J. Math. Anal. Appl. 435 (2016) 1160–1182

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Flux-approximation limits of solutions to the relativistic Euler equations for polytropic gas ☆

Yu Zhang, Hanchun Yang *

Department of Mathematics, Yunnan University, Kunming 650091, PR China

A R T I C L E   I N F O

Article history:
Received 4 March 2015
Available online 10 November 2015
Submitted by H. Liu

Keywords:
Relativistic Euler equations
Pressureless relativistic Euler equations
Riemann problem
Delta shock wave
Vacuum
Flux approximation

A B S T R A C T

The flux-approximation problem of the relativistic Euler equations for polytropic gas in special relativity is studied. At first, we solve the Riemann problem of the pressureless relativistic Euler equations with a flux approximation, and obtain two kinds of solutions involving a family of delta shock wave and pseudo-vacuum state. Then, as the flux approximation vanishes, we show that the limits of the family of delta-shock and pseudo-vacuum solutions are exactly the delta-shock and vacuum state solutions of the pressureless relativistic Euler equations, respectively. Next, the Riemann problem of the relativistic Euler equations with a double parameter flux approximation including pressure is solved analytically. Furthermore, it is rigorously proved that, as the double parameter flux perturbation vanishes, any two-shock Riemann solution tends to a delta-shock solution to the pressureless relativistic Euler equations; any two-rarefaction Riemann solution tends to a two-contact-discontinuity solution to the pressureless relativistic Euler equations and the nonvacuum intermediate state in between tends to a vacuum state.

1. Introduction

The well-known Euler system of conservation laws of energy and momentum in special relativity reads

$$\begin{cases} \left((p+\rho c^2)\dfrac{v^2}{c^2(c^2-v^2)}+\rho\right)_t + \left((p+\rho c^2)\dfrac{v}{c^2-v^2}\right)_x = 0, \\ \left((p+\rho c^2)\dfrac{v}{c^2-v^2}\right)_t + \left((p+\rho c^2)\dfrac{v^2}{c^2-v^2}+p\right)_x = 0, \end{cases} \tag{1.1}$$

where ρ, v and p represent the proper energy density, particle speed and pressure, c is the speed of light, and the physically relevant region for solution is $\{(\rho, v) \mid \rho \geq 0, |v| < c\}$. The system (1.1) models the dynamics of plane waves in special relativity fluids, see [19–22] in a two-dimensional Minkowski space–time (x^0, x^1)

☆ This work is supported by the NSF of China 11361073.
* Corresponding author.
E-mail address: hyang@ynu.edu.cn (H. Yang).

http://dx.doi.org/10.1016/j.jmaa.2015.11.012
0022-247X/

$$\operatorname{div} T = 0,$$

with the stress-energy tensor for a fluid

$$T^{ij} = (p + \rho c^2)u^i u^j + p\eta^{ij},$$

where all indices run from 0 to 1 with $x^0 = ct$, $\eta^{ij} = \eta_{ij} = diag(-1, 1)$ denotes the flat Minkowski metric, u the 2-velocity of the fluid particle, and ρ the mass-energy density of the fluid as measured in units of mass in a frame moving with the fluid particle.

In general, the solution to the system (1.1) strongly depends on the state equation $p = p(\rho)$. In this paper, we are concerned with the polytropic gas, whose state equation can be formulated as

$$p(\rho) = \kappa^2\rho^\gamma, \qquad \gamma > 1, \tag{1.2}$$

where κ is a positive constant satisfying $\kappa < c$. When $\gamma = 1$, (1.2) models an isothermal gas, which corresponds to the extremely relativistic gases, when the temperature is very high and the particles move near the speed of light. This case will be studied in the future.

The study of the relativistic Euler equations (1.1) has been attracting more and more challenging attention of mathematics and physics researchers due to its importance and extreme complexity. Smoller and Temple [16] studied the Riemann problem and Cauchy problem of the system (1.1) when $\gamma = 1$. While for the case $\gamma > 1$, Chen [4] analyzed the properties of elementary waves, and solved the Riemann problem and Cauchy problem. Further, Chen and Li [1] established the uniqueness of Riemann solutions in the class of entropy solutions with arbitrarily large oscillation. Li, Feng and Wang [13] established the global existence of the entropy solutions with a class of large initial data which involve the interaction of shock waves and rarefaction waves. Recently, Ding and Li [6] studied a kind of multidimensional piston problem for (1.1). They established the local existence of shock front solutions to the spherically symmetric piston problem, as well as the convergence of the local solution as $c \to \infty$ to the corresponding solution of the classical non-relativistic Euler equations. Cheng and Yang [5] solved the Riemann problem of (1.1) for the Chaplygin gas.

As the pressure vanishes, that is $\kappa \to 0$, the limit system of (1.1) formally becomes the following pressureless relativistic Euler equations

$$\begin{cases} \left(\dfrac{\rho}{c^2 - v^2}\right)_t + \left(\dfrac{\rho v}{c^2 - v^2}\right)_x = 0, \\ \left(\dfrac{\rho v}{c^2 - v^2}\right)_t + \left(\dfrac{\rho v^2}{c^2 - v^2}\right)_x = 0, \end{cases} \tag{1.3}$$

which are fully linearly degenerate. The classical elementary waves only involve contact discontinuities. Interestingly, delta shock waves and vacuum states do occur in solutions. As for the delta shock waves, there have been rich results for various strictly or nonstrictly hyperbolic systems of conservation laws, see [7–10,12,14,15,17,18,23,27,28] and the reference cited therein.

In the past decade, the vanishing pressure limit method has been introduced to explore the phenomena of concentration and cavitation and the formation of delta shock waves and vacuum states in solutions, say Li [11] for the compressible Euler equations with zero temperature, Chen and Liu [2,3] for the isentropic and nonisentropic fluids. Recently, Yin and Sheng [29,30] for the system (1.1) and the relativistic fluid dynamics, Yin and Song [31] for the Chaplygin gas, and Yang and Wang [26] for the modified Chaplygin gas, etc. It is noticed that all these works on this topic are only focused on the pressure level.

In the present paper, by introducing a flux approximation in (1.1), we consider the following perturbed Euler system

$$\begin{cases} \left((p+\rho c^2)\dfrac{v^2}{c^2(c^2-v^2)}+\rho\right)_t + \left((p+\rho c^2)\dfrac{v}{c^2-v^2}-2\varepsilon v\right)_x = 0, \\ \left((p+\rho c^2)\dfrac{v}{c^2-v^2}\right)_t + \left((p+\rho c^2)\dfrac{v^2}{c^2-v^2}-\varepsilon v^2+p\right)_x = 0, \end{cases} \tag{1.4}$$

in the physical region $\{(\rho,v) \mid \rho \geq 2\varepsilon(1-v^2/c^2), |v|<c\}$, where ε and κ in (1.2) are small scaling parameters modeling the strength of flux and pressure, respectively. It is a physically reasonable perturbation which result from the small external shear forces imposed on the fluids. So it can be used to govern some dynamical behaviors of fluids. The flux approximation approach, in contrast to the previous works in [2,3, 11,26,29–31], which contains the pressure perturbation portion, was proposed by Yang and Liu [24] to study the isentropic Euler equations of gas dynamics. Furthermore, in [25] they also studied the Euler equations for nonisentropic fluids. One of the main objectives of this paper is to show rigorously that the occurrence of delta-shocks and vacuum states can be regarded as a singular flux-function limit of entropy solutions to the perturbed Euler system (1.4).

As a special situation, when the pressure vanishes, the system (1.4) becomes the following family of pressureless relativistic Euler equations

$$\begin{cases} \left(\dfrac{\rho c^2}{c^2-v^2}\right)_t + \left(\dfrac{\rho v c^2}{c^2-v^2}-2\varepsilon v\right)_x = 0, \\ \left(\dfrac{\rho v c^2}{c^2-v^2}\right)_t + \left(\dfrac{\rho v^2 c^2}{c^2-v^2}-\varepsilon v^2\right)_x = 0, \end{cases} \tag{1.5}$$

which is a pure flux approximation of special curiosity. We first solve the Riemann problem of (1.5) with the initial data

$$(\rho,v)(0,x) = (\rho_\pm, v_\pm), \quad \pm x>0, \tag{1.6}$$

where $\rho_\pm$ and $v_\pm$ are arbitrary constants. There are two kinds of Riemann solutions of somewhat surprising features. One consists of two contact discontinuities and a pseudo-vacuum state besides two constant states, another one is a family of delta-shock solution. More precisely, one can observe that, compared with the pressureless relativistic Euler equations, the vacuum state is moved into a curve, while for the delta shock wave, both the propagation speed and the weight decrease. Also, there is a great similarity to the classical nonrelativistic Euler system in which a constant-density state and a parameterized delta shock wave develop in solutions [24]. These results show that the flux perturbation works in the pressureless relativistic Euler equations.

Then, we show that, as the flux approximation vanishes, that is, the parameter $\varepsilon \to 0$, the limits of the pseudo-vacuum state and the parameterized delta-shock solutions of (1.5) are exactly the corresponding vacuum state and the delta-shock solutions of (1.3) with the same initial data, respectively.

Second, we consider the system (1.4) together with the state equation (1.2). The elementary waves of (1.4) consist of backward rarefaction wave $\overleftarrow{R}$, forward rarefaction wave $\overrightarrow{R}$, backward shock wave $\overleftarrow{S}$ and forward shock wave $\overrightarrow{S}$. With the phase plane analysis method and Lorentz transformation technique, by analyzing the properties of the elementary wave curves, we construct five different structures of Riemann solutions depending on the two parameters $\varepsilon, \kappa > 0$.

Furthermore, we study the limits of Riemann solutions of (1.4), (1.6) as the flux perturbation vanishes, that is, the two parameters $\varepsilon, \kappa \to 0$. It is shown that, as $\varepsilon, \kappa \to 0$, any Riemann solution containing two shock waves to the perturbed Euler system (1.4) tends to a delta-shock solution to the pressureless relativistic Euler equations (1.3), and the intermediate density between the two shocks tends to a weighted δ-measure that forms a delta shock wave. By contrast, it is also shown that any Riemann solution involving two rarefaction waves to the perturbed Euler system (1.4) converges to a two-contact-discontinuity solution to the pressureless relativistic Euler equations (1.3), whose intermediate state in between tends to a vacuum state as $\varepsilon, \kappa \to 0$.

The above results show that the delta-shock and vacuum state of the pressureless relativistic Euler equations can be obtained as flux-approximation limits of Riemann solutions to the relativistic Euler equations for polytropic gas. Therefore, both the delta shock wave and vacuum state are stable for the pressureless relativistic Euler equations under some flux small perturbations. Moreover, the above results also identify a fact of interest, that is, the flux approximations of difference have their respective effects on the formation of delta-shock and vacuum state in relativistic fluids. In this regard, it differs from those only in pressure level [2,3,11,26,29–31]. Our work therefore extends in some sense the previous results and proofs in [29].

This paper is organized as follows. In Section 2, we discuss the delta-shocks and vacuum states for the pressureless relativistic Euler equations. In Section 3, we solve the Riemann problem of the perturbed pressureless relativistic Euler equations, and analyze the limits of Riemann solutions. Section 4 solves the Riemann problem (1.4), (1.6), and examines the dependence of Riemann solutions on parameters ε and κ. At last, we consider the limits of Riemann solutions of (1.4), (1.6) by letting $\varepsilon, \kappa \to 0$, which will be shown in Sections 5 and 6, respectively.

2. Delta-shocks and vacuum states

In this section, we discuss the delta shock waves and vacuum states in Riemann solutions to the pressureless relativistic Euler equations (1.3).

The system (1.3) has a repeated eigenvalue $\lambda = v$ and only one associated right eigenvector $r = (1, 0)^T$. It is obviously linearly degenerate by $\nabla\lambda \cdot r \equiv 0$. By seeking the self-similar solution $(\rho, v)(t, x) = (\rho, v)(\xi)$, $(\xi = x/t)$, the Riemann solution of (1.3), (1.6) can be constructed in the following two cases.

When $v_- < v_+$, the solution consists of two contact discontinuities and a vacuum state between them and can be expressed as

$$(\rho, v)(\xi) = \begin{cases} (\rho_-, v_-), & -\infty < \xi \le v_-, \\ (0, \xi), & v_- \le \xi \le v_+, \\ (\rho_+, v_+), & v_+ \le \xi < +\infty. \end{cases} \tag{2.1}$$

When $v_- > v_+$, a solution with Dirac delta distribution can be constructed. To do so, let us define a weighted delta function supported on a curve as follows.

