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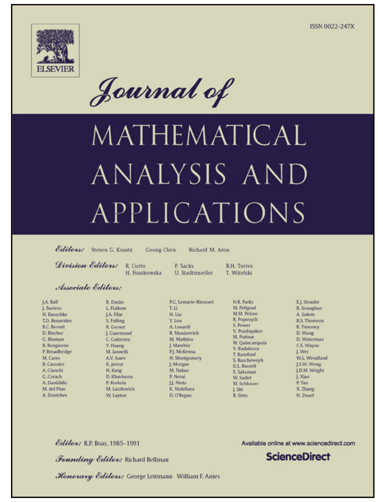

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Ground state of Kirchhoff type fractional Schrödinger equations with critical growth*

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Abstract In this paper, we study the critical Kirchhoff type fractional Schrödinger equation:

$$\left(1+\alpha\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u(x)-u(y)|^2}{|x-y|^{3+2s}}\mathrm{d}x\mathrm{d}y\right)(-\Delta)^s u+u=\beta f(u)+u^{2_s^*-1} \quad \text{in } \mathbb{R}^3, \tag{0.1}$$

where $s\in(0,1)$ and $2_s^*=\frac{6}{3-2s}$. We establish the Pohozăev type identity of (0.1). When $s\in[\frac{3}{4},1)$, under some conditions on α, β and $f(u)$, we obtain some results on the existence of ground state solutions. When $s\in(0,\frac{3}{4}]$, we also prove the non-existence result. In particular, when $\alpha=0$, we obtain an existence result.

Keywords: Fractional Schrödinger equation; Kirchhoff type; Critical growth; Variational method

1 Introduction

In this paper, we study the Kirchhoff type fractional Schrödinger equation with critical growth:

$$\left(1+\alpha\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u(x)-u(y)|^2}{|x-y|^{3+2s}}\mathrm{d}x\mathrm{d}y\right)(-\Delta)^s u+u=\beta f(u)+u^{2_s^*-1} \quad \text{in } \mathbb{R}^3, \tag{1.1}$$

where $\alpha\geq 0$, $\beta>0$ are parameters, $s\in(0,1)$, $2_s^*=\frac{6}{3-2s}$ is the critical exponent. The fractional Schrödinger equation is formulated by Laskin [21, 22]. It is a

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fundamental of fractional quantum mechanics. Also, it appears in various areas such as plasma physics, optimization, finance, free boundary obstacle problems, population dynamics and minimal surfaces.

Finding ground state solutions is a very classical problem. The ground state solutions are those solutions whose energy level is minimized. When $\alpha = 0$, problem (1.1) reduces to the equation

$$(-\Delta)^s u = g(u) \quad \text{in} \quad \mathbb{R}^N. \tag{1.2}$$

When $s = 1$, in the celebrated paper [5, 6], the authors gave almost necessary and sufficient conditions of the existence of ground state solutions of the subcritical problem

$$-\Delta u = g(u) \quad \text{in} \quad \mathbb{R}^N. \tag{1.3}$$

Precisely, they assumed the following conditions:

(g_1) $g(u) \in C(\mathbb{R}, \mathbb{R})$ is continuous and odd;

(g_2) $-\infty < \liminf_{u\to 0} \frac{g(u)}{u} \leq \limsup_{u\to 0} \frac{g(u)}{u} = -a < 0$ when $N \geq 3$ and $\lim_{u\to 0} \frac{g(u)}{u} = -a < 0$ when $N = 2$;

(g_3) when $N \geq 3$, $\limsup_{u\to\infty} \frac{g(u)}{|u|^{\frac{N+2}{N-2}}} \leq 0$; when $N = 2$, for any $\alpha > 0$, there exists $C_\alpha > 0$ such that $g(u) \leq C_\alpha \exp(\alpha u^2)$ for all $u > 0$;

(g_4) there exists $\xi_0 > 0$ such that $G(\xi_0) = \int_0^{\xi_0} g(s)\mathrm{d}s > 0$.

Motivated by [5, 6], Chang and Wang [11] considered the fractional subcritical problem

$$(-\Delta)^s u = g(u), \quad u \in H^1(\mathbb{R}^N)\ (N \geq 2). \tag{1.4}$$

They proved that when $s \in (0, 1)$ and $g \in C^1(\mathbb{R}, \mathbb{R})$ satisfying (g_1)-(g_4), problem (1.4) had ground state solutions. For the critical case, by using the Nehari manifold method, the authors in [19, 28, 30] also obtained ground state solutions of (1.4). We point out that, in [19, 28, 30], the Ambrosetti-Rabinowitz condition and the monotonicity of $u \to f(u)/u$ are essential. Recently, without these conditions, the authors in [34] proved the existence of ground state solutions of the fractional Schrödinger-Poisson system. A natural question is whether similar results hold for the Kirchhoff type fractional Schrödinger equation (1.1).

In the last few years, many papers focused on the Kirchhoff type equation, which occurs in various branches of mathematical physics. On the contrary, the results of Kirchhoff type fractional equations are relatively few. When the domain is bounded in $\mathbb{R}^N$, Fiscella and Valdinoci [16] established a stationary Kirchhoff type problem and proved the existence, asymptotic behavior of nontrivial solutions. Subsequently, Autuori, Fiscella and Pucci [4] extended the results of [16] to a more general case. By using the truncation argument and the genus theory, Fiscella [17] obtained infinitely many solutions of a critical Kirchhoff type problem involving a fractional operator. When the domain is unbounded, Pucci, Xiang and Zhang [25] considered the nonhomogeneous fractional p-Laplacian equation of Schrödinger-Kirchhoff type in $\mathbb{R}^N$. By using the Ambrosetti-Rabinowitz condition, they obtained at least two nontrivial solutions. For the critical case, Pucci and Saldi [24] proved the existence and multiplicity of solutions of a critical Kirchhoff type eigenvalue problem with a fractional Laplacian in $\mathbb{R}^N$. However, there are still few things about the Kirchhoff type fractional equation with critical growth. Recently, the authors in [23] studied ground state solutions of the critical fractional Kirchhoff equation for the case $s \in (\frac{3}{4}, 1)$. However, when $s \in (0, \frac{3}{4}]$, to our best knowledge, there are no results on the corresponding critical problem.

Motivated by this fact, in this paper, we study the existence of ground state solutions of (1.1). To solve the problem, we must deal with the lack of the compactness caused by the critical term. Also, we have to consider the interaction between the Kirchhoff term and the critical term, which is crucial when we seek solutions. Moreover, when $s \in (0, \frac{3}{4}]$, since $2_s^* \leq 4$, it is difficult to derive the geometric structure of the functional and the boundness, convergence of the Palais-Smale sequences. In this paper, we first consider the case $s \in (\frac{3}{4}, 1)$. We assume the following conditions:

(f_1) $f \in C^1(\mathbb{R}, \mathbb{R})$ and $\lim_{u\to 0} \frac{f(u)}{u} = \lim_{u\to\infty} \frac{f(u)}{u^{2_s^*-1}} = 0$.

(f_2) $F(u) = \int_0^u f(s)\mathrm{d}s \geq 0$ for $u \in \mathbb{R}$. Moreover, there exist D_0, $R_0 > 0$ and $p_0 \in (2, 2_s^*)$ such that $F(u) \geq \frac{D_0}{R_0^{p_0}}|u|^{p_0}$ for $u \geq R_0$.

Theorem 1.1. *Let $s \in (\frac{3}{4}, 1)$. If (f_1)-(f_2) hold, then there exists $\beta_0 > 0$ such that for $\beta > \beta_0$, problem (1.1) has a radial ground state solution.*

Remark 1.1. *When replacing (f_2) with the stronger condition:*

(f_2') *there exist $D > 0$ and $q \in (2, 2_s^*)$ such that $f(u) \geq Du^{q-1}$ for $u \geq 0$,*

the authors in [23] proved the existence of ground state solutions of the critical fractional Kirchhoff equation with D large enough. The condition (f_2') was introduced in [35] and played an important role in estimating the upper bound of the energy. In this paper, instead of (f_2'), we use a more general condition (f_2), which involves more nonlinearities. Moreover, instead of using the monotonicity trick developed by Jeanjean [20], we give a direct proof here.

Remark 1.2. *Without the Ambrosetti-Rabinowitz condition and the monotonicity of $u \to \frac{f(u)}{u}$, we cannot use the Nehari approach to obtain ground state solutions of (1.1). In particular, we cannot prove the boundedness of the Palais-Smale sequences easily. To overcome the difficulties, we establish the Pohožaev type identity.*

Now we consider the case $s \in (0, \frac{3}{4}]$. Denote the best Sobolev constant:

$$S(s) = \inf_{u \in D^{s,2}(\mathbb{R}^3)\setminus\{0\}} \frac{\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x}{\left(\int_{\mathbb{R}^3} |u|^{2_s^*} \mathrm{d}x\right)^{\frac{2}{2_s^*}}}.$$

We make the following assumptions:

(f_1') $f \in C(\mathbb{R}, \mathbb{R})$ and $\lim_{u \to 0} \frac{f(u)}{u} = \lim_{u \to \infty} \frac{f(u)}{u^{2_s^*-1}} = 0$.

(f_3) There exists $\xi > 0$ such that $F(\xi) = \int_0^\xi f(s)\mathrm{d}s > 0$.

Theorem 1.2. *Let $s = \frac{3}{4}$, $\alpha < \frac{1}{S(s)^2}$, $\beta > 0$. If (f_1)-(f_2) hold, then problem (1.1) has a radial ground state solution.*

Theorem 1.3. *Let $s = \frac{3}{4}$, $\alpha > \frac{1}{S(s)^2}$. Then*

(i) there exists $\beta_1 > 0$ such that for $\beta \in (\beta_1, +\infty)$, problem (1.1) has a radial ground state solution if (f_1) and (f_3) hold;

(ii) there exists $\beta_2 \in (0, \beta_1)$ such that for $\beta \in (0, \beta_2)$, problem (1.1) has no nontrivial solutions if (f_1') holds.

Theorem 1.4. *Let $s \in (0, \frac{3}{4})$, $\alpha > \frac{2s(3-4s)^{\frac{3-4s}{2s}}}{[S(s)]^{\frac{3}{2s}}(3-2s)^{\frac{3-2s}{2s}}}$. Then there exists $\beta_3 > 0$ such that for $\beta \in (0, \beta_3)$, problem (1.1) has no nontrivial solutions if (f_1') holds.*

Remark 1.3. *Because of the presence of the Kirchhoff term and the growth of the nonlinearity, it is difficult to deal with the geometric structure of the functional and the convergence of the Palais-Smale sequences. To solve the problem, the condition (f_2) or (f_3) is essential.*

The condition $f \in C^1(\mathbb{R}, \mathbb{R})$ is only used to guarantee the Pohožaev type identity. When $\alpha = 0$, problem (1.1) reduces to

$$(-\Delta)^s u + u = \beta f(u) + u^{2_s^*-1} \quad \text{in} \quad \mathbb{R}^3. \tag{1.5}$$

By replacing the condition (f_1) with (f_1'), we get the following result, which improves the existing results.

Theorem 1.5. *Let $\beta > 0$. If (f_1') and (f_2) hold with $s \in (0, \frac{3}{4}]$, $p_0 \in (2, 2_s^*)$, or $s \in (\frac{3}{4}, 1)$, $p_0 \in (2_s^* - 2, 2_s^*)$, then problem (1.5) has a radial nontrivial solution.*

Before closing this section, we say a few words on the equation

$$(-\Delta)^s u + V(x)u = f(x, u) \quad \text{in} \quad \mathbb{R}^N. \tag{1.6}$$

A valid tool to deal with the fractional problem is due to Caffarelli and Silvestre. In the remarkable paper [9], they expressed the nonlocal operator $(-\Delta)^s$ as a

Dirichlet-Neumann map for a certain elliptic boundary value problem with local differential operators defined on the upper half space. Subsequently, this common approach is widely used to deal with the fractional equation in the respects of regularity and variational methods. When $V(x) \equiv 1$ and $f(x,u)$ has subcritical growth satisfying the Ambrosetti-Rabinowitz condition, Felmer, Quaas and Tan [18] proved the existence, regularity, decay, and symmetry properties of positive solutions of (1.6). When $\lim_{|x|\to\infty} V(x) = +\infty$ and $f(x,u) = |u|^{p-1}u$, the authors in [12] obtained the existence of ground state solutions of (1.6). Later, by using the Nehari manifold method, Simone [31] provided a generalization of the main result in [12]. Under weaker assumptions of the behavior of the potential V at infinity, he also gave some existence results. When $f(x,u)$ is asymptotically linear with respect to u at infinity, Chang [10] proved the existence of ground state solutions. By using the method of Pohozaev-Nehari manifold, the monotonic trick and the global compactness principle, Teng [32] obtained ground state solutions of the fractional Schrödinger-Poisson system with critical growth. Recently, the semiclassical limit of problem (1.6) was also discussed. See [1, 2, 19, 29, 30] and the references therein.

The outline of this paper is as follows: in Section 2, we give some important lemmas; in Section 3, we prove Theorem 1.1; in Section 4, we prove Theorems 1.2-1.3; in Section 5, we prove Theorem 1.4; in Section 6, we prove Theorem 1.5.

2 Preliminary Lemmas

We introduce some definitions. Let Φ be the Schwartz space of rapidly decaying C^∞ functions in $\mathbb{R}^3$. For any $s \in (0,1)$, define the functional Laplacian $(-\Delta)^s$ by $\mathcal{F}((-\Delta)^s\phi)(\xi) = |\xi|^{2s}\mathcal{F}(\phi)(\xi)$, where $\phi \in \Phi$, $\xi \in \mathbb{R}^3$, $\mathcal{F}$ is the Fourier transform. In particular, if ϕ is smooth, then

$$(-\Delta)^s\phi(x) = C_s \mathrm{P.V.} \int_{\mathbb{R}^3} \frac{|\phi(x)-\phi(y)|^2}{|x-y|^{3+2s}} \mathrm{d}x,$$

where P.V. is the Cauchy principle value, C_s is the normalization constant.

