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A note on the BMO-Teichmüller space

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ABSTRACT

Astala–Zinsmeister [2] introduced a new topology on the set of all strongly quasisymmetric homeomorphisms of the unit circle denoted by $\mathrm{SQS}(S^1)$ by means of BMO norm. Under this new topology, we prove that $\mathrm{SQS}(S^1)$ is a partial topological group, and the characteristic topological subgroup of $\mathrm{SQS}(S^1)$ consists of strongly symmetric homeomorphisms.

1. Introduction

A sense preserving self-homeomorphism h of the unit circle S^1 is quasisymmetric if there exists a constant $C(h) > 0$ such that

$$|h(I^*)| \leqslant C(h)|h(I)|$$

for any interval $I \subset S^1$ with $|I| \leqslant \pi$, where I^* is the interval with same center as I but with double length and $|\cdot|$ denotes the Lebesgue measure. Beurling–Ahlfors [3] proved that a sense preserving self-homeomorphism h is quasisymmetric if and only if there exists some quasiconformal homeomorphism of the unit disk Δ onto itself which has boundary values h. Later Douady–Earle [7] gave a quasiconformal extension of h to the unit disk Δ which is also conformally invariant and bi-Lipschitz continuous for the hyperbolic metric.

The universal Teichmüller space T is a universal parameter space for all Riemann surfaces and one of its models can be defined as the space of all normalized quasisymmetric homeomorphisms on the unit circle S^1, namely, $T = \mathrm{QS}(S^1)/\mathrm{Möb}(S^1)$. Here $\mathrm{QS}(S^1)$ denotes the group of all quasisymmetric homeomorphisms of the unit circle S^1, and $\mathrm{Möb}(S^1)$ the subgroup of Möbius transformations of the unit disk Δ. It is known that the universal Teichmüller space plays a significant role in Teichmüller theory, and it is also a fundamental object in mathematics and in mathematical physics. In addition, several subclasses of quasisymmetric

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homeomorphisms and their Teichmüller spaces were introduced and studied for various purposes in the literature. We refer to the books [1,13,14,16] and the papers [2,4,9,11,19,20] for an introduction to the subject and more details. In the following we shall list two of them, which were introduced and investigated in our recent paper [19].

A quasisymmetric homeomorphism h is said to be strongly quasisymmetric if for each $\varepsilon > 0$ there is a $\delta > 0$ such that

$$|E| \leqslant \delta|I| \Rightarrow |h(E)| \leqslant \varepsilon h(I)$$

whenever $I \subset S^1$ is an interval and $E \subset I$ a measurable subset. In other words, h is strongly quasisymmetric if and only if h is absolutely continuous with density h' belonging to the class of weights A^∞ introduced by Muckenhoupt (see [10]), in particular, $\log h' \in \mathrm{BMO}(S^1)$, the space of integrable functions on the unit circle S^1 of bounded mean oscillation (see [10]). Cui–Zinsmeister [4] proved that the complex dilatation of the Douady–Earle extension of a strongly quasisymmetric homeomorphism produces a Carleson measure $\lambda_\mu(z) = |\mu(z)|^2/(1-|z|) \in \mathrm{CM}(\Delta)$. This sub-class of quasisymmetric homeomorphisms was much investigated because of its great importance in the application to harmonic analysis (see [5,8,12,18]). Let $\mathrm{SQS}(S^1)$ denote the set of all strongly quasisymmetric homeomorphisms of the unit circle S^1. Then $T_b = \mathrm{SQS}(S^1)/\mathrm{Möb}(S^1)$ is a model of the BMOA-Teichmüller space. We say a quasisymmetric homeomorphism h is strongly symmetric if it is absolutely continuous such that $\log h' \in \mathrm{VMO}(S^1)$, the space of integrable functions on the unit circle S^1 of vanishing mean oscillation (see [10,16,17,21]). We denote by $\mathrm{SS}(S^1)$ the set of all strongly symmetric homeomorphisms of the unit circle S^1. Then $T_v = \mathrm{SS}(S^1)/\mathrm{Möb}(S^1)$ is a model of the VMOA-Teichmüller space.