Definition 2.1. A two-dimensional weighted delta function $w(s)\delta_S$ supported on a smooth curve S parameterized as $t = t(s)$, $x = x(s)$ $(a \le s \le b)$ is defined by

$$\big\langle w(t(s))\delta_S, \varphi(t(s), x(s))\big\rangle = \int_a^b w(t(s))\varphi(t(s), x(s))\sqrt{x'(s)^2 + t'(s)^2}ds \tag{2.2}$$

for all test functions $\varphi(t, x) \in C_0^\infty(R^+ \times R^1)$.

Using this definition, a family of delta-shock solutions with the parameter σ can be introduced to construct the solution of (1.3), which is

$$\rho(t, x) = \rho_0(t, x) + w(t)\delta_S, \quad v(t, x) = v_0(t, x), \tag{2.3}$$

where $S = \{(t, \sigma t) : 0 \leq t < \infty\}$, and

$$\begin{cases} \rho_0(t,x) = \rho_- + [\rho]\chi(x - \sigma t), \\ v_0(t,x) = v_- + [v]\chi(x - \sigma t), \\ w(t) = \dfrac{(c^2-\sigma^2)t}{\sqrt{1+\sigma^2}} \left(\sigma \left[\dfrac{\rho}{c^2 - v^2} \right] - \left[\dfrac{\rho v}{c^2 - v^2} \right] \right), \end{cases} \tag{2.4}$$

in which $[h] = h_+ - h_-$ denotes the jump of function h across the discontinuity, σ is the velocity of the delta shock wave, and $\chi(x)$ the characteristic function that is 0 when $x < 0$ and 1 when $x > 0$.

Similar to [12,15], for the system (1.3), the definition of solutions in the sense of distributions is introduced as follows.

Definition 2.2. A pair (ρ, v) consists of a solution of (1.3) in the sense of distributions if it satisfies

$$\begin{aligned} \left\langle \frac{\rho}{c^2 - v^2}, \varphi_t \right\rangle + \left\langle \frac{\rho v}{c^2 - v^2}, \varphi_x \right\rangle = 0, \\ \left\langle \frac{\rho v}{c^2 - v^2}, \varphi_t \right\rangle + \left\langle \frac{\rho v^2}{c^2 - v^2}, \varphi_x \right\rangle = 0, \end{aligned} \tag{2.5}$$

where

$$\begin{aligned} \left\langle \frac{\rho}{c^2 - v^2}, \varphi \right\rangle &= \int_0^{+\infty} \int_{-\infty}^{+\infty} \frac{\rho_0}{c^2 - v_0^2} \varphi dx dt + \langle \frac{w}{c^2 - \sigma^2} \delta_S, \varphi \rangle, \\ \left\langle \frac{\rho v}{c^2 - v^2}, \varphi \right\rangle &= \int_0^{+\infty} \int_{-\infty}^{+\infty} \frac{\rho_0 v_0}{c^2 - v_0^2} \varphi dx dt + \langle \frac{\sigma w}{c^2 - \sigma^2} \delta_S, \varphi \rangle, \end{aligned} \tag{2.6}$$

for any $\varphi(t,x) \in C_0^\infty(R^+ \times R^1)$.

Then, a unique solution of (1.3), (1.6) involving a δ-measure with the parameter σ can be constructed as

$$(\rho, v)(t,x) = \begin{cases} (\rho_-, v_-), & x < x(t), \\ \big(w(t)\delta(x - x(t)), \sigma\big), & x = x(t), \\ (\rho_+, v_+), & x > x(t), \end{cases} \tag{2.7}$$

in which $x(t)$, σ and $w(t)$ satisfy the generalized Rankine–Hugoniot relation

$$\begin{cases} \dfrac{dx}{dt} = \sigma, \\ \dfrac{d}{dt}\left(\dfrac{w(t)\sqrt{1+\sigma^2}}{c^2 - \sigma^2} \right) = \sigma \left[\dfrac{\rho}{c^2 - v^2} \right] - \left[\dfrac{\rho v}{c^2 - v^2} \right], \\ \dfrac{d}{dt}\left(\dfrac{w(t)\sigma\sqrt{1+\sigma^2}}{c^2 - \sigma^2} \right) = \sigma \left[\dfrac{\rho v}{c^2 - v^2} \right] - \left[\dfrac{\rho v^2}{c^2 - v^2} \right], \end{cases} \tag{2.8}$$

which reflects the relationship among the location, weight and propagation speed of the delta shock wave, and the entropy condition

$$v_+ < \sigma < v_-, \tag{2.9}$$

which means that all characteristics on both sides of the discontinuity are in-coming.

Under the entropy condition (2.9), by solving the generalized Rankine–Hugoniot relation (2.8) with the initial data $w(0) = 0$ and $x(0) = 0$, we have

$$\sigma = \frac{v_-\sqrt{\frac{\rho_-}{c^2-v_-^2}} + v_+\sqrt{\frac{\rho_+}{c^2-v_+^2}}}{\sqrt{\frac{\rho_-}{c^2-v_-^2}} + \sqrt{\frac{\rho_+}{c^2-v_+^2}}}, \quad w(t) = \sqrt{\frac{\rho_-\rho_+}{(c^2-v_-^2)(c^2-v_+^2)}}\frac{(v_- - v_+)(c^2-\sigma^2)t}{\sqrt{1+\sigma^2}}. \tag{2.10}$$

Thus we obtain the delta-shock solution defined by (2.2) with (2.3) and (2.10).

3. Riemann solutions and limit analysis of (1.5) as $\varepsilon \to 0$

This section solves the Riemann problem (1.5), (1.6), and studies the limit of solutions.

3.1. Riemann problem of the system (1.5)

The system (1.5) has a double eigenvalue $\lambda^\varepsilon = v$ and only one right eigenvector $r^\varepsilon = (1, 0)^T$. The system is also obviously linearly degenerate by $\nabla\lambda^\varepsilon \cdot r^\varepsilon \equiv 0$. Considering the self-similar solution, then the Riemann problem (1.5), (1.6) is transformed into a boundary value problem

$$\begin{cases} -\xi\Big(\dfrac{\rho c^2}{c^2-v^2}\Big)_\xi + \Big(\dfrac{\rho v c^2}{c^2-v^2} - 2\varepsilon v\Big)_\xi = 0, \\ -\xi\Big(\dfrac{\rho v c^2}{c^2-v^2}\Big)_\xi + \Big(\dfrac{\rho v^2 c^2}{c^2-v^2} - \varepsilon v^2\Big)_\xi = 0, \end{cases} \tag{3.1}$$

and

$$(\rho, v)(\pm\infty) = (\rho_\pm, v_\pm). \tag{3.2}$$

For any smooth solution, (3.1) can be rewritten as

$$\begin{pmatrix} \dfrac{vc^2 - \xi c^2}{c^2-v^2} & \dfrac{\rho c^2(c^2+v^2-2\xi v)}{(c^2-v^2)^2} - 2\varepsilon \\ \dfrac{v^2c^2-\xi vc^2}{c^2-v^2} & \dfrac{2\rho v c^4 - \xi\rho c^2(c^2+v^2)}{(c^2-v^2)^2} - 2\varepsilon v \end{pmatrix}\begin{pmatrix} d\rho \\ dv \end{pmatrix} = 0,$$

which provides, besides the constant state, the singular solution

$$\begin{cases} \rho = 2\varepsilon(1 - v^2/c^2), \\ v = \xi, \end{cases} \tag{3.3}$$

which is called as a pseudo-vacuum state. Noticing that $\rho_v = -4\varepsilon v/c^2$ and $\rho_{vv} = -4\varepsilon/c^2$ in (3.3), one can easily check the following conclusions which show the properties of the pseudo-vacuum solution.

Lemma 3.1. *Let L be the curve determined by the equation*

$$\rho = 2\varepsilon(1 - v^2/c^2), \tag{3.4}$$

then L: (1) $\lim_{v\to\pm c}\rho = 0$; (2) passes through the point $(0, 2\varepsilon)$ in (v, ρ)-plane; (3) monotonely increases as $-c < v < 0$, and monotonely decreases as $0 < v < c$; (4) is convex.

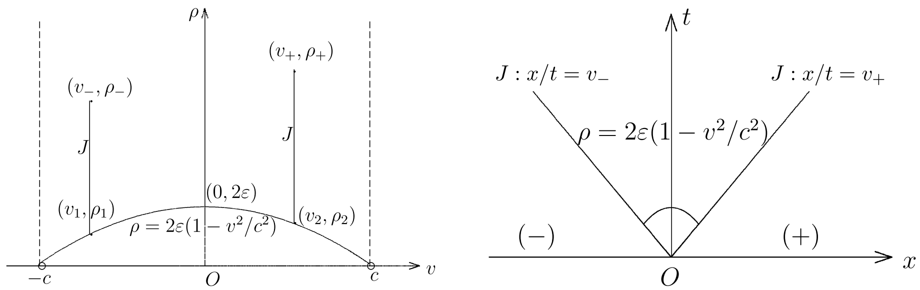

Fig. 1. Pseudo-vacuum state.

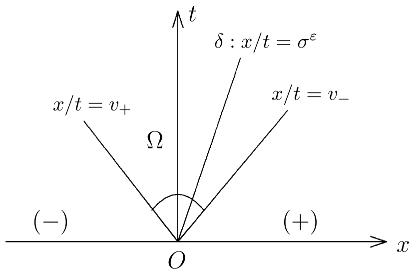

Fig. 2. Characteristic analysis of delta shock wave.

For a bounded discontinuity at $\xi = \omega$, the Rankine–Hugoniot condition

$$\begin{cases} \omega\left[\dfrac{\rho c^2}{c^2 - v^2}\right] = \left[\dfrac{\rho v c^2}{c^2 - v^2} - 2\varepsilon v\right], \\ \omega\left[\dfrac{\rho v c^2}{c^2 - v^2}\right] = \left[\dfrac{\rho v^2 c^2}{c^2 - v^2} - \varepsilon v^2\right] \end{cases} \tag{3.5}$$

holds. By solving (3.5), one can obtain the contact discontinuity

$$J: \quad \omega = \xi = v_- = v_+,$$

which is characterized by $x/t = v_- = v_+$ in the (x, t)-plane. Any two states (ρ_-, v_-) and (ρ_+, v_+) can be connected by J if and only if they are located on the line $v = v_- = v_+$.

Now, with the constant state, pseudo-vacuum state and contact discontinuity, we construct the solutions of the Riemann problem (1.5), (1.6) by two cases.

For the case $v_- < v_+$, the solution consists of two contact discontinuities and a pseudo-vacuum state besides two constant states, that is

$$(\rho, v)(\xi) = \begin{cases} (\rho_-, v_-), & -\infty < \xi < v_-, \\ \big(2\varepsilon(1 - \xi^2/c^2), \xi\big), & v_- \le \xi \le v_+, \\ (\rho_+, v_+), & v_+ < \xi < +\infty, \end{cases} \tag{3.6}$$

as shown in Fig. 1, where the intermediate states connecting two contact discontinuities and the pseudo-vacuum state are $(\rho_1, v_1) = \big(2\varepsilon(1 - v_-^2/c^2), v_-\big)$ and $(\rho_2, v_2) = \big(2\varepsilon(1 - v_+^2/c^2), v_+\big)$, respectively.

In the case $v_- > v_+$, since the characteristic lines will overlap in the region Ω as indicated in Fig. 2, so the singularity of solutions must develop in this region and the delta shock wave will occur in solution.

Similarly, we seek the parameterized delta-shock solution in the form

$$(\rho, v)(t,x) = \begin{cases} (\rho_-, v_-), & x < x(t), \\ \Big(w^\varepsilon(t)\delta\big(x - x(t)\big), \sigma^\varepsilon\Big), & x = x(t), \\ (\rho_+, v_+), & x > x(t), \end{cases} \tag{3.7}$$

where $w^\varepsilon(t)$ and σ^ε are the weight and velocity of the parameterized delta shock wave, respectively. The parameterized delta shock wave $(\rho^\varepsilon, v^\varepsilon, \sigma^\varepsilon, w^\varepsilon)$ of the form (2.2), (2.3) is subjected to the generalized Rankine–Hugoniot relation

$$\begin{cases} \dfrac{dx}{dt} = \sigma^\varepsilon, \\ \dfrac{d}{dt}\left(\dfrac{w^\varepsilon(t)c^2\sqrt{1+(\sigma^\varepsilon)^2}}{c^2-(\sigma^\varepsilon)^2}\right) = \sigma^\varepsilon\left[\dfrac{\rho c^2}{c^2-v^2}\right] - \left[\dfrac{\rho v c^2}{c^2-v^2} - 2\varepsilon v\right], \\ \dfrac{d}{dt}\left(\dfrac{w^\varepsilon(t)c^2\sigma^\varepsilon\sqrt{1+(\sigma^\varepsilon)^2}}{c^2-(\sigma^\varepsilon)^2}\right) = \sigma^\varepsilon\left[\dfrac{\rho v c^2}{c^2-v^2}\right] - \left[\dfrac{\rho v^2 c^2}{c^2-v^2} - \varepsilon v^2\right], \end{cases} \tag{3.8}$$

and the entropy condition

$$v_+ < \sigma^\varepsilon < v_-. \tag{3.9}$$

In what follows, the Riemann problem (1.5), (1.6) for the case $v_- > v_+$ is reduced to solving the generalized Rankine–Hugoniot relation (3.8) with initial conditions $t = 0 : x(0) = 0, w^\varepsilon(0) = 0$.