Let $\hat{u} = \mathcal{F}(u)$. For any $s \in (0,1)$, define the fractional Sobolev space

$$H^s(\mathbb{R}^3) = \left\{ u \in L^2(\mathbb{R}^3) : \int_{\mathbb{R}^3} (|\xi|^{2s}\hat{u}^2 + \hat{u}^2)\mathrm{d}\xi < \infty \right\}$$

with the norm $\|u\|_{H^s(\mathbb{R}^3)} = \left(\int_{\mathbb{R}^3}(|\xi|^{2s}\hat{u}^2 + \hat{u}^2)\mathrm{d}\xi\right)^{\frac{1}{2}}$. From Plancherel's theorem, for any $u \in H^s(\mathbb{R}^3)$,

$$\int_{\mathbb{R}^3} |u|^2\mathrm{d}x = \int_{\mathbb{R}^3} |\hat{u}|^2\mathrm{d}x, \quad \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}u|^2\mathrm{d}x = \int_{\mathbb{R}^3} |\xi|^{2s}\hat{u}^2\mathrm{d}\xi.$$

Then

$$\|u\|_{H^s(\mathbb{R}^3)} = \left(\int_{\mathbb{R}^3} (|(-\Delta)^{\frac{s}{2}}u|^2 + u^2)\mathrm{d}x\right)^{\frac{1}{2}}.$$

From [15],

$$\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 \mathrm{d}x = \frac{1}{2} C_s \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u(x)-u(y)|^2}{|x-y|^{3+2s}} \mathrm{d}x\mathrm{d}y. \tag{2.1}$$

Without loss of generality, we assume $C_s = 2$. By [15], the embedding $H^s(\mathbb{R}^3) \hookrightarrow L^p(\mathbb{R}^3)$ is continuous for $p \in [2, 2_s^*]$, and is locally compact for $p \in [2, 2_s^*)$. For simplicity, we denote $\|.\|_{H^s(\mathbb{R}^3)}$ and $\|.\|_{L^p(\mathbb{R}^3)}$ by $\|.\|$ and $\|.\|_p$, respectively. Let

$$H_r^s(\mathbb{R}^3) = \{u \in H^s(\mathbb{R}^3) : u(x) = u(|x|)\}.$$

Lemma 2.1. *([11]) Assume that $P \in C(\mathbb{R}, \mathbb{R})$ satisfies*

$$\lim_{t\to 0} \frac{P(t)}{|t|^2} = \lim_{t\to\infty} \frac{P(t)}{|t|^{2_s^*}} = 0$$

and there exists a bounded sequence $\{u_n\} \subset H_r^s(\mathbb{R}^3)$, $v \in L^1(\mathbb{R}^3)$ such that $\lim_{n\to\infty} P(u_n(x)) = v(x)$ a.e. $x \in \mathbb{R}^3$. Then $P(u_n) \to v$ in $L^1(\mathbb{R}^3)$ as $n \to \infty$.

We denote $D^{s,2}(\mathbb{R}^3)$ the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm

$$\|u\|_{D^{s,2}(\mathbb{R}^3)} = \left(\int_{\mathbb{R}^3} |\xi|^{2s} \hat{u}^2 \mathrm{d}\xi\right)^{\frac{1}{2}} = \left(\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 \mathrm{d}x\right)^{\frac{1}{2}}.$$

Define the best Sobolev constant:

$$S(s) = \inf_{u \in D^{s,2}(\mathbb{R}^3)\setminus\{0\}} \frac{\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 \mathrm{d}x}{\left(\int_{\mathbb{R}^3} |u(x)|^{2_s^*} \mathrm{d}x\right)^{\frac{2}{2_s^*}}}.$$

By [14, 27], $S(s)$ is attained by the functions $\tilde{u}(x) = \frac{\kappa}{(\mu^2+|x|^2)^{\frac{3-2s}{2}}}$, where $\kappa \in \mathbb{R} \setminus \{0\}$, $\mu > 0$. Let $\bar{u}(x) = \frac{\tilde{u}(x)}{\|\tilde{u}\|_{2_s^*}}$, we have $S(s) = \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} \bar{u}(x)|^2 \mathrm{d}x$. Let $u^*(x) = \bar{u}\left(\frac{x}{S(s)^{\frac{1}{2s}}}\right)$. Then $\|u^*\|_{2_s^*}^{2_s^*} = [S(s)]^{\frac{3}{2s}}$ and $(-\Delta)^s u^* = |u^*|^{2_s^*-2} u^*$ in $\mathbb{R}^3$. For any $\varepsilon > 0$, define $U_\varepsilon(x) = \varepsilon^{-\frac{3-2s}{2}} u^*(\frac{x}{\varepsilon})$. Then $\|U_\varepsilon\|_{2_s^*}^{2_s^*} = [S(s)]^{\frac{3}{2s}}$ and $(-\Delta)^s U_\varepsilon = |U_\varepsilon|^{2_s^*-2} U_\varepsilon$ in $\mathbb{R}^3$. Define $\psi \in C_0^\infty(B_{2r}(0))$ such that $\psi(x) = 1$ on $B_r(0)$ and $0 \le \psi(x) \le 1$. Let $u_\varepsilon(x) = \psi(x) U_\varepsilon(x)$. By [27],

$$\begin{aligned} &\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_\varepsilon|^2 \mathrm{d}x \le [S(s)]^{\frac{3}{2s}} + O(\varepsilon^{3-2s}), \\ &\int_{\mathbb{R}^3} |u_\varepsilon|^{2_s^*} \mathrm{d}x = [S(s)]^{\frac{3}{2s}} + O(\varepsilon^3). \end{aligned} \tag{2.2}$$

By the direct calculation, there exists $\kappa_0 \neq 0$ such that

$$\begin{aligned} U_\varepsilon(x) =& \varepsilon^{-\frac{3-2s}{2}} \bar{u}\left(\frac{x}{\varepsilon [S(s)]^{\frac{1}{2s}}}\right) \\ =& \varepsilon^{-\frac{3-2s}{2}} \frac{\kappa_0}{\left(\mu^2 + \left|\frac{x}{\varepsilon[S(s)]^{\frac{1}{2s}}}\right|^2\right)^{\frac{3-2s}{2}}}. \end{aligned}$$

Thus,

$$\|u_\varepsilon\|_2^2 \le \int_{B_{2r}(0)} |U_\varepsilon(x)|^2 \mathrm{d}x = \varepsilon^{-(3-2s)} \int_{B_{2r}(0)} \frac{\kappa_0^2}{\left(\mu^2 + \left|\frac{x}{\varepsilon[S(s)]^{\frac{1}{2s}}}\right|^2\right)^{3-2s}} \mathrm{d}x$$

$$\le C\varepsilon^{2s} \int_0^{\frac{2r}{\mu[S(s)]^{\frac{1}{2s}}\varepsilon}} \frac{t^2}{(1+t^2)^{3-2s}} \mathrm{d}t = \begin{cases} O(\varepsilon^{2s}), & s \in (0, \frac{3}{4}), \\ O(\varepsilon^{2s}|\log\varepsilon|), & s = \frac{3}{4}, \\ O(\varepsilon^{3-2s}), & s \in (\frac{3}{4}, 1). \end{cases} \tag{2.3}$$

Define the functional on $H^s(\mathbb{R}^3)$ by

$$I(u) = \frac{1}{2}\|u\|^2 + \frac{\alpha}{4}\left(\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 \mathrm{d}x\right)^2 - \beta \int_{\mathbb{R}^3} F(u)\mathrm{d}x - \frac{1}{2_s^*}\int_{\mathbb{R}^3} |u|^{2_s^*}\mathrm{d}x. \tag{2.4}$$

Clearly, $I \in C^1(H^s(\mathbb{R}^3), \mathbb{R})$ and

$$\langle I'(u), v\rangle = \left(1 + \alpha \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 \mathrm{d}x\right) \int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} v \mathrm{d}x + \int_{\mathbb{R}^3} uv\mathrm{d}x$$
$$- \beta \int_{\mathbb{R}^3} f(u)v\mathrm{d}x - \int_{\mathbb{R}^3} |u|^{2_s^*-2} uv\mathrm{d}x, \quad \forall\, v \in H^s(\mathbb{R}^3).$$

Then critical points of I are weak solutions of (1.1).

We introduce the s-harmonic extension technique in [9] to establish the Pohožaev identity. Define the weighted space $X^s(\mathbb{R}^{3+1}_+)$ the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm

$$\|w\|_{X^s(\mathbb{R}^{3+1}_+)} = \left(k_s \int_{\mathbb{R}+} \int_{\mathbb{R}^3} y^{1-2s} |\nabla w|^2 \mathrm{d}x\mathrm{d}y\right)^{\frac{1}{2}},$$

where k_s is a constant. For any $u \in D^{s,2}(\mathbb{R}^3)$, we call its s-harmonic extension $w = E_s(u)$ by the solution of the problem

$$\begin{cases} -\operatorname{div}(y^{1-2s}\nabla w) = 0, & \text{in} \quad \mathbb{R}^{3+1}_+, \\ w(x,0) = u, & \text{on} \quad \mathbb{R}^3. \end{cases}$$

By [7], we have $\|w\|_{X^s(\mathbb{R}^{3+1}_+)} = \|u\|_{D^{s,2}(\mathbb{R}^3)}$. Moreover, by [8], the fractional Laplacian operator $(-\Delta)^s$ can be defined by the Dirichlet-to-Neumann map:

$$(-\Delta)^s u(x) = \lim_{y\to 0+} y^{1-2s} \frac{\partial w}{\partial y}(x,y), \quad \forall\, u \in D^{s,2}(\mathbb{R}^3).$$

Let $g(u) = \beta f(u) + u^{2_s^*-1} - u$. Then (1.1) reduces to the local problem

$$\begin{cases} -\left(1 + \alpha \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 \mathrm{d}x\right) \operatorname{div}(y^{1-2s}\nabla w) = 0, & \text{in} \quad \mathbb{R}^{3+1}_+, \\ \left(1 + \alpha \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 \mathrm{d}x\right) \partial_\nu^s w = g(w), & \text{on} \quad \mathbb{R}^3, \end{cases} \tag{2.5}$$

where $\partial_\nu^s w(x,0) = -\frac{1}{k_s}\lim_{y\to 0+} y^{1-2s}\frac{\partial w}{\partial y}(x,y)$, $\forall\ x\in\mathbb{R}^3$. Obviously, if w is a weak solution of (2.5), then $u = w(.,0) := \mathrm{Tr}(w)$ is a weak solution of (1.1). Without loss of generality, we assume $k_s = 1$. Now we establish the Pohožaev identity for the Kirchhoff type problem (1.1).

Lemma 2.2. *Assume that (f_1). If $u\in H^s(\mathbb{R}^3)$ be a weak solution of (1.1), then*

$$\frac{3-2s}{2}\left(1+\alpha\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2\mathrm{d}x\right)\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2\mathrm{d}x = 3\int_{\mathbb{R}^3}G(u)\mathrm{d}x,$$

where $G(u)=\int_0^u g(s)\mathrm{d}s = \beta F(u)+\frac{1}{2_s^*}|u|^{2_s^*}-\frac{1}{2}|u|^2$.

Proof. We use the idea in [11]. The standard argument shows that $w = E_s(u)\in C^2(\mathbb{R}_+^{3+1})$. For any $R>0$ and $r\in(0,R)$, define

$$D_{R,r} = \{z=(x,y)\in\mathbb{R}^3\times[r,+\infty) : |z|\le R\}.$$

Let

$$\partial D_{R,r}^1 = \{z=(x,y)\in\mathbb{R}^3\times\{y=r\} : |x|^2\le R^2-r^2\},$$
$$\partial D_{R,r}^2 = \{z=(x,y)\in\mathbb{R}^3\times[r,+\infty) : |z| = R\}.$$

Then $\partial D_{R,r} = \partial D_{R,r}^1\cup\partial D_{R,r}^2$. Let $\vec{n}$ be the unit outward normal vector on $\partial D_{R,r}$. Then $\vec{n} = (0,\dots,0,-1)$ on $\partial D_{R,r}^1$ and $\vec{n} = \frac{z}{R}$ on $\partial D_{R,r}^2$. Note that $\|w\|^2_{X^s(\mathbb{R}_+^{3+1})} = \int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2\mathrm{d}x$. So

$$\begin{aligned}&\left(1+\alpha\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2\mathrm{d}x\right)\mathrm{div}(y^{1-2s}\nabla w)(z,\nabla w)\\ &= \mathrm{div}\left[\left(1+\alpha\|w\|^2_{X^s(\mathbb{R}_+^{3+1})}\right)y^{1-2s}(z,\nabla w)\nabla w\right]\\ &\quad - y^{1-2s}\nabla w\nabla(z,\nabla w) - \alpha y^{1-2s}\nabla w\nabla\left[\|w\|^2_{X^s(\mathbb{R}_+^{3+1})}(z,\nabla w)\right].\end{aligned}$$

By the direct calculation, we get

$$y^{1-2s}\nabla w\nabla(z,\nabla w) = y^{1-2s}\left[|\nabla w|^2 + z\nabla\left(\frac{|\nabla w|^2}{2}\right)\right].$$

Also,

$$\begin{aligned}&y^{1-2s}\nabla w\nabla\left[\|w\|^2_{X^s(\mathbb{R}_+^{3+1})}(z,\nabla w)\right]\\ &= \|w\|^2_{X^s(\mathbb{R}_+^{3+1})}y^{1-2s}\nabla w\nabla(z,\nabla w) + y^{1-2s}\nabla w\nabla\left(\|w\|^2_{X^s(\mathbb{R}_+^{3+1})}\right)(z,\nabla w)\\ &= \|w\|^2_{X^s(\mathbb{R}_+^{3+1})}y^{1-2s}\nabla w\nabla(z,\nabla w).\end{aligned}$$