Recall that a partial topological group is a group with a neighborhood system at the identity which is respected by composition and inverse in the following sense: given any two mappings h_1 and h_2 near to the identity, then the product $h_1 \circ h_2$ and the inverse h_1^{-1} are also near to the identity (see [9] and also see Section 3 for details). For any partial topological group the subgroup of elements f for which the adjoint action by f is continuous is called the characteristic topological subgroup (see [9]). We know that the adjoint action in $\mathrm{QS}(S^1)$ is not continuous at the identity, so the group $\mathrm{QS}(S^1)$ is not a topological group. However, Gardiner–Sullivan [9] proved that it is a partial topological group, and the characteristic topological subgroup $\mathrm{S}(S^1)$ of the group $\mathrm{QS}(S^1)$, in the Teichmüller metric, consists of those mappings which have vanishing ratio distortion.

It is known that $\mathrm{SS}(S^1)$ is a subgroup of $\mathrm{S}(S^1)$. So $\mathrm{SS}(S^1)$ is a topological group for the Teichmüller metric. Astala–Zinsmeister [2] introduced a new topology in $\mathrm{SQS}(S^1)$ via $d(h,k) = \|\log(h') - \log(k')\|_*$, where $\|\cdot\|_*$ denotes the BMO norm. In the following we call it the A–Z topology. In our previous paper [19], we showed that the A–Z topology is stronger than the topology induced by Teichmüller metric. A natural problem is whether $\mathrm{SS}(S^1)$ is a topological group under the A–Z topology.

The purpose of this short paper is to follow the same idea as Gardiner–Sullivan [9] to prove that $\mathrm{SQS}(S^1)$ is a partial topological group, and $\mathrm{SS}(S^1)$ is the characteristic topological subgroup of $\mathrm{SQS}(S^1)$ for the A–Z topology.

2. Some lemmas

In this section, we recall some basic results which will be needed in the following sections. Here and in what follows, C will denote a positive constant which may vary from line to line.

Let $M(\Delta)$ denote the open unit ball of the Banach space $L^\infty(\Delta)$ of essentially bounded measurable functions in Δ. For any $\mu \in M(\Delta)$, there is a unique quasiconformal map f_μ of Δ whose complex dilatation is μ and normalized by fixing 1, -1 and i. Denote by $\sigma(\mu)$ the Beltrami coefficient of the Douady–Earle extension $E(f_\mu|_{S^1})$. Let $\rho(\mu)$ and $\chi(\mu)$ be the Beltrami coefficients of $E(f_\mu^{-1}|_{S^1})^{-1}$ and $E(f_\mu^{-1}|_{S^1})$, respectively.

Lemma 2.1. *Let f be a conformal mapping in Δ and $h = f^{-1} \circ g$ be the corresponding quasisymmetric conformal welding. Then the following statements are equivalent:*

(1) $\log f' \in \mathrm{BMOA}(\Delta)$ *with a small norm,*
(2) $f(\partial\Delta)$ *is a Lavrentiev curve with a small norm,*
(3) $\log h' \in \mathrm{BMO}(S^1)$ *with a small norm,*
(4) f *has a quasiconformal extension with complex dilatation μ such that $|\mu|^2/(|z|-1)$ is a Carleson measure with a small norm.*

Remark 2.1. The equivalence (1) ⇔ (2) is due to Pommerenke [15]. (3) ⇔ (4) was proved by David [6]. (4) ⇒ (2), (2) ⇒ (3) and (2) ⇒ (1) were proved by Semmes [18]. (2) ⇒ (1) was also proved by Astala–Zinsmeister [2]. The above lemma implies that at least in a neighborhood of the origin BMO-Teichmüller theory deals with bi-Lipschitz geometry.

Lemma 2.2. *(See [18].) Let f be a quasiconformal map of the unit disk Δ onto itself that satisfies*

(1) $f|_{S^1} \in \mathrm{SQS}(S^1)$,
(2) f *is bi-Lipschitz continuous for the hyperbolic metric in the unit disk Δ.*

If $\lambda \in \mathrm{CM}(\Delta)$, then $\lambda \circ f|\partial f| \in \mathrm{CM}(\Delta)$, and $\|\lambda \circ f|\partial f|\|_c \leqslant C\|\lambda\|_c$.

