsilon_j/2) = 2\delta^j(1 - \varepsilon_j/2) \geq \delta^j$ and $|y| < r_j = \theta(\delta^j)$; hence $|y| < \theta(\delta^j) \leq \theta(1 - x)$ and $z \in \Omega$. On the other hand, $c_{j+1} - c_j = 2(\delta^j - \delta^{j+1}) = 2(1 - \delta)\delta^j \geq 2\varepsilon_1 \delta^j \geq 2\varepsilon_j \delta^j = 2r_j > r_j + r_{j+1}$; hence $\Delta_j \cap \Delta_{j+1} = \emptyset$. $\square$

We will next need a description of S_μ.

Lemma 3.4. *For every $g \in L^2(\mu)$ and every $z \in \mathbb{D}$:*

$$I_\mu^* g(z) = \int_\Omega \frac{g(w)}{(1 - \overline{w}z)^2} \, dA(w) \tag{3.8}$$

$$S_\mu g(z) = \left(\int_\Omega \frac{g(w)}{(1 - \overline{w}z)^2} \, dA(w) \right) \mathbb{1}_\Omega(z). \tag{3.9}$$

Proof. K_w being the reproducing kernel of $\mathfrak{B}$, we have for any pair of functions $f \in \mathfrak{B}$ and $g \in L^2(\mu)$:

$$\left\langle I_\mu^* g, f \right\rangle_{\mathfrak{B}} = \left\langle g, I_\mu f \right\rangle_{L^2(\mu)} = \int_\Omega g(w) \overline{f(w)} \, dA(w) = \int_\Omega g(w) \langle K_w, f \rangle_{\mathfrak{B}} \, dA(w)$$

$$= \left\langle \int_\Omega g(w) K_w \, dA(w), f \right\rangle_{\mathfrak{B}},$$

so that $I_\mu^* g = \int_\Omega g(w) K_w \, dA(w)$, giving the result. $\square$

In the rest of the proof, we fix a positive integer n and put:

$$f_j = \frac{1}{r_j} \mathbb{1}_{\Delta_j}, \quad j = 1, \ldots, n. \tag{3.10}$$

Let:

$$E = \operatorname{span}(f_1, \ldots, f_n).$$

This is an n-dimensional subspace of $L^2(\mu)$.

The Δ_j's being disjoint, the sequence $(f_1, \ldots, f_n)$ is orthonormal in $L^2(\mu)$. Indeed, those functions have disjoint supports, so are orthogonal, and:

$$\int f_j^2 \, d\mu = \int f_j^2 n_\varphi \, dA = \int_{\Delta_j} \frac{1}{r_j^2} \, dA = 1.$$

We now estimate from below the Bernstein numbers of I_μ^*. To that effect, we compute the scalar products $m_{i,j} = \langle I_\mu^*(f_i), I_\mu^*(f_j) \rangle$. One has:

$$m_{i,j} = \left\langle f_i, S_\mu(f_j) \right\rangle = \int_\Omega f_i(z) \overline{S_\mu f_j(z)} \, dA(z)$$

$$= \iint_{\Omega \times \Omega} \frac{f_i(z) \overline{f_j(w)}}{(1 - w\overline{z})^2} \, dA(z) \, dA(w)$$

$$= \frac{1}{r_i r_j} \iint_{\Delta_i \times \Delta_j} \frac{1}{(1 - w\overline{z})^2} \, dA(z) \, dA(w).$$

Lemma 3.5. *We have*

$$m_{i,i} \geq \frac{\varepsilon_i^2}{32}, \quad and \quad |m_{i,j}| \leq \varepsilon_i \varepsilon_j \delta^{j-i} \quad for \ i < j. \tag{3.11}$$

Proof. Set $\varepsilon_i' = \frac{r_i}{1-c_i^2} = \frac{\varepsilon_i}{2(1+c_i)}$. One has $\frac{\varepsilon_i}{4} \leq \varepsilon_i' \leq \frac{\varepsilon_i}{2}$. We observe that (recall that $A(\Delta_i) = r_i^2$):

$$m_{i,i} - \varepsilon_i'^2 = \frac{1}{r_i^2} \iint\limits_{\Delta_i \times \Delta_i} \left[\frac{1}{(1-w\bar{z})^2} - \frac{1}{(1-c_i^2)^2} \right] dA(z) \, dA(w).$$