Thus,

$$\begin{aligned}
&\left(1+\alpha\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2\mathrm{d}x\right)\mathrm{div}(y^{1-2s}\nabla w)(z,\nabla w)\\
&=\mathrm{div}\left[\left(1+\alpha\|w\|^2_{X^s(\mathbb{R}^{3+1}_+)}\right)y^{1-2s}(z,\nabla w)\nabla w\right]\\
&\quad-\left(1+\alpha\|w\|^2_{X^s(\mathbb{R}^{3+1}_+)}\right)y^{1-2s}\left[|\nabla w|^2+z\nabla\left(\frac{|\nabla w|^2}{2}\right)\right].
\end{aligned}\tag{2.6}$$

By the direct calculation,

$$\begin{aligned}
&y^{1-2s}\left[|\nabla w|^2+z\nabla\left(\frac{|\nabla w|^2}{2}\right)\right]\\
&=y^{1-2s}|\nabla w|^2+\mathrm{div}\left(y^{1-2s}z\frac{|\nabla w|^2}{2}\right)-\mathrm{div}\left(y^{1-2s}z\right)\frac{|\nabla w|^2}{2}\\
&=\mathrm{div}\left(y^{1-2s}z\frac{|\nabla w|^2}{2}\right)+\frac{2s-3}{2}y^{1-2s}|\nabla w|^2.
\end{aligned}$$

Also,

$$\begin{aligned}
&\|w\|^2_{X^s(\mathbb{R}^{3+1}_+)}y^{1-2s}\left[|\nabla w|^2+z\nabla\left(\frac{|\nabla w|^2}{2}\right)\right]\\
&=\|w\|^2_{X^s(\mathbb{R}^{3+1}_+)}y^{1-2s}|\nabla w|^2+\mathrm{div}\left[\|w\|^2_{X^s(\mathbb{R}^{3+1}_+)}y^{1-2s}z\frac{|\nabla w|^2}{2}\right]\\
&\quad-\nabla\left(\|w\|^2_{X^s(\mathbb{R}^{3+1}_+)}y^{1-2s}z\right)\frac{|\nabla w|^2}{2}\\
&=\|w\|^2_{X^s(\mathbb{R}^{3+1}_+)}y^{1-2s}|\nabla w|^2+\mathrm{div}\left[\|w\|^2_{X^s(\mathbb{R}^{3+1}_+)}y^{1-2s}z\frac{|\nabla w|^2}{2}\right]\\
&\quad-\mathrm{div}\left(y^{1-2s}z\right)\|w\|^2_{X^s(\mathbb{R}^{3+1}_+)}\frac{|\nabla w|^2}{2}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\left(1+\alpha\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2\mathrm{d}x\right)\mathrm{div}(y^{1-2s}\nabla w)(z,\nabla w)\\
&=\mathrm{div}\left[\left(1+\alpha\|w\|^2_{X^s(\mathbb{R}^{3+1}_+)}\right)y^{1-2s}(z,\nabla w)\nabla w\right]\\
&\quad-\mathrm{div}\left[\left(1+\alpha\|w\|^2_{X^s(\mathbb{R}^{3+1}_+)}\right)y^{1-2s}z\frac{|\nabla w|^2}{2}\right]\\
&\quad+\frac{3-2s}{2}\left(1+\alpha\|w\|^2_{X^s(\mathbb{R}^{3+1}_+)}\right)y^{1-2s}|\nabla w|^2.
\end{aligned}\tag{2.7}$$

By (2.5) and (2.7),

$$
\begin{aligned}
0 &= \int_{D_{R,r}} \left(1+\alpha\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}u|^2\mathrm{d}x\right) \operatorname{div}(y^{1-2s}\nabla w)(z,\nabla w)\mathrm{d}x\mathrm{d}y \\
&= \int_{\partial D_{R,r}} \left(1+\alpha\|w\|^2_{X^s(\mathbb{R}^{3+1}_+)}\right) y^{1-2s}(z,\nabla w)(\nabla w,\vec{n})\mathrm{d}\sigma \\
&\quad - \int_{\partial D_{R,r}} \left(1+\alpha\|w\|^2_{X^s(\mathbb{R}^{3+1}_+)}\right) y^{1-2s}(z,\vec{n})\frac{|\nabla w|^2}{2}\mathrm{d}\sigma \\
&\quad + \frac{3-2s}{2}\left(1+\alpha\|w\|^2_{X^s(\mathbb{R}^{3+1}_+)}\right)\int_{D_{R,r}} y^{1-2s}|\nabla w|^2\mathrm{d}x\mathrm{d}y. \qquad (2.8)
\end{aligned}
$$

Similar to the argument of [11], we prove that there exists $R_n \to \infty$ such that

$$
\begin{aligned}
&\lim_{n\to\infty}\lim_{r\to 0+}\int_{\partial D_{R_n,r}} \left(1+\alpha\|w\|^2_{X^s(\mathbb{R}^{3+1}_+)}\right) y^{1-2s}\left[(z,\nabla w)(\nabla w,\vec{n}) - (z,\vec{n})\frac{|\nabla w|^2}{2}\right] \\
&= -3\int_{\mathbb{R}^3} G(u)\mathrm{d}x, \qquad (2.9)
\end{aligned}
$$

and

$$
\begin{aligned}
&\lim_{n\to\infty}\lim_{r\to 0+}\frac{3-2s}{2}\left(1+\alpha\|w\|^2_{X^s(\mathbb{R}^{3+1}_+)}\right)\int_{D_{R,r}} y^{1-2s}|\nabla w|^2\mathrm{d}x\mathrm{d}y \\
&= \frac{3-2s}{2}\left(1+\alpha\|w\|^2_{X^s(\mathbb{R}^{3+1}_+)}\right)\int_{\mathbb{R}^{3+1}_+} y^{1-2s}|\nabla w|^2\mathrm{d}x\mathrm{d}y \\
&= \frac{3-2s}{2}\left(1+\alpha\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2\mathrm{d}x\right)\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2\mathrm{d}x. \qquad (2.10)
\end{aligned}
$$

Combining (2.8)-(2.10), we get Lemma 2.2. □

Let X be the Banach space. Recall that a sequence $\{u_n\} \subset X$ is a $(C)_c$ sequence for the functional I if $I(u_n) \to c$ and $(1+\|u_n\|_X)\|I'(u_n)\| \to 0$ as $n \to \infty$.

Theorem 2.1. *([26]) Let X be a real Banach space and suppose that $I \in C^1(X,\mathbb{R})$ satisfying*

$$\max\{I(0), I(u_1)\} \le \alpha_2 < \alpha_1 \le \inf_{\|u\|=\rho} I(u)$$

for some $\rho > 0$ and $u_1 \in X$ with $\|u_1\| > \rho$. Let $c = \inf_{\gamma\in\Gamma}\max_{0\le t\le 1} I(\gamma(t))$, where $\Gamma = \{\gamma \in C([0,1],X) : \gamma(0) = 0, \gamma(1) = u_1\}$. Then there exists a $(C)_c$ sequence $\{u_n\}$ for the functional I satisfying $c \ge \alpha_1$.

Theorem 2.2. *([20]) Let $(X, \|.\|_X)$ be a Banach space and let $J \subset \mathbb{R}^+$ be an interval. Consider a family $(J_\lambda)_{\lambda\in J}$ of C^1 functionals on X of the form*

$$J_\lambda(u) = A(u) - \lambda B(u), \quad \forall\ \lambda \in J,$$

where $B(u) \geq 0$ for any $u \in X$ and either $A(u) \to +\infty$ or $B(u) \to +\infty$ as $\|u\|_X \to \infty$. Assume there exist two points v_1, v_2 in X such that

$$c_\lambda = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J_\lambda(\gamma(t)) > \max\{J_\lambda(v_1), J_\lambda(v_2)\}, \quad \forall\ \lambda \in J,$$

where $\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = v_1, \gamma(1) = v_2\}$. Then for almost every $\lambda \in J$, there is a sequence $\{v_n\} \subset X$ such that $\{v_n\}$ is bounded, $J_\lambda(v_n) \to c_\lambda$ and $J'_\lambda(v_n) \to 0$ in X^{-1}. Moreover, the map $\lambda \to c_\lambda$ is continuous from the left.

3 The Case $s \in (\frac{3}{4}, 1)$

Let $H(u) = \beta F(u) + \frac{1}{2_s^*}|u|^{2_s^*}$. Then

$$I(u) = \frac{1}{2}\|u\|^2 + \frac{\alpha}{4}\left(\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 \mathrm{d}x\right)^2 - \int_{\mathbb{R}^3} H(u)\mathrm{d}x. \tag{3.1}$$

Let

$$\begin{aligned}\tilde{c} =& \frac{s}{3}\min\left\{(2\alpha)^{\frac{3-2s}{4s-3}}\,[S(s)]^{\frac{3}{4s-3}}, 2^{\frac{3-2s}{2s}}\,[S(s)]^{\frac{3}{2s}}\right\} \\ &+ \frac{(4s-3)\alpha}{12}\left[\min\left\{(2\alpha)^{\frac{3-2s}{4s-3}}\,[S(s)]^{\frac{3}{4s-3}}, 2^{\frac{3-2s}{2s}}\,[S(s)]^{\frac{3}{2s}}\right\}\right]^2.\end{aligned}$$

Lemma 3.1. *Let $\varepsilon = \frac{1}{\beta^{\frac{1}{2s}}}$. Then there exists $\beta_0 > 0$ such that $\sup_{t\geq 0} I(u_\varepsilon(\frac{x}{t})) < \tilde{c}$ for $\beta > \beta_0$.*

Proof. By the direct calculation, we have

$$\begin{aligned}I\left(u_\varepsilon\left(\frac{x}{t}\right)\right) =& \frac{t^{3-2s}}{2}\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_\varepsilon|^2 \mathrm{d}x + \frac{\alpha t^{2(3-2s)}}{4}\left(\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_\varepsilon|^2 \mathrm{d}x\right)^2 \\ &+ \frac{t^3}{2}\int_{\mathbb{R}^3} |u_\varepsilon|^2 \mathrm{d}x - t^3 \int_{\mathbb{R}^3} H(u_\varepsilon)\mathrm{d}x.\end{aligned} \tag{3.2}$$

By (2.2)-(2.3), there exists ε_1, $C_1 > 0$ such that $\int_{\mathbb{R}^3} |u_\varepsilon|^2 \mathrm{d}x \leq C_1 \varepsilon^{3-2s}$ and $\|u_\varepsilon\|^2 \leq \frac{3[S(s)]^{\frac{3}{2s}}}{2}$ for $\varepsilon \in (0, \varepsilon_1)$. Let

$$\beta > \beta_1 := \max\left\{1, \frac{1}{\varepsilon_1^{2s}}, \left(\frac{\mu[S(s)]^{\frac{1}{2s}}}{r}\right)^{2s}, \left(\frac{2\mu^2 R_0^{\frac{2}{3-2s}}}{\kappa_0^{\frac{2}{3-2s}}}\right)^{2s}\right\}.$$

Then by (3.2),

$$\begin{aligned}\sup_{t\in[0,\frac{1}{\beta}]} I\left(u_\varepsilon\left(\frac{x}{t}\right)\right) \leq& \frac{1}{2\beta^{3-2s}}\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_\varepsilon|^2 \mathrm{d}x + \frac{1}{2\beta^3}\int_{\mathbb{R}^3} |u_\varepsilon|^2 \mathrm{d}x \\ &+ \frac{\alpha}{4\beta^{2(3-2s)}}\left(\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_\varepsilon|^2 \mathrm{d}x\right)^2 \\ \leq& \left(\frac{3[S(s)]^{\frac{3}{2s}}}{4} + \frac{9\alpha[S(s)]^{\frac{3}{s}}}{16}\right)\frac{1}{\beta^{3-2s}}.\end{aligned} \tag{3.3}$$

Note that $u_\varepsilon(x) = \varepsilon^{-\frac{3-2s}{2}} \frac{\kappa_0}{\left(\mu^2 + \left|\frac{x}{\varepsilon[S(s)]^{\frac{1}{2s}}}\right|^2\right)^{\frac{3-2s}{2}}}$ for $|x| \le r$. By $\beta = \frac{1}{\varepsilon^{2s}} > \beta_1$, we get $u_\varepsilon(x) \ge \frac{\kappa_0 \varepsilon^{-\frac{3-2s}{2}}}{(2\mu^2)^{\frac{3-2s}{2}}} \ge R_0$ for $|x| \le \mu[S(s)]^{\frac{1}{2s}}\varepsilon \le r$. Then by (f_2) and $\varepsilon = \frac{1}{\beta^{\frac{1}{2s}}}$,

$$
\begin{aligned}
\beta \int_{\mathbb{R}^3} F(u_\varepsilon)\mathrm{d}x \ge & \frac{D_0\beta}{R_0^{p_0}} \int_{|x|\le\mu[S(s)]^{\frac{1}{2s}}\varepsilon} |u_\varepsilon(x)|^{p_0}\mathrm{d}x \\
\ge & \frac{D_0\beta}{R_0^{p_0}} \int_{|x|\le\mu[S(s)]^{\frac{1}{2s}}\varepsilon} \frac{\kappa_0^{p_0}\varepsilon^{-\frac{(3-2s)p_0}{2}}}{(2\mu^2)^{\frac{(3-2s)p_0}{2}}}\mathrm{d}x \\
:= & B_0\beta\varepsilon^{3-\frac{(3-2s)p_0}{2}} = B_0\beta^{1-\frac{3}{2s}+\frac{(3-2s)p_0}{4s}}. \qquad (3.4)
\end{aligned}
$$

