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Integral estimates of conformal derivatives and spectral properties of the Neumann-Laplacian

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ABSTRACT

In this paper we study integral estimates of derivatives of conformal mappings $\varphi : \mathbb{D} \to \Omega$ of the unit disc $\mathbb{D} \subset \mathbb{C}$ onto bounded domains Ω that satisfy the Ahlfors condition. These integral estimates lead to estimates of constants in Sobolev–Poincaré inequalities, and by the Rayleigh quotient we obtain spectral estimates of the Neumann–Laplace operator in non-Lipschitz domains (quasidiscs) in terms of the (quasi)conformal geometry of the domains. Specifically, the lower estimates of the first non-trivial eigenvalues of the Neumann–Laplace operator in some fractal type domains (snowflakes) were obtained.

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1. Introduction

1.1. Estimates of conformal derivatives

In the work [27] we obtained lower estimates of the first non-trivial eigenvalues of the Neumann–Laplace operator in the terms of integrals of complex derivatives (i.e. hyperbolic metrics) of conformal mappings $\varphi : \mathbb{D} \to \Omega$. Let us recall that the classical Koebe distortion theorem [10] gives the following estimates of the complex derivatives in the case of univalent analytic functions (conformal homeomorphisms): $\varphi : \mathbb{D} \to \Omega$ normalized so that $\varphi(0) = 0$ and $\varphi'(0) = 1$:

$$\frac{1-|z|}{(1+|z|)^3} \leq |\varphi'(z)| \leq \frac{1+|z|}{(1-|z|)^3}.$$

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The example of the Koebe function

$$\varphi(z) = \frac{z}{(1-z)^2}$$

which maps the unit disc $\mathbb{D}$ onto $\Omega = \mathbb{C} \setminus (-\infty, -1/4]$ shows that these estimates don't give even square integrability of the complex derivatives in arbitrary simply connected planar domains. But if $\Omega \subset \mathbb{C}$ is a simply connected planar domain of finite measure then by the simple calculation

$$\iint\limits_{\mathbb{D}} |\varphi'(z)|^2 \, dxdy = \iint\limits_{\mathbb{D}} J(z,\varphi) \, dxdy = |\Omega| < \infty.$$

(We identify the complex plane $\mathbb{C}$ and the real plane $\mathbb{R}^2$: $\mathbb{C} \ni z = x + iy = (x,y) \in \mathbb{R}^2$.)

Hence in special classes of domains we have better integral estimates of the complex derivatives than by the Koebe distortion theorem.

In the present work we study integral estimates of the complex derivatives in domains bounded by Jordan curves that satisfy the Ahlfors three point condition [2]. For this study we introduce a notion of hyperbolic (integral) α-dilatation of Ω:

$$Q(\alpha,\Omega) := \iint\limits_{\mathbb{D}} |\varphi'(z)|^{\alpha} \, dxdy = \iint\limits_{\Omega} \left|\left(\varphi^{-1}\right)'(w)\right|^{2-\alpha} dudv.$$

The finiteness of the hyperbolic α-dilatation and its convergence hyperbolic interval

$$\mathrm{HI}(\Omega) := \{\alpha \in \mathbb{R} : Q(\alpha,\Omega) < \infty\}$$

doesn't depend on choice of a conformal mapping $\varphi : \mathbb{D} \to \Omega$ and can be reformulated in terms of the hyperbolic metrics [14]. Namely

$$\begin{aligned}\iint\limits_{\mathbb{D}} |\varphi'(z)|^{\alpha} \, dxdy &= \iint\limits_{\mathbb{D}} \left(\frac{\lambda_{\mathbb{D}}(z)}{\lambda_{\Omega}(\varphi(z))}\right)^{\alpha} dxdy \\ &= \iint\limits_{\Omega} \left|\left(\varphi^{-1}\right)'(w)\right|^{2-\alpha} dudv = \iint\limits_{\Omega} \left(\frac{\lambda_{\mathbb{D}}\left(\varphi^{-1}(w)\right)}{\lambda_{\Omega}(w)}\right)^{2-\alpha} dudv\end{aligned}$$

where $\lambda_{\mathbb{D}}$ and λ_{Ω} are hyperbolic metrics in $\mathbb{D}$ and Ω [9]. Let us recall that the hyperbolic metrics generated by $\lambda_{\mathbb{D}}\left(\varphi^{-1}(w)\right)$ are equivalent for different choice of conformal homeomorphism $\varphi : \Omega \to \mathbb{D}$ because any other conformal homeomorphism $\psi^{-1} : \Omega \to \mathbb{D}$ is a composition of φ^{-1} and a Möbius homeomorphism (that is an isometry of the hyperbolic metric).

Remark 1.1. For any bounded simply connected domain $(-1.78, 2] \subset \mathrm{HI}(\alpha,\Omega)$ [26,28]. A more detailed discussion about the hyperbolic α-dilatation and its convergence hyperbolic interval can be found in Appendix.

In [26] we proved that if a number $\alpha > 2$ belongs to $\mathrm{HI}(\Omega)$ then Ω has finite geodesic diameter. By this reason we call domains that satisfy to a property $2 < \alpha \in \mathrm{HI}(\Omega)$ as conformal α-regular domains [14].

In this paper we obtain estimates of $Q(\alpha,\Omega)$ for quasidiscs, that represent a large subclass of conformal α-regular domains, with the help of the exact inverse Hölder inequality for Jacobians of quasiconformal mappings. Using the estimates for $Q(\alpha,\Omega)$ we obtain estimates of constants for Sobolev–Poincaré inequalities and as a result we obtain lower estimates for first non-trivial eigenvalues of the Laplace operator with the Neumann boundary condition.

