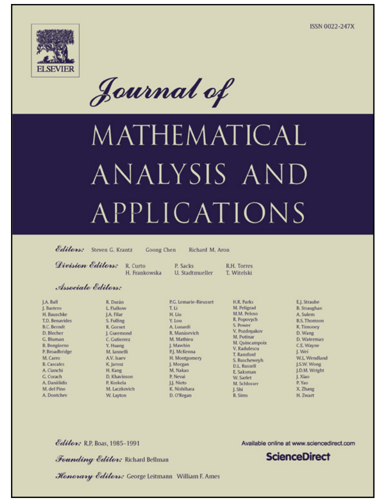

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Finite time blow-up for a thin-film equation with initial data at arbitrary energy level

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Abstract

In this paper, we consider the initial boundary value problem for a class of thin-film equations in $\mathbb{R}^n$ with a p-Laplace term and a nonlocal source term $|u|^{q-2}u - \frac{1}{|\Omega|}\int_\Omega |u|^{q-2}u\mathrm{dx}$. We prove that there exist weak solutions for the problem with arbitrarily initial energy that blow up in finite time. We also obtain the upper bounds for the blow-up time.

Keywords: Thin-film equation, blow-up, upper bound for blow-up time, arbitrary initial energy, nonlocal source

MR Subject Classification 2010: 35K35, 35K92.

1 Introduction

The thin-film equation

$$u_t - \Delta^2 u + \nabla \cdot (f(\nabla u)) = g$$

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can be used to describe the evolution of the epitaxial growth of nanoscale thin films [1–3]. Various mathematical aspects of the thin-film equation have been studied by many researchers in recent years, as reported in [4–11] and the references therein. In [12], Qu and Zhou studied the following thin-film equation with a nonlocal source term

$$\begin{cases} u_t + u_{xxxx} = |u|^{p-1}u - \frac{1}{|\Omega|}\int_\Omega |u|^{p-1}u\mathrm{dx}, & (x,t) \in \Omega \times (0,T), \\ u_x = u_{xxx} = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega = (0,a)$, $p > 1$, and $u_0 \in H^2(\Omega)$ with $\int_\Omega u_0\mathrm{dx} = 0$ and $u_0 \not\equiv 0$. By using the potential well method, the authors obtained a threshold result for the global existence and finite time blow-up of the weak solutions with initial data at low energy level (*i.e.* $J(u_0) \leq d$). They also studied the extinction of the solutions for the problem under some conditions. In [13], Li et al. studied the following thin-film equation with the same initial and boundary conditions

$$u_t + u_{xxxx} - \left(|u_x|^{p-2}u_x\right)_x = |u|^{q-1}u - \frac{1}{|\Omega|}\int_\Omega |u|^{q-1}u\mathrm{dx}, \quad (\mathrm{x},\mathrm{t}) \in (0,\mathrm{a}) \times (0,\mathrm{T}),$$

where $p > 1$, $q > \max\{1, p-1\}$, and obtained similar results by using the potential well method. In [14], Zhou considered problem (1.1) and established a new blow-up result for the case that the initial energy is positive but upper bounded. He also gave an estimate for the upper bound of the blow-up time.

To our knowledge, no results have been obtained about the global existence, extinction or finite time blow-up for the solutions of the thin-film equation in $\mathbb{R}^n$ with initial data at high energy level. In this paper, we are concerned with the finite time blow-up for the weak solutions of the following initial-boundary value problem

$$\begin{cases} u_t + \Delta^2 u - \Delta_p u = |u|^{q-2}u - \frac{1}{|\Omega|}\int_\Omega |u|^{q-2}u\mathrm{dx}, & (x,t) \in \Omega \times (0,T), \\ \frac{\partial u}{\partial n} = \frac{\partial(\Delta u)}{\partial n} = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.2}$$

where Ω is a bounded domain of $\mathbb{R}^n$ ($n \geq 1$) with smooth boundary $\partial\Omega$, $p > 1$, $q > 2$, $\Delta_p u = \mathrm{div}\left(|\nabla \mathrm{u}|^{\mathrm{p}-2}\nabla \mathrm{u}\right)$, $u_0 \in H^2(\Omega)$ such that $\int_\Omega u_0\mathrm{dx} = 0$ and $u_0 \not\equiv 0$. In addition, we assume p, q satisfy the following conditions:

(H1) $2 < q < +\infty$, if $n \leq 4$; $2 < q < \frac{2n}{n-4}$, if $n > 4$.

(H2) $1 < p < +\infty$, if $n \leq 2$; $1 < p < \frac{2n}{n-2}$, if $n > 2$.

(H3) $q > \max\{2, p\}$.

By applying the technique similar to that in [9, 15] and using Levine's concavity method [16], we show that there are weak solutions of problem (1.2) with arbitrarily initial energy that blow up in finite time. We also obtain the upper bounds for the blow-up time.

2 Preliminaries

For convenience, we denote the L^r-norm ($1 \leq r \leq \infty$) by $|\cdot|_r$ and the usual norm of $H^2(\Omega)$ by $\|u\|_{H^2} = \left(\int_\Omega (|u|^2 + |\nabla u|^2 + |\Delta u|^2)\,\mathrm{dx}\right)^{\frac{1}{2}}$. We also denote $(\cdot,\cdot)$ as the inner product on the Hilbert space $L^2(\Omega)$.