Let

$$
\beta > \beta_2 := \max\left\{\beta_1, \left(\frac{C_1}{B_0}\right)^{\frac{4s}{(3-2s)p_0}}\right\}.
$$

By $\varepsilon = \frac{1}{\beta^{\frac{1}{2s}}}$ and (3.4),

$$
\int_{\mathbb{R}^3} |u_\varepsilon|^2\mathrm{d}x \le C_1\varepsilon^{3-2s} = \frac{C_1}{\beta^{\frac{3-2s}{2s}}} \le \beta\int_{\mathbb{R}^3} F(u_\varepsilon)\mathrm{d}x. \qquad (3.5)
$$

Combining (3.2) and (3.4)-(3.5),

$$
\begin{aligned}
\sup_{t\ge\frac{1}{\beta}} I\left(u_\varepsilon\left(\frac{x}{t}\right)\right) \le & \sup_{t\ge 0}\left[\frac{t^{3-2s}}{2}\|u_\varepsilon\|^2 - \frac{\beta t^3}{4}\int_{\mathbb{R}^3} F(u_\varepsilon)\mathrm{d}x\right] \\
& + \sup_{t\ge 0}\left[\frac{\alpha t^{2(3-2s)}}{4}\left(\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}u_\varepsilon|^2\mathrm{d}x\right)^2 - \frac{\beta t^3}{4}\int_{\mathbb{R}^3} F(u_\varepsilon)\mathrm{d}x\right] \\
\le & \sup_{t\ge 0}\left[\frac{3[S(s)]^{\frac{3}{2s}}}{4}t^{3-2s} - \frac{B_0\beta^{1-\frac{3}{2s}+\frac{(3-2s)p_0}{4s}}t^3}{4}\right] \\
& + \sup_{t\ge 0}\left(\frac{9\alpha[S(s)]^{\frac{3}{s}}t^{2(3-2s)}}{16} - \frac{B_0\beta^{1-\frac{3}{2s}+\frac{(3-2s)p_0}{4s}}t^3}{4}\right) \\
= & \frac{s[S(s)]^{\frac{3}{2s}}}{2}\left(\frac{(3-2s)[S(s)]^{\frac{3}{2s}}}{B_0\beta^{1-\frac{3}{2s}+\frac{(3-2s)p_0}{4s}}}\right)^{\frac{3-2s}{2s}} \\
& + \frac{3\alpha(4s-3)[S(s)]^{\frac{3}{s}}}{16}\left(\frac{3\alpha(3-2s)[S(s)]^{\frac{3}{s}}}{2B_0\beta^{1-\frac{3}{2s}+\frac{(3-2s)p_0}{4s}}}\right)^{\frac{2(3-2s)}{4s-3}}. \qquad (3.6)
\end{aligned}
$$

From (3.3) and (3.6), we derive there exists $\beta_0 > 0$ such that $\sup_{t\ge 0} I(u_\varepsilon(\frac{x}{t})) < \tilde{c}$ for $\beta > \beta_0$.

$\square$

Lemma 3.2. *Let $\beta > \beta_0$. If $\{u_n\} \subset H_r^s(\mathbb{R}^3)$ is a sequence such that $I(u_n) \to c$ and $(1 + \|u_n\|)\|I'(u_n)\| \to 0$, then $\|u_n\|$ is bounded.*

Proof. Otherwise, we have $\|u_n\| \to \infty$. Let $v_n = \frac{u_n}{\|u_n\|}$. Then $v_n \rightharpoonup v$ weakly in $H_r^s(\mathbb{R}^3)$ and $v_n(x) \to v(x)$ a.e. $x \in \mathbb{R}^3$.

Case 1. $v(x) = 0$ a.e. $x \in \mathbb{R}^3$. Let $\theta \in (4, 2_s^*)$. By (f_1), for $\varepsilon \in \left(0, \frac{1}{\beta}\left(\frac{1}{\theta} - \frac{1}{2_s^*}\right)\right)$, there exists $C_\varepsilon > 0$ such that

$$\left|\frac{1}{\theta} f(u_n)u_n - F(u_n)\right| \leq \varepsilon |u_n|^{2_s^*} + C_\varepsilon |u_n|^2.$$

Then

$$\left(I(u_n) - \frac{1}{\theta}\left(I'(u_n), u_n\right)\right) \geq \left(\frac{1}{2} - \frac{1}{\theta}\right)\|u_n\|^2 - \beta C_\varepsilon \int_{\mathbb{R}^3} |u_n|^2 \mathrm{d}x. \tag{3.7}$$

From (3.7), we derive that

$$\frac{1}{\|u_n\|^2}\left(I(u_n) - \frac{1}{\theta}\left(I'(u_n), u_n\right)\right) \geq \left(\frac{1}{2} - \frac{1}{\theta}\right) - \beta C_\varepsilon \int_{\mathbb{R}^3} |v_n|^2 \mathrm{d}x.$$

By $\int_{\mathbb{R}^3} |v_n|^2 \mathrm{d}x \to 0$, $I(u_n) \to c$, $(I'(u_n), u_n) \to 0$ and $\|u_n\| \to \infty$, we get $0 \geq \left(\frac{1}{2} - \frac{1}{\theta}\right)$, a contradiction.

Case 2. $v(x) \neq 0$. Let $\Omega = \{x \in \mathbb{R}^N : v(x) \neq 0\}$. Then the measure of Ω is positive. For $x \in \Omega$, by $v_n(x) = \frac{u_n(x)}{\|u_n\|} \to v(x)$, we get $\lim_{n\to\infty} u_n(x) = \infty$. Let $q \in (4, 2_s^*)$. By (f_2) and Fatou's Lemma,

$$\lim_{n\to\infty} \int_\Omega \frac{\beta F(u_n) + \frac{1}{2_s^*}|u_n|^{2_s^*}}{\|u_n\|^q} \mathrm{d}x = +\infty. \tag{3.8}$$

By (f_1), for $\varepsilon \in \left(0, \frac{1}{\beta 2_s^*}\right)$, there exists $C_\varepsilon > 0$ such that $F(u_n) \leq \varepsilon |u_n|^{2_s^*} + C_\varepsilon |u_n|^2$. Then

$$\int_{\mathbb{R}^3\setminus\Omega} \beta F(u_n)\mathrm{d}x + \frac{1}{2_s^*}\int_{\mathbb{R}^3\setminus\Omega} |u_n|^{2_s^*}\mathrm{d}x \geq -\beta C_\varepsilon \int_{\mathbb{R}^3\setminus\Omega} |u_n|^2 \mathrm{d}x,$$

from which we get

$$\lim_{n\to\infty} \int_{\mathbb{R}^3\setminus\Omega} \frac{\beta F(u_n) + \frac{1}{2_s^*}|u_n|^{2_s^*}}{\|u_n\|^q} \mathrm{d}x \geq 0. \tag{3.9}$$

By (3.8)-(3.9), we derive that

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} \frac{\beta F(u_n) + \frac{1}{2_s^*}|u_n|^{2_s^*}}{\|u_n\|^q} \mathrm{d}x = +\infty. \tag{3.10}$$

On the other hand, we have

$$I(u_n) + \int_{\mathbb{R}^3} \left(\beta F(u_n) + \frac{1}{2_s^*}|u_n|^{2_s^*}\right) \mathrm{d}x = \frac{1}{2}\|u_n\|^2 + \frac{\alpha}{4}\left(\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 \mathrm{d}x\right)^2.$$

So $\lim_{n\to\infty} \int_{\mathbb{R}^3} \frac{\beta F(u_n)+\frac{1}{2_s^*}|u_n|^{2_s^*}}{\|u_n\|^q} \mathrm{d}x = 0$, a contradiction with (3.10). □

Lemma 3.3. *Assume that $\{u_n\} \subset H_r^s(\mathbb{R}^3)$ is a sequence such that $\|u_n\|$ is bounded, $I(u_n) \to c \in (0, \tilde{c})$ and $I'(u_n) \to 0$. Then $\{u_n\}$ converges strongly in $H_r^s(\mathbb{R}^3)$ up to a subsequence.*

Proof. Since $\|u_n\|$ is bounded, we assume $u_n \rightharpoonup u$ weakly in $H_r^s(\mathbb{R}^3)$. Let $A = \lim_{n\to\infty} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_n|^2 \mathrm{d}x$. Define

$$\begin{aligned}
\acute{I}(u) =& \frac{1}{2}\|u\|^2 + \frac{\alpha A}{4}\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 \mathrm{d}x - \beta \int_{\mathbb{R}^3} F(u)\mathrm{d}x - \frac{\lambda}{2_s^*}\int_{\mathbb{R}^3} |u|^{2_s^*} \mathrm{d}x, \\
\grave{I}(u) =& \frac{1}{2}\|u\|^2 + \frac{\alpha A}{2}\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 \mathrm{d}x - \beta \int_{\mathbb{R}^3} F(u)\mathrm{d}x - \frac{\lambda}{2_s^*}\int_{\mathbb{R}^3} |u|^{2_s^*} \mathrm{d}x,
\end{aligned}$$

where $u \in H_r^s(\mathbb{R}^3)$. Then $\acute{I}(u_n) \to c$ and $\grave{I}'(u_n) \to 0$. By $u_n \rightharpoonup u$ weakly in $H_r^s(\mathbb{R}^3)$, we have $\grave{I}'(u) = 0$. By (f_1) and Lemma 2.1, we derive that

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} F(u_n)\mathrm{d}x = \int_{\mathbb{R}^3} F(u)\mathrm{d}x, \quad \lim_{n\to\infty} \int_{\mathbb{R}^3} f(u_n)u_n \mathrm{d}x = \int_{\mathbb{R}^3} f(u)u \mathrm{d}x. \tag{3.11}$$

Set $v_n = u_n - u$. By the Brezis-Lieb Lemma in [33], we have

$$\begin{aligned}
&\|v_n\|^2 = \|u_n\|^2 - \|u\|^2 + o_n(1), \\
&\int_{\mathbb{R}^3} |v_n|^2 \mathrm{d}x = \int_{\mathbb{R}^3} |u_n|^2 \mathrm{d}x - \int_{\mathbb{R}^3} |u|^2 \mathrm{d}x + o_n(1), \\
&\int_{\mathbb{R}^3} |v_n|^{2_s^*} \mathrm{d}x = \int_{\mathbb{R}^3} |u_n|^{2_s^*} \mathrm{d}x - \int_{\mathbb{R}^3} |u|^{2_s^*} \mathrm{d}x + o_n(1).
\end{aligned} \tag{3.12}$$

Then by $\acute{I}(u_n) - \acute{I}(u) \to c - \acute{I}(u)$ and $\left(\grave{I}'(u_n), u_n\right) - \left(\grave{I}'(u), u\right) \to 0$,

$$\begin{aligned}
&c - \acute{I}(u) = \frac{1}{2}\|v_n\|^2 + \frac{\alpha A}{4}\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} v_n|^2 \mathrm{d}x - \frac{1}{2_s^*}\int_{\mathbb{R}^3} |v_n|^{2_s^*} \mathrm{d}x + o_n(1), \\
&o_n(1) = \|v_n\|^2 + \alpha A \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} v_n|^2 \mathrm{d}x - \int_{\mathbb{R}^3} |v_n|^{2_s^*} \mathrm{d}x.
\end{aligned} \tag{3.13}$$

Assume that $\lim_{n\to\infty} \int_{\mathbb{R}^3} |v_n|^{2_s^*} \mathrm{d}x = l$. By (3.12), we get

$$A \geq \lim_{n\to\infty} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} v_n|^2 \mathrm{d}x.$$

Then by (3.13) and $S(s) \leq \frac{\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} v_n|^2 \mathrm{d}x}{\left(\int_{\mathbb{R}^3} |v_n|^{2_s^*} \mathrm{d}x\right)^{\frac{2}{2_s^*}}}$,

$$\begin{aligned}\lim_{n\to\infty} \int_{\mathbb{R}^3} |v_n|^{2_s^*} \mathrm{d}x \geq & S(s) \left(\lim_{n\to\infty} \int_{\mathbb{R}^3} |v_n|^{2_s^*} \mathrm{d}x\right)^{\frac{2}{2_s^*}} \\ & + \alpha[S(s)]^2 \left(\lim_{n\to\infty} \int_{\mathbb{R}^3} |v_n|^{2_s^*} \mathrm{d}x\right)^{\frac{4}{2_s^*}}.\end{aligned}$$

If $l > 0$ and

$$S(s) \left(\lim_{n\to\infty} \int_{\mathbb{R}^3} |v_n|^{2_s^*} \mathrm{d}x\right)^{\frac{2}{2_s^*}} \geq \alpha[S(s)]^2 \left(\lim_{n\to\infty} \int_{\mathbb{R}^3} |v_n|^{2_s^*} \mathrm{d}x\right)^{\frac{4}{2_s^*}},$$

then

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} |v_n|^{2_s^*} \mathrm{d}x \geq 2\alpha[S(s)]^2 \left(\lim_{n\to\infty} \int_{\mathbb{R}^3} |v_n|^{2_s^*} \mathrm{d}x\right)^{\frac{4}{2_s^*}}.$$

So $l \geq \left(2\alpha[S(s)]^2\right)^{\frac{3}{4s-3}}$. If $l > 0$ and

$$S(s) \left(\lim_{n\to\infty} \int_{\mathbb{R}^3} |v_n|^{2_s^*} \mathrm{d}x\right)^{\frac{2}{2_s^*}} \leq \alpha[S(s)]^2 \left(\lim_{n\to\infty} \int_{\mathbb{R}^3} |v_n|^{2_s^*} \mathrm{d}x\right)^{\frac{4}{2_s^*}},$$

then

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} |v_n|^{2_s^*} \mathrm{d}x \geq 2S(s) \left(\lim_{n\to\infty} \int_{\mathbb{R}^3} |v_n|^{2_s^*} \mathrm{d}x\right)^{\frac{2}{2_s^*}}.$$

So $l \geq (2S(s))^{\frac{3}{2s}}$. Thus,

$$l \geq \min\left\{\left(2\alpha[S(s)]^2\right)^{\frac{3}{4s-3}}, (2S(s))^{\frac{3}{2s}}\right\},$$

from which we get

$$\begin{aligned}A &\geq \lim_{n\to\infty} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} v_n|^2 \mathrm{d}x \\ &\geq \min\left\{(2\alpha)^{\frac{3-2s}{4s-3}} [S(s)]^{\frac{3}{4s-3}}, 2^{\frac{3-2s}{2s}} [S(s)]^{\frac{3}{2s}}\right\}.\end{aligned}$$

So by (3.13), we have

$$\begin{aligned}c - I(u) \geq & \frac{s}{3} \lim_{n\to\infty} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} v_n|^2 \mathrm{d}x \\ & + \frac{(4s-3)\alpha}{12} \left(\lim_{n\to\infty} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} v_n|^2 \mathrm{d}x\right)^2 \geq \tilde{c}.\end{aligned}$$

Since $\grave{I}'(u) = 0$, by Lemma 2.2,

$$\frac{3-2s}{2}(1+\alpha A)\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2\mathrm{d}x = 3\int_{\mathbb{R}^3}\left(\beta F(u) + \frac{1}{2_s^*}|u|^{2_s^*} - \frac{1}{2}|u|^2\right)\mathrm{d}x.$$

Then $\acute{I}(u) \geq 0$. Thus, we have $c \geq \tilde{c}$, a contradiction. So $l = 0$. By (3.13), we get $v_n \to 0$ in $H_r^s(\mathbb{R}^3)$.