From (3.8), we have

$$\begin{cases} \dfrac{w^\varepsilon(t)c^2\sqrt{1+(\sigma^\varepsilon)^2}}{c^2-(\sigma^\varepsilon)^2} = \left[\dfrac{\rho c^2}{c^2-v^2}\right]x - \left[\dfrac{\rho v c^2}{c^2-v^2} - 2\varepsilon v\right]t, \\ \dfrac{w^\varepsilon(t)c^2\sigma^\varepsilon\sqrt{1+(\sigma^\varepsilon)^2}}{c^2-(\sigma^\varepsilon)^2} = \left[\dfrac{\rho v c^2}{c^2-v^2}\right]x - \left[\dfrac{\rho v^2 c^2}{c^2-v^2} - \varepsilon v^2\right]t. \end{cases} \tag{3.10}$$

Noticing that σ^ε is a constant, multiplying the first equation by σ^ε and together with the second equation gives

$$\left[\frac{\rho c^2}{c^2-v^2}\right](\sigma^\varepsilon)^2 - \left(\left[\frac{\rho v c^2}{c^2-v^2} - 2\varepsilon v\right] + \left[\frac{\rho v c^2}{c^2-v^2}\right]\right)\sigma^\varepsilon + \left[\frac{\rho v^2 c^2}{c^2-v^2} - \varepsilon v^2\right] = 0. \tag{3.11}$$

When $\left[\frac{\rho c^2}{c^2-v^2}\right] \neq 0$, (3.11) is a quadratic equation. Since

$$\Delta^\varepsilon = 4(v_- - v_+)^2\left(\frac{\rho_- c^2}{c^2-v_-^2} - \varepsilon\right)\left(\frac{\rho_+ c^2}{c^2-v_+^2} - \varepsilon\right) > 0$$

in the relevant region of solutions, we obtain

$$\sigma^\varepsilon = \frac{\left[\dfrac{\rho v c^2}{c^2-v^2} - \varepsilon v\right] + \sqrt{\left(\dfrac{\rho_- c^2}{c^2-v_-^2} - \varepsilon\right)\left(\dfrac{\rho_+ c^2}{c^2-v_+^2} - \varepsilon\right)}(v_- - v_+)}{\left[\dfrac{\rho c^2}{c^2-v^2}\right]}, \tag{3.12}$$

under the entropy condition (3.9). As a result, we can get from (3.10) that

$$w^\varepsilon(t) = \frac{[\varepsilon v] + \sqrt{\left(\frac{\rho_- c^2}{c^2 - v_-^2} - \varepsilon\right)\left(\frac{\rho_+ c^2}{c^2 - v_+^2} - \varepsilon\right)}(v_- - v_+)}{c^2\sqrt{1 + (\sigma^\varepsilon)^2}}\big(c^2 - (\sigma^\varepsilon)^2\big)t. \tag{3.13}$$

When $\left[\frac{\rho c^2}{c^2 - v^2}\right] = 0$, we have

$$\sigma^\varepsilon = \frac{v_- + v_+}{2}, \quad w^\varepsilon(t) = \frac{\left[2\varepsilon v - \frac{\rho v c^2}{c^2 - v^2}\right]}{c^2\sqrt{1 + (\sigma^\varepsilon)^2}}\big(c^2 - (\sigma^\varepsilon)^2\big)t. \tag{3.14}$$

Then we reach the following result.

Theorem 3.2. *For every fixed $\varepsilon > 0$, there exists a unique entropy solution of Riemann problem (1.5), (1.6), which consists of two contact discontinuities and a pseudo-vacuum state when $v_- < v_+$ and a parameterized delta shock wave when $v_- > v_+$.*

3.2. Limiting behavior of Riemann solutions to the system (1.5) as $\varepsilon \to 0$

Now we proceed to study the limiting behavior of the solution of (1.5), (1.6) as $\varepsilon \to 0$ for the situation $\left[\frac{\rho c^2}{c^2 - v^2}\right] \neq 0$ by two cases.

For the case $v_- < v_+$, by letting $\varepsilon \to 0$ in (3.3) and (3.6), one can obviously find that the pseudo-vacuum state of the perturbed system (1.5) converges to the vacuum state of the pressureless relativistic Euler equations (1.3).

While for the case $v_- > v_+$, we calculate the limits of σ^ε and w^ε as $\varepsilon \to 0$ from (3.12) and (3.13), then

$$\lim_{\varepsilon \to 0} \sigma^\varepsilon = \frac{v_- \sqrt{\frac{\rho_-}{c^2 - v_-^2}} + v_+ \sqrt{\frac{\rho_+}{c^2 - v_+^2}}}{\sqrt{\frac{\rho_-}{c^2 - v_-^2}} + \sqrt{\frac{\rho_+}{c^2 - v_+^2}}} = \sigma, \tag{3.15}$$

$$\lim_{\varepsilon \to 0} w^\varepsilon(t) = \sqrt{\frac{\rho_- \rho_+}{(c^2 - v_-^2)(c^2 - v_+)}}\frac{(v_- - v_+)(c^2 - \sigma^2)t}{\sqrt{1 + \sigma^2}} = w(t). \tag{3.16}$$

In a simple way similar to that in [2,26,29], the following results can be easily concluded.

Theorem 3.3. *Let $v_- > v_+$. For each fixed $\varepsilon > 0$, assume that $(\rho^\varepsilon, v^\varepsilon)$ is a delta-shock solution of (1.5), (1.6). Then, when $\varepsilon \to 0$, the pair of limit functions (ρ, v) is a delta-shock solution of (1.3), (1.6). Moreover, the limit functions $\frac{\rho}{c^2 - v^2}$ and $\frac{\rho v}{c^2 - v^2}$ are all the sums of a step function and a Dirac delta function with weights*

$$\frac{t}{\sqrt{1 + \sigma^2}}\left(\sigma\left[\frac{\rho}{c^2 - v^2}\right] - \left[\frac{\rho v}{c^2 - v^2}\right]\right) \text{ and } \frac{t}{\sqrt{1 + \sigma^2}}\left(\sigma\left[\frac{\rho v}{c^2 - v^2}\right] - \left[\frac{\rho v^2}{c^2 - v^2}\right]\right),$$

respectively.

4. Riemann problem (1.4), (1.6)

In this section, we put the Riemann problem (1.4), (1.6) and examine the dependence of elementary waves on the parameters ε and κ.

The system (1.4) can be rewritten as

$$A\begin{pmatrix}\rho\\ v\end{pmatrix}_t+B\begin{pmatrix}\rho\\ v\end{pmatrix}_x=0,$$

where

$$A=\begin{pmatrix}\dfrac{c^4+p'(\rho)v^2}{c^2(c^2-v^2)} & \dfrac{2(p(\rho)+\rho c^2)v}{(c^2-v^2)^2}\\ \dfrac{(p'(\rho)+c^2)v}{c^2-v^2} & \dfrac{(p(\rho)+\rho c^2)(c^2+v^2)}{(c^2-v^2)^2}\end{pmatrix},$$

$$B=\begin{pmatrix}\dfrac{(p'(\rho)+c^2)v}{c^2-v^2} & \dfrac{(p(\rho)+\rho c^2)(c^2+v^2)}{(c^2-v^2)^2}-2\varepsilon\\ \dfrac{c^2(p'(\rho)+v^2)}{c^2-v^2} & \dfrac{2c^2(p(\rho)+\rho c^2)v}{(c^2-v^2)^2}-2\varepsilon v\end{pmatrix}.$$

A routine computation shows that the system (1.4) has two eigenvalues

$$\lambda_1^{\varepsilon\kappa}=\frac{2vc^2(c^2-p')-\sqrt{Q}+Mvp'}{2(c^4-v^2p')},\quad \lambda_2^{\varepsilon\kappa}=\frac{2vc^2(c^2-p')+\sqrt{Q}+Mvp'}{2(c^4-v^2p')},$$

and the associated right eigenvectors are

$$r_1^{\varepsilon\kappa}=\left(\frac{-1}{c^2-v^2},\frac{2c^2p'(c^2-v^2)}{(\sqrt{Q}+Mvp')(p+\rho c^2)}\right)^T,\quad r_2^{\varepsilon\kappa}=\left(\frac{1}{c^2-v^2},\frac{2c^2p'(c^2-v^2)}{(\sqrt{Q}-Mvp')(p+\rho c^2)}\right)^T,$$

where

$$M(\rho,v)=\frac{2\varepsilon(c^2-v^2)^2}{p+\rho c^2},\quad Q(\rho,v)=4c^4p'(c^2-v^2)(c^2-v^2-M)+M^2v^2p'^{\,2}.$$

It can be verified that $\nabla\lambda_i^{\varepsilon\kappa}\cdot r_i^{\varepsilon\kappa}\neq 0$, $(i=1,2)$. Therefore, the system (1.4) is strictly hyperbolic and the characteristics are genuinely nonlinear.

Looking for the self-similar solutions $(\rho,v)(t,x)=(\rho,v)(\xi)(\xi=x/t)$, the Riemann problem (1.4), (1.6) is reduced to a two-point boundary value problem

$$\begin{cases}-\xi\left((p+\rho c^2)\dfrac{v^2}{c^2(c^2-v^2)}+\rho\right)_\xi+\left((p+\rho c^2)\dfrac{v}{c^2-v^2}-2\varepsilon v\right)_\xi=0,\\ -\xi\left((p+\rho c^2)\dfrac{v}{c^2-v^2}\right)_\xi+\left((p+\rho c^2)\dfrac{v^2}{c^2-v^2}-\varepsilon v^2+p\right)_\xi=0,\end{cases}\tag{4.1}$$

with boundary condition (3.2).

For any smooth solution, (4.1) satisfies

$$\begin{pmatrix}\frac{(p'(\rho)+c^2)v}{c^2-v^2}-\xi\frac{c^4+p'(\rho)v^2}{c^2(c^2-v^2)} & \frac{(p(\rho)+\rho c^2)(c^2+v^2)}{(c^2-v^2)^2}-2\varepsilon-\xi\frac{2(p(\rho)+\rho c^2)v}{(c^2-v^2)^2}\\ \frac{c^2(p'(\rho)+v^2)}{c^2-v^2}-\xi\frac{(p'(\rho)+c^2)v}{c^2-v^2} & \frac{2c^2(p(\rho)+\rho c^2)v}{(c^2-v^2)^2}-2\varepsilon v-\xi\frac{(p(\rho)+\rho c^2)(c^2+v^2)}{(c^2-v^2)^2}\end{pmatrix}\begin{pmatrix}d\rho\\ dv\end{pmatrix}=0,$$

which provides the general solution (constant state) and the backward rarefaction wave $\overleftarrow{R}(\rho_-,v_-)$:

$$\begin{cases} \lambda_1^{\varepsilon\kappa} = \dfrac{2vc^2(c^2-p') - \sqrt{Q} + Mvp'}{2(c^4 - v^2p')}, \\ \dfrac{1}{2c}\ln\dfrac{c+v}{c-v} + \displaystyle\int_{\rho_-}^{\rho} \dfrac{2c^2p'(s)(c^2-v^2)}{\Big(\sqrt{Q(s,v)} + M(s,v)vp'(s)\Big)\Big(p(s)+sc^2\Big)}ds = C, \ \ \rho < \rho_-, \end{cases} \tag{4.2}$$

and the forward rarefaction wave $\overrightarrow{R}(\rho_-, v_-)$:

$$\begin{cases} \lambda_2^{\varepsilon\kappa} = \dfrac{2vc^2(c^2-p') + \sqrt{Q} + Mvp'}{2(c^4 - v^2p')}, \\ \dfrac{1}{2c}\ln\dfrac{c+v}{c-v} - \displaystyle\int_{\rho_-}^{\rho} \dfrac{2c^2p'(s)(c^2-v^2)}{\Big(\sqrt{Q(s,v)} - M(s,v)vp'(s)\Big)\Big(p(s)+sc^2\Big)}ds = C, \ \ \rho > \rho_-. \end{cases} \tag{4.3}$$

The rarefaction wave curves possess the following geometric properties.