Lemma 2.3. *(See [4].) Let $(1+\|\mu\|_\infty)/(1-\|\mu\|_\infty) \leqslant K$. Then for any $w \in \Delta$,*

$$\frac{|\rho(\mu)(w)|^2}{1-|\rho(\mu)(w)|^2} \leqslant C \iint\limits_{\Delta} \frac{|\mu(z)|^2}{1-|\mu(z)|^2} \frac{(1-|w|^2)^2}{|1-\bar{w}z|^4} dxdy.$$

Lemma 2.4. *(See [19].) Let $\alpha > 0$, $\beta > 0$. For a positive measure λ in Δ, set*

$$\tilde{\lambda}(z) = \iint\limits_{\Delta} \frac{(1-|z|^2)^\alpha (1-|w|^2)^\beta}{|1-\bar{z}w|^{\alpha+\beta+2}} \lambda(w) dudv. \tag{2.1}$$

Then $\tilde{\lambda} \in \mathrm{CM}(\Delta)$ if $\lambda \in \mathrm{CM}(\Delta)$, and $\|\tilde{\lambda}\|_c \leqslant C\|\lambda\|_c$, while $\tilde{\lambda} \in \mathrm{CM}_0(\Delta)$ if $\lambda \in \mathrm{CM}_0(\Delta)$.

For a quasisymmetric homeomorphism h, the kernel function

$$\phi_h(\zeta, z) = \frac{1}{2\pi i} \int\limits_{S^1} \frac{h(w)}{(1-\zeta w)^2(1-zh(w))} dw, \quad (\zeta, z) \in \Delta \times \Delta$$

was introduced in [11]. Set

$$\phi_h(z) = \Big(\frac{1}{\pi} \iint\limits_{\Delta} |\phi_h(\zeta, z)|^2 \, d\xi d\eta\Big)^{\frac{1}{2}}, \quad z \in \Delta.$$

Hu–Shen [11] proved that

Lemma 2.5. *Let $E(h)$ denote the Douady–Earle extension of h, and $\nu(h)$ denote the Beltrami coefficient of the inverse mapping $E^{-1}(h)$. Then for any $w \in \Delta$,*

$$\frac{|\nu(h)(w)|^2}{1-|\nu(h)(w)|^2} \leq C\phi_h^2(\bar{w})(1-|w|^2)^2.$$

Lemma 2.6. *(See [4].) Let g be bi-Lipschitz for the hyperbolic metric in the unit disk Δ, and μ be the complex dilatation of g. If*

$$\lambda_\mu(z) = \frac{|\mu(z)|^2}{1-|z|} \in \mathrm{CM}(\Delta),$$

then the same is true for $\lambda_{\mu^{-1}}$, and $\|\lambda_{\mu^{-1}}\|_c \leqslant C\|\lambda_\mu\|_c$, while $\lambda_{\mu^{-1}} \in \mathrm{CM}_0(\Delta)$ if $\lambda_\mu \in \mathrm{CM}_0(\Delta)$, where μ^{-1} is defined through g^{-1}.

Lemma 2.7. *Let $(1+\|\mu\|_\infty)/(1-\|\mu\|_\infty) \leqslant K$. If*

$$\lambda_\mu(z) = \frac{|\mu(z)|^2}{1-|z|} \in \mathrm{CM}(\Delta),$$

then the same is true for $\lambda_{\sigma(\mu)}$, and $\|\lambda_{\sigma(\mu)}\|_c \leqslant C\|\lambda_\mu\|_c$, while $\lambda_{\sigma(\mu)} \in \mathrm{CM}_0(\Delta)$ if $\lambda_\mu \in \mathrm{CM}_0(\Delta)$.

Proof. The first statement was proved by Cui–Zinsmeister [4] by means of real and harmonic analysis theory. We will give a new and brief proof by above some lemmas. On the other hand, we also give the proof of the second statement.