Therefore, using the fact that, for $z \in \Delta_i$ and $w \in \mathbb{D}$:

$$|1 - w\bar{z}| \geq 1 - |z| \geq 1 - c_i - r_i = 1 - c_i - \varepsilon_i \left(\frac{1-c_i}{2} \right) \geq (1-c_i)\left(1 - \frac{\varepsilon_i}{2} \right) \geq \frac{1-c_i}{2}$$

and then the mean-value theorem, we get:

$$\begin{aligned}
\left| m_{i,i} - \varepsilon_i'^2 \right| &\leq \frac{1}{r_i^2} \iint\limits_{\Delta_i \times \Delta_i} \left| \frac{1}{(1-w\bar{z})^2} - \frac{1}{(1-c_i^2)^2} \right| dA(z) \, dA(w) \\
&\leq \frac{1}{r_i^2} \iint\limits_{\Delta_i \times \Delta_i} \frac{32 r_i}{(1-c_i)^3} \, dA(z) \, dA(w) \\
&= \frac{32 r_i^3}{(1-c_i)^3} \leq 32 \times 8 \varepsilon_i'^3 \leq \frac{\varepsilon_i'^2}{2},
\end{aligned}$$

since $\varepsilon_i \leq \varepsilon_1 \leq 2^{-8}$ implies that $\varepsilon_i' \leq 1/(32 \times 16)$. This gives us the lower bound $m_{i,i} \geq \varepsilon_i'^2/2 \geq \varepsilon_i^2/32$.

Next, for $i < j$:

$$\begin{aligned}
|m_{i,j}| &\leq \frac{1}{r_i r_j} \iint\limits_{\Delta_i \times \Delta_j} \left| \frac{1}{(1-w\bar{z})^2} \right| dA(z) \, dA(w) \leq \frac{1}{r_i r_j} \frac{4}{(1-c_i)^2} r_i^2 r_j^2 \\
&= \frac{4 \varepsilon_i \varepsilon_j \delta^{i+j}}{4 \delta^{2i}} = \varepsilon_i \varepsilon_j \delta^{j-i},
\end{aligned}$$

and that ends the proof of Lemma 3.5.   □

We further write the $n \times n$ matrix $M = (m_{i,j})_{1 \leq i,j \leq n}$ as $M = D + R$ where D is the diagonal matrix $m_i = m_{i,i}$ with $m_i \geq \frac{\varepsilon_i^2}{32}$, $1 \leq i \leq n$. Observe that M is nothing but the matrix of S_μ on the orthonormal basis $(f_1, \ldots, f_n)$ of E, so that we can identify M and S_μ on E.

Now the following lemma will end the proof of Theorem 3.1.

Lemma 3.6. *If $\delta \leq 1/200$, we have:*

$$\left\| D^{-1} R \right\| \leq 1/2. \tag{3.12}$$

Indeed, by the ideal property of Bernstein numbers, Neumann's lemma and the relations:

$$M = D(I + D^{-1}R), \quad and \quad D = MQ \quad with \ \|Q\| \leq 2,$$

we have $b_n(D) \leq b_n(M)\|Q\| \leq 2b_n(M)$, that is:

Doctopic: Functional Analysis
ARTICLE IN PRESS
YJMAA:19189

D. Li et al. / J. Math. Anal. Appl. ••• (••••) •••–•••
9

$$a_n(S_\mu) = b_n(S_\mu) \geq b_n(M) \geq \frac{b_n(D)}{2} = \frac{m_{n,n}}{2} \geq \frac{\varepsilon_n^2}{64},$$

since the n first approximation numbers of the diagonal matrix D (the matrices being viewed as well as operators on the Hilbertian space $\mathbb{C}^n$ with its canonical basis) are $m_{1,1}, \ldots, m_{n,n}$. It follows that, using (3.6):

$$a_n(I_\mu) = a_n(I_\mu^*) = \sqrt{a_n(S_\mu)} \geq \frac{\varepsilon_n}{8}. \tag{3.13}$$

In view of (3.6), we have as well $a_n(C_\varphi) \geq \varepsilon_n/8$, and we are done. $\quad\square$