□

Proof of Theorem 1.1. Let $\beta > \beta_0$. By (f_1), for $\varepsilon = \frac{1}{4}$, there exists $C_\varepsilon = C_{\frac{1}{4}} > 0$ such that $H(u) \leq \frac{1}{4}|u|^2 + C_{\frac{1}{4}}|u|^{2_s^*}$. Then by the Sobolev embedding theorem,

$$I(u) \geq \frac{1}{2}\|u\|^2 - \frac{1}{4}\int_{\mathbb{R}^3}|u|^2\mathrm{d}x - C_{\frac{1}{4}}\int_{\mathbb{R}^3}|u|^{2_s^*}\mathrm{d}x \geq \frac{1}{4}\|u\|^2 - \frac{C_{\frac{1}{4}}\|u\|^{2_s^*}}{[S(s)]^{\frac{2_s^*}{2}}}.$$

So there exists ρ_0, $\gamma_0 > 0$ such that $I(u) \geq \gamma_0$ for $\|u\| = \rho_0$. Let $\varepsilon = \frac{1}{\beta^{\frac{1}{2s}}}$. By (3.2) and (3.5), we have

$$\begin{aligned} I\left(u_\varepsilon\left(\frac{x}{t}\right)\right) \leq & \frac{t^{3-2s}}{2}\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u_\varepsilon|^2\mathrm{d}x + \frac{\alpha t^{2(3-2s)}}{4}\left(\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u_\varepsilon|^2\mathrm{d}x\right)^2 \\ & - \frac{\beta t^3}{2}\int_{\mathbb{R}^3}F(u_\varepsilon)\mathrm{d}x. \end{aligned}$$

Then $\lim_{t\to+\infty} I\left(u_\varepsilon\left(\frac{x}{t}\right)\right) = -\infty$. We also have $I(0) = 0$. Let $\gamma(t)(x) = u_\varepsilon(\frac{x}{t})$ for $t > 0$ and $\gamma(t)(x) = 0$ for $t = 0$. By the direct calculation, we have

$$\|\gamma(t)\|^2 = t^{3-2s}\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u_\varepsilon|^2\mathrm{d}x + t^3\int_{\mathbb{R}^3}|u_\varepsilon|^2\mathrm{d}x.$$

Then $\gamma(t) \in C([0,\infty), H_r^s(\mathbb{R}^3))$. By Theorem 2.1 and Lemma 3.2, there exists a bounded sequence $\{u_n\} \subset H_r^s(\mathbb{R}^3)$ such that $I(u_n) \to c > 0$ and $(1 + \|u_n\|)\|I'(u_n)\| \to 0$. Assume $u_n \rightharpoonup u_0$ weakly in $H_r^s(\mathbb{R}^3)$. By the definition of c and Lemma 3.1, we get $c < \tilde{c}$. From Lemma 3.3, we know $u_n \to u_0$ in $H_r^s(\mathbb{R}^3)$. So $I(u_0) = c \in [\gamma_0, \tilde{c})$ and $I'(u_0) = 0$.

Let

$$m = \inf\{I(u) : u \in H_r^s(\mathbb{R}^3) \setminus \{0\} : I'(u) = 0\}.$$

Since $I'(u_0) = 0$, we get $m \leq I(u_0) = c < \tilde{c}$. By Lemma 2.2, we have $m \geq 0$. So $0 \leq m \leq I(u_0) < \tilde{c}$. By the definition of m, there exists $\{u_n\} \subset H_r^s(\mathbb{R}^3) \setminus \{0\}$ such that $I(u_n) \to m$ and $I'(u_n) = 0$. Then by Lemma 2.2, for n large enough,

$$\frac{s}{3}\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u_n|^2\mathrm{d}x + \frac{(4s-3)\alpha}{12}\left(\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u_n|^2\mathrm{d}x\right)^2 = I(u_n) \leq 2m. \tag{3.14}$$

By (f_1), for $\varepsilon = \frac{1}{2}$, there exists $C_\varepsilon = C_{\frac{1}{2}} > 0$ such that $|\beta f(u_n)u_n| \le \frac{1}{2}|u_n|^2 + C_{\frac{1}{2}}|u_n|^{2_s^*}$. Then by $(I'(u_n), u_n) = 0$,

$$\|u_n\|^2 + \alpha \left(\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_n|^2 \mathrm{d}x \right)^2 \le \frac{1}{2} \int_{\mathbb{R}^3} |u_n|^2 \mathrm{d}x + (C_{\frac{1}{2}} + 1) \int_{\mathbb{R}^3} |u_n|^{2_s^*} \mathrm{d}x. \tag{3.15}$$

From (3.14)-(3.15) and the Sobolev embedding theorem,

$$\begin{aligned} \|u_n\|^2 \le & 2(C_{\frac{1}{2}} + 1) \int_{\mathbb{R}^3} |u_n|^{2_s^*} \mathrm{d}x \\ \le & \frac{2(C_{\frac{1}{2}} + 1)}{S(s)^{\frac{2_s^*}{2}}} \left(\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_n|^2 \mathrm{d}x \right)^{\frac{2_s^*}{2}} \le \frac{2(C_{\frac{1}{2}} + 1)}{S(s)^{\frac{2_s^*}{2}}} \left(\frac{6m}{s} \right)^{\frac{2_s^*}{2}}. \end{aligned}$$

Also, by (3.15),

$$\frac{S(s)}{2} \left(\int_{\mathbb{R}^3} |u_n|^{2_s^*} \mathrm{d}x \right)^{\frac{2}{2_s^*}} \le \|u_n\|^2 - \frac{1}{2} \int_{\mathbb{R}^3} |u_n|^2 \mathrm{d}x \le (C_{\frac{1}{2}} + 1) \int_{\mathbb{R}^3} |u_n|^{2_s^*} \mathrm{d}x,$$

from which we derive that $\int_{\mathbb{R}^3} |u_n|^{2_s^*} \mathrm{d}x \ge \left[\frac{S(s)}{2(C_{\frac{1}{2}}+1)} \right]^{\frac{3}{2s}}$. Assume that $u_n \rightharpoonup u$ weakly in $H_r^s(\mathbb{R}^3)$. Similar to the proof of Lemma 3.3, we derive from $I(u_n) \to m \in [0, \tilde{c})$ and $I'(u_n) = 0$ that $u_n \to u$ in $H_r^s(\mathbb{R}^3)$. Then $I(u) = m$ and $I'(u) = 0$ with $u \ne 0$. So m is attained by u.

□

4 The Case $s = \frac{3}{4}$

By $s = \frac{3}{4}$, we have $2_s^* = \frac{6}{3-2s} = 4$. We first consider the case $\alpha < \frac{1}{[S(s)]^2}$. Choose $\lambda_1 \in (0, 1)$ such that $\alpha[S(s)]^4 < \lambda[S(s)]^2$ for $\lambda \in [\lambda_1, 1]$. For $\lambda \in [\lambda_1, 1]$, define

$$J_\lambda(u) = \frac{1}{2}\|u\|^2 + \frac{\alpha}{4} \left(\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 \mathrm{d}x \right)^2 - \lambda \int_{\mathbb{R}^3} H(u) \mathrm{d}x, \quad u \in H_r^s(\mathbb{R}^3). \tag{4.1}$$

Lemma 4.1. *Let $\{u_n\} \subset H_r^s(\mathbb{R}^3)$ be a sequence such that $J_{\lambda_n}(u_n) < \frac{[S(s)]^2}{4(\lambda - \alpha[S(s)]^2)}$ and $J'_{\lambda_n}(u_n) = 0$, where $\lambda_n \in [\lambda_1, 1]$. Then there exists $M_0 > 0$ independent of n such that $\|u_n\| \le M_0$.*

Proof. By Lemma 2.2, we have

$$\left(1 + \alpha \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_n|^2 \mathrm{d}x \right) \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_n|^2 \mathrm{d}x = \int_{\mathbb{R}^3} \left(4\lambda H(u_n) - 2|u_n|^2 \right) \mathrm{d}x.$$

Then

$$\frac{1}{4}\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u_n|^2\mathrm{d}x = J_{\lambda_n}(u_n) < \frac{[S(s)]^2}{4(\lambda-\alpha[S(s)]^2)}. \tag{4.2}$$

By (f_1), for $\varepsilon = \frac{1}{2}$, there exists $C_\varepsilon = C_{\frac{1}{2}} > 0$ such that $|\beta f(u_n)u_n| \le \frac{1}{2}|u_n|^2 + C_{\frac{1}{2}}|u_n|^4$. Then by $\left(J'_{\lambda_n}(u_n), u_n\right) = 0$,

$$\begin{aligned}\|u_n\|^2 + \alpha\left(\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u_n|^2\mathrm{d}x\right)^2 &= \lambda_n\int_{\mathbb{R}^3}h(u_n)u_n\mathrm{d}x\\ &\le \frac{1}{2}\int_{\mathbb{R}^3}|u_n|^2\mathrm{d}x + (C_{\frac{1}{2}}+1)\int_{\mathbb{R}^3}|u_n|^4\mathrm{d}x.\end{aligned} \tag{4.3}$$

From (4.2)-(4.3),

$$\begin{aligned}\|u_n\|^2 \le& 2(C_{\frac{1}{2}}+1)\int_{\mathbb{R}^3}|u_n|^4\mathrm{d}x\\ \le& \frac{2(C_{\frac{1}{2}}+1)}{[S(s)]^2}\left(\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u_n|^2\mathrm{d}x\right)^2 \le \frac{2(C_{\frac{1}{2}}+1)}{[S(s)]^2}\frac{[S(s)]^4}{(\lambda-\alpha[S(s)]^2)^2} := M_0.\end{aligned}$$

□

Lemma 4.2. *There exists $\alpha_0 > 0$ such that for almost every $\lambda \in [\lambda_1, 1]$, there exists a sequence $\{u_n\} \subset H^s_r(\mathbb{R}^3)$ satisfying $\|u_n\|$ is bounded, $J_\lambda(u_n) \to c_\lambda \ge \alpha_0$ and $J'_\lambda(u_n) \to 0$. Moreover, $c_\lambda < \frac{[S(s)]^2}{4(\lambda-\alpha[S(s)]^2)}$ for any $\lambda \in [\lambda_1, 1]$ and the map $\lambda \to c_\lambda$ is continuous from the left.*

Proof. Let $J = [\lambda_1, 1]$, $A(u) = \frac{1}{2}\|u\|^2 + \frac{\alpha}{4}\left(\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2\mathrm{d}x\right)^2$ and $B(u) = \int_{\mathbb{R}^3}H(u)\mathrm{d}x$. Then $B(u) \ge 0$ and $A(u) \to +\infty$ as $\|u\| \to \infty$. By (f_1), for $\varepsilon = \frac{1}{4\beta}$, there exists $C_\varepsilon = C_{\frac{1}{4\beta}} > 0$ such that $|F(u)| \le \frac{1}{4\beta}|u|^2 + C_{\frac{1}{4\beta}}|u|^4$. Then

$$\begin{aligned}J_\lambda(u) &\ge \frac{1}{2}\|u\|^2 - \frac{1}{4}\int_{\mathbb{R}^3}|u|^2\mathrm{d}x - \left(\beta C_{\frac{1}{4\beta}} + \frac{1}{4}\right)\int_{\mathbb{R}^3}|u|^4\mathrm{d}x\\ &\ge \frac{1}{4}\|u\|^2 - \frac{\beta C_{\frac{1}{4\beta}} + \frac{1}{4}}{[S(s)]^2}\|u\|^4.\end{aligned}$$

So there exists r_0, $\alpha_0 > 0$ independent of λ such that $J_\lambda(u) \ge \alpha_0$ for $\|u\| = r_0$. For $\lambda \in [\lambda_1, 1]$,

$$J_\lambda(tu_\varepsilon) \le \frac{t^2}{2}\|u_\varepsilon\|^2 - \frac{t^4}{4}\left[\lambda_1\int_{\mathbb{R}^3}|u_\varepsilon|^4\mathrm{d}x - \alpha\left(\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u_\varepsilon|^2\mathrm{d}x\right)^2\right]. \tag{4.4}$$

Since $\alpha[S(s)]^4 < \lambda[S(s)]^2$ for $\lambda \in [\lambda_1, 1]$, by (2.2)-(2.3), we derive that there exists $\varepsilon_1 > 0$ such that for $\varepsilon \in (0, \varepsilon_1)$,

$$\begin{aligned} &\|u_\varepsilon\|^2 \le \frac{3[S(s)]^2}{2}, \\ &\lambda_1 \int_{\mathbb{R}^3} |u_\varepsilon|^4 \mathrm{d}x - \alpha \left(\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_\varepsilon|^2 \mathrm{d}x \right)^2 \ge \frac{\lambda_1 [S(s)]^2 - \alpha [S(s)]^4}{2}. \end{aligned} \tag{4.5}$$

From (4.4)-(4.5), we have $\lim_{t\to+\infty} J_\lambda(tu_\varepsilon) = -\infty$. We also have $J_\lambda(0) = 0$. By Theorem 2.2, for almost every $\lambda \in [\lambda_1, 1]$, there exists $\{u_n\} \subset H^s_r(\mathbb{R}^3)$ such that $\|u_n\|$ is bounded, $J_\lambda(u_n) \to c_\lambda \ge \alpha_0$ and $J'_\lambda(u_n) \to 0$. Moreover, the map $\lambda \to c_\lambda$ is continuous from the left.