Recalling that $u_0 \in H^2(\Omega)$ with $\int_\Omega u_0 \mathrm{dx} = 0$ and considering the homogenous Neumann boundary condition of problem (1.2), we introduce the following space

$$H_N^2(\Omega) := \left\{ u \in H^2(\Omega) \,:\, \frac{\partial u}{\partial n}\Big|_{\partial\Omega} = 0 \text{ and } \int_\Omega u\mathrm{dx} = 0 \right\}.$$

For every $u \in H_N^2(\Omega)$, we have

$$|\nabla u|_2^2 = \left| \int_\Omega \nabla u \cdot \nabla u \mathrm{dx} \right| = \left| \int_{\partial\Omega} u \frac{\partial u}{\partial n} \mathrm{dS} - \int_\Omega \mathrm{u}\Delta \mathrm{udx} \right| = \left| \int_\Omega u \Delta u \mathrm{dx} \right| \leq |u|_2 |\Delta u|_2. \tag{2.1}$$

Let $\lambda_N > 0$ be the first nontrivial eigenvalue of $-\Delta$ in Ω with homogeneous Neumann boundary condition $\frac{\partial u}{\partial n}\big|_{\partial\Omega} = 0$. By (2.1), we have

$$\lambda_N^2 |u|_2^2 \leq \lambda_N |\nabla u|_2^2 \leq |\Delta u|_2^2$$

for all $u \in H_N^2(\Omega)$. Moreover, following from [4] and [17], $H_N^2(\Omega)$ is a Hilbert space with the inner product $(u,v)_N := \int_\Omega \Delta u \Delta v \mathrm{dx}$ and the norm $\|u\| := (u,u)_N^{\frac{1}{2}} = |\Delta u|_2$ is equivalent to the usual norm $\|\cdot\|_{H^2}$ in $H_N^2(\Omega)$.

In this paper, we consider the weak solutions as defined below:

Definition 2.1 *A function $u \in L^\infty(0,T;H_N^2(\Omega))$ with $u_t \in L^2(0,T;H_N^2(\Omega))$ is said to be a weak solution of problem (1.2) if $u(0)=u_0 \in H_N^2(\Omega)$ and*

$$\int_0^t \int_\Omega \left[u_t\phi + \Delta u \Delta \phi + |\nabla u|^{p-2}\nabla u \cdot \nabla \phi - \left(|u|^{q-2}u - \frac{1}{|\Omega|}\int_\Omega |u|^{q-2}u\mathrm{dy} \right)\phi \right] \mathrm{dxd}\tau = 0$$

for all $\phi \in L^2(0,T;H_N^2(\Omega))$.

By using the argument similar to that in [4], we can obtain the local existence, uniqueness and regularity for the weak solutions of problem (1.2).

Suppose that u is a weak solution of problem (1.2) with initial data u_0 and T^* is the maximal existence time of u. Integrating equation (1.2) over Ω, in view of the boundary condition, we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{dt}}\int_\Omega u\mathrm{dx} = \int_\Omega \mathrm{u_t dx} &= \int_\Omega \left[-\Delta^2 u + \Delta_p u + |u|^{q-2}u - \left(\frac{1}{|\Omega|}\int_\Omega |u|^{q-2}u\mathrm{dy} \right) \right] \mathrm{dx} \\ &= -\int_{\partial\Omega} \frac{\partial(\Delta u)}{\partial n}\mathrm{ds} + \int_{\partial\Omega} |\nabla \mathrm{u}|^{\mathrm{p}-2}\frac{\partial \mathrm{u}}{\partial \mathrm{n}}\mathrm{ds} \\ &\quad + \int_\Omega |u|^{p-2}u\mathrm{dx} - \left(\frac{1}{|\Omega|}\int_\Omega |\mathrm{u}|^{\mathrm{q}-2}\mathrm{udy} \right) \int_\Omega \mathrm{dx} \\ &= 0, \end{aligned} \tag{2.2}$$

so $\int_\Omega u(t)\mathrm{dx}$ is a constant for all $t \in [0,T^*)$. Since $u_0 \in H^2(\Omega)$, $u_0 \not\equiv 0$ and $\frac{1}{|\Omega|}\int_\Omega u_0 \mathrm{dx} = 0$, we obtain that

$$\int_\Omega u(t)\mathrm{dx} = \int_\Omega \mathrm{u_0 dx} = 0 \quad \text{for all } \mathrm{t} \in [0,\mathrm{T}^*). \tag{2.3}$$

Since u is a nontrivial solution of problem (1.2), (2.3) implies that u is a sign-changing solution.

Multiplying equation (1.2) with u_t and then integrating the equation over Ω by parts, in view of the boundary condition, we have

$$\begin{aligned} & (|u|^{q-2}u, u_t) - \left(\frac{1}{|\Omega|}\int_\Omega |u|^{q-2}u\mathrm{dy}, \mathrm{u_t} \right) \\ = \;& |u_t|_2^2 + (\Delta^2 u, u_t) - (\Delta_p u, u_t) \\ = \;& |u_t|_2^2 + \left(\int_{\partial\Omega} u_t \frac{\partial(\Delta u)}{\partial n}\mathrm{ds} - \int_{\partial\Omega} \Delta \mathrm{u}\frac{\partial \mathrm{u_t}}{\partial \mathrm{n}}\mathrm{ds} + \int_\Omega \Delta \mathrm{u}\ \mathrm{div}(\nabla \mathrm{u_t})\mathrm{dx} \right) \\ & - \left(\int_{\partial\Omega} u_t|\nabla u|^{p-2}\frac{\partial u}{\partial n}\mathrm{ds} - \int_\Omega |\nabla \mathrm{u}|^{\mathrm{p}-2}\nabla \mathrm{u}\cdot\nabla \mathrm{u_t dx} \right) \\ = \;& |u_t|_2^2 + \frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{1}{2}|\Delta u|_2^2\mathrm{dx} \right) + \frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{1}{p}|\nabla u|_p^p \right). \end{aligned}$$