By Lemma 2.3,

$$\frac{|\rho(\mu)(w)|^2}{1-|\rho(\mu)(w)|^2} \leqslant C \iint\limits_{\Delta} \frac{|\mu(z)|^2}{1-|\mu(z)|^2} \frac{(1-|w|^2)^2}{|1-\bar{w}z|^4} dxdy.$$

Then

$$\frac{|\rho(\mu)(w)|^2}{1-|w|} \leqslant C \iint\limits_{\Delta} \frac{|\mu(z)|^2}{1-|z|^2} \frac{(1-|w|^2)(1-|z|^2)}{|1-\bar{w}z|^4} dxdy.$$

It follows from Lemma 2.4 that $\lambda_{\rho(\mu)} \in \mathrm{CM}(\Delta)$, and $\|\lambda_{\rho(\mu)}\|_c \leqslant C\|\lambda_\mu\|_c$. It is known that $f_\mu|_{S^1} \in \mathrm{SQS}(S^1)$ if $\lambda_\mu \in \mathrm{CM}(\Delta)$, so $f_\mu^{-1}|_{S^1} \in \mathrm{SQS}(S^1)$. Combining the bi-Lipschitz continuity of $E(f_\mu^{-1}|_{S^1})$ for the hyperbolic metric, Lemma 2.2 implies that $\lambda_{\rho(\mu)} \circ E(f_\mu^{-1}|_{S^1})|\partial E(f_\mu^{-1}|_{S^1})| \in \mathrm{CM}(\Delta)$, and $\|\lambda_{\rho(\mu)} \circ E(f_\mu^{-1}|_{S^1})|\partial E(f_\mu^{-1}|_{S^1})|\|_c \leqslant C\|\lambda_{\rho(\mu)}\|_c$. On the other hand,

$$\begin{aligned}
\lambda_{\rho(\mu)} \circ E(f_\mu^{-1}|_{S^1})|\partial E(f_\mu^{-1}|_{S^1})| &= \frac{|\rho(\mu)\circ E(f_\mu^{-1}|_{S^1})(z)|^2}{1-|E(f_\mu^{-1}|_{S^1})|}|\partial E(f_\mu^{-1}|_{S^1})| \\
&= \frac{|\chi(\mu)(z)|^2}{1-|E(f_\mu^{-1}|_{S^1})|}\frac{|dE(f_\mu^{-1}|_{S^1})(z)|}{|dE(f_\mu^{-1}|_{S^1})(z)|}|\partial E(f_\mu^{-1}|_{S^1})| \\
&\geqslant C|\chi(\mu)(z)|^2 \frac{|dz|}{1-|z|}\frac{|\partial E(f_\mu^{-1}|_{S^1})|}{|dE(f_\mu^{-1}|_{S^1})(z)|} \\
&\geqslant C\frac{|\chi(\mu)(z)|^2}{1-|z|}\frac{|dz||\partial E(f_\mu^{-1}|_{S^1})|}{|dz||\partial E(f_\mu^{-1}|_{S^1})|(1+\|\chi(\mu)\|_\infty)} \\
&= C\frac{|\chi(\mu)(z)|^2}{1-|z|}.
\end{aligned}$$

That is, $\lambda_{\chi(\mu)} \in \mathrm{CM}(\Delta)$, and $\|\lambda_{\chi(\mu)}\|_c \leqslant C\|\lambda_{\rho(\mu)} \circ E(f_\mu^{-1}|_{S^1})|\partial E(f_\mu^{-1}|_{S^1})|\|_c$. Note that $\sigma(\mu) = \chi\circ\chi(\mu)$. This implies $\lambda_{\sigma(\mu)} \in \mathrm{CM}(\Delta)$, and $\|\lambda_{\sigma(\mu)}\|_c \leqslant C\|\lambda_\mu\|_c$. Consequently, the first statement is proved.