Proof of Lemma 3.6. Write $M = (m_{i,j}) = D(I + N)$ with $N = D^{-1}R$. One has:

$$N = (\nu_{i,j}), \quad \text{with } \nu_{i,i} = 0 \text{ and } \nu_{i,j} = \frac{m_{i,j}}{m_{i,i}} \text{ for } j \neq i. \tag{3.14}$$

We shall show that $\|N\| \leq 1/2$ by using the (unweighted) Schur test, which we recall [5, Problem 45]:

Proposition 3.7. *Let $(a_{i,j})_{1 \leq i,j \leq n}$ be a matrix of complex numbers. Suppose that there exist two positive numbers $\alpha, \beta > 0$ such that:*

1. $\sum_{j=1}^{n} |a_{i,j}| \leq \alpha$ *for all i;*
2. $\sum_{i=1}^{n} |a_{i,j}| \leq \beta$ *for all j.*

Then, the (Hilbertian) norm of this matrix satisfies $\|A\| \leq \sqrt{\alpha\beta}$.

It is essential for our purpose to note that:

$$i < j \quad \Longrightarrow \quad |\nu_{i,j}| \leq 32\delta^{j-i}, \tag{3.15}$$

$$i > j \quad \Longrightarrow \quad |\nu_{i,j}| \leq 32(2\delta)^{i-j}. \tag{3.16}$$

Indeed, we see from (3.11) and (3.14) that, for $i < j$:

$$|\nu_{i,j}| = \frac{|m_{i,j}|}{m_{i,i}} \leq 32\varepsilon_i\varepsilon_j\varepsilon_i^{-2}\delta^{j-i} \leq 32\delta^{j-i}$$

since $\varepsilon_j \leq \varepsilon_i$. Secondly, using $\varepsilon_j/\varepsilon_i \leq 2^{i-j}$ for $i > j$ (recall that we assumed that $\varepsilon_{k+1} \geq \varepsilon_k/2$), as well as $|m_{i,j}| = |m_{j,i}|$, we have, for $i > j$:

$$|\nu_{i,j}| = \frac{|m_{j,i}|}{m_{i,i}} \leq 32\frac{\varepsilon_j}{\varepsilon_i}\delta^{i-j} \leq 32(2\delta)^{i-j}.$$

Now, for fixed i, (3.15) gives:

$$\sum_{j=1}^{n} |\nu_{i,j}| = \sum_{j>i} |\nu_{i,j}| + \sum_{j<i} |\nu_{i,j}| \leq 32\left(\sum_{j>i} \delta^{j-i} + \sum_{j<i} (2\delta)^{i-j}\right)$$

$$\leq 32\left(\frac{\delta}{1-\delta} + \frac{2\delta}{1-2\delta}\right) \leq 32\frac{3\delta}{1-2\delta} \leq \frac{96}{198} \leq \frac{1}{2},$$

since $\delta \leq 1/200$. Hence:

Doctopic: Functional Analysis

ARTICLE IN PRESS

YJMAA:19189

10

D. Li et al. / J. Math. Anal. Appl. • • • (• • • •) • • • – • • •

$$\sup_i \left(\sum_j |\nu_{i,j}| \right) \leq 1/2. \tag{3.17}$$

In the same manner, but using (3.16) instead of (3.15), one has:

$$\sup_j \left(\sum_i |\nu_{i,j}| \right) \leq 1/2. \tag{3.18}$$

Now, (3.17), (3.18) and the Schur criterion recalled above give:

$$\|N\| \leq \sqrt{1/2 \times 1/2} = 1/2,$$

as claimed. □

Remark. We could reverse the point of view in the preceding proof: start from θ and see what lower bound for $a_n(C_\varphi)$ emerges. For example, if $\theta(h) \approx h$ as is the case for lens maps (see [8]), we find again that $a_n(C_\varphi) \geq \delta_0 > 0$ and that C_φ is not compact. But if $\theta(h) \approx h^{1+\alpha}$ with $\alpha > 0$, the method only gives $a_n(C_\varphi) \gtrsim e^{-\alpha n}$ (which is always true: see [8, Theorem 2.1]), whereas the methods of [8] easily give $a_n(C_\varphi) \gtrsim e^{-\alpha\sqrt{n}}$. Therefore, this μ-method seems to be sharp when we are close to non-compactness, and to be beaten by those of [8] for "strongly compact" composition operators.