On the other hand, by (2.2) we get

$$\left(|u|^{q-2}u, u_t\right) - \left(\frac{1}{|\Omega|}\int_\Omega |u|^{q-2}u\mathrm{dy}, \mathrm{u_t}\right) = \left(|u|^{q-2}u, u_t\right) - \frac{1}{|\Omega|}\int_\Omega |u|^{q-2}u\mathrm{dy}\int_\Omega \mathrm{u_t dx} = \frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{1}{\mathrm{q}}|\mathrm{u}|_\mathrm{q}^\mathrm{q}\right).$$

So

$$\frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{1}{2}|\Delta u|_2^2 + \frac{1}{p}|\nabla u|_p^p - \frac{1}{q}|u|_q^q\right) = -|u_t|_2^2. \tag{2.4}$$

Motivated by the calculation above, we introduce the energy functional J and the Nehari functional I on $H_N^2(\Omega)$ for problem (1.2) by

$$J(u) = \frac{1}{2}|\Delta u|_2^2 + \frac{1}{p}|\nabla u|_p^p - \frac{1}{q}|u|_q^q, \tag{2.5}$$

$$I(u) = |\Delta u|_2^2 + |\nabla u|_p^p - |u|_q^q. \tag{2.6}$$

Since p, q satisfy the assumptions $(H1)$ and $(H2)$, the functionals J and I are well-defined and continuous on $H_N^2(\Omega)$. Furthermore, simple calculation shows that

$$J(u) = \frac{q-2}{2q}|\Delta u|_2^2 + \frac{q-p}{pq}|\nabla u|_p^p + \frac{1}{q}I(u). \tag{2.7}$$

By using the Fountain Theorem, we obtain the following lemma.

Lemma 2.2 *Assume that p, q satisfy $(H1)$-$(H3)$. Then the functional J on $H_0^2(\Omega) \cap H_N^2(\Omega)$ has a sequence of critical points $\{u_k\}$ such that*

$$J(u_k) = \frac{1}{2}|\Delta u_k|_2^2 + \frac{1}{p}|\nabla u_k|_p^p - \frac{1}{q}|u_k|_p^p \to +\infty, \quad \text{as } k \to \infty.$$

Proof. The proof is similar to that of Theorem 1.1 in [17] with a few modifications. □

In view of (2.4), we have the following lemma.

Lemma 2.3 *Assume that p, q satisfy $(H1)$ and $(H2)$ and u is a solution of problem (1.2) with initial data u_0. Then $\frac{\mathrm{d}}{\mathrm{dt}}J(u(t)) \le 0$ and*

$$J(u(t)) + \int_0^t |u_t(\tau)|_2^2 \mathrm{d}\tau = \mathrm{J(u_0)} \quad \text{for all } \mathrm{t} \in [0, \mathrm{T}^*). \tag{2.8}$$

3 Main results

Theorem 3.1 *Assume that p, q satisfy (H1)-(H3) and u is a weak solution of problem (1.2) with initial data $u_0 \in H_N^2(\Omega)$. Suppose that one of the following statements holds:*

(i) $J(u_0) < 0$;

(ii) $0 \leq J(u_0) < \frac{C^*}{2q}|u_0|_2^2$, *where* $C^* = (q-2)\lambda_N^2$.

Then $T^* < \infty$, which means that u blows up in finite time. Moreover, the upper bound for T^* is estimated as follows:

- *In case (i),*

$$T^* \leq \frac{|u_0|_2^2}{(2-q)qJ(u_0)}.$$

- *In case (ii),*

$$T^* \leq \frac{8q|u_0|_2^2}{(q-2)^2\left(C^*|u_0|_2^2 - 2qJ(u_0)\right)}.$$

Proof.

(i) For the case of $J(u_0) < 0$, let

$$\theta(t) = \frac{1}{2}|u(t)|_2^2, \quad \eta(t) = -J(u(t)),$$

then $\theta(0) > 0$, $\eta(0) > 0$. By (2.7) and Lemma 2.3, we have

$$\eta'(t) = -\frac{\mathrm{d}}{\mathrm{dt}}J(u(t)) = |u_t(t)|_2^2 \geq 0,$$

then $\eta(t) \geq \eta(0) > 0$ for all $t \in [0, T^*)$. Since $q > \max\{2, p\}$ and in view of the boundary condition, (2.3), (2.6) and (2.7), it holds that

$$\begin{aligned}
\theta'(t) &= \Big(u(t), u_t(t)\Big) \\
&= \Big(u(t),\ -\Delta^2 u(t) + \Delta_p u(t) + |u(t)|^{q-2}u(t) - \frac{1}{|\Omega|}\int_\Omega |u|^{q-2}u\mathrm{dy}\Big) \\
&= -|\Delta u(t)|_2^2 - |\nabla u(t)|_p^p + |u(t)|_q^q \\
&= -I(u(t)) \\
&= \frac{q-2}{2}|\Delta u(t)|_2^2 + \frac{q-p}{p}|\nabla u(t)|_p^p - qJ(u(t))
\end{aligned}$$

$$\geq \quad q\eta(t) \tag{3.1}$$

for all $t \in [0, T^*)$. Using the Cauchy-Schwartz inequality, we obtain that

$$\theta(t)\eta'(t) \geq \frac{1}{2}|u(t)|_2^2|u_t(t)|_2^2 \geq \frac{1}{2}\Big((u(t), u_t(t))\Big)^2 = \frac{1}{2}\left(\theta'(t)\right)^2 \geq \frac{q}{2}\theta'(t)\eta(t), \tag{3.2}$$

By (3.2) and through simple calculation, we have

$$\left(\eta\theta^{-\frac{q}{2}}\right)' = \theta^{-\frac{q}{2}-1}\left(\theta\eta' - \frac{q}{2}\eta\theta'\right) \geq 0,$$

so

$$0 < M := \eta(0)\theta^{-\frac{q}{2}}(0) \leq \eta(t)\theta^{-\frac{q}{2}}(t) \leq \frac{1}{q}\theta'(t)\theta^{-\frac{q}{2}}(t) = \frac{2}{(2-q)q}\left(\theta^{-\frac{q-2}{2}}(t)\right)'. \tag{3.3}$$