Lemma 4.1. *For the back and forward rarefaction waves based on the given left state (ρ_-, v_-), we have*

$$\overleftarrow{R}(\rho_-, v_-): \frac{dv}{d\rho} < 0, \quad \lim_{\rho\to 0^+} v = l_a \le c,$$
$$\overrightarrow{R}(\rho_-, v_-): \frac{dv}{d\rho} > 0, \quad \lim_{\rho\to +\infty} v = l_b \le c.$$

Proof. From the second equations of (4.2), (4.3), by differentiating v with respect to ρ, respectively, it follows that

$$\overleftarrow{R}(\rho_-, v_-): \frac{dv}{d\rho} = -\frac{2c^2(c^2-v^2)^2p'}{(\sqrt{Q}+Mvp')(p+\rho c^2)} < 0,$$

and

$$\overrightarrow{R}(\rho_-, v_-): \frac{dv}{d\rho} = \frac{2c^2(c^2-v^2)^2p'}{(\sqrt{Q}-Mvp')(p+\rho c^2)} > 0$$

for ε small.

In addition, on the $\overleftarrow{R}(\rho_-, v_-)$, we can calculate that

$$v = \frac{c\Big((c+v_-)e^{-2cI(\rho,v)} - (c-v_-)\Big)}{(c+v_-)e^{-2cI(\rho,v)} + (c-v_-)},$$

where

$$I(\rho, v) = \int_{\rho_-}^{\rho} f(s,v)ds \quad \text{and} \quad f(\rho,v) = \frac{2c^2(c^2-v^2)p'(\rho)}{\Big(\sqrt{Q(\rho,v)} + M(\rho,v)vp'(\rho)\Big)\Big(p(\rho)+\rho c^2\Big)}.$$

For the integral

$$\tilde{I}(\rho) = \int_{\rho_-}^{\rho} g(s)ds, \ \ g(\rho) = \frac{\sqrt{p'(\rho)}}{p(\rho)+\rho c^2} = \frac{\kappa\sqrt{\gamma\rho^{\gamma-1}}}{\kappa^2\rho^\gamma + \rho c^2},$$

we have

$$\tilde{I}(\rho) = \frac{1}{c}\frac{2\sqrt{\gamma}}{\gamma-1}\arctan\left(\frac{\kappa}{c}\rho^{\frac{\gamma-1}{2}}\right) - \frac{1}{c}\frac{2\sqrt{\gamma}}{\gamma-1}\arctan\left(\frac{\kappa}{c}\rho_-^{\frac{\gamma-1}{2}}\right).$$

Since $\lim_{\rho\to 0^+}\frac{f(\rho,v)}{g(\rho)} = 0$, then according to the comparison test, we know that $I(\rho, v)$ is convergent if $\tilde{I}(\rho)$ converges as $\rho \to 0^+$. As a result, we conclude that $\lim_{\rho\to 0^+} v$ exists, denoted by l_a. Obviously, $l_a \le c$.

Similarly, the limit $\lim_{\rho\to+\infty}\frac{f(\rho,v)}{g(\rho)} = 1$ shows that $I(\rho, v)$ and $\tilde{I}(\rho)$ have the same properties of convergence and divergence as $\rho \to +\infty$. Then, we can verify that $\lim_{\rho\to+\infty} v = l_b \le c$ on $\overrightarrow{R}(\rho_-, v_-)$. So this lemma is true. □

For a bounded discontinuity at $\xi = \sigma^{\varepsilon\kappa}$, the Rankine–Hugoniot condition

$$\begin{cases} \sigma^{\varepsilon\kappa}\left[(p+\rho c^2)\dfrac{v^2}{c^2(c^2-v^2)} + \rho\right] = \left[(p+\rho c^2)\dfrac{v}{c^2-v^2} - 2\varepsilon v\right], \\ \sigma^{\varepsilon\kappa}\left[(p+\rho c^2)\dfrac{v}{c^2-v^2}\right] = \left[(p+\rho c^2)\dfrac{v^2}{c^2-v^2} - \varepsilon v^2 + p\right] \end{cases} \tag{4.4}$$

holds, where $\sigma^{\varepsilon\kappa}$ denotes the velocity of the shock.

To solve the shock wave curves for the system (1.4), we will make the most of the Lorentz transformation properties of the system, see also [4,22], etc. In fact, we can find, under any Lorentz transformation $(t, x) \to (\bar{t}, \bar{x})$, an identical system in the barred coordinates once the velocity states are renamed in terms of the coordinate velocities as measured in the barred coordinate system. Particularly, $\rho(t, x)$ is a scalar invariant under the Lorentz transformations and it owns the same value in the barred and unbarred coordinates that name the same geometric point in the background space–time manifold. However, since the velocity v is formed from the entries of the vector quantity (u^0, u^1), so it is not a scalar.

In the following, we will exploit the transformation law for velocities by two steps. At first, we calculate the shock waves and shock speeds in a frame in which the particle velocity v is zero. Then we apply the Lorentz transformation law for velocities to obtain these curves in an arbitrary frame. So, by introducing a velocity transformation law, we can prove the following lemma which parameterizes the shock wave curves.

Lemma 4.2. *The shock wave curves based on the left state (ρ_-, v_-) are given by*

$$\frac{v-v_-}{c^2-v_-^2} = -\frac{\sqrt{\Theta(\rho,\rho_-)}}{c^2 - v_-\sqrt{\Theta(\rho,\rho_-)}}, \qquad v < v_-, \tag{4.5}$$

with $\rho > \rho_-$ for $\overleftarrow{S}(\rho_-, v_-)$, and $\rho < \rho_-$ for $\overrightarrow{S}(\rho_-, v_-)$, where

$$E(\rho,\rho_-) = 2c^2(p-p_-)(\rho-\rho_-), \qquad G(\rho,\rho_-) = 4\varepsilon(\rho_- c^2 + p)(p-p_-)(\rho-\rho_-),$$
$$F(\rho,\rho_-) = \frac{(p_- + \rho c^2)(p+\rho_- c^2)}{c^2} - \varepsilon(2p + \rho c^2 + \rho_- c^2), \qquad \Theta(\rho,\rho_-) = \frac{E}{F + \sqrt{F^2+G}}$$

with $p_- = p(\rho_-)$.

Proof. Due to the Lorentz transformation, if the barred coordinate $(\bar{t}, \bar{x})$ moves with the velocity τ measured in the unbarred coordinate (t, x), and the corresponding states are denoted by $(\bar{\rho}_-, \bar{v}_-) = (\rho_-, \bar{v}_-)$ for the left state, $(\bar{\rho}_+, \bar{v}_+) = (\rho, \bar{v})$ for the right state measured in the coordinate $(\bar{t}, \bar{x})$, which satisfy

$$\bar{t} = \frac{t - \tau x/c^2}{\sqrt{1-\tau^2/c^2}}, \quad \bar{x} = \frac{x-\tau t}{\sqrt{1-\tau^2/c^2}}, \quad v = \frac{\tau + \bar{v}}{1+\tau\bar{v}/c^2}. \tag{4.6}$$

By taking $\tau = v_-$, we have $\bar{v}_- = 0$. Then we can use the above transformations to obtain the corresponding shock wave curves in arbitrary coordinate. From the Rankine–Hugoniot condition (4.4), we get

$$\begin{cases} \sigma^{\varepsilon\kappa}\left((p+\rho c^2)\dfrac{\bar{v}^2}{c^2(c^2-\bar{v}^2)} + \rho - \rho_-\right) = (p+\rho c^2)\dfrac{\bar{v}}{c^2-\bar{v}^2} - 2\varepsilon\bar{v}, \\ \sigma^{\varepsilon\kappa}\left((p+\rho c^2)\dfrac{\bar{v}}{c^2-\bar{v}^2}\right) = (p+\rho c^2)\dfrac{\bar{v}^2}{c^2-\bar{v}^2} + p - p_- - \varepsilon\bar{v}^2. \end{cases} \tag{4.7}$$

Eliminating $\sigma^{\varepsilon\kappa}$ in (4.7) yields

$$\bar{v}^2 = \frac{E}{F+\sqrt{F^2+G}} := \Theta(\rho,\rho_-). \tag{4.8}$$

For shock waves, the speed on the right is always less than the speed on the left. Thus $\bar{v} < \bar{v}_- = 0$ on the shock curves, and $\bar{v} = -\sqrt{\Theta(\rho,\rho_-)}$. Then, (4.5) follows immediately from the Lorentz transformation (4.6).

From (4.8) we see that there are two branches of shock wave curves, with $\rho > \rho_-$ on one and $\rho < \rho_-$ on the other. We call the branch with $\rho > \rho_-$ backward shock wave curve, denoted by $\overleftarrow{S}$, the other forward shock wave curve, denoted by $\overrightarrow{S}$.

It can be proved as done in [4] that, for small ε, the shock wave $\overleftarrow{S}$ associating with $\lambda_1^{\varepsilon\kappa}$ has to be satisfied with the Lax shock condition

$$\sigma^{\varepsilon\kappa} < \lambda_1^{\varepsilon\kappa}(\rho_-,v_-) < \lambda_2^{\varepsilon\kappa}(\rho_-,v_-), \qquad \lambda_1^{\varepsilon\kappa}(\rho,v) < \sigma^{\varepsilon\kappa} < \lambda_2^{\varepsilon\kappa}(\rho,v), \tag{4.9}$$

and $\overrightarrow{S}$ associating with $\lambda_2^{\varepsilon\kappa}$ should satisfy

$$\lambda_1^{\varepsilon\kappa}(\rho_-,v_-) < \sigma^{\varepsilon\kappa} < \lambda_2^{\varepsilon\kappa}(\rho_-,v_-), \qquad \lambda_1^{\varepsilon\kappa}(\rho,v) < \lambda_2^{\varepsilon\kappa}(\rho,v) < \sigma^{\varepsilon\kappa}. \tag{4.10}$$

The proof of Lemma 4.2 is complete. $\square$

The following lemma shows the geometric properties of the shock wave curves.

Lemma 4.3. *For the shock wave curves based on the given left state (ρ_-,v_-), we have*

$$\overleftarrow{S}(\rho_-,v_-): \frac{dv}{d\rho} < 0, \qquad \lim_{\rho\to+\infty} v = -c, \tag{4.11}$$

$$\overrightarrow{S}(\rho_-,v_-): \frac{dv}{d\rho} > 0, \qquad \lim_{\rho\to 0^+} v = -c. \tag{4.12}$$

Proof. Due to the Lorentz invariance, it is enough to prove those in a frame with $v_- = 0$. Then, we can get from (4.8) that

$$\bar{v}^2 = \frac{c^4(p-p_-)(\rho-\rho_-)}{(p_-+\rho c^2)(p+\rho_- c^2)} - \varepsilon\cdot h(\rho,\rho_-,\bar{v}), \tag{4.13}$$

where

$$h(\rho,\rho_-,\bar{v}) = \frac{\bar{v}^4}{p_-+\rho c^2} - \frac{c^2(2p+\rho c^2+\rho_- c^2)\bar{v}^2}{(p_-+\rho c^2)(p+\rho_- c^2)}.$$

Differentiating $\bar{v}$ with respect to ρ in (4.13) yields

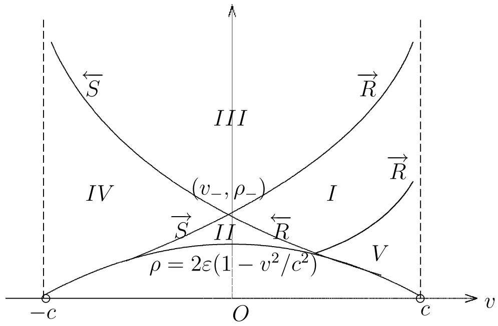

Fig. 3. Curves of elementary waves.

$$2\bar{v}\frac{d\bar{v}}{d\rho} = c^4(p_- + \rho_- c^2) \cdot \frac{p'(p_- + \rho c^2)(\rho - \rho_-) + (p - p_-)(p + \rho_- c^2)}{\Big((p_- + \rho c^2)(p + \rho_- c^2)\Big)^2} - \varepsilon \cdot h'(\rho, \rho_-, \bar{v}). \tag{4.14}$$

Noticing that $\bar{v} < 0$, for ε small, we can check from (4.14) that $\frac{d\bar{v}}{d\rho} < 0$ on $\overleftarrow{S}$ since $\rho > \rho_-, p > p_-$, and $\frac{d\bar{v}}{d\rho} > 0$ on $\overrightarrow{S}$ since $\rho < \rho_-, p < p_-$.