3.1. Optimality of the EKSY result

El-Fallah, Kellay, Shabankhah and Youssfi proved in [4] the following: if φ is a Schur function such that $\varphi \in \mathcal{D}$ and $\|\varphi^p\|_{\mathcal{D}} = O(1)$ as $p \to \infty$, then φ is a symbol on $\mathcal{D}$. We have the following theorem, already stated in the Introduction, which shows the optimality of their result.

Theorem 3.8. *Let $(M_p)_{p \geq 1}$ be an arbitrary sequence of positive numbers such that $\lim_{p \to \infty} M_p = \infty$. Then, there exists a Schur function $\varphi \in \mathcal{D}$ such that:*

1) *$\|\varphi^p\|_{\mathcal{D}} = O(M_p)$ as $p \to \infty$;*
2) *φ is not a symbol on $\mathcal{D}$.*

Remark. We first observe that we cannot replace lim by lim sup in Theorem 3.8. Indeed, since $\varphi \in \mathcal{D}$, the measure $\mu = n_\varphi \, dA$ is finite, and

$$\|\varphi^p\|_{\mathcal{D}}^2 = p^2 \int_{\mathbb{D}} |w|^{2p-2} \, d\mu(w) \geq cp^2 \left(\int_{\mathbb{D}} |w|^2 \, d\mu(w) \right)^{p-1} \geq c\delta^p,$$

where c and δ are positive constants.

Proof of Theorem 3.8. We may, and do, assume that (M_p) is non-decreasing and integer-valued. Let $(l_n)_{n \geq 1}$ be a non-decreasing sequence of positive integers tending to infinity, to be adjusted. Let Ω be the subdomain of the right half-plane $\mathbb{C}_0$ defined as follows. We set:

$$\varepsilon_n = -\log(1 - 2^{-n}) \sim 2^{-n},$$

and we consider the (essentially) disjoint boxes $(k = 0, 1, \ldots)$:

$$B_{k,n} = B_{0,n} + 2k\pi i,$$

Doctopic: Functional Analysis

ARTICLE IN PRESS

YJMAA:19189

D. Li et al. / J. Math. Anal. Appl. • • • (• • • •) • • • – • • •

11

with:

$$B_{0,n} = \left\{ u \in \mathbb{C};\ \varepsilon_{n+1} \leq \mathfrak{Re}\, u \leq \varepsilon_n \text{ and } |\mathfrak{Im}\, u| \leq 2^{-n}\pi \right\},$$

as well as the union

$$T_n = \bigcup_{0 < k < l_n} B_{k,2n},$$

which is a kind of broken tower above the "basis" $B_{0,2n}$ of even index.

We also consider, for $1 \leq k \leq l_n - 1$, very thin vertical pipes $P_{k,n}$ connecting $B_{k,2n}$ and $B_{k-1,2n}$, of side lengths 4^{-2n} and $2\pi(1 - 2^{-2n})$ respectively:
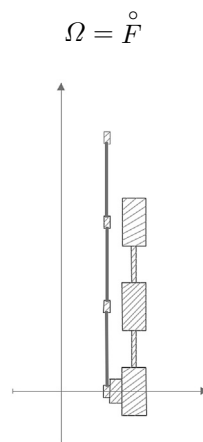
$$P_{k,n} = P_{0,n} + 2k\pi i,$$

and we set:

$$P_n = \bigcup_{1 \leq k < l_n} P_{k,n}$$

Finally, we set:

$$F = \left(\bigcup_{n=2}^{\infty} B_{0,n} \right) \cup \left(\bigcup_{n=1}^{\infty} T_n \right) \cup \left(\bigcup_{n=1}^{\infty} P_n \right)$$

and:

$$\Omega = \overset{\circ}{F}$$

Then Ω is a simply connected domain. Indeed, it is connected thanks to the $B_{0,n}$ and the P_n, since the $P_{k,n}$ were added to ensure that. Secondly, its unbounded complement is connected as well, since we take one value of n out of two in the union of sets $B_{k,n}$ defining F.

Let now $f \colon \mathbb{D} \to \Omega$ be a Riemann map, and $\varphi = e^{-f} \colon \mathbb{D} \to \mathbb{D}$.