Since $q > 2$, integrating (3.3) yields

$$0 \leq \theta^{-\frac{q-2}{2}}(t) \leq -\frac{(q-2)q}{2}Mt + \theta^{-\frac{q-2}{2}}(0), \quad t \in [0, T^*(u_0)). \tag{3.4}$$

Note that the above inequality (3.4) does not hold for all $t > 0$. So $T^* < +\infty$. Moreover, by (3.4), we get

$$T^* \leq \frac{2}{(q-2)qM}\theta^{-\frac{q-2}{2}}(0) = \frac{|u_0|_2^2}{(2-q)qJ(u_0)}.$$

(ii) For the case of $0 \leq J(u_0) < \frac{C^*}{2q}|u_0|_2^2$, suppose that the solution u of problem (1.2) with initial data u_0 is global.

Since

$$\int_0^t |u_t(\tau)|_2 \mathrm{d}\tau \geq \left|\int_0^{\mathrm{t}} \mathrm{u_t}(\tau)\mathrm{d}\tau\right|_2 = |\mathrm{u(t)} - \mathrm{u_0}|_2 \geq |\mathrm{u(t)}|_2 - |\mathrm{u_0}|_2, \quad \mathrm{t} \in [0, \infty),$$

by the Hölder's inequality and (2.8), we obtain that

$$|u(t)|_2 \leq |u_0|_2 + t^{\frac{1}{2}}\left(\int_0^t |u_t(\tau)|_2^2 \mathrm{d}\tau\right)^{\frac{1}{2}} = |u_0|_2 + t^{\frac{1}{2}}\Big(J(u_0) - J(u(t))\Big)^{\frac{1}{2}}. \tag{3.5}$$

Since u is a global solution of problem (1.2), we have $J(u(t)) \geq 0$ for all $t \in [0, \infty)$. Otherwise, there exists $t_0 \in (0, \infty)$ such that $J(u(t_0)) < 0$. From the first part of the proof, it follows that u blows up in finite time, which is a contradiction. Finally, (3.5) implies that

$$|u(t)|_2 \leq |u_0|_2 + t^{\frac{1}{2}}\Big(J(u_0) - J(u(t))\Big)^{\frac{1}{2}} \leq |u_0|_2 + t^{\frac{1}{2}}\Big(J(u_0)\Big)^{\frac{1}{2}} \quad \text{for all } t \in [0, \infty). \tag{3.6}$$

On the other hand, from (3.1), it follows that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{1}{2}|u(t)|_2^2\right) &= \theta'(t) = \frac{q-2}{2}|\Delta u(t)|_2^2 + \frac{q-p}{p}|\nabla u(t)|_p^p - qJ(u(t)) \\ &\geq \frac{q-2}{2}\lambda_N^2 |u(t)|_2^2 - qJ(u(t)) \\ &= C^*\left(\frac{1}{2}|u(t)|_2^2 - \frac{q}{C^*}J(u(t))\right). \end{aligned}$$

Since $\frac{\mathrm{d}}{\mathrm{dt}}J(u(t)) \leq 0$, we have

$$\frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{1}{2}|u(t)|_2^2 - \frac{q}{C^*}J(u(t))\right) \geq \frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{1}{2}|u(t)|_2^2\right) \geq C^*\left(\frac{1}{2}|u(t)|_2^2 - \frac{q}{C^*}J(u(t))\right).$$

Let $H(t) = \frac{1}{2}|u(t)|_2^2 - \frac{q}{C^*}J(u(t))$, then $\frac{\mathrm{d}}{\mathrm{dt}}H(t) \geq C^*H(t)$ for all $t \in [0, \infty)$. By using the Gronwall's inequality, we get $H(t) \geq e^{C^*t}H(0)$, hence

$$|u(t)|_2^2 \geq \frac{2q}{C^*}J(u(t)) + 2e^{C^*t}H(0), \quad t \in [0, \infty).$$

Since $0 \leq J(u_0) < \frac{C^*}{2q}|u_0|_2^2$, we have $H(0) > 0$. Recall that $0 \leq J(u(t)) \leq J(u_0)$ for all $t \in [0, \infty)$, we have $|u(t)|_2^2 \geq 2e^{C^*t}H(0)$, *i.e.*,

$$|u(t)|_2 \geq (2H(0))^{\frac{1}{2}}\, e^{\frac{C^*}{2}t} \quad \text{for all } t \in [0, \infty), \tag{3.7}$$

which contradicts (3.6) for t sufficiently large. So $T^* < +\infty$.

Next, we will find an upper bound for T^*.

By (2.7), we have

$$I(u_0) = qJ(u_0) - \frac{q-2}{2}|\Delta u_0|_2^2 - \frac{q-p}{p}|\nabla u_0|_p^p$$

$$
\begin{aligned}
&= q\left(J(u_0)-\frac{C^*}{2q}|u_0|_2^2\right)-\frac{q-2}{2}\left(|\Delta u_0|_2^2-\lambda_N^2|u_0|_2^2\right)-\frac{q-p}{p}|\nabla u_0|_p^p \\
&< 0.
\end{aligned}
$$

We claim that $I(u(t))<0$ for all $t\in[0,T^*)$. Otherwise, there exists $t_0\in(0,T^*)$ such that

$$
I(u(t_0))=0,\quad I(u(t))<0,\quad t\in[0,t_0).
$$

By (3.1), we have $\theta'(t)=-I(u(t))$, then $\theta(t)$ is strictly increasing on $[0,t_0)$, hence

$$
0\le J(u_0)<\frac{C^*}{2q}|u_0|_2^2<\frac{C^*}{2q}|u(t_0)|_2^2. \tag{3.8}
$$

On the other hand, by (2.7) and Lemma 2.3, we have

$$
\begin{aligned}
J(u_0)\ge J(u(t_0)) &= \frac{q-2}{2q}|\Delta u(t_0)|_2^2+\frac{q-p}{pq}|\nabla u(t_0)|_p^p+\frac{1}{q}I(u(t_0)) \\
&\ge \frac{q-2}{2q}|\Delta u(t_0)|_2^2\ge\frac{C^*}{2q}|u(t_0)|_2^2,
\end{aligned}
$$

which contradicts (3.8). So, $I(u(t))<0$ for all $t\in[0,T^*)$ and $\theta(t)$ is strictly increasing on $[0,T^*)$.