We claim $c_\lambda < \frac{[S(s)]^2}{4(\lambda - \alpha[S(s)]^2)}$ for $\lambda \in [\lambda_1, 1]$. By (4.4)-(4.5), there exists a small $t_1 > 0$ and a large $t_2 > 0$ independent of ε and λ satisfying

$$\sup_{t \in [0, t_1] \cup [t_2, +\infty)} J_\lambda(tu_\varepsilon) < \frac{[S(s)]^2}{4(\lambda - \alpha[S(s)]^2)}. \tag{4.6}$$

Choose $\varepsilon > 0$ small such that $\mu[S(s)]^{\frac{1}{2s}} \varepsilon \le r$. Then $u_\varepsilon(x) \ge \frac{\kappa_0 \varepsilon^{-\frac{3-2s}{2}}}{(2\mu^2)^{\frac{3-2s}{2}}}$ for $|x| \le \mu[S(s)]^{\frac{1}{2s}} \varepsilon$. By (f_2), we derive that for $\varepsilon > 0$ small, $F(tu_\varepsilon) \ge \frac{D_0 t_1^{p_0} u_\varepsilon^{p_0}}{R_0^{p_0}} \ge \frac{\kappa_0^{p_0} D_0 t_1^{p_0} \varepsilon^{-\frac{p_0(3-2s)}{2}}}{R_0^{p_0} (2\mu^2)^{\frac{p_0(3-2s)}{2}}}$ for $t \in [t_1, t_2]$ and $|x| \le \mu[S(s)]^{\frac{1}{2s}} \varepsilon$. Thus,

$$\begin{aligned} \inf_{t \in [t_1, t_2]} \int_{\mathbb{R}^3} F(tu_\varepsilon) \mathrm{d}x \ge& \inf_{t \in [t_1, t_2]} \int_{|x| \le \mu[S(s)]^{\frac{1}{2s}} \varepsilon} F(tu_\varepsilon) \mathrm{d}x \\ \ge& \frac{\kappa_0^{p_0} D_0 t_1^{p_0} \varepsilon^{-\frac{p_0(3-2s)}{2}}}{R_0^{p_0} (2\mu^2)^{\frac{p_0(3-2s)}{2}}} \int_{|x| \le \mu[S(s)]^{\frac{1}{2s}} \varepsilon} \mathrm{d}x \\ =& \frac{\kappa_0^{p_0} D_0 t_1^{p_0} [S(s)]^{\frac{3}{2s}} \varepsilon^{-\frac{p_0(3-2s)}{2}+3}}{R_0^{p_0} 2^{\frac{p_0(3-2s)}{2}} \mu^{p_0(3-2s)-3}} \int_{|x|\le 1} \mathrm{d}x. \end{aligned} \tag{4.7}$$

Since $s = \frac{3}{4}$, by (2.2)-(2.3) and (4.7), we derive that for $\varepsilon > 0$ small,

$$\begin{aligned} &\sup_{t \in [t_1, t_2]} J_\lambda(tu_\varepsilon) \\ &\le \sup_{t \ge 0} \left[\frac{t^2}{2} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_\varepsilon|^2 \mathrm{d}x - \frac{t^4}{4} \left(\lambda \int_{\mathbb{R}^3} |u_\varepsilon|^4 \mathrm{d}x - \alpha \left(\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_\varepsilon|^2 \mathrm{d}x \right)^2 \right) \right] \\ &\quad + \frac{t_2^2}{2} \int_{\mathbb{R}^3} |u_\varepsilon|^2 \mathrm{d}x - \frac{\beta \kappa_0^{p_0} D_0 t_1^{p_0} [S(s)]^{\frac{3}{2s}} \varepsilon^{-\frac{p_0(3-2s)}{2}+3}}{R_0^{p_0} 2^{\frac{p_0(3-2s)}{2}} \mu^{p_0(3-2s)-3}} \int_{|x|\le 1} \mathrm{d}x \\ &= \frac{\left(\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_\varepsilon|^2 \mathrm{d}x \right)^2}{4 \left(\lambda \int_{\mathbb{R}^3} |u_\varepsilon|^4 \mathrm{d}x - \alpha \left(\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_\varepsilon|^2 \mathrm{d}x \right)^2 \right)} + O(\varepsilon^{\frac{3}{2}} |\log \varepsilon|) - C_0 \varepsilon^{3 - \frac{3p_0}{4}} \\ &\le \frac{[S(s)]^2}{4(\lambda - \alpha[S(s)]^2)} + O(\varepsilon^{\frac{3}{2}} |\log \varepsilon|) - C_0 \varepsilon^{3 - \frac{3p_0}{4}}. \end{aligned}$$

By $p_0 \in (2,4)$, we get $3-\frac{3p_0}{4} < \frac{3}{2}$, from which we derive that $\sup_{t\in[t_1,t_2]} J_\lambda(tu_\varepsilon) < \frac{a^2[S(s)]^2}{4(\lambda-b[S(s)]^2)}$ for $\varepsilon > 0$ small. Together with (4.6) and the definition of c_λ, we get $c_\lambda < \frac{[S(s)]^2}{4(\lambda-\alpha[S(s)]^2)}$. □

Lemma 4.3. *Let $\lambda \in [\lambda_1, 1]$. Assume that $\{u_n\} \subset H_r^s(\mathbb{R}^3)$ is a sequence such that $\|u_n\|$ is bounded, $J_\lambda(u_n) \to c_\lambda \in \left(0, \frac{[S(s)]^2}{4(\lambda-\alpha[S(s)]^2)}\right)$ and $J'_\lambda(u_n) \to 0$. Then $\{u_n\}$ converges strongly in $H_r^s(\mathbb{R}^3)$ up to a subsequence.*

Proof. Assume $u_n \rightharpoonup u_\lambda$ weakly in $H_r^s(\mathbb{R}^3)$. Let $A = \lim_{n\to\infty} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_n|^2 \mathrm{d}x$. Define

$$\hat{J}_\lambda(u) = \frac{1}{2}\|u\|^2 + \frac{\alpha A}{4}\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 \mathrm{d}x - \lambda \int_{\mathbb{R}^3} H(u)\mathrm{d}x,$$
$$\tilde{J}_\lambda(u) = \frac{1}{2}\|u\|^2 + \frac{\alpha A}{2}\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 \mathrm{d}x - \lambda \int_{\mathbb{R}^3} H(u)\mathrm{d}x, \quad u \in H_r^s(\mathbb{R}^3).$$

Then $\hat{J}_\lambda(u_n) \to c_\lambda$ and $\tilde{J}'_\lambda(u_n) \to 0$. By $u_n \rightharpoonup u_\lambda$ weakly in $H_r^s(\mathbb{R}^3)$, we have $\tilde{J}'_\lambda(u_\lambda) = 0$. By Lemma 2.1, we get

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} F(u_n)\mathrm{d}x = \int_{\mathbb{R}^3} F(u_\lambda)\mathrm{d}x, \quad \lim_{n\to\infty} \int_{\mathbb{R}^3} f(u_n)u_n\mathrm{d}x = \int_{\mathbb{R}^3} f(u_\lambda)u_\lambda\mathrm{d}x. \tag{4.8}$$

Set $v_n = u_n - u_\lambda$. By the Brezis-Lieb Lemma in [33], we have

$$\|v_n\|^2 = \|u_n\|^2 - \|u_\lambda\|^2 + o_n(1),$$
$$\int_{\mathbb{R}^3} |v_n|^2\mathrm{d}x = \int_{\mathbb{R}^3} |u_n|^2\mathrm{d}x - \int_{\mathbb{R}^3} |u_\lambda|^2\mathrm{d}x + o_n(1),$$
$$\int_{\mathbb{R}^3} |v_n|^4\mathrm{d}x = \int_{\mathbb{R}^3} |u_n|^4\mathrm{d}x - \int_{\mathbb{R}^3} |u_\lambda|^4\mathrm{d}x + o_n(1). \tag{4.9}$$

Combining (4.8)-(4.9),

$$\begin{aligned} c_\lambda - \hat{J}_\lambda(u_\lambda) &= \hat{J}_\lambda(u_n) - \hat{J}_\lambda(u_\lambda) + o_n(1) \\ &= \frac{1}{2}\|v_n\|^2 + \frac{\alpha A}{4}\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} v_n|^2\mathrm{d}x - \frac{\lambda}{4}\int_{\mathbb{R}^3} |v_n|^4\mathrm{d}x + o_n(1). \end{aligned} \tag{4.10}$$

Also,

$$\begin{aligned} o_n(1) &= \left(\tilde{J}'_\lambda(u_n), u_n\right) - \left(\tilde{J}'_\lambda(u_\lambda), u_\lambda\right) + o_n(1) \\ &= \|v_n\|^2 + \alpha A \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} v_n|^2\mathrm{d}x - \lambda \int_{\mathbb{R}^3} |v_n|^4\mathrm{d}x + o_n(1). \end{aligned} \tag{4.11}$$

Assume that $\lim_{n\to\infty} \int_{\mathbb{R}^3} |v_n|^4\mathrm{d}x = l$. We claim $l = 0$. Otherwise, we have $l > 0$. By (4.9), we get $A \geq \lim_{n\to\infty} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} v_n|^2\mathrm{d}x$. Then by (4.11) and

$S(s) \leq \frac{\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} v_n|^2 \mathrm{d}x}{\left(\int_{\mathbb{R}^3} |v_n|^4 \mathrm{d}x\right)^{\frac{1}{2}}}$, we get

$$\lambda \lim_{n\to\infty} \int_{\mathbb{R}^3} |v_n|^4 \mathrm{d}x \geq S(s) \left(\lim_{n\to\infty} \int_{\mathbb{R}^3} |v_n|^4 \mathrm{d}x \right)^{\frac{1}{2}} + \alpha [S(s)]^2 \lim_{n\to\infty} \int_{\mathbb{R}^3} |v_n|^4 \mathrm{d}x,$$

from which we derive that $l \geq \frac{[S(s)]^2}{(\lambda - \alpha[S(s)]^2)^2}$. So

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} v_n|^2 \mathrm{d}x \geq \frac{[S(s)]^2}{\lambda - b[S(s)]^2}.$$

Together with (4.10)-(4.11), we get

$$c_\lambda - \hat{J}_\lambda(u_\lambda) \geq \frac{1}{4} \lim_{n\to\infty} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} v_n|^2 \mathrm{d}x \geq \frac{[S(s)]^2}{4(\lambda - \alpha[S(s)]^2)}. \tag{4.12}$$

Since $\tilde{J}'_\lambda(u_\lambda) = 0$, similar to Lemma 2.2, we have

$$\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_\lambda|^2 \mathrm{d}x + 2 \int_{\mathbb{R}^3} |u_\lambda|^2 \mathrm{d}x + \alpha A \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_\lambda|^2 \mathrm{d}x = 4\lambda \int_{\mathbb{R}^3} H(u_\lambda) \mathrm{d}x.$$

Then

$$\begin{aligned} \hat{J}_\lambda(u_\lambda) =& \hat{J}_\lambda(u_\lambda) - \frac{1}{4} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_\lambda|^2 \mathrm{d}x - \frac{1}{2} \int_{\mathbb{R}^3} |u_\lambda|^2 \mathrm{d}x - \frac{\alpha A}{4} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_\lambda|^2 \mathrm{d}x \\ &+ \lambda \int_{\mathbb{R}^3} H(u_\lambda) \mathrm{d}x \geq 0. \end{aligned} \tag{4.13}$$

By (4.12)-(4.13), we have $c_\lambda \geq \frac{[S(s)]^2}{4(\lambda - \alpha[S(s)]^2)}$, a contradiction. So $l = 0$. By (4.11), we get $v_n \to 0$ in $H^s_r(\mathbb{R}^3)$.