We introduce the Carleson window $W = W(1, h)$ defined as:

$$W(1, h) = \left\{ z \in \mathbb{D};\ 1 - h \leq |z| < 1 \text{ and } |\arg z| < \pi h \right\}.$$

This is a variant of the sets $S(1, h)$ of Section 2. We also introduce the Hastings–Luecking half-windows W'_n defined by:

$$W_n' = \big\{z \in \mathbb{D};\ 1 - 2^{-n} < |z| < 1 - 2^{-n-1} \text{ and } |\arg z| < \pi 2^{-n}\big\}.$$

We will also need the sets:

$$E_n = \mathrm{e}^{-(T_n \cup B_{0,2n+1} \cup P_n)} = \mathrm{e}^{-(B_{0,2n} \cup B_{0,2n+1} \cup P_{0,n})},$$

for which one has:

$$\varphi(\mathbb{D}) \subseteq \bigcup_{n=1}^{\infty} E_n.$$

Next, we consider the measure $\mu = n_\varphi \, dA$, and a Carleson window $W = W(1, h)$ with $h = 2^{-2N}$. We observe that $W_{2N}' \subseteq W$ and claim that:

Lemma 3.9. *One has:*

1) $w \in W_{2N}' \implies n_\varphi(w) \geq l_N$;
2) $\big\| \varphi^p \big\|_{\mathcal{D}}^2 \lesssim p^2 \sum_{n=1}^{\infty} l_n 16^{-n} \mathrm{e}^{-p4^{-n}}$.

Proof. 1) Let $w = re^{i\theta} \in W_{2N}'$ with $1 - 2^{-2N} < r < 1 - 2^{-2N-1}$ and $|\theta| < \pi 2^{-2N}$. As $-(\log r + i\theta) \in B_{0,2N}$, one has $-(\log r + i\theta) = f(z_0)$ for some $z_0 \in \mathbb{D}$. Similarly, $-(\log r + i\theta) + 2k\pi i$, for $1 \leq k < l_N$, belongs to $B_{k,2N}$ and can be written as $f(z_k)$, with $z_k \in \mathbb{D}$. The z_k's, $0 \leq k < l_N$, are distinct and satisfy $\varphi(z_k) = \mathrm{e}^{-f(z_k)} = \mathrm{e}^{-f(z_0)} = w$ for $0 \leq k < l_N$, thanks to the $2\pi i$-periodicity of the exponential function.

2) We have $A(E_n) \lesssim \mathrm{e}^{-2\varepsilon_{2n+2}} 4^{-2n} \leq 4^{-2n}$ (the term $\mathrm{e}^{-2\varepsilon_{2n+2}}$ coming from the Jacobian of e^{-z}) and we observe that

$$w \in E_n \implies |w|^{2p-2} \leq \big(1 - 2^{-2n-1}\big)^{2p-2} \lesssim \mathrm{e}^{-p4^{-n}}.$$

It is easy to see that $n_\varphi(w) \leq l_n$ for $w \in E_n$; thus we obtain, forgetting the constant term $|\varphi(0)|^{2p} \leq 1$, using (2.5) and keeping in mind the fact that $n_\varphi(w) = 0$ for $w \notin \varphi(\mathbb{D})$:

$$\big\| \varphi^p \big\|_{\mathcal{D}}^2 = p^2 \int_{\varphi(\mathbb{D})} |w|^{2p-2} n_\varphi(w) \, dA(w)$$

$$\leq p^2 \left(\sum_{n=1}^{\infty} \int_{E_n} |w|^{2p-2} n_\varphi(w) \, dA(w) \right)$$

$$\leq p^2 \left(\sum_{n=1}^{\infty} \int_{E_n} |w|^{2p-2} l_n \, dA(w) \right)$$

$$\lesssim p^2 \sum_{n=1}^{\infty} l_n 16^{-n} \mathrm{e}^{-p4^{-n}},$$

ending the proof of Lemma 3.9. $\quad\square$

End of the proof of Theorem 3.8. Note that, as a consequence of the first part of the proof of Lemma 3.9, one has

$$\mu(W) \geq \mu\big(W_{2N}'\big) = \int_{W_{2N}'} n_\varphi \, dA \geq l_N A\big(W_{2N}'\big) \gtrsim l_N h^2,$$

Doctopic: Functional Analysis

ARTICLE IN PRESS

YJMAA:19189

D. Li et al. / J. Math. Anal. Appl. • • • (• • • •) • • • − • • •

13

which implies that $\sup_{0<h<1} h^{-2}\mu[W(1,h)] = +\infty$ and shows that C_φ is not bounded on $\mathcal{D}$ by Zorboska's criterion [12, Theorem 1], recalled in (2.7).