For any $T\in(0,T^*)$, $\alpha>0$, $\beta>0$ and $\sigma>0$, define the function

$$
F(t)=\int_0^t|u(\tau)|_2^2\mathrm{d}\tau+(\mathrm{T}^*-\mathrm{t})|\mathrm{u}_0|_2^2+\beta(\mathrm{t}+\sigma)^2,\quad \mathrm{t}\in[0,\mathrm{T}].
$$

In view of (2.7), (2.8) and (3.1), we have

$$
\begin{aligned}
F'(t) &= |u(t)|_2^2-|u_0|_2^2+2\beta(t+\sigma) \\
&= \int_0^t\frac{\mathrm{d}}{\mathrm{d}\tau}|u(\tau)|_2^2\mathrm{d}\tau+2\beta(\mathrm{t}+\sigma) \\
&= 2\int_0^t(u,u_t)\mathrm{d}\tau+2\beta(\mathrm{t}+\sigma), \qquad (3.9)\\
F''(t) &= 2(u,u_t)+2\beta \\
&= -2I(u(t))+2\beta \\
&= -2qJ(u(t))+(q-2)|\Delta u|_2^2+\frac{2(q-p)}{p}|\nabla u|_p^p+2\beta
\end{aligned}
$$

$$= \quad -2qJ(u_0) + 2q\int_0^t |u_t|_2^2 \mathrm{d}\tau + (\mathrm{q}-2)|\Delta \mathrm{u}|_2^2 + \frac{2(\mathrm{q}-\mathrm{p})}{\mathrm{p}}|\nabla \mathrm{u}|_\mathrm{p}^\mathrm{p} + 2\beta \tag{3.10}$$

for all $t \in [0,T]$. Notice that $F(0) = T^*|u_0|_2^2 + \beta\sigma^2 > 0$ and $F'(0) = 2\beta\sigma > 0$. Recalling that $\theta(t)$ is strictly increasing on $[0,T^*)$, it follows that $F'(t) > 0$ on $[0,T]$, that is, $F(t)$ is strictly increasing on $[0,T]$. By using the Cauchy-Schwartz inequality and Young's inequality, we have

$$\xi(t) := \left(\int_0^t |u|_2^2 \mathrm{d}\tau + \beta(\mathrm{t}+\sigma)^2\right)\left(\int_0^t |u_t|_2^2 \mathrm{d}\tau + \beta\right) - \left(\int_0^t (u,u_t)\mathrm{d}\tau + \beta(\mathrm{t}+\sigma)\right)^2 \geq 0 \tag{3.11}$$

for all $t \in [0,T]$. Therefore, in view of (3.9)-(3.11), for any $\alpha > 0$, we get

$$\begin{aligned}
& FF'' - \alpha\,(F')^2 \\
= \; & FF'' - 4\alpha\left(\int_0^t (u,u_t)\mathrm{d}\tau + \beta(\mathrm{t}+\sigma)\right)^2 \\
= \; & FF'' + 4\alpha\Bigg[\left(\int_0^t |u|_2^2 \mathrm{d}\tau + \beta(\mathrm{t}+\sigma)^2\right)\left(\int_0^t |u_t|_2^2 \mathrm{d}\tau + \beta\right) \\
& - \left(\int_0^t (u,u_t)\mathrm{d}\tau + \beta(\mathrm{t}+\sigma)\right)^2 - \left(F - (T^*-t)|u_0|_2^2\right)\left(\int_0^t |u_t|_2^2 \mathrm{d}\tau + \beta\right)\Bigg] \\
= \; & FF'' + 4\alpha\xi(t) + 4\alpha(T^*-t)|u_0|_2^2\left(\int_0^t |u_t|_2^2 \mathrm{d}\tau + \beta\right) - 4\alpha F\left(\int_0^t |u_t|_2^2 \mathrm{d}\tau + \beta\right) \\
\geq \; & F\left(F'' - 4\alpha\int_0^t |u_t|_2^2 \mathrm{d}\tau - 4\alpha\beta\right) \\
= \; & F\Bigg[-2qJ(u_0) + 2q\int_0^t |u_t|_2^2 \mathrm{d}\tau + (\mathrm{q}-2)|\Delta \mathrm{u}|_2^2 + \frac{2(\mathrm{q}-\mathrm{p})}{\mathrm{p}}|\nabla \mathrm{u}|_\mathrm{p}^\mathrm{p} \\
& -4\alpha\int_0^t |u_t|_2^2 \mathrm{d}\tau - 4\alpha\beta\Bigg] \\
\geq \; & F\left[-2qJ(u_0) + (q-2)\lambda_N^2|u(t)|_2^2 + (2q-4\alpha)\int_0^t |u_t|_2^2 \mathrm{d}\tau - 4\alpha\beta\right] \\
\geq \; & F\left[-2qJ(u_0) + (q-2)\lambda_N^2|u_0|_2^2 + (2q-4\alpha)\int_0^t |u_t|_2^2 \mathrm{d}\tau - 4\alpha\beta\right] \\
= \; & F\left[2q\left(\frac{C^*}{2q}|u_0|_2^2 - J(u_0)\right) + (2q-4\alpha)\int_0^t |u_t|_2^2 \mathrm{d}\tau - 4\alpha\beta\right]
\end{aligned} \tag{3.12}$$