□

Proof of Theorem 1.2. By Lemma 4.2, for almost every $\lambda \in [\lambda_1, 1]$, there exists $\{u_n\} \subset H^s_r(\mathbb{R}^3)$ such that $\|u_n\|$ is bounded, $J_\lambda(u_n) \to c_\lambda \in \left[\alpha_0, \frac{[S(s)]^2}{4(\lambda - \alpha[S(s)]^2)}\right)$ and $J'_\lambda(u_n) \to 0$. By Lemma 4.3, we have $u_n \to u_\lambda$ in $H^s_r(\mathbb{R}^3)$. So $J_\lambda(u_\lambda) = c_\lambda \in \left[\alpha_0, \frac{[S(s)]^2}{4(\lambda - \alpha[S(s)]^2)}\right)$ and $J'_\lambda(u_\lambda) = 0$. Thus, there exists $\{\lambda_n\} \subset [\lambda_1, 1]$ and $\{u_{\lambda_n}\} \subset H^s_r(\mathbb{R}^3) \setminus \{0\}$ satisfying $\lambda_n \to 1$, $J'_{\lambda_n}(u_{\lambda_n}) = 0$ and $J_{\lambda_n}(u_{\lambda_n}) = c_{\lambda_n} \in \left[\alpha_0, \frac{[S(s)]^2}{4(\lambda_n - \alpha[S(s)]^2)}\right)$. From Lemma 4.2, we have $\lim_{n\to\infty} c_{\lambda_n} = c_1 \in \left[\alpha_0, \frac{[S(s)]^2}{4(1 - \alpha[S(s)]^2)}\right)$. Since $I(u_{\lambda_n}) = J_{\lambda_n}(u_{\lambda_n}) + (\lambda_n - 1) \int_{\mathbb{R}^3} H(u_{\lambda_n}) \mathrm{d}x$, by Lemma 4.1, we derive that $I(u_{\lambda_n}) \to c_1 \in \left[\alpha_0, \frac{[S(s)]^2}{4(1 - \alpha[S(s)]^2)}\right)$ and $I'(u_{\lambda_n}) \to 0$. By Lemma 4.3, we get $u_{\lambda_n} \to v_0$ in $H^s_r(\mathbb{R}^3)$. So $I(v_0) = c_1 > 0$ and $I'(v_0) = 0$. Let

$$m = \inf\{I(u) : u \in H^s_r(\mathbb{R}^3) \setminus \{0\} : I'(u) = 0\}.$$

From $I'(v_0) = 0$ and Lemma 2.2, we have $0 \le m \le I(v_0) < \frac{[S(s)]^2}{4(1-\alpha[S(s)]^2)}$. By the definition of m, there exists $\{u_n\} \subset H^s_r(\mathbb{R}^3) \setminus \{0\}$ such that $I(u_n) \to m$ and $I'(u_n) = 0$. Since $I'(u_n) = 0$, by the standard argument, we derive that there exists $\varrho_0 > 0$ such that $\int_{\mathbb{R}^3} |u_n|^4 \mathrm{d}x \ge \varrho_0 > 0$. By Lemma 4.1 and the argument of Lemma 4.3, we get $u_n \to u$ in $H^s_r(\mathbb{R}^3)$. Then $I(u) = m$ and $I'(u) = 0$ with $u \ne 0$. So m is attained by u.

□

Now we consider the case $\alpha > \frac{1}{[S(s)]^2}$. By $S(s) \le \frac{\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 \mathrm{d}x}{\left(\int_{\mathbb{R}^3} |u|^4 \mathrm{d}x\right)^{\frac{1}{2}}}$, we have

$$\alpha \left(\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 \mathrm{d}x \right)^2 - \int_{\mathbb{R}^3} |u|^4 \mathrm{d}x \ge \left(\alpha - \frac{1}{[S(s)]^2} \right) \left(\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 \mathrm{d}x \right)^2. \tag{4.14}$$

We prove Theorem 1.3 (i). For $\lambda \in [\frac{1}{2}, 1]$, define

$$I_\lambda(u) = \frac{1}{2}\|u\|^2 + \frac{\alpha}{4} \left(\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 \mathrm{d}x \right)^2 - \frac{1}{4} \int_{\mathbb{R}^3} |u|^4 \mathrm{d}x - \lambda\beta \int_{\mathbb{R}^3} F(u) \mathrm{d}x, \tag{4.15}$$

where $u \in H^s_r(\mathbb{R}^3)$.

Lemma 4.4. *Let $\lambda \in \left[\frac{1}{2}, 1\right]$. Assume that $\{u_n\} \subset H^s_r(\mathbb{R}^3)$ is a sequence such that $\|u_n\|$ is bounded, $I_\lambda(u_n) \to c_\lambda$ and $I'_\lambda(u_n) \to 0$. Then $\{u_n\}$ converges strongly in $H^s_r(\mathbb{R}^3)$ up to a subsequence.*

Proof. Assume $u_n \rightharpoonup u_\lambda$ weakly in $H^s_r(\mathbb{R}^3)$. Let $\hat{A} = \lim_{n\to\infty} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_n|^2 \mathrm{d}x$. Define

$$\tilde{I}_\lambda(u) = \frac{1}{2}\|u\|^2 + \frac{\alpha \hat{A}}{2} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 \mathrm{d}x - \frac{1}{4} \int_{\mathbb{R}^3} |u|^4 \mathrm{d}x - \lambda\beta \int_{\mathbb{R}^3} F(u) \mathrm{d}x,$$

where $u \in H^s_r(\mathbb{R}^3)$. Then $\tilde{I}'_\lambda(u_n) \to 0$. By $u_n \rightharpoonup u_\lambda$ weakly in $H^s_r(\mathbb{R}^3)$, we have $\tilde{I}'_\lambda(u_\lambda) = 0$. Set $v_n = u_n - u_\lambda$. By (4.8)-(4.9), we have

$$\begin{aligned} o_n(1) &= \left(\tilde{I}'_\lambda(u_n), u_n \right) - \left(\tilde{I}'_\lambda(u_\lambda), u_\lambda \right) + o_n(1) \\ &= \|v_n\|^2 + \alpha \hat{A} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} v_n|^2 \mathrm{d}x - \int_{\mathbb{R}^3} |v_n|^4 \mathrm{d}x + o_n(1). \end{aligned} \tag{4.16}$$

We also have $\hat{A} \ge \lim_{n\to\infty} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} v_n|^2 \mathrm{d}x$. Then by (4.14), (4.16) and $\alpha > \frac{1}{[S(s)]^2}$, we get

$$\begin{aligned} 0 &\ge \lim_{n\to\infty} \|v_n\|^2 + \alpha \lim_{n\to\infty} \left(\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} v_n|^2 \mathrm{d}x \right)^2 - \lim_{n\to\infty} \int_{\mathbb{R}^3} |v_n|^4 \mathrm{d}x \\ &\ge \lim_{n\to\infty} \|v_n\|^2 + \left(\alpha - \frac{1}{[S(s)]^2} \right) \lim_{n\to\infty} \left(\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} v_n|^2 \mathrm{d}x \right)^2 \ge \lim_{n\to\infty} \|v_n\|^2. \end{aligned}$$

So $u_n \to u_\lambda$ in $H^s_r(\mathbb{R}^3)$. □

Lemma 4.5. *There exists β_1, $\hat{c} > 0$ such that for $\beta > \beta_1$ and almost every $\lambda \in [\frac{1}{2}, 1]$, there is a sequence $\{u_n\} \subset H^s_r(\mathbb{R}^3)$ satisfying $\|u_n\|$ is bounded, $I_\lambda(u_n) \to c_\lambda \geq \hat{c}$ and $I'_\lambda(u_n) \to 0$. Moreover, the map $\lambda \to c_\lambda$ is continuous from the left.*

Proof. Let $J = [\frac{1}{2}, 1]$, $A(u) = \frac{1}{2}\|u\|^2 + \frac{\alpha}{4}\left(\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}u|^2 \mathrm{d}x\right)^2 - \frac{1}{4}\int_{\mathbb{R}^3} |u|^4 \mathrm{d}x$, $B(u) = \beta \int_{\mathbb{R}^3} F(u)\mathrm{d}x$. Then $B(u) \geq 0$ and $A(u) \to +\infty$ as $\|u\| \to \infty$ in view of (4.14). For $R > 0$, define $w(x) = \xi$ for $|x| \leq R$, $w(x) = 0$ for $|x| \geq R+1$ and $w(x) = \xi(R+1-|x|)$ for $R \leq |x| \leq R+1$. Then $w \in H^s_r(\mathbb{R}^3)$. Moreover, by choosing $R > 0$ large, we can derive that $\int_{\mathbb{R}^3} F(w)\mathrm{d}x > 0$. Thus, there exists $\beta_1 > 0$ such that $I_\lambda(w) \leq I_{\frac{1}{2}}(w) < 0$ for $\beta > \beta_1$. Let $\beta > \beta_1$. By (f_1), there exists $C_{\frac{1}{4\beta}} > 0$ such that $|F(u)| \leq \frac{1}{4\beta}|u|^2 + C_{\frac{1}{4\beta}}|u|^4$. Then by the Sobolev embedding theorem, we obtain that for $\lambda \in [\frac{1}{2}, 1]$,

$$\begin{aligned} I_\lambda(u) &\geq \frac{1}{2}\|u\|^2 - \frac{1}{4}\int_{\mathbb{R}^3} |u|^2 \mathrm{d}x - \left(\beta C_{\frac{1}{4\beta}} + \frac{1}{4}\right)\int_{\mathbb{R}^3} |u|^4 \mathrm{d}x \\ &\geq \frac{1}{4}\|u\|^2 - \frac{\beta C_{\frac{1}{4\beta}} + \frac{1}{4}}{[S(s)]^2}\|u\|^4. \end{aligned}$$

Let

$$\rho_0 = \min\left\{\frac{S(s)}{\left[8\left(\beta C_{\frac{1}{4\beta}} + \frac{1}{4}\right)\right]^{\frac{1}{2}}}, \frac{1}{2}\|w\|\right\}.$$

Then $I_\lambda(u) \geq \frac{1}{8}\|u\|^2 = \frac{1}{8}\rho_0^2 := \hat{c}$ for $\|u\| = \rho_0$. Since $I_\lambda(0) = 0$ and $I_\lambda(w) < 0$, by Theorem 2.2, for almost every $\lambda \in [\frac{1}{2}, 1]$, there is $\{u_n\} \subset H^s_r(\mathbb{R}^3)$ such that $\|u_n\|$ is bounded, $I_\lambda(u_n) \to c_\lambda \geq \hat{c}$ and $I'_\lambda(u_n) \to 0$. Moreover, the map $\lambda \to c_\lambda$ is continuous from the left.

□

Proof of Theorem 1.3 (i)**.** Let $\beta > \beta_1$. By Lemma 4.5, for almost every $\lambda \in \left[\frac{1}{2}, 1\right]$, there is $\{u_n\} \subset H^s_r(\mathbb{R}^3)$ such that $\|u_n\|$ is bounded, $I_\lambda(u_n) \to c_\lambda \geq \hat{c} > 0$ and $I'_\lambda(u_n) \to 0$. By Lemma 4.4, we get $u_n \to u_\lambda$ in $H^s_r(\mathbb{R}^3)$. So $I_\lambda(u_\lambda) = c_\lambda > 0$ and $I'_\lambda(u_\lambda) = 0$. Then there exists $\lambda_n \in [\frac{1}{2}, 1]$ and $\{u_{\lambda_n}\} \subset H^s_r(\mathbb{R}^3) \setminus \{0\}$ such that $\lambda_n \to 1$, $I'_{\lambda_n}(u_{\lambda_n}) = 0$ and $I_{\lambda_n}(u_{\lambda_n}) = c_{\lambda_n} \geq \hat{c}$. By Lemma 4.5, we have $c_{\lambda_n} \to c_1 \geq \hat{c}$. Then there exists $\breve{c} > 0$ such that $c_{\lambda_n} \leq \breve{c}$. Similar to the argument of Lemma 4.1, we derive that $\|u_n\|$ is bounded. By $I(u_{\lambda_n}) = I_{\lambda_n}(u_{\lambda_n}) + \beta(\lambda_n - 1)\int_{\mathbb{R}^3} F(u_{\lambda_n})\mathrm{d}x$, we get $I(u_{\lambda_n}) \to c_1 \geq \hat{c}$ and $I'(u_{\lambda_n}) \to 0$. By Lemma 4.4, we have $u_{\lambda_n} \to w$ in $H^s_r(\mathbb{R}^3)$. So $I(w) = c_1 > 0$ and $I'(w) = 0$. Let

$$m = \inf\{I(u) : u \in H^s_r(\mathbb{R}^3) \setminus \{0\} : I'(u) = 0\}.$$

Similar to the argument of Theorem 1.2, we can derive that m is attained. We omit the proof here.

□

Now we prove Theorem 1.3 (ii).

Proof of Theorem 1.3 *(ii)***.** Assume that (1.1) has a solution u. Then $(I'(u),u)=0$. Since $\alpha>\frac{1}{S(s)^2}$, by (4.14),

$$\|u\|^2+\left(\alpha-\frac{1}{S(s)^2}\right)\left(\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2\mathrm{d}x\right)^2\leq\beta\int_{\mathbb{R}^3}f(u)u\mathrm{d}x. \tag{4.17}$$

By (f_1'), for $\varepsilon=\frac{1}{2}$, there exists $C_\varepsilon=C_{\frac{1}{2}}>0$ such that $|f(u)u|\leq\frac{1}{2}|u|^2+C_{\frac{1}{2}}|u|^4$. Then by $S(s)\leq\frac{\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2\mathrm{d}x}{\left(\int_{\mathbb{R}^3}|u|^4\mathrm{d}x\right)^{\frac{1}{2}}}$, we derive that for $\beta\in(0,1)$,

$$\begin{aligned}&\|u\|^2+\left(\alpha-\frac{1}{[S(s)]^2}\right)\left(\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2\mathrm{d}x\right)^2\\&\leq\frac{1}{2}\int_{\mathbb{R}^3}|u|^2\mathrm{d}x+\frac{C_{\frac{1}{2}}\beta}{[S(s)]^2}\left(\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2\mathrm{d}x\right)^2.\end{aligned}$$

So

$$\frac{1}{2}\|u\|^2+\left(\alpha-\frac{1}{[S(s)]^2}-\frac{C_{\frac{1}{2}}\beta}{[S(s)]^2}\right)\left(\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2\mathrm{d}x\right)^2\leq 0.$$

Thus, if $0<\beta<\beta_2\in\left(0,\min\left\{1,\frac{\alpha[S(s)]^2-1}{C_{\frac{1}{2}}},\beta_1\right\}\right)$, we get $\|u\|=0$, that is, $u=0$.