It remains now to show that we can adjust the non-decreasing sequence of integers (l_n) so as to have $\|\varphi^p\|_\mathcal{D} = O(M_p)$. To this effect, we first observe that, if one sets $F(x) = x^2 \mathrm{e}^{-x}$, we have:

$$p^2 \sum_{n=1}^\infty 16^{-n} \mathrm{e}^{-p4^{-n}} = \sum_{n=1}^\infty F\left(\frac{p}{4^n}\right) \lesssim 1.$$

Indeed, let s be the integer such that $4^s \le p < 4^{s+1}$. We have:

$$\sum_{n=1}^\infty F\left(\frac{p}{4^n}\right) \lesssim \sum_{n=1}^s \frac{4^n}{p} + \sum_{n>s} F\left(4^{-(n-s-1)}\right) \lesssim 1 + \sum_{n=0}^\infty F\left(4^{-n}\right) < \infty,$$

where we used that F is increasing on $(0,1)$ and satisfies $F(x) \lesssim \min(x^2, 1/x)$ for $x > 0$. We finally choose the non-decreasing sequence (l_n) of integers as:

$$l_n = \min\left(n, M_n^2\right).$$

In view of Lemma 3.9 and of the previous observation, we obtain:

$$\left\|\varphi^p\right\|_\mathcal{D}^2 \lesssim p^2 \sum_{n=1}^\infty 16^{-n} \mathrm{e}^{-p4^{-n}} l_n$$

$$\le p^2 \sum_{n=1}^p 16^{-n} \mathrm{e}^{-p4^{-n}} l_p + p^2 \sum_{n>p} 16^{-n} l_n$$

$$\lesssim l_p + p^2 \sum_{n>p} 4^{-n} \lesssim l_p + p^2 4^{-p} \lesssim M_p^2,$$

as desired. This choice of (l_n) gives us an unbounded composition operator on $\mathcal{D}$ such that $\|\varphi^p\|_\mathcal{D} = O(M_p)$, which ends the proof of Theorem 3.8. $\square$

References

[1] N. Arcozzi, R. Rochberg, E.T. Sawyer, B.D. Wick, The Dirichlet space: a survey, New York J. Math. 17A (2011) 45–86.
[2] T. Carroll, C. Cowen, Compact composition operators not in the Schatten classes, J. Operator Theory 26 (1991) 109–120.
[3] C. Cowen, B. MacCluer, Composition Operators on Spaces of Analytic Functions, CRC Press, 1994.
[4] O. El-Fallah, K. Kellay, M. Shabankhah, H. Youssfi, Level sets and composition operators on the Dirichlet space, J. Funct. Anal. 260 (6) (2011) 1721–1733.
[5] P. Halmos, A Hilbert Space Problem Book, second edition, Grad. Texts in Math., vol. 19, Springer-Verlag, 1982.
[6] W.H. Hastings, A Carleson theorem for Bergman spaces, Proc. Amer. Math. Soc. 52 (1975) 237–241.
[7] P. Lefèvre, D. Li, H. Queffélec, L. Rodríguez-Piazza, Compact composition operators on the Dirichlet space and capacity of sets of contact points, J. Funct. Anal. 264 (4) (2013) 895–919.
[8] P. Lefèvre, D. Li, H. Queffélec, L. Rodríguez-Piazza, Approximation numbers of composition operators on the Dirichlet space, Ark. Mat. (2014), http://dx.doi.org/10.1007/s11512-013-0194-z.
[9] D. Li, H. Queffélec, L. Rodríguez-Piazza, On approximation numbers of composition operators, J. Approx. Theory 164 (4) (2012) 431–459.
[10] B. MacCluer, J. Shapiro, Angular derivatives and compact composition operators on the Hardy and Bergman spaces, Canad. J. Math. 38 (4) (1986) 878–906.
[11] W.T. Ross, The classical Dirichlet space, in: Recent Advances in Operator-Related Function Theory, in: Contemp. Math., vol. 393, Amer. Math. Soc., Providence, RI, 2006, pp. 171–197.
[12] N. Zorboska, Composition operators on weighted Dirichlet spaces, Proc. Amer. Math. Soc. 126 (7) (1998) 2013–2023.