for any $t \in [0,T]$. Taking $\alpha = \frac{q}{2}$ in (3.12), we have

$$FF'' - \frac{q}{2}\,(F')^2 \geq 2qF\left[\left(\frac{C^*}{2q}|u_0|_2^2 - J(u_0)\right) - \beta\right] \geq 0 \tag{3.13}$$

for any $t \in [0,T]$ and $\beta \in \left(0, \frac{C^*}{2q}|u_0|_2^2 - J(u_0)\right]$. Let $G(t) = F^{1-\frac{q}{2}}(t)$ for $t \in [0,T]$. Through simple calculation, we obtain

$$
\begin{aligned}
G'(t) &= \left(1 - \frac{q}{2}\right) F^{-\frac{q}{2}}(t)F'(t), \\
G''(t) &= \left(1 - \frac{q}{2}\right) F^{-\frac{q}{2}-1}(t) \left(F(t)F''(t) - \frac{q}{2}\left(F'(t)\right)^2\right)
\end{aligned}
\tag{3.14}
$$

for any $t \in [0,T]$. By (3.13) and since $q > 2$, we have $G''(t) \le 0$ for all $t \in [0,T]$. This means that $G(t)$ is concave on $[0,T]$. Then, it holds that

$$
G(T) \le G(0) + G'(0)T. \tag{3.15}
$$

Since $F(0) > 0$ and $F(t)$ is strictly increasing on $[0,T]$, we have

$$
G(0) = F^{1-\frac{q}{2}}(0) > 0 \tag{3.16}
$$

and

$$
G(T) = F^{1-\frac{q}{2}}(T) > 0 \tag{3.17}
$$

for any $T \in (0,T^*)$. In addition, recalling that $F'(0) > 0$, $p > 2$ and using (3.14), we have

$$
G'(0) = \left(1 - \frac{q}{2}\right) F^{-\frac{q}{2}}(0)F'(0) = \left(1 - \frac{q}{2}\right) G(0)\frac{F'(0)}{F(0)} < 0. \tag{3.18}
$$

Therefore, from (3.15)-(3.18), it follows that

$$
T \le -\frac{G(0)}{G'(0)} = \frac{2F(0)}{(q-2)F'(0)} = \frac{T^*|u_0|_2^2 + \beta\sigma^2}{(q-2)\beta\sigma} = \frac{|u_0|_2^2}{(q-2)\beta\sigma}T^* + \frac{\sigma}{q-2}
$$

for any $T \in (0,T^*)$, so

$$
T^* \le \frac{|u_0|_2^2}{(q-2)\beta\sigma}T^* + \frac{\sigma}{q-2} \tag{3.19}
$$

for any $\beta \in \left(0, \frac{C^*}{2q}|u_0|_2^2 - J(u_0)\right]$ and $\sigma > 0$. Fixing an arbitrary $\beta_0 \in \left(0, \frac{C^*}{2q}|u_0|_2^2 - J(u_0)\right]$, it holds that

$$
0 < \frac{|u_0|_2^2}{(q-2)\beta_0\sigma} < 1
$$

for any $\sigma \in \left(\frac{|u_0|_2^2}{(q-2)\beta_0}, +\infty\right)$. Then, by (3.19), we obtain

$$T^* \leq \frac{\sigma}{q-2}\left(1 - \frac{|u_0|_2^2}{(q-2)\beta_0\sigma}\right)^{-1} = \frac{\beta_0\sigma^2}{(q-2)\beta_0\sigma - |u_0|_2^2} \tag{3.20}$$

for any $\sigma \in \left(\frac{|u_0|_2^2}{(q-2)\beta_0}, +\infty\right)$. Define a function $T_{\beta_0}(\sigma)$ by

$$T_{\beta_0}(\sigma) = \frac{\beta_0\sigma^2}{(q-2)\beta_0\sigma - |u_0|_2^2}, \quad \sigma \in \left(\frac{|u_0|_2^2}{(q-2)\beta_0}, +\infty\right).$$

It is easy to verify that $T_{\beta_0}(\sigma)$ has a unique minimum at $\sigma_{\beta_0} := \frac{2|u_0|_2^2}{(q-2)\beta_0} \in \left(\frac{|u_0|_2^2}{(q-2)\beta_0}, +\infty\right)$. Then, in view of (3.20), we have

$$T^* \leq \inf_{\sigma\in\left(\frac{|u_0|_2^2}{(q-2)\beta_0}, +\infty\right)} T_{\beta_0}(\sigma) = T_{\beta_0}(\sigma_{\beta_0}) = \frac{4|u_0|_2^2}{(q-2)^2\beta_0} \tag{3.21}$$

for any $\beta_0 \in \left(0, \frac{C^*}{2q}|u_0|_2^2 - J(u_0)\right]$. Finally, it holds that

$$T^* \leq \inf_{\beta_0\in\left(0,\frac{C^*}{2q}|u_0|_2^2 - J(u_0)\right]} \frac{4|u_0|_2^2}{(q-2)^2\beta_0} = \frac{8q|u_0|_2^2}{(q-2)^2\left(C^*|u_0|_2^2 - 2qJ(u_0)\right)}.$$

□

Corollary 3.2 *Assume that p, q satisfy (H1)-(H2). Then there exists weak solution for problem (1.2) with initial data at arbitrary high energy level that blows up in finite time .*

Proof. Let Ω_1, Ω_2 be two disjoint open subdomain of Ω and v be an arbitrary nonzero function in $C_0^\infty(\Omega_1) \cap H_N^2(\Omega)$. For any $R > 0$, there exists $r_1 > 0$ sufficiently large such that

$$|r_1 v|_2^2 = r_1^2 \int_\Omega |v|^2 \mathrm{dx} = \mathrm{r}_1^2 \int_{\Omega_1} |\mathrm{v}|^2 \mathrm{dx} > \frac{2\mathrm{q}}{\mathrm{C}^*}\mathrm{R}. \tag{3.22}$$