Furthermore, we obtain from (4.8) that

$$\lim_{\rho \to +\infty} \bar{v}^2 = \lim_{\rho \to +\infty} \frac{2(\rho - \rho_-)c^2}{\frac{p_- + \rho c^2}{c^2} - 2\varepsilon + \sqrt{\Big(\frac{p_- + \rho c^2}{c^2} - 2\varepsilon\Big)^2 + 4\varepsilon(\rho - \rho_-)}} = c^2,$$

and

$$\lim_{\rho \to 0+} \bar{v}^2 = \lim_{\rho \to 0+} \frac{2c^2 \rho_- p_-}{\rho_- p_- - \varepsilon \rho_- c^2 + \sqrt{(\rho_- p_- - \varepsilon \rho_- c^2)^2 + 4\varepsilon c^2 \rho_-^2 p_-}} = c^2.$$

Therefore, $\lim_{\rho \to +\infty} \bar{v} = -c$ on $\overleftarrow{S}$, and $\lim_{\rho \to 0+} \bar{v} = -c$ on $\overrightarrow{S}$. Then (4.11) and (4.12) hold due to the Lorentz invariance. This completes the proof of Lemma 4.3. □

From Lemmas 3.1, 4.1 and 4.3, it follows that the backward rarefaction wave curve $\overleftarrow{R}$ and the forward shock wave curve $\overrightarrow{S}$ must intersect with the pseudo-vacuum state curve ($\rho = 2\varepsilon(1 - v^2/c^2)$) at two different points. Therefore, given a left state (ρ_-, v_-), all the possible states can be connected on the right by a backward (forward) rarefaction wave $\overleftarrow{R}$ ($\overrightarrow{R}$) or a backward (forward) shock wave $\overleftarrow{S}$ ($\overrightarrow{S}$). So the phase plane can be divided into five regions, as illustrated in Fig. 3.

According to the right state (ρ_+, v_+) in the different regions, one can obtain five kinds of configurations of solutions. Particularly, when $(\rho_+, v_+) \in \overleftarrow{S}\overrightarrow{S}(\rho_-, v_-)$, the Riemann solution of (1.4), (1.6) involves a backward shock wave $\overleftarrow{S}$, a forward shock wave $\overrightarrow{S}$, and a nonvacuum intermediate constant state whose density may become singular as $\varepsilon, \kappa \to 0$; When $(\rho_+, v_+) \in \overleftarrow{R}\overrightarrow{R}(\rho_-, v_-)$, the solution contains a backward rarefaction wave $\overleftarrow{R}$, a forward rarefaction wave $\overrightarrow{R}$, and an intermediate constant state, maybe a pseudo-vacuum state solution ($\rho = 2\varepsilon(1 - v^2/c^2)$). As both ε and κ drop to zero, it is sufficient for us to consider only the limit process for the above two cases, because the regions $\overleftarrow{S}\overrightarrow{R}(\rho_-, v_-)$ and $\overleftarrow{R}\overrightarrow{S}(\rho_-, v_-)$ both have empty interior.

5. Formation of delta shock wave for the system (1.4)

In this section, as $\varepsilon, \kappa \to 0$, we analyze the formation of delta shock waves in the Riemann solutions of (1.4), (1.6) for the case $(\rho_+, v_+) \in \overleftarrow{S}\overrightarrow{S}(\rho_-, v_-)$ with $v_- > v_+$.

5.1. Limit behavior of the Riemann solutions as $\varepsilon, \kappa \to 0$

When $(\rho_+, v_+) \in \overleftarrow{S}\overrightarrow{S}(\rho_-, v_-)$, for each pair of fixed ε and κ, suppose that (ρ_-, v_-) and $(\rho_*^{\varepsilon\kappa}, v_*^{\varepsilon\kappa})$ are connected by a $\overleftarrow{S}$ with speed $\sigma_1^{\varepsilon\kappa}$, and that $(\rho_*^{\varepsilon\kappa}, v_*^{\varepsilon\kappa})$ and (ρ_+, v_+) are connected by a $\overrightarrow{S}$ with speed $\sigma_2^{\varepsilon\kappa}$. Then, from (4.5), $(\rho_*^{\varepsilon\kappa}, v_*^{\varepsilon\kappa})$ are determined by

$$\frac{v_*^{\varepsilon\kappa} - v_-}{c^2 - v_-^2} = -\frac{\sqrt{\Theta(\rho_*^{\varepsilon\kappa}, \rho_-)}}{c^2 - v_-\sqrt{\Theta(\rho_*^{\varepsilon\kappa}, \rho_-)}}, \quad \rho_*^{\varepsilon\kappa} > \rho_-, \tag{5.1}$$

and

$$\frac{v_+ - v_*^{\varepsilon\kappa}}{c^2 - (v_*^{\varepsilon\kappa})^2} = -\frac{\sqrt{\Theta(\rho_+, \rho_*^{\varepsilon\kappa})}}{c^2 - v_*^{\varepsilon\kappa}\sqrt{\Theta(\rho_+, \rho_*^{\varepsilon\kappa})}}, \quad \rho_*^{\varepsilon\kappa} > \rho_+. \tag{5.2}$$

Combining (5.1) and (5.2), we have

$$\frac{v_- - v_+}{c^2 - v_- v_+} = \frac{\sqrt{\Theta(\rho_*^{\varepsilon\kappa}, \rho_-)} + \sqrt{\Theta(\rho_+, \rho_*^{\varepsilon\kappa})}}{c^2 + \sqrt{\Theta(\rho_*^{\varepsilon\kappa}, \rho_-)\Theta(\rho_+, \rho_*^{\varepsilon\kappa})}}. \tag{5.3}$$

As a start, we assert that

Lemma 5.1. $\lim_{\substack{\varepsilon\to 0\\ \kappa\to 0}} \rho_*^{\varepsilon\kappa} = +\infty$.

In fact, if $\lim_{\substack{\varepsilon\to 0\\ \kappa\to 0}} \rho_*^{\varepsilon\kappa} = \tilde{\rho} \in (\max(\rho_-, \rho_+), +\infty)$, then by taking the limit of (5.3) as $\varepsilon, \kappa \to 0$, one can easily obtain that $v_- = v_+$, which contradicts with $v_- > v_+$.

By Lemma 5.1, letting $\varepsilon, \kappa \to 0$ in (5.3), we get immediately the following result.

Lemma 5.2.

$$\lim_{\substack{\varepsilon\to 0\\ \kappa\to 0}} \kappa^2(\rho_*^{\varepsilon\kappa})^\gamma = \frac{\rho_-\rho_+ c^4(v_- - v_+)^2}{(\rho_- + \rho_+)(c^2 - v_-^2)(c^2 - v_+^2) + 2(c^2 - v_- v_+)\sqrt{\rho_-\rho_+(c^2 - v_-^2)(c^2 - v_+^2)}}.$$

Lemma 5.3. *Set* $\lim_{\substack{\varepsilon\to 0\\ \kappa\to 0}} v_*^{\varepsilon\kappa} = \sigma$, *then*

$$\lim_{\substack{\varepsilon\to 0\\ \kappa\to 0}} v_*^{\varepsilon\kappa} = \lim_{\substack{\varepsilon\to 0\\ \kappa\to 0}} \sigma_1^{\varepsilon\kappa} = \lim_{\substack{\varepsilon\to 0\\ \kappa\to 0}} \sigma_2^{\varepsilon\kappa} = \sigma \in (v_+, v_-). \tag{5.4}$$

Proof. From the first equation of (4.4) for $\overleftarrow{S}$, we get

$$\lim_{\substack{\varepsilon\to 0\\ \kappa\to 0}} \sigma_1^{\varepsilon\kappa} = \lim_{\substack{\varepsilon\to 0\\ \kappa\to 0}} \frac{\left(\kappa^2(\rho_*^{\varepsilon\kappa})^\gamma + \rho_*^{\varepsilon\kappa}c^2\right)\frac{v_*^{\varepsilon\kappa}}{c^2 - (v_*^{\varepsilon\kappa})^2} - 2\varepsilon v_*^{\varepsilon\kappa} - (\kappa^2\rho_-^\gamma + \rho_- c^2)\frac{v_-}{c^2 - v_-^2} + 2\varepsilon v_-}{\frac{\kappa^2(\rho_*^{\varepsilon\kappa})^\gamma + \rho_*^{\varepsilon\kappa}c^2}{c^2}\frac{(v_*^{\varepsilon\kappa})^2}{c^2 - (v_*^{\varepsilon\kappa})^2} + \rho_*^{\varepsilon\kappa} - \frac{\kappa^2\rho_-^\gamma + \rho_- c^2}{c^2}\frac{v_-^2}{c^2 - v_-^2} - \rho_-}.$$

By Lemma 5.1, it is easy to know that the limit above is nothing but σ, which means that $\lim_{\substack{\varepsilon\to 0\\ \kappa\to 0}} \sigma_1^{\varepsilon\kappa} = \lim_{\substack{\varepsilon\to 0\\ \kappa\to 0}} v_*^{\varepsilon\kappa} = \sigma$. The equality $\lim_{\substack{\varepsilon\to 0\\ \kappa\to 0}} \sigma_2^{\varepsilon\kappa} = \lim_{\substack{\varepsilon\to 0\\ \kappa\to 0}} v_*^{\varepsilon\kappa} = \sigma$ can be similarly obtained from the first equation of (4.4) for $\overrightarrow{S}$.

On the other hand, passing to the limit $\varepsilon, \kappa \to 0$ in (5.1) and (5.2) and noticing Lemma 5.1, we have $v_+ < \sigma = \lim_{\substack{\varepsilon\to 0\\ \kappa\to 0}} v_*^{\varepsilon\kappa} < v_-$. Therefore, Lemma 5.3 holds. □

Combining the Rankine–Hugoniot conditions (4.4) with Lemmas 5.1–5.3, we can prove the following lemma.

Lemma 5.4.

$$\lim_{\substack{\varepsilon\to0\\ \kappa\to0}}\int_{\sigma_1^{\varepsilon\kappa}}^{\sigma_2^{\varepsilon\kappa}}\frac{\rho_*^{\varepsilon\kappa}}{c^2-(v_*^{\varepsilon\kappa})^2}d\xi=\lim_{\substack{\varepsilon\to0\\ \kappa\to0}}\frac{\rho_*^{\varepsilon\kappa}}{c^2-(v_*^{\varepsilon\kappa})^2}(\sigma_2^{\varepsilon\kappa}-\sigma_1^{\varepsilon\kappa})=\sigma\left[\frac{\rho}{c^2-v^2}\right]-\left[\frac{\rho v}{c^2-v^2}\right], \tag{5.5}$$

and

$$\lim_{\substack{\varepsilon\to0\\ \kappa\to0}}\int_{\sigma_1^{\varepsilon\kappa}}^{\sigma_2^{\varepsilon\kappa}}\frac{\rho_*^{\varepsilon\kappa}v_*^{\varepsilon\kappa}}{c^2-(v_*^{\varepsilon\kappa})^2}d\xi=\lim_{\substack{\varepsilon\to0\\ \kappa\to0}}\frac{\rho_*^{\varepsilon\kappa}v_*^{\varepsilon\kappa}}{c^2-(v_*^{\varepsilon\kappa})^2}(\sigma_2^{\varepsilon\kappa}-\sigma_1^{\varepsilon\kappa})=\sigma\left[\frac{\rho v}{c^2-v^2}\right]-\left[\frac{\rho v^2}{c^2-v^2}\right]. \tag{5.6}$$