□

5 The Case $s\in(0,\frac{3}{4})$

Proof of Theorem 1.4**.** Assume that (1.1) has a solution u. Then $(I'(u),u)=0$. Since $\alpha>\frac{2s(3-4s)^{\frac{3-4s}{2s}}}{[S(s)]^{\frac{3}{2s}}(3-2s)^{\frac{3-2s}{2s}}}$, we can choose $\delta\in(0,1)$ small such that

$$\alpha>\frac{2s(3-4s)^{\frac{3-4s}{2s}}}{(1-\delta)^{\frac{3-4s}{2s}}[S(s)]^{\frac{3}{2s}}(3-2s)^{\frac{3-2s}{2s}}}.$$

Note that

$$\begin{aligned}&\left(\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2\mathrm{d}x\right)^{\frac{2_s^*}{2}}\\&=\left((1-\delta)[S(s)]^{\frac{2_s^*}{2}}\right)^{\frac{3-4s}{3-2s}}\left(\frac{3-2s}{3-4s}\right)^{\frac{3-4s}{3-2s}}\left(\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2\mathrm{d}x\right)^{\frac{3-4s}{3-2s}}\\&\quad\times\left(\frac{1}{(1-\delta)[S(s)]^{\frac{2_s^*}{2}}}\right)^{\frac{3-4s}{3-2s}}\left(\frac{3-4s}{3-2s}\right)^{\frac{3-4s}{3-2s}}\left(\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2\mathrm{d}x\right)^{\frac{4s}{3-2s}}.\end{aligned}$$

By the Young's inequality,

$$
\begin{aligned}
&\left(\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2\mathrm{d}x\right)^{\frac{2_s^*}{2}}\\
&\leq \frac{3-4s}{3-2s}(1-\delta)[S(s)]^{\frac{2_s^*}{2}}\frac{3-2s}{3-4s}\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2\mathrm{d}x\\
&\quad+\frac{2s}{3-2s}\left(\frac{1}{(1-\delta)[S(s)]^{\frac{2_s^*}{2}}}\right)^{\frac{3-4s}{2s}}\left(\frac{3-4s}{3-2s}\right)^{\frac{3-4s}{2s}}\left(\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2\mathrm{d}x\right)^2.
\end{aligned}
$$

Then

$$
\begin{aligned}
\int_{\mathbb{R}^3}|u|^{2_s^*}\mathrm{d}x &\leq \frac{1}{[S(s)]^{\frac{2_s^*}{2}}}\left(\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2\mathrm{d}x\right)^{\frac{2_s^*}{2}}\\
&\leq (1-\delta)\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2\mathrm{d}x\\
&\quad+\frac{2s(3-4s)^{\frac{3-4s}{2s}}}{(1-\delta)^{\frac{3-4s}{2s}}[S(s)]^{\frac{3}{2s}}(3-2s)^{\frac{3-2s}{2s}}}\left(\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2\mathrm{d}x\right)^2. \qquad (5.1)
\end{aligned}
$$

Since $(I'(u),u)=0$, by (5.1), we get

$$
\begin{aligned}
&\|u\|^2+\alpha\left(\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2\mathrm{d}x\right)^2\\
&\leq \beta\int_{\mathbb{R}^3}f(u)u\mathrm{d}x+(1-\delta)\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2\mathrm{d}x\\
&\quad+\frac{2s(3-4s)^{\frac{3-4s}{2s}}}{(1-\delta)^{\frac{3-4s}{2s}}[S(s)]^{\frac{3}{2s}}(3-2s)^{\frac{3-2s}{2s}}}\left(\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2\mathrm{d}x\right)^2. \qquad (5.2)
\end{aligned}
$$

By (f_1'), for $\varepsilon=\frac{1}{2}$, there exists $C_\varepsilon=C_{\frac{1}{2}}>0$ such that $|f(u)u|\leq\frac{1}{2}|u|^2+C_{\frac{1}{2}}|u|^{2_s^*}$. Together with (5.1)-(5.2), we derive that for $\beta\in(0,1)$,

$$
\begin{aligned}
&\delta\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2\mathrm{d}x+\frac{1}{2}\int_{\mathbb{R}^3}|u|^2\mathrm{d}x\\
&+\left(\alpha-\frac{2s(3-4s)^{\frac{3-4s}{2s}}}{(1-\delta)^{\frac{3-4s}{2s}}[S(s)]^{\frac{3}{2s}}(3-2s)^{\frac{3-2s}{2s}}}\right)\left(\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2\mathrm{d}x\right)^2\\
&\leq \beta C_{\frac{1}{2}}(1-\delta)\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2\mathrm{d}x\\
&\quad+\beta C_{\frac{1}{2}}\frac{2s(3-4s)^{\frac{3-4s}{2s}}}{(1-\delta)^{\frac{3-4s}{2s}}[S(s)]^{\frac{3}{2s}}(3-2s)^{\frac{3-2s}{2s}}}\left(\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2\mathrm{d}x\right)^2.
\end{aligned}
$$

Let

$$\beta_3 \in \left(0, \min\left\{1, \frac{\delta}{(1-\delta)C_{\frac{1}{2}}}, \frac{\alpha - \frac{2s(3-4s)^{\frac{3-4s}{2s}}}{(1-\delta)^{\frac{3-4s}{2s}}[S(s)]^{\frac{3}{2s}}(3-2s)^{\frac{3-2s}{2s}}}}{\frac{2s(3-4s)^{\frac{3-4s}{2s}}C_{\frac{1}{2}}}{(1-\delta)^{\frac{3-4s}{2s}}[S(s)]^{\frac{3}{2s}}(3-2s)^{\frac{3-2s}{2s}}}}\right\}\right).$$

Then for $\beta \in (0, \beta_3)$, we have

$$\left[\delta - \beta C_{\frac{1}{2}}(1-\delta)\right]\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}u|^2 \mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^3} |u|^2 \mathrm{d}x \le 0.$$

So $\|u\| = 0$, that is, $u = 0$.

□

6 The Case $\alpha = 0$

Let

$$J(u) = \frac{1}{2}\|u\|^2 - \beta\int_{\mathbb{R}^3} F(u)\mathrm{d}x - \frac{1}{2_s^*}\int_{\mathbb{R}^3} |u|^{2_s^*}\mathrm{d}x, \quad u \in H_r^s(\mathbb{R}^3). \tag{6.1}$$

Then critical points of J are weak solutions of (1.5).

Lemma 6.1. *Assume that $\{u_n\} \subset H_r^s(\mathbb{R}^3)$ is a sequence such that $\|u_n\|$ is bounded, $J(u_n) \to c \in \left(0, \frac{s}{3}[S(s)]^{\frac{3}{2s}}\right)$ and $J'(u_n) \to 0$. Then $u_n \rightharpoonup u \neq 0$ weakly in $H_r^s(\mathbb{R}^3)$.*

Proof. Otherwise, $u_n \rightharpoonup 0$ weakly in $H_r^s(\mathbb{R}^3)$. By (f_1) and Lemma 2.1, we have $\int_{\mathbb{R}^3} F(u_n)\mathrm{d}x = \int_{\mathbb{R}^3} f(u_n)u_n\mathrm{d}x = o_n(1)$. Then $c + o_n(1) = \frac{1}{2}\|u_n\|^2 - \frac{1}{2_s^*}\int_{\mathbb{R}^3}|u_n|^{2_s^*}\mathrm{d}x$ and $o_n(1) = \|u_n\|^2 - \int_{\mathbb{R}^3}|u_n|^{2_s^*}\mathrm{d}x$. Since $c > 0$, we assume that $\|u_n\|^2 \to l$. By the definition of $S(s)$, we get $l \ge [S(s)]^{\frac{3}{2s}}$. Then $c \ge \frac{s}{3}[S(s)]^{\frac{3}{2s}}$, a contradiction. So $u_n \rightharpoonup u \neq 0$ weakly in $H_r^s(\mathbb{R}^3)$.

□

Proof of Theorem 1.5. Similar to the argument of Theorem 1.1, we derive that there exists a sequence $\{u_n\} \subset H_r^s(\mathbb{R}^3)$ such that $J(u_n) \to c > 0$ and $(1 + \|u_n\|)\|J'(u_n)\| \to 0$. By the definition of c, we have $c \le \sup_{t\ge 0} J(tu_\varepsilon)$. By (f_2), we have $J(tu_\varepsilon) \le \frac{t^2}{2}\|u_\varepsilon\|^2 - \frac{t^{2_s^*}}{2_s^*}\int_{\mathbb{R}^3}|u_\varepsilon|^{2_s^*}\mathrm{d}x$. From (2.2)-(2.3), there exists $\varepsilon' > 0$ such that $\|u_\varepsilon\|^2 \le \frac{3[S(s)]^2}{2}$ and $\int_{\mathbb{R}^3}|u_\varepsilon|^{2_s^*}\mathrm{d}x \ge \frac{[S(s)]^2}{2}$ for $\varepsilon \in (0, \varepsilon')$. So there exists a small $t' > 0$ and a large $t'' > 0$ independent of $\varepsilon \in (0, \varepsilon')$ such that $\sup_{t\in[0,t']\cup[t'',+\infty)} J(tu_\varepsilon) < \frac{s}{3}[S(s)]^{\frac{3}{2s}}$. Similar to (4.7), we have

$$\inf_{t\in[t',t'']}\int_{\mathbb{R}^3} F(tu_\varepsilon)\mathrm{d}x \ge \frac{\kappa_0^{p_0} D_0(t')^{p_0}[S(s)]^{\frac{3}{2s}}\varepsilon^{-\frac{p_0(3-2s)}{2}+3}}{R_0^{p_0}2^{\frac{p_0(3-2s)}{2}}\mu^{p_0(3-2s)-3}}\int_{|x|\le 1}\mathrm{d}x. \tag{6.2}$$

Combining (2.2)-(2.3) and (6.2), we derive that for $\varepsilon > 0$ small,

$$\begin{aligned}
&\sup_{t\in[t',t'']} J(tu_\varepsilon) \\
&\leq \sup_{t\geq 0}\left[\frac{t^2}{2}\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u_\varepsilon|^2\mathrm{d}x - \frac{t^{2_s^*}}{2_s^*}\int_{\mathbb{R}^3}|u_\varepsilon|^{2_s^*}\mathrm{d}x\right] + \frac{(t'')^2}{2}\int_{\mathbb{R}^3}|u_\varepsilon|^2\mathrm{d}x \\
&\quad - \frac{\beta\kappa_0^{p_0}D_0(t')^{p_0}[S(s)]^{\frac{3}{2s}}\varepsilon^{-\frac{p_0(3-2s)}{2}+3}}{R_0^{p_0}2^{\frac{p_0(3-2s)}{2}}\mu^{p_0(3-2s)-3}}\int_{|x|\leq 1}\mathrm{d}x \\
&= \frac{s}{3}\left[\frac{\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u_\varepsilon|^2\mathrm{d}x}{\left(\int_{\mathbb{R}^3}|u_\varepsilon|^{2_s^*}\mathrm{d}x\right)^{\frac{2}{2_s^*}}}\right]^{\frac{2_s^*}{2_s^*-2}} - C'\varepsilon^{-\frac{p_0(3-2s)}{2}+3} + \begin{cases} O(\varepsilon^{2s}), & s\in(0,\frac{3}{4}) \\ O(\varepsilon^{2s}|\log\varepsilon|), & s=\frac{3}{4} \\ O(\varepsilon^{3-2s}), & s\in(\frac{3}{4},1) \end{cases} \\
&\leq \frac{s}{3}[S(s)]^{\frac{3}{2s}} + O(\varepsilon^{3-2s}) - C'\varepsilon^{-\frac{p_0(3-2s)}{2}+3} + \begin{cases} O(\varepsilon^{2s}), & s\in(0,\frac{3}{4}) \\ O(\varepsilon^{2s}|\log\varepsilon|), & s=\frac{3}{4} \\ O(\varepsilon^{3-2s}), & s\in(\frac{3}{4},1) \end{cases}.
\end{aligned}$$

If $s\in(0,\frac{3}{4}]$, by $p_0\in(2,2_s^*)$, we get $-\frac{p_0(3-2s)}{2}+3<2s$. Then for $\varepsilon>0$ small,

$$\sup_{t\in[t',t'']} J(tu_\varepsilon) \leq \frac{s}{3}[S(s)]^{\frac{3}{2s}} + O(\varepsilon^{2s}|\log\varepsilon|) - C'\varepsilon^{-\frac{p_0(3-2s)}{2}+3} < \frac{s}{3}[S(s)]^{\frac{3}{2s}}.$$

If $s\in(\frac{3}{4},1)$, by $p_0\in(2_s^*-2,2_s^*)$, we get $-\frac{p_0(3-2s)}{2}+3<3-2s$. Then for $\varepsilon>0$ small,

$$\sup_{t\in[t',t'']} J(tu_\varepsilon) \leq \frac{s}{3}[S(s)]^{\frac{3}{2s}} + O(\varepsilon^{3-2s}) - C'\varepsilon^{-\frac{p_0(3-2s)}{2}+3} < \frac{s}{3}[S(s)]^{\frac{3}{2s}}.$$

Recall that $\sup_{t\in[0,t']\cup[t'',+\infty)} J(tu_\varepsilon) < \frac{s}{3}[S(s)]^{\frac{3}{2s}}$. So $c<\frac{s}{3}[S(s)]^{\frac{3}{2s}}$. Let θ, $q\in(2,2_s^*)$ in Lemma 3.2, we can prove that $\|u_n\|$ bounded. By Lemma 6.1, we have $u_n \rightharpoonup u\neq 0$ weakly in $H_r^s(\mathbb{R}^3)$. So $J'(u)=0$.

□

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