We claim that there exist $\tilde{w} \in H_N^2(\Omega)$ and $r > r_1$ such that $J(\tilde{w}) = R - J(rv)$. In fact, due to Lemma 2.2, there exists a sequence $\{w_k\} \subset H_0^2(\Omega_2) \cap H_N^2(\Omega_2)$ such that

$$\frac{1}{2}\int_{\Omega_2} |\Delta w_k|^2 \mathrm{dx} + \frac{1}{\mathrm{p}}\int_{\Omega_2} |\nabla \mathrm{w_k}|^{\mathrm{p}} \mathrm{dx} - \frac{1}{\mathrm{q}}\int_{\Omega_2} |\mathrm{w_k}|^{\mathrm{q}} \mathrm{dx} \to +\infty, \quad \text{as } \mathrm{k} \to \infty.$$

On the other hand, since $q > \max\{2, p\}$, it holds that

$$R - J(rv) = R - \frac{1}{2}r^2|\Delta v|_2^2 - \frac{1}{p}r^p|\nabla v|_p^p + \frac{1}{q}r^q|v|_q^q \to +\infty, \quad \text{as } r \to +\infty.$$

So, there exist $k \in \mathbb{N}$ and $r > r_1$ both sufficiently large such that

$$R - J(rv) = \frac{1}{2}\int_{\Omega_2} |\Delta w_k|^2 \mathrm{dx} + \frac{1}{\mathrm{p}}\int_{\Omega_2} |\nabla \mathrm{w_k}|^{\mathrm{p}} \mathrm{dx} - \frac{1}{\mathrm{q}}\int_{\Omega_2} |\mathrm{w_k}|^{\mathrm{q}} \mathrm{dx}. \tag{3.23}$$

Now, extend $w = w_k \in H_0^2(\Omega_2) \cap H_N^2(\Omega_2)$ to be $\tilde{w} \in H_N^2(\Omega)$ such that

$$\tilde{w} = \begin{cases} w, & x \in \Omega_2, \\ 0, & x \in \Omega \setminus \Omega_2. \end{cases}$$

By (3.23), we obtain

$$\begin{aligned} R - J(rv) &= \frac{1}{2}\int_{\Omega_2} |\Delta w|^2 \mathrm{dx} + \frac{1}{\mathrm{p}}\int_{\Omega_2} |\nabla \mathrm{w}|^{\mathrm{p}} \mathrm{dx} - \frac{1}{\mathrm{q}}\int_{\Omega_2} |\mathrm{w}|^{\mathrm{q}} \mathrm{dx} \\ &= \frac{1}{2}\left(\int_{\Omega_2} |\Delta \tilde{w}|^2 \mathrm{dx} + \int_{\Omega\setminus\Omega_2} |\Delta \tilde{\mathrm{w}}|^2 \mathrm{dx}\right) \\ &\quad + \frac{1}{p}\left(\int_{\Omega_2} |\nabla \tilde{w}|^p \mathrm{dx} + \int_{\Omega\setminus\Omega_2} |\nabla \tilde{\mathrm{w}}|^{\mathrm{p}} \mathrm{dx}\right) \\ &\quad - \frac{1}{q}\left(\int_{\Omega_2} |\tilde{w}|^q \mathrm{dx} + \int_{\Omega\setminus\Omega_2} |\tilde{\mathrm{w}}|^{\mathrm{q}} \mathrm{dx}\right) \\ &= J(\tilde{w}). \end{aligned} \tag{3.24}$$

Let $u_0 := rv + \tilde{w}$. In view of (3.22) and (3.24), we have $u_0 \in H_N^2(\Omega)$,

$$|u_0|_2^2 = \int_\Omega |u_0|^2 \mathrm{dx} \geq \int_{\Omega_1} |\mathrm{rv}|^2 \mathrm{dx} > \frac{2\mathrm{q}}{\mathrm{C}^*}\mathrm{R}$$

and

$$\begin{aligned} J(u_0) &= \frac{1}{2}\int_\Omega |\Delta u_0|^2 \mathrm{dx} + \frac{1}{\mathrm{p}}\int_\Omega |\mathrm{u_0}|^{\mathrm{p}} \mathrm{dx} - \frac{1}{\mathrm{q}}\int_\Omega |\mathrm{u_0}|^{\mathrm{q}} \mathrm{dx} \\ &= \int_{\Omega_1} \left(\frac{1}{2}|r\Delta v|^2 + \frac{1}{p}|rv|^p - \frac{1}{q}|rv|^q\right) \mathrm{dx} \\ &\quad + \int_{\Omega_2} \left(\frac{1}{2}|\Delta w|^2 + \frac{1}{p}|w|^p - \frac{1}{q}|w|^q\right) \mathrm{dx} \end{aligned}$$

$$
\begin{aligned}
&= J(rv) + J(\tilde{w}) \\
&= R < \frac{C^*}{2q}|u_0|_2^2.
\end{aligned}
$$

According to Theorem 3.1, the solution u of problem (1.2) with initial data u_0 blows up in finite time. This completes the proof. $\square$

Remark 3.3 *If $J(u_0) < 0$, we have $I(u_0) < 0$. If $u_0 \in H_N^2(\Omega)$ satisfies the statement (ii) in Theorem 3.1, in view of (2.7), it also holds that $I(u_0) < 0$. It is a natural question that whether negative initial Nehari energy $I(u_0) < 0$ is sufficient for the finite time blow-up of the weak solution to problem (1.2) with initial data u_0. For the heat equation*

$$u_t - \Delta u = |u|^{p-1}u,$$

where $1 < p < \frac{n+2}{n-2}$, Dickstein et al. proved that negative initial Nehari energy $I(u_0) := |\nabla u_0|_2^2 - |u_0|_{p+1}^{p+1} < 0$ is not sufficient for the finite time blow-up of the solution [18]. While, in [19], the authors studied the following pseudo-parabolic equation

$$u_t - \Delta u - \Delta u_t + u = |u|^{p-2}u,$$

where $2 < p < 2^*$, and proved that negative initial Nehari energy $I(u_0) := |u_0|_2^2 + |\nabla u_0|_2^2 - |u_0|_p^p < 0$ is sufficient for the finite time blow-up of the solution.