Proof. From (4.4) for $\overleftarrow{S}$ and $\overrightarrow{S}$, we have

$$\begin{cases}\sigma_1^{\varepsilon\kappa}\left(\dfrac{\kappa^2(\rho_*^{\varepsilon\kappa})^\gamma+\rho_*^{\varepsilon\kappa}c^2}{c^2}\dfrac{(v_*^{\varepsilon\kappa})^2}{c^2-(v_*^{\varepsilon\kappa})^2}+\rho_*^{\varepsilon\kappa}-\dfrac{\kappa^2\rho_-^\gamma+\rho_-c^2}{c^2}\dfrac{v_-^2}{c^2-v_-^2}-\rho_-\right)\\ \quad=\left(\kappa^2(\rho_*^{\varepsilon\kappa})^\gamma+\rho_*^{\varepsilon\kappa}c^2\right)\dfrac{v_*^{\varepsilon\kappa}}{c^2-(v_*^{\varepsilon\kappa})^2}-2\varepsilon v_*^{\varepsilon\kappa}-\left(\kappa^2\rho_-^\gamma+\rho_-c^2\right)\dfrac{v_-}{c^2-v_-^2}+2\varepsilon v_-,\\ \sigma_2^{\varepsilon\kappa}\left(\dfrac{\kappa^2\rho_+^\gamma+\rho_+c^2}{c^2}\dfrac{v_+^2}{c^2-v_+^2}+\rho_+-\dfrac{\kappa^2(\rho_*^{\varepsilon\kappa})^\gamma+\rho_*^{\varepsilon\kappa}c^2}{c^2}\dfrac{(v_*^{\varepsilon\kappa})^2}{c^2-(v_*^{\varepsilon\kappa})^2}-\rho_*^{\varepsilon\kappa}\right)\\ \quad=\left(\kappa^2\rho_+^\gamma+\rho_+c^2\right)\dfrac{v_+}{c^2-v_+^2}-2\varepsilon v_+-\left(\kappa^2(\rho_*^{\varepsilon\kappa})^\gamma+\rho_*^{\varepsilon\kappa}c^2\right)\dfrac{v_*^{\varepsilon\kappa}}{c^2-(v_*^{\varepsilon\kappa})^2}+2\varepsilon v_*^{\varepsilon\kappa},\end{cases}$$

which yields

$$\begin{aligned}&\frac{\kappa^2(\rho_*^{\varepsilon\kappa})^\gamma(v_*^{\varepsilon\kappa})^2}{c^2(c^2-(v_*^{\varepsilon\kappa})^2)}(\sigma_1^{\varepsilon\kappa}-\sigma_2^{\varepsilon\kappa})+\frac{\rho_*^{\varepsilon\kappa}c^2}{c^2-(v_*^{\varepsilon\kappa})^2}(\sigma_1^{\varepsilon\kappa}-\sigma_2^{\varepsilon\kappa})\\ &\quad=\frac{\kappa^2\rho_-^\gamma v_-^2}{c^2(c^2-v_-^2)}\sigma_1^{\varepsilon\kappa}+\frac{\rho_-c^2}{c^2-v_-^2}\sigma_1^{\varepsilon\kappa}-\frac{(\kappa^2\rho_-^\gamma+\rho_-c^2)v_-}{c^2-v_-^2}+2\varepsilon v_-\\ &\qquad-\frac{\kappa^2\rho_+^\gamma v_+^2}{c^2(c^2-v_+^2)}\sigma_2^{\varepsilon\kappa}-\frac{\rho_+c^2}{c^2-v_+^2}\sigma_2^{\varepsilon\kappa}+\frac{(\kappa^2\rho_+^\gamma+\rho_+c^2)v_+}{c^2-v_+^2}-2\varepsilon v_+.\end{aligned}$$

Letting $\varepsilon,\kappa\to0$, we get

$$\lim_{\substack{\varepsilon\to0\\ \kappa\to0}}\frac{\rho_*^{\varepsilon\kappa}}{c^2-(v_*^{\varepsilon\kappa})^2}(\sigma_2^{\varepsilon\kappa}-\sigma_1^{\varepsilon\kappa})=\sigma\left[\frac{\rho}{c^2-v^2}\right]-\left[\frac{\rho v}{c^2-v^2}\right]. \tag{5.7}$$

Similarly, the second equations of (4.4) for $\overleftarrow{S}$ and $\overrightarrow{S}$ yields

$$\begin{aligned}&\frac{\kappa^2(\rho_*^{\varepsilon\kappa})^\gamma v_*^{\varepsilon\kappa}}{c^2-(v_*^{\varepsilon\kappa})^2}(\sigma_1^{\varepsilon\kappa}-\sigma_2^{\varepsilon\kappa})+\frac{\rho_*^{\varepsilon\kappa}v_*^{\varepsilon\kappa}c^2}{c^2-(v_*^{\varepsilon\kappa})^2}(\sigma_1^{\varepsilon\kappa}-\sigma_2^{\varepsilon\kappa})\\ &\quad=\frac{(\kappa^2\rho_-^\gamma+\rho_-c^2)v_-}{c^2-v_-^2}\sigma_1^{\varepsilon\kappa}-\frac{(\kappa^2\rho_-^\gamma+\rho_-c^2)v_-^2}{c^2-v_-^2}-\kappa^2\rho_-^\gamma+\varepsilon v_-^2\\ &\qquad-\frac{(\kappa^2\rho_+^\gamma+\rho_+c^2)v_+}{c^2-v_+^2}\sigma_2^{\varepsilon\kappa}+\frac{(\kappa^2\rho_+^\gamma+\rho_+c^2)v_+^2}{c^2-v_+^2}+\kappa^2\rho_+^\gamma-\varepsilon v_+^2.\end{aligned}$$

Taking the limit $\varepsilon,\kappa\to0$, it follows that

$$\lim_{\substack{\varepsilon\to0\\ \kappa\to0}} \frac{\rho_*^{\varepsilon\kappa} v_*^{\varepsilon\kappa}}{c^2-(v_*^{\varepsilon\kappa})^2}(\sigma_2^{\varepsilon\kappa}-\sigma_1^{\varepsilon\kappa}) = \sigma\left[\frac{\rho v}{c^2-v^2}\right] - \left[\frac{\rho v^2}{c^2-v^2}\right]. \tag{5.8}$$

Therefore, (5.5) and (5.6) can be immediately obtained from (5.7) and (5.8). □

Lemma 5.5. *For the σ mentioned above, we have*

$$\sigma = \frac{\left[\frac{\rho v}{c^2-v^2}\right] + \sqrt{\left[\frac{\rho v}{c^2-v^2}\right]^2 - \left[\frac{\rho}{c^2-v^2}\right]\left[\frac{\rho v^2}{c^2-v^2}\right]}}{\left[\frac{\rho}{c^2-v^2}\right]} = \frac{v_-\sqrt{\frac{\rho_-}{c^2-v_-^2}} + v_+\sqrt{\frac{\rho_+}{c^2-v_+^2}}}{\sqrt{\frac{\rho_-}{c^2-v_-^2}} + \sqrt{\frac{\rho_+}{c^2-v_+^2}}}$$

when $\left[\frac{\rho}{c^2-v^2}\right] \neq 0$, *and*

$$\sigma = \frac{v_- + v_+}{2}$$

when $\left[\frac{\rho}{c^2-v^2}\right] = 0$.

Proof. It is clear according to Lemmas 5.3–5.4 that

$$\lim_{\substack{\varepsilon\to0\\ \kappa\to0}} \frac{\rho_*^{\varepsilon\kappa} v_*^{\varepsilon\kappa}}{c^2-(v_*^{\varepsilon\kappa})^2}(\sigma_2^{\varepsilon\kappa}-\sigma_1^{\varepsilon\kappa}) = \lim_{\substack{\varepsilon\to0\\ \kappa\to0}} \frac{\rho_*^{\varepsilon\kappa}}{c^2-(v_*^{\varepsilon\kappa})^2}(\sigma_2^{\varepsilon\kappa}-\sigma_1^{\varepsilon\kappa}) \cdot \lim_{\substack{\varepsilon\to0\\ \kappa\to0}} v_*^{\varepsilon\kappa}.$$

Thus we have

$$\sigma\left[\frac{\rho v}{c^2-v^2}\right] - \left[\frac{\rho v^2}{c^2-v^2}\right] = \left(\sigma\left[\frac{\rho}{c^2-v^2}\right] - \left[\frac{\rho v}{c^2-v^2}\right]\right)\cdot\sigma,$$

from which, the desired conclusions are easily obtained thanks to $\sigma \in (v_+, v_-)$. □

From Lemma 5.1 and Lemmas 5.3–5.5, one can observe that, when $\varepsilon,\kappa \to 0$, the velocities of shocks $\overleftarrow{S}$ and $\overrightarrow{S}$ and the intermediate velocity $v_*^{\varepsilon\kappa}$ of solution of (1.4), (1.6) approach to σ. This implies $\overleftarrow{S}$ and $\overrightarrow{S}$ coincide, and the intermediate density $\rho_*^{\varepsilon\kappa}$ becomes singular which determines the delta-shock solution of (1.3), (1.6).

5.2. Delta shock waves

Now, in the case $v_- > v_+$, the following theorem gives a very nice depiction of the limit.

Theorem 5.6. *Let $v_- > v_+$. For each fixed pair $\varepsilon,\kappa > 0$, assume that $(\rho^{\varepsilon\kappa}, v^{\varepsilon\kappa})$ is a solution containing two shocks $\overleftarrow{S}$ and $\overrightarrow{S}$ of (1.4), (1.6) constructed in Section 4. Then $(\rho^{\varepsilon\kappa}, v^{\varepsilon\kappa})$ converges in the sense of distributions as $\varepsilon,\kappa \to 0$, and the limit functions $\frac{\rho}{c^2-v^2}$ and $\frac{\rho v}{c^2-v^2}$ are all the sums of a step function and a Dirac delta function with weights*

$$\frac{t}{\sqrt{1+\sigma^2}}\left(\sigma\left[\frac{\rho}{c^2-v^2}\right] - \left[\frac{\rho v}{c^2-v^2}\right]\right) \text{ and } \frac{t}{\sqrt{1+\sigma^2}}\left(\sigma\left[\frac{\rho v}{c^2-v^2}\right] - \left[\frac{\rho v^2}{c^2-v^2}\right]\right),$$

respectively, which form the delta-shock solution of (1.3), (1.6).

Proof. (i). For each fixed pair $\varepsilon, \kappa > 0$, the two-shock Riemann solution to the perturbed Euler system (1.4) can be expressed as

$$(\rho^{\varepsilon\kappa}, v^{\varepsilon\kappa})(\xi) = \begin{cases} (\rho_-, v_-), & \xi < \sigma_1^{\varepsilon\kappa}, \\ (\rho_*^{\varepsilon\kappa}(\xi), v_*^{\varepsilon\kappa}(\xi)), & \sigma_1^{\varepsilon\kappa} < \xi < \sigma_2^{\varepsilon\kappa}, \\ (\rho_+, v_+), & \xi > \sigma_2^{\varepsilon\kappa}, \end{cases}$$

which, for any $\phi \in C_0^\infty(-\infty, +\infty)$, satisfies the following weak formulations

$$\begin{aligned} &\int_{-\infty}^{+\infty} \left(\frac{\rho^{\varepsilon\kappa} v^{\varepsilon\kappa} c^2}{c^2 - (v^{\varepsilon\kappa})^2} - \frac{\xi \rho^{\varepsilon\kappa} c^2}{c^2 - (v^{\varepsilon\kappa})^2} - 2\varepsilon v^{\varepsilon\kappa} \right) \phi' d\xi - \int_{-\infty}^{+\infty} \frac{\rho^{\varepsilon\kappa} c^2}{c^2 - (v^{\varepsilon\kappa})^2} \phi d\xi \\ &= \int_{-\infty}^{+\infty} \frac{\kappa^2 (\rho^{\varepsilon\kappa})^\gamma (v^{\varepsilon\kappa})^2}{c^2 \big(c^2 - (v^{\varepsilon\kappa})^2\big)} \xi \phi' d\xi - \int_{-\infty}^{+\infty} \frac{\kappa^2 (\rho^{\varepsilon\kappa})^\gamma v^{\varepsilon\kappa}}{c^2 - (v^{\varepsilon\kappa})^2} \phi' d\xi + \int_{-\infty}^{+\infty} \frac{\kappa^2 (\rho^{\varepsilon\kappa})^\gamma (v^{\varepsilon\kappa})^2}{c^2 \big(c^2 - (v^{\varepsilon\kappa})^2\big)} \phi d\xi, \end{aligned} \tag{5.9}$$

and

$$\begin{aligned} &\int_{-\infty}^{+\infty} \left(\frac{\rho^{\varepsilon\kappa} (v^{\varepsilon\kappa})^2 c^2}{c^2 - (v^{\varepsilon\kappa})^2} - \frac{\xi \rho^{\varepsilon\kappa} v^{\varepsilon\kappa} c^2}{c^2 - (v^{\varepsilon\kappa})^2} - \varepsilon (v^{\varepsilon\kappa})^2 \right) \phi' d\xi - \int_{-\infty}^{+\infty} \frac{\rho^{\varepsilon\kappa} v^{\varepsilon\kappa} c^2}{c^2 - (v^{\varepsilon\kappa})^2} \phi d\xi \\ &= \int_{-\infty}^{+\infty} \frac{\kappa^2 (\rho^{\varepsilon\kappa})^\gamma v^{\varepsilon\kappa}}{c^2 - (v^{\varepsilon\kappa})^2} \xi \phi' d\xi - \int_{-\infty}^{+\infty} \frac{\kappa^2 (\rho^{\varepsilon\kappa})^\gamma c^2}{c^2 - (v^{\varepsilon\kappa})^2} \phi' d\xi + \int_{-\infty}^{+\infty} \frac{\kappa^2 (\rho^{\varepsilon\kappa})^\gamma v^{\varepsilon\kappa}}{c^2 - (v^{\varepsilon\kappa})^2} \phi d\xi. \end{aligned} \tag{5.10}$$