For $\lambda_\mu \in \mathrm{CM}_0(\Delta)$, denote by $h = f_\mu|_{S^1}$. By Theorem 4.1 in our previous paper [19], $h \in \mathrm{SS}(S^1)$ and $\phi_h^2(\bar{w})(1-|w|^2) \in \mathrm{CM}_0(\Delta)$. Then by Lemma 2.5, we have

$$\frac{|\nu(h)(w)|^2}{1-|w|} \in \mathrm{CM}_0(\Delta).$$

By the bi-Lipschitz continuity of $w = E(h)(z)$ for the hyperbolic metric and Lemma 2.6,

$$\frac{|\sigma(\mu)(z)|^2}{1-|z|} \in \mathrm{CM}_0(\Delta).$$

Namely, $\lambda_{\sigma(\mu)} \in \mathrm{CM}_0(\Delta)$. This completes the proof of Lemma 2.7. $\square$

Lemma 2.8. *(See [10].) For a function $\varphi \in \mathrm{BMO}(\mathbb{R})$, the following conditions are equivalent:*

(1) $\varphi \in \mathrm{VMO}(\mathbb{R})$.
(2) *If $\varphi_x(t) = \varphi(t-x)$ is the translation of φ by x units, then $\lim_{x\to 0} \|\varphi_x - \varphi\|_* = 0$,*

where $\|\cdot\|_*$ denotes the BMO norm.

Lemma 2.9. *(See [9].) The characteristic topological subgroup of partial topological group is a closed topological subgroup.*

3. Strongly quasisymmetric homeomorphisms

In this section, we shall prove that the A–Z topology induces a structure of partial topological group in $\mathrm{SQS}(S^1)$.

First we introduce two systems of neighborhoods of the identity in $\mathrm{SQS}(S^1)$ in the following way. Let the neighborhood $U(\epsilon)$ consist of all strongly quasisymmetric maps h for which

(1) $\sup_{|z|=1}\{|h(z)-z|, |h^{-1}(z)-z|\} < \epsilon$, and
(2) h is absolutely continuous with $\log h' \in \mathrm{BMO}(S^1)$, and $\|\log h'\|_* < \epsilon$.

Let the neighborhood $V(\epsilon)$ consist of all strongly quasisymmetric maps h for which

(1a) $\sup_{|z|=1}\{|h(z)-z|, |h^{-1}(z)-z|\} < \epsilon$, and
(2a) h has a quasiconformal extension with complex dilatation μ such that $|\mu(z)|^2/(1-|z|)$ is a Carleson measure, with norm dominated by ϵ.

These systems are Hausdorff, that is, the intersection over all $\epsilon > 0$ of the sets $U(\epsilon)$ consists of the identity, and the same statement is true with U replaced by V. Moreover, by Lemma 2.1, the two systems $U(\epsilon)$ and $V(\epsilon)$ are cofinal in the sense that for every $\epsilon > 0$ there is a $\delta > 0$ such that $U(\delta) \subset V(\epsilon)$ and the same statement is true with U and V interchanged. Moreover, we will prove

Theorem 3.1. *Both systems are compatible with the group structure in $\mathrm{SQS}(S^1)$ in the following sense:*

(a) *for every $\epsilon > 0$, there exists $\delta > 0$, such that $V(\delta) \circ V(\delta) \subset V(\epsilon)$, and*
(b) *for every $\epsilon > 0$, there exists $\delta > 0$, such that $(V(\delta))^{-1} \subset V(\epsilon)$,*

and the same two statements are true with V replaced by U.

Proof. For $h_1,\ h_2 \in \mathrm{SQS}(S^1)$, let μ_i be the complex dilatation of the Douady–Earle extension f_i of h_i, $i = 1, 2$. It is known that μ_i induces a Carleson measure, $\lambda_{\mu_i}(z) = |\mu_i(z)|^2/(1 - |z|)$. Note that

$$\mu_1 \circ \mu_2 = \frac{\mu_2 + (\mu_1 \circ f_2)\tau}{1 + \bar{\mu_2}(\mu_1 \circ f_2)\tau}, \qquad \tau = \frac{\overline{(f_2)_z}}{(f_2)_z},$$

which is defined through $f_1 \circ f_2$. Then, by the fact that f_2 is bi-Lipschitz for the hyperbolic metric,

$$\lambda_{\mu_1 \circ \mu_2}(z) \leqslant C\lambda_{\mu_2}(z) + C\lambda_{\mu_1}(f)|\partial f|.$$