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References

[1] A. Zangwill, Some causes and a consequence of epitaxial roughening, Journal of Crystal Growth 163 (1) (1996) 8–21.

doi:http://dx.doi.org/10.1016/0022-0248(95)01048-3.

[2] M. Ortiz, E. A. Repetto, H. Si, A continuum model of kinetic roughening and coarsening in thin films, Journal of the Mechanics and Physics of Solids 47 (4) (1999) 697–730. doi:http://dx.doi.org/10.1016/S0022-5096(98)00102-1.

[3] T. P. Schulze, R. V. Kohn, A geometric model for coarsening during spiral-mode growth of thin films, Physica D: Nonlinear Phenomena 132 (4) (1999) 520–542. doi:http://dx.doi.org/10.1016/S0167-2789(99)00108-6.

[4] B. B. King, O. Stein, M. Winkler, A fourth-order parabolic equation modeling epitaxial thin film growth, Journal of Mathematical Analysis and Applications 286 (2) (2003) 459–490. doi:http://dx.doi.org/10.1016/S0022-247X(03)00474-8.

[5] C. Liu, A fourth order parabolic equation with nonlinear principal part, Nonlinear Analysis: Theory, Methods & Applications 68 (2) (2008) 393–401. doi:http://dx.doi.org/10.1016/j.na.2006.11.005.

[6] M. Winkler, Global solutions in higher dimensions to a fourth-order parabolic equation modeling epitaxial thin-film growth, Zeitschrift für angewandte Mathematik und Physik 62 (4) (2011) 575–608. doi:10.1007/s00033-011-0128-1.

[7] C. Qu, X. Bai, S. Zheng, Blow-up versus extinction in a nonlocal p–Laplace equation with Neumann boundary conditions, Journal of Mathematical Analysis and Applications 412 (1) (2014) 326–333. doi:http://dx.doi.org/10.1016/j.jmaa.2013.10.040.

[8] B. Guo, W. Gao, Non-extinction of solutions to a fast diffusive p-Laplace equation with Neumann boundary conditions, Journal of Mathematical Analysis and Applications 422 (2) (2015) 1527–1531. doi:http://dx.doi.org/10.1016/j.jmaa.2014.09.006.

[9] G. A. Philippin, S. Vernier Piro, Behaviour in time of solutions to a class of fourth order evolution equations, Journal of Mathematical Analysis and Applications 436 (2) (2016) 718–728. doi:http://dx.doi.org/10.1016/j.jmaa.2015.11.066.

[10] A. Hao, J. Zhou, Blowup, extinction and non-extinction for a nonlocal p–biharmonic parabolic equation, Applied Mathematics Letters 64 (2017) 198–204. doi:http://dx.doi.org/10.1016/j.aml.2016.09.007.

[11] D. Li, Z. Qiao, T. Tang, Gradient bounds for a thin film epitaxy equation, Journal of Differential Equations 262 (3) (2017) 1720–1746. doi:http://dx.doi.org/10.1016/j.jde.2016.10.025.

[12] C. Qu, W. Zhou, Blow-up and extinction for a thin-film equation with initial-boundary value conditions, Journal of Mathematical Analysis and Applications 436 (2) (2016) 796–809. doi:http://dx.doi.org/10.1016/j.jmaa.2015.11.075.

[13] Q. Li, W. Gao, Y. Han, Global existence blow up and extinction for a class of thin–film equation, Nonlinear Analysis: Theory, Methods & Applications 147 (2016) 96–109. doi:http://dx.doi.org/10.1016/j.na.2016.08.021.

[14] J. Zhou, Blow-up for a thin-film equation with positive initial energy, Journal of Mathematical Analysis and Applications 446 (1) (2017) 1133–1138. doi:https://doi.org/10.1016/j.jmaa.2016.09.026.

[15] H. Song, Blow up of arbitrarily positive initial energy solutions for a viscoelastic wave equation, Nonlinear Analysis: Real World Applications 26 (2015) 306–314. doi:http://dx.doi.org/10.1016/j.nonrwa.2015.05.015.

[16] H. A. Levine, Some nonexistence and instability theorems for solutions of formally parabolic equations of the form $Pu_t = -Au + F(u)$, Archive for Rational Mechanics and Analysis 51 (5) (1973) 371–386. doi:10.1007/bf00263041.

[17] F. Sun, L. Liu, Y. Wu, Infinitely many sign-changing solutions for a class of biharmonic equation with p-Laplacian and Neumann boundary condition, Applied Mathematics Letters 73 (2017) 128–135. doi:https://doi.org/10.1016/j.aml.2017.05.001.

[18] F. Dickstein, N. Mizoguchi, P. Souplet, F. Weissler, Transversality of stable and Nehari manifolds for a semilinear heat equation, Calculus of Variations and Partial Differential Equations 42 (3) (2011) 547–562. doi:10.1007/s00526-011-0397-8.

[19] X. Zhu, F. Li, Y. Li, Some sharp result about the global existence and blow up of solutions to a class of pseudo-parabolic equations, Proceedings of the Royal Society of Edinburgh, Section: A Mathematics. prepint.