(ii). Consider the limits of $\frac{\rho^{\varepsilon\kappa}}{c^2 - (v^{\varepsilon\kappa})^2}$ and $\frac{\rho^{\varepsilon\kappa} v^{\varepsilon\kappa}}{c^2 - (v^{\varepsilon\kappa})^2}$ depending on ξ. The first integral on the left hand side of (5.9) can be rewritten as

$$\left(\int_{-\infty}^{\sigma_1^{\varepsilon\kappa}} + \int_{\sigma_1^{\varepsilon\kappa}}^{\sigma_2^{\varepsilon\kappa}} + \int_{\sigma_2^{\varepsilon\kappa}}^{+\infty} \right) \left(\frac{\rho^{\varepsilon\kappa} v^{\varepsilon\kappa} c^2}{c^2 - (v^{\varepsilon\kappa})^2} - \frac{\xi \rho^{\varepsilon\kappa} c^2}{c^2 - (v^{\varepsilon\kappa})^2} - 2\varepsilon v^{\varepsilon\kappa} \right) \phi' d\xi. \tag{5.11}$$

The limit of the sum of the first and last term of (5.11) is

$$\begin{aligned} &\lim_{\substack{\varepsilon \to 0 \\ \kappa \to 0}} \int_{-\infty}^{\sigma_1^{\varepsilon\kappa}} \left(\frac{\rho_- v_- c^2}{c^2 - v_-^2} - \frac{\xi \rho_- c^2}{c^2 - v_-^2} - 2\varepsilon v_- \right) \phi' d\xi + \lim_{\substack{\varepsilon \to 0 \\ \kappa \to 0}} \int_{\sigma_2^{\varepsilon\kappa}}^{+\infty} \left(\frac{\rho_+ v_+ c^2}{c^2 - v_+^2} - \frac{\xi \rho_+ c^2}{c^2 - v_+^2} - 2\varepsilon v_+ \right) \phi' d\xi \\ &= c^2 \left(\sigma \left[\frac{\rho}{c^2 - v^2} \right] - \left[\frac{\rho v}{c^2 - v^2} \right] \right) \phi(\sigma) + c^2 \int_{-\infty}^{+\infty} H(\xi - \sigma) \phi d\xi, \end{aligned} \tag{5.12}$$

where

$$H(\xi - \sigma) = \begin{cases} \dfrac{\rho_-}{c^2 - v_-^2}, & \xi < \sigma, \\ \dfrac{\rho_+}{c^2 - v_+^2}, & \xi > \sigma. \end{cases}$$

For the second term of (5.11), it equals

$$
\begin{aligned}
&\lim_{\substack{\varepsilon\to 0\\ \kappa\to 0}} \int_{\sigma_1^{\varepsilon\kappa}}^{\sigma_2^{\varepsilon\kappa}} \left(\frac{\rho_*^{\varepsilon\kappa} v_*^{\varepsilon\kappa} c^2}{c^2-(v_*^{\varepsilon\kappa})^2} - \frac{\xi \rho_*^{\varepsilon\kappa} c^2}{c^2-(v_*^{\varepsilon\kappa})^2} - 2\varepsilon v_*^{\varepsilon\kappa} \right) \phi' d\xi \\
&= \lim_{\substack{\varepsilon\to 0\\ \kappa\to 0}} \frac{\rho_*^{\varepsilon\kappa} c^2}{c^2-(v_*^{\varepsilon\kappa})^2} (\sigma_2^{\varepsilon\kappa}-\sigma_1^{\varepsilon\kappa}) \left(\frac{\phi(\sigma_2^{\varepsilon\kappa})-\phi(\sigma_1^{\varepsilon\kappa})}{\sigma_2^{\varepsilon\kappa}-\sigma_1^{\varepsilon\kappa}} v_*^{\varepsilon\kappa} - \frac{\sigma_2^{\varepsilon\kappa}\phi(\sigma_2^{\varepsilon\kappa})-\sigma_1^{\varepsilon\kappa}\phi(\sigma_1^{\varepsilon\kappa})}{\sigma_2^{\varepsilon\kappa}-\sigma_1^{\varepsilon\kappa}} \right. \\
&\quad \left. + \frac{1}{\sigma_2^{\varepsilon\kappa}-\sigma_1^{\varepsilon\kappa}} \int_{\sigma_1^{\varepsilon\kappa}}^{\sigma_2^{\varepsilon\kappa}} \phi d\xi \right) - \lim_{\substack{\varepsilon\to 0\\ \kappa\to 0}} 2\varepsilon v_*^{\varepsilon\kappa} \Big(\phi(\sigma_2^{\varepsilon\kappa}) - \phi(\sigma_1^{\varepsilon\kappa}) \Big) \\
&= c^2 \left(\sigma \left[\frac{\rho}{c^2-v^2} \right] - \left[\frac{\rho v}{c^2-v^2} \right] \right) \Big(\sigma\phi'(\sigma) - \sigma\phi'(\sigma) - \phi(\sigma) + \phi(\sigma) \Big) \\
&= 0.
\end{aligned} \tag{5.13}
$$

Meanwhile, the first and the second integral in (5.9) on the right can be composed into

$$
\begin{aligned}
&\left(\int_{-\infty}^{\sigma_1^{\varepsilon\kappa}} + \int_{\sigma_1^{\varepsilon\kappa}}^{\sigma_2^{\varepsilon\kappa}} + \int_{\sigma_2^{\varepsilon\kappa}}^{+\infty} \right) \left(\frac{\kappa^2(\rho^{\varepsilon\kappa})^\gamma (v^{\varepsilon\kappa})^2}{c^2\big(c^2-(v^{\varepsilon\kappa})^2\big)} \xi - \frac{\kappa^2(\rho^{\varepsilon\kappa})^\gamma v^{\varepsilon\kappa}}{c^2-(v^{\varepsilon\kappa})^2} \right) \phi' d\xi \\
&= \frac{\kappa^2\rho_-^\gamma v_-^2}{c^2\big(c^2-v_-^2\big)} \sigma_1^{\varepsilon\kappa}\phi(\sigma_1^{\varepsilon\kappa}) - \frac{\kappa^2\rho_+^\gamma v_+^2}{c^2\big(c^2-v_+^2\big)} \sigma_2^{\varepsilon\kappa}\phi(\sigma_2^{\varepsilon\kappa}) - \frac{\kappa^2\rho_-^\gamma v_-}{c^2-v_-^2} \phi(\sigma_1^{\varepsilon\kappa}) \\
&\quad + \frac{\kappa^2\rho_+^\gamma v_+}{c^2-v_+^2} \phi(\sigma_2^{\varepsilon\kappa}) + \frac{\kappa^2(\rho_*^{\varepsilon\kappa})^\gamma (v_*^{\varepsilon\kappa})^2}{c^2\big(c^2-(v_*^{\varepsilon\kappa})^2\big)} \Big(\sigma_2^{\varepsilon\kappa}\phi(\sigma_2^{\varepsilon\kappa}) - \sigma_1^{\varepsilon\kappa}\phi(\sigma_1^{\varepsilon\kappa}) \Big) \\
&\quad - \frac{\kappa^2(\rho_*^{\varepsilon\kappa})^\gamma v_*^{\varepsilon\kappa}}{c^2-(v_*^{\varepsilon\kappa})^2} \Big(\phi(\sigma_2^{\varepsilon\kappa}) - \phi(\sigma_1^{\varepsilon\kappa}) \Big) - \frac{\kappa^2\rho_-^\gamma v_-^2}{c^2\big(c^2-v_-^2\big)} \int_{-\infty}^{\sigma_1^{\varepsilon\kappa}} \phi d\xi \\
&\quad - \frac{\kappa^2\rho_+^\gamma v_+^2}{c^2\big(c^2-v_+^2\big)} \int_{\sigma_2^{\varepsilon\kappa}}^{+\infty} \phi d\xi - \frac{\kappa^2(\rho_*^{\varepsilon\kappa})^\gamma (v_*^{\varepsilon\kappa})^2}{c^2\big(c^2-(v_*^{\varepsilon\kappa})^2\big)} \int_{\sigma_1^{\varepsilon\kappa}}^{\sigma_2^{\varepsilon\kappa}} \phi d\xi.
\end{aligned} \tag{5.14}
$$

While for the last term of (5.9) on the right, we have

$$
\begin{aligned}
&\int_{-\infty}^{+\infty} \frac{\kappa^2(\rho^{\varepsilon\kappa})^\gamma (v^{\varepsilon\kappa})^2}{c^2\big(c^2-(v^{\varepsilon\kappa})^2\big)} \phi d\xi \\
&= \frac{\kappa^2\rho_-^\gamma v_-^2}{c^2\big(c^2-v_-^2\big)} \int_{-\infty}^{\sigma_1^{\varepsilon\kappa}} \phi d\xi + \frac{\kappa^2\rho_+^\gamma v_+^2}{c^2\big(c^2-v_+^2\big)} \int_{\sigma_2^{\varepsilon\kappa}}^{+\infty} \phi d\xi + \frac{\kappa^2(\rho_*^{\varepsilon\kappa})^\gamma (v_*^{\varepsilon\kappa})^2}{c^2\big(c^2-(v_*^{\varepsilon\kappa})^2\big)} \int_{\sigma_1^{\varepsilon\kappa}}^{\sigma_2^{\varepsilon\kappa}} \phi d\xi.
\end{aligned} \tag{5.15}
$$

Owing to (5.14) and (5.15), we deduce that the expression of the right-hand of (5.9) converges to 0 as $\varepsilon, \kappa \to 0$.

Returning to (5.9), we immediately obtain that

$$
\lim_{\substack{\varepsilon\to 0\\ \kappa\to 0}} \int_{-\infty}^{+\infty} \left(\frac{\rho^{\varepsilon\kappa}}{c^2-(v^{\varepsilon\kappa})^2} - H(\xi-\sigma) \right) \phi d\xi = \left(\sigma \left[\frac{\rho}{c^2-v^2} \right] - \left[\frac{\rho v}{c^2-v^2} \right] \right) \phi(\sigma), \tag{5.16}
$$

for any test function $\phi \in C_0^\infty(-\infty, +\infty)$.

Let us proceed to consider the limit of $\frac{\rho^{\varepsilon\kappa}v^{\varepsilon\kappa}}{c^2-(v^{\varepsilon\kappa})^2}$. In a similar way as before, noticing the facts that $\kappa^2(\rho_*^{\varepsilon\kappa})^\gamma$ is bounded and $\lim_{\substack{\varepsilon\to 0\\ \kappa\to 0}} \sigma_1^{\varepsilon\kappa} = \lim_{\substack{\varepsilon\to 0\\ \kappa\to 0}} \sigma_2^{\varepsilon\kappa} = \sigma$, one can get that

$$\lim_{\substack{\varepsilon\to 0\\ \kappa\to 0}} \int_{-\infty}^{+\infty} \left(\frac{\rho^{\varepsilon\kappa}v^{\varepsilon\kappa}}{c^2-(v^{\varepsilon\kappa})^2} - \widetilde{H}(\xi-\sigma) \right)\phi d\xi = \left(\sigma \left[\frac{\rho v}{c^2-v^2} \right] - \left[\frac{\rho v^2}{c^2-v^2} \right] \right)\phi(\sigma), \tag{5.17}$$

where

$$\widetilde{H}(\xi-\sigma) = \begin{cases} \dfrac{\rho_- v_-}{c^2-v_-^2}, & \xi<\sigma, \\ \dfrac{\rho_+ v_+}{c^2-v_+^2}, & \xi>\sigma. \end{cases}$$