By Lemma 2.2,

$$\lambda_{\mu_1}(f)|\partial f| \in \mathrm{CM}(\Delta)$$

and $\|\lambda_{\mu_1}(f)|\partial f|\|_c \leqslant C\|\lambda_{\mu_1}\|_c$. Consequently, we can conclude that

$$\|\lambda_{\mu_1 \circ \mu_2}\|_c \leqslant C(\|\lambda_{\mu_1}\|_c + \|\lambda_{\mu_2}\|_c).$$

Lemma 2.7 implies that if a strongly quasisymmetric homeomorphism has a quasiconformal extension with complex dilatation μ such that λ_μ is a Carleson measure with a small norm, then the same is true for the Douady–Earle extension. Thus, by Lemma 2.1, it is possible that $\|\lambda_{\mu_1}\|_c$ and $\|\lambda_{\mu_2}\|_c$ are sufficiently small. On the other hand, for each $z \in S^1$,

$$|h_1 \circ h_2(z) - z| \leqslant |h_1 \circ h_2(z) - h_2(z)| + |h_2(z) - z|$$

and

$$|h_2^{-1} \circ h_1^{-1}(z) - z| \leqslant |h_2^{-1} \circ h_1^{-1}(z) - h_1^{-1}(z)| + |h_1^{-1}(z) - z|.$$

Consequently, for any $\epsilon > 0$, we can choose $\delta = \max(\frac{\epsilon}{2C}, \frac{\epsilon}{2})$, such that $h_1 \circ h_2 \in V(\epsilon)$ if $h_1 \in V(\delta)$ and $h_1 \in V(\delta)$.

Now we prove the statement (b) is valid. By bi-Lipschitz continuity of f_1 and the fact that $\lambda_{\mu_1} \in \mathrm{CM}(\Delta)$, Lemma 2.6 implies that

$$\lambda_{\mu_1^{-1}}(z) = \frac{|\mu_1^{-1}(z)|^2}{1 - |z|} \in \mathrm{CM}(\Delta),$$

where μ_1^{-1} is defined through f_1^{-1}, in particular,

$$\|\lambda_{\mu_1^{-1}}\|_c \leqslant C\|\lambda_{\mu_1}\|_c.$$

On the other hand, for each $z \in S^1$,

$$|h_1^{-1}(z) - z| = |h_1 \circ h_1^{-1}(z) - h_1^{-1}(z)|.$$

Thus, for any $\epsilon > 0$, we can choose $\delta = \max(\frac{\epsilon}{C}, \epsilon)$, such that $h_1^{-1} \in V(\epsilon)$ if $h_1 \in V(\delta)$.

Since the two systems are cofinal, the fact that the two properties hold for $V(\epsilon)$ implies that they also hold for $U(\epsilon)$. $\square$

According to the following

Definition 3.1. A partial topological group is a group with a Hausdorff system of neighborhoods of the identity satisfying (a) and (b) above.

We have proved that $\mathrm{SQS}(S^1)$ is a partial topological group of strongly quasisymmetric homeomorphisms of the unit circle under the A–Z topology.

4. Strongly symmetric homeomorphisms

In this section, our objective is to identify the characteristic topological subgroup of $\mathrm{SQS}(S^1)$.

Theorem 4.1. *Let* $\mathrm{SQS}(S^1)$ *be the partial topological group of strongly quasisymmetric homeomorphisms of the unit circle. Then* $\mathrm{SS}(S^1)$ *is the characteristic topological subgroup of* $\mathrm{SQS}(S^1)$*.*

Combining the theorem with Lemma 2.9 yields the following

Corollary 4.1. *The group of strongly symmetric homeomorphisms is a closed topological subgroup of* $\mathrm{SQS}(S^1)$*.*

In order to prove the theorem, we use a result about topological group by Gardiner–Sullivan [9]. It says that the following conditions on a partial topological group are equivalent: it is a topological group with the given neighborhood system of the identity if and only if the adjoint map $f \mapsto h \circ f \circ h^{-1}$ is continuous at the identity for every h in the group. Thus, to prove the theorem we must show the following two statements.