(iii). Examine the limits of $\frac{\rho^{\varepsilon\kappa}}{c^2-(v^{\varepsilon\kappa})^2}$ and $\frac{\rho^{\varepsilon\kappa}v^{\varepsilon\kappa}}{c^2-(v^{\varepsilon\kappa})^2}$ depending on time t. Let $\psi(t,x)\in C_0^\infty(R^+\times R^1)$. Noticing (5.16), we obtain that

$$\begin{aligned}
&\lim_{\substack{\varepsilon\to 0\\ \kappa\to 0}} \int_0^{+\infty}\int_{-\infty}^{+\infty} \frac{\rho^{\varepsilon\kappa}(x/t)}{c^2-(v^{\varepsilon\kappa}(x/t))^2}\psi(t,x)dxdt \\
&= \lim_{\substack{\varepsilon\to 0\\ \kappa\to 0}} \int_0^{+\infty} t\Big(\int_{-\infty}^{+\infty} \frac{\rho^{\varepsilon\kappa}(\xi)}{c^2-(v^{\varepsilon\kappa}(\xi))^2}\psi(t,\xi t)d\xi \Big)dt \\
&= \int_0^{+\infty} t\left(\left(\sigma\left[\frac{\rho}{c^2-v^2} \right] - \left[\frac{\rho v}{c^2-v^2} \right] \right)\psi(t,\sigma t) + \int_{-\infty}^{+\infty} H(\xi-\sigma)\psi(t,\xi t)d\xi \right)dt \\
&= \int_0^{+\infty} t\left(\int_{-\infty}^{+\infty} H(\xi-\sigma)\psi(t,\xi t)d\xi \right)dt + \int_0^{+\infty} t\left(\sigma\left[\frac{\rho}{c^2-v^2} \right] - \left[\frac{\rho v}{c^2-v^2} \right] \right)\psi(t,\sigma t)dt \\
&= \int_0^{+\infty}\int_{-\infty}^{+\infty} H(x-\sigma t)\psi(t,x)dxdt + \int_0^{+\infty} \left(\sigma\left[\frac{\rho}{c^2-v^2} \right] - \left[\frac{\rho v}{c^2-v^2} \right] \right)t\psi(t,\sigma t)dt.
\end{aligned} \tag{5.18}$$

Then by Definition 2.1, for the last term on the right-hand side of (5.18), we have

$$\int_0^{+\infty} \left(\sigma\left[\frac{\rho}{c^2-v^2} \right] - \left[\frac{\rho v}{c^2-v^2} \right] \right)t\psi(t,\sigma t)dt = \big\langle w_1(\cdot)\delta_S, \psi(\cdot,\cdot) \big\rangle$$

where

$$w_1(t) = \frac{(c^2-\sigma^2)t}{\sqrt{1+\sigma^2}} \left(\sigma\left[\frac{\rho}{c^2-v^2} \right] - \left[\frac{\rho v}{c^2-v^2} \right] \right).$$

Similarly, we can show that

$$\begin{aligned}
&\lim_{\substack{\varepsilon\to 0\\ \kappa\to 0}} \int_0^{+\infty}\int_{-\infty}^{+\infty} \frac{\rho^{\varepsilon\kappa}(x/t)v^{\varepsilon\kappa}(x/t)}{c^2-(v^{\varepsilon\kappa}(x/t))^2}\psi(t,x)dxdt \\
&= \int_0^{+\infty}\int_{-\infty}^{+\infty} \widetilde{H}(x-\sigma t)\psi(t,x)dxdt + \int_0^{+\infty} \left(\sigma\left[\frac{\rho v}{c^2-v^2} \right] - \left[\frac{\rho v^2}{c^2-v^2} \right] \right)t\psi(t,\sigma t)dt,
\end{aligned} \tag{5.19}$$

in which

$$\int_0^{+\infty}\left(\sigma\left[\frac{\rho v}{c^2-v^2}\right]-\left[\frac{\rho v^2}{c^2-v^2}\right]\right)t\psi(t,\sigma t)dt=\langle w_2(\cdot)\delta_S,\psi(\cdot,\cdot)\rangle,$$

where

$$w_2(t)=\frac{(c^2-\sigma^2)t}{\sqrt{1+\sigma^2}}\left(\sigma\left[\frac{\rho v}{c^2-v^2}\right]-\left[\frac{\rho v^2}{c^2-v^2}\right]\right).$$

The proof of Theorem 5.6 is finished. □

6. Formation of vacuum states for the system (1.4)

This section focuses on the limit behavior of the solution of Riemann problem (1.4), (1.6) in the case $(\rho_+,v_+)\in\overleftarrow{R}\overrightarrow{R}(\rho_-,v_-)$ with $v_-<v_+$ as $\varepsilon,\kappa\to 0$.

According to Section 4, we know that on the rarefaction waves the solution satisfies

$$\xi=\frac{2v^{\varepsilon\kappa}c^2\big(c^2-p'(\rho^{\varepsilon\kappa})\big)-\sqrt{Q(\rho^{\varepsilon\kappa},v^{\varepsilon\kappa})}+M(\rho^{\varepsilon\kappa},v^{\varepsilon\kappa})v^{\varepsilon\kappa}p'(\rho^{\varepsilon\kappa})}{2\big(c^4-(v^{\varepsilon\kappa})^2p'(\rho^{\varepsilon\kappa})\big)}:=K(\rho^{\varepsilon\kappa},v^{\varepsilon\kappa}),$$

and

$$\xi=\frac{2v^{\varepsilon\kappa}c^2\big(c^2-p'(\rho^{\varepsilon\kappa})\big)+\sqrt{Q(\rho^{\varepsilon\kappa},v^{\varepsilon\kappa})}+M(\rho^{\varepsilon\kappa},v^{\varepsilon\kappa})v^{\varepsilon\kappa}p'(\rho^{\varepsilon\kappa})}{2\big(c^4-(v^{\varepsilon\kappa})^2p'(\rho^{\varepsilon\kappa})\big)}:=L(\rho^{\varepsilon\kappa},v^{\varepsilon\kappa}),$$

for each fixed $\varepsilon,\kappa>0$, respectively. Precisely, on the backward rarefaction wave, the solution satisfies

$$\xi=K(\rho^{\varepsilon\kappa},v^{\varepsilon\kappa}),\quad K(\rho_-,v_-)<\xi<K(\rho_*^{\varepsilon\kappa},v_*^{\varepsilon\kappa}),\quad \rho_*^{\varepsilon\kappa}<\rho_-,\tag{6.1}$$

and, on the forward rarefaction wave,

$$\xi=L(\rho^{\varepsilon\kappa},v^{\varepsilon\kappa}),\quad L(\rho_*^{\varepsilon\kappa},v_*^{\varepsilon\kappa})<\xi<L(\rho_+,v_+),\quad \rho_*^{\varepsilon\kappa}<\rho_+.\tag{6.2}$$

Now, we conclude the following theorem.

Theorem 6.1. *Let $v_-<v_+$. For each fixed pair $\varepsilon,\kappa>0$, assume that $(\rho^{\varepsilon\kappa},v^{\varepsilon\kappa})$ is a two-rarefaction wave solution of (1.4), (1.6) constructed in Section 4. Then, there exists $\varepsilon_0>0$, as $0<\varepsilon<\varepsilon_0$ and $0<\kappa<\varepsilon_0$, the pseudo-vacuum state solution $\rho=2\varepsilon(1-v^2/c^2)$ appears in solution. And as $\varepsilon,\kappa\to 0$, the two-rarefaction waves become two contact discontinuities connecting the constant states $(\rho_\pm,v_\pm)$ and the vacuum state $\rho=0$, which form a vacuum solution of (1.3), (1.6).*

Proof. Set $\varepsilon=\kappa=\varepsilon_0$. Since $(\rho_*^{\varepsilon\kappa},v_*^{\varepsilon\kappa})$ is on the curve $\overleftarrow{R}(\rho_-,v_-)$, then we have

$$\ln\frac{c+v_*^{\varepsilon\kappa}}{c-v_*^{\varepsilon\kappa}}=\ln\frac{c+v_-}{c-v_-}-2c\int_{\rho_-}^{\rho_*^{\varepsilon\kappa}}f(s,v_*^{\varepsilon\kappa})ds\le\ln\frac{c+v_-}{c-v_-}+2c\int_{2\varepsilon(1-(v_*^{\varepsilon\kappa})^2/c^2)}^{\rho_-}f(s,v_*^{\varepsilon\kappa})ds:=A^{\varepsilon_0}.$$

When $\ln\frac{c+v_-}{c-v_-}<\ln\frac{c+v_+}{c-v_+}<A^{\varepsilon_0}$, there is no pseudo-vacuum state in the solution. That is, there exists ε_{01} such that $(\rho_+,v_+)\in I(\rho_-,v_-)$ when $\ln\frac{c+v_-}{c-v_-}<\ln\frac{c+v_+}{c-v_+}<A^{\varepsilon_{01}}$.

However, when $A^{\varepsilon_0} < \ln \frac{c+v_+}{c-v_+}$, the pseudo-vacuum state solution appears, which implies that there exists ε_{02} such that $(\rho_+, v_+) \in V(\rho_-, v_-)$ when $A^{\varepsilon_{02}} < \ln \frac{c+v_+}{c-v_+}$.

Let

$$J(\varepsilon; v_*^{\varepsilon\kappa}) = 2c \int\limits_{2\varepsilon(1-(v_*^{\varepsilon\kappa})^2/c^2)}^{\rho_-} f(s, v_*^{\varepsilon\kappa})ds + \ln \frac{c+v_-}{c-v_-} - \ln \frac{c+v_+}{c-v_+}.$$

As done in Section 4, we can deduce that the integral $\int\limits_{2\varepsilon(1-(v_*^{\varepsilon\kappa})^2/c^2)}^{\rho_-} f(s, v_*^{\varepsilon\kappa})ds$ is uniformly convergent in ε with $v_*^{\varepsilon\kappa}$ a parameter, then the function $J(\varepsilon; v_*^{\varepsilon\kappa})$ is continuous with respect to ε and $J(\varepsilon_{01}; v_*^{\varepsilon\kappa})J(\varepsilon_{02}; v_*^{\varepsilon\kappa}) < 0$. Thus, there exists $\varepsilon_0 \in [\varepsilon_{02}, \varepsilon_{01}]$ such that $J(\varepsilon_0; v_*^{\varepsilon\kappa}) = 0$.

As a result, as $0 < \varepsilon < \varepsilon_0$ and $0 < \kappa < \varepsilon_0$, the density of the intermediate state becomes a pseudo-vacuum state with

$$(\rho_*^{\varepsilon\kappa}, v_*^{\varepsilon\kappa})(\xi) = \big(2\varepsilon(1 - v_*^{\varepsilon\kappa}/c^2), \xi\big), \quad v_1^{\varepsilon\kappa} \le \xi \le v_2^{\varepsilon\kappa}, \tag{6.3}$$

where

$$v_1^{\varepsilon\kappa} = \frac{c\left((c+v_-)\exp\left(2c \int\limits_{2\varepsilon(1-(v_*^{\varepsilon\kappa})^2/c^2)}^{\rho_-} f(s, v_*^{\varepsilon\kappa})ds\right) - (c-v_-)\right)}{(c+v_-)\exp\left(2c \int\limits_{2\varepsilon(1-(v_*^{\varepsilon\kappa})^2/c^2)}^{\rho_-} f(s, v_*^{\varepsilon\kappa})ds\right) + (c-v_-)}, \tag{6.4}$$

and

$$v_2^{\varepsilon\kappa} = \frac{c\left((c+v_+)\exp\left(-2c \int\limits_{2\varepsilon(1-(v_*^{\varepsilon\kappa})^2/c^2)}^{\rho_+} \bar{f}(s, v_*^{\varepsilon\kappa})ds\right) - (c-v_+)\right)}{(c+v_+)\exp\left(-2c \int\limits_{2\varepsilon(1-(v_*^{\varepsilon\kappa})^2/c^2)}^{\rho_+} \bar{f}(s, v_*^{\varepsilon\kappa})ds\right) + (c-v_+)}, \tag{6.5}$$

where

$$\bar{f}(\rho, v) = \frac{2c^2(c^2 - v^2)^2 p'(\rho)}{\left(\sqrt{Q(\rho, v)} - M(\rho, v)vp'(\rho)\right)\left(p(\rho) + \rho c^2\right)}.$$

Thus, letting $\varepsilon, \kappa \to 0$, one can find that $\lim_{\varepsilon,\kappa\to 0} \rho_*^{\varepsilon\kappa} = 0$. Using the boundedness of $\rho^{\varepsilon\kappa}$ with respect to ε and κ, it follows that

$$\lim_{\substack{\varepsilon\to 0\\ \kappa\to 0}} v_1^{\varepsilon\kappa} = v_-, \quad \lim_{\substack{\varepsilon\to 0\\ \kappa\to 0}} v_2^{\varepsilon\kappa} = v_+.$$

$$\lim_{\substack{\varepsilon\to 0\\ \kappa\to 0}} v^{\varepsilon\kappa}(\xi) = \xi \quad \text{for} \quad \xi \in (v_-, v_+).$$

In summary, the limit solution can be expressed as (2.1), which is a solution of (1.3) containing two contact discontinuities $\xi = x/t = v_\pm$ and a vacuum state in between. This completes the proof of Theorem 6.1. □

Acknowledgments

The authors specially thank the referee for many valuable suggestions to revise this paper.

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