Lemma 4.1. *The following statements hold:*

(1) *If* $h \in \mathrm{SS}(S^1)$ *and if* $f \in \mathrm{SQS}(S^1)$ *and near the identity, then* $h \circ f \circ h^{-1}$ *is near the identity.*
(2) *If conjugation by* h*,* $f \mapsto h \circ f \circ h^{-1}$*, is continuous at the identity in* $\mathrm{SQS}(S^1)$*, then* $h \in \mathrm{SS}(S^1)$*.*

Proof. In order to simplify the notations, we prove the analogous statements for the real line $\mathbb{R}$. For any $f \in U(\delta)$, by the definition, $\lim_{\delta\to 0} \|\log f'\|_* = 0$. Note that

$$\|\log(h \circ f \circ h^{-1})'\|_* \leqslant \|\log h'(f) - \log h'\|_* + \|\log f'\|_*.$$

In order to prove $\lim_{\delta\to 0} \|\log(h \circ f \circ h^{-1})'\|_* = 0$, we use an approximation process to prove $\lim_{\delta\to 0} \|\log h'(f) - \log h'\|_* = 0$. We let UC denote the space of uniformly continuous functions on $\mathbb{R}$. Recall that $\mathrm{VMO}(\mathbb{R})$ is the closed subspace of $\mathrm{BMO}(\mathbb{R})$ which is the closure of $\mathrm{UC} \cap \mathrm{BMO}(\mathbb{R})$ under the BMO norm (see [17]). By $\log h' \in \mathrm{VMO}(\mathbb{R})$, there exists a sequence $\psi_n \in \mathrm{UC} \cap \mathrm{BMO}(\mathbb{R})$ such that $\|\psi_n - \log h'\|_* \to 0$ as $n \to \infty$. Noting that

$$\begin{aligned}\|\log h'(f) - \log h'\|_* &\leqslant \|\psi_n \circ f - \log h'(f)\|_* + \|\psi_n - \log h'\|_* + \|\psi_n \circ f - \psi_n\|_* \\ &\leqslant 2\|\psi_n - \log h'\|_* + \|\psi_n \circ f - \psi_n\|_{L^\infty},\end{aligned}$$

by the inequality $|f(x) - x| < \delta$ for any $x \in \mathbb{R}$ and the uniform continuity of ψ_n in $\mathbb{R}$, we have $\lim_{\delta\to 0} \|\log h'(f) - \log h'\|_* = 0$. On the other hand, by the Hölder continuity of h, there exist positive numbers C, α, such that for any $x \in \mathbb{R}$,

$$\begin{aligned}|h \circ f \circ h^{-1}(x) - x| &= |h \circ f \circ h^{-1}(x) - h \circ h^{-1}(x)| \\ &\leqslant C|f \circ h^{-1}(x) - h^{-1}(x)|^\alpha \\ &\leqslant C\delta^\alpha.\end{aligned}$$

Similarly, $|h \circ f^{-1} \circ h^{-1}(x) - x| \leqslant C\delta^\alpha$. Thus, the assumptions that f and f^{-1} are uniformly near to the identity imply that $h \circ f \circ h^{-1}$ and $(h \circ f \circ h^{-1})^{-1}$ are uniformly near the identity. Consequently, the statement (1) is valid.

Now we prove the statement (2) is valid. Let $f(x) = x - t \in U(\delta_1)$, $h \circ f \circ h^{-1} \in U(\delta_2)$ and $\delta_2 \to 0$ as $\delta_1 \to 0$. By the definition, $\|\log f'\|_* < \delta_1$ and $\|\log(h \circ f \circ h^{-1})'\|_* < \delta_2$. Note that

$$\|\log h'(f) - \log h'\|_* \leqslant \|\log(h \circ f \circ h^{-1})'\|_* + \|\log f'\|_*.$$

We can conclude that $\lim_{\delta_1 \to 0} \|\log h'(f) - \log h'\|_* = \lim_{t \to 0} \|\log h'(x-t) - \log h'(x)\|_* = 0$. By Lemma 2.8, $\log h' \in \mathrm{VMO}(\mathbb{R})$. Thus, $h \in \mathrm{SS}(\mathbb{R})$. $\square$

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