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1.2. Spectral estimates of the Neumann-Laplacian

Let $\Omega \subset \mathbb{C}$ be a bounded Lipschitz domain (an open connected set). The Neumann eigenvalue problem for the Laplace operator is:

$$\begin{cases} -\operatorname{div}(\nabla u) = \mu u & \text{in } \Omega \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases}$$

The weak statement of this spectral problem is as follows: a function u solves the previous problem iff $u \in W_2^1(\Omega)$ and

$$\iint\limits_{\Omega} \nabla u(x,y) \cdot \nabla v(x,y)\, dxdy = \mu \iint\limits_{\Omega} u(x,y)v(x,y)\, dxdy$$

for all $v \in W_2^1(\Omega)$. This statement is correct in any bounded domain $\Omega \subset \mathbb{C}$.

Let us give a short historical review about eigenvalues estimates for the Neumann–Laplace operator. The classical upper estimate of the first non-trivial Neumann eigenvalue

$$\mu_1(\Omega) \le \mu_1(\Omega^*) = \frac{p_{n/2}^2}{R_*^2}$$

was proved by G. Szegö [39] for simply connected planar domains and by H. F. Weinberger [43] for domains in $\mathbb{R}^n$. In this inequality $p_{n/2}$ denotes the first positive zero of the function $(t^{1-n/2}J_{n/2}(t))'$, and Ω^* is an n-ball of the same n-volume as Ω with R_* as its radius. In particular, if $n = 2$, $p_1 = j'_{1,1} \approx 1.84118$ where $j'_{1,1}$ denotes the first positive zero of the derivative of the Bessel function J_1.

In 1961 G. Polya [35] obtained upper estimates for eigenvalues in so-called plane-covering domains. Namely, for the first non-trivial eigenvalue $\mu_1(\Omega)$ it is:

$$\mu_1(\Omega) \le \frac{4\pi}{|\Omega|}.$$

The upper estimate of the Laplace eigenvalues, with the help of different techniques, were intensively studied in recent decades, see, for example, [5–7,15,32].

The lower estimates for the $\mu_1(\Omega)$ for convex domains were obtained in the classical work [34]. It was proved that if Ω is convex with diameter $d(\Omega)$ (see, also [16,19,41]), then

$$\mu_1(\Omega) \ge \frac{\pi^2}{d(\Omega)^2}.$$

Unfortunately in non-convex domains $\mu_1(\Omega)$ can not be estimated in the terms of Euclidean diameters. It can be seen by considering a domain consisting of two identical squares connected by a thin corridor [12]. In [27] was proved, on the basis of the geometric theory of composition operators on Sobolev spaces [23,40] with applications to the (weighted) Poincaré–Sobolev inequalities [22,25], that if $\Omega \subset \mathbb{C}$ be a conformal α-regular domain, then the spectrum of Neumann–Laplace operator in Ω is discrete, can be written in the form of a non-decreasing sequence

$$0 = \mu_0(\Omega) < \mu_1(\Omega) \le \mu_2(\Omega) \le \ldots \le \mu_n(\Omega) \le \ldots,$$

and

$$1/\mu_1(\Omega) \le \frac{4}{\sqrt[\alpha]{\pi^2}} \left(\frac{2\alpha - 2}{\alpha - 2}\right)^{\frac{2\alpha-2}{\alpha}} Q(\alpha,\Omega)^{2/\alpha}. \tag{1.1}$$

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In the present article we suggest the approach which is based on the geometric theory of composition operators on Sobolev spaces and estimates of hyperbolic metric in quasidiscs. Recall that a domain $\Omega \subset \mathbb{C}$ is called a K-quasidisc if it is the image of the unit disc $\mathbb{D}$ under a K-quasiconformal homeomorphism of the complex plane $\mathbb{C}$ onto itself. Note that quasidiscs represent large class domains including fractal type domains like snowflakes. The Hausdorff dimensions of the quasidisc's boundary can be any number in $[1, 2)$ [30].

Following [3] a homeomorphism $\varphi : \Omega \to \Omega'$ between planar domains is called K-quasiconformal if it preserves orientation, belongs to the Sobolev class $W^1_{2,\mathrm{loc}}(\Omega)$ and its directional derivatives ∂_ξ satisfy the distortion inequality

$$\max_{\xi} |\partial_\xi \varphi| \le K \min_{\xi} |\partial_\xi \varphi| \ \text{ a.e. in } \ \Omega\,.$$

If Ω is a K-quasidisc, then a conformal mapping $\varphi : \mathbb{D} \to \Omega$ allows a K^2-quasiconformal reflection. It is well known that Jacobians of quasiconformal mappings satisfy the weak inverse Hölder inequality [11]. On the basis of the weak inverse Hölder inequality and the estimates of the constants in doubling conditions for measures generated by Jacobians of quasiconformal mappings (Proposition 3.6) we obtain

Theorem A. *Let $\Omega \subset \mathbb{C}$ be a K-quasidisc. Then the spectrum of Neumann–Laplace operator in Ω is discrete, can be written in the form of a non-decreasing sequence*

$$0 = \mu_0(\Omega) < \mu_1(\Omega) \le \mu_2(\Omega) \le \ldots \le \mu_n(\Omega) \le \ldots,$$

and

$$\frac{1}{\mu_1(\Omega)} \le \frac{K^2 C_\alpha^2}{\pi} \left(\frac{2\alpha - 2}{\alpha - 2}\right)^{\frac{2\alpha-2}{\alpha}} \exp\left\{\frac{K^2\pi^2(2+\pi^4)^2}{2\log 3}\right\} \cdot \big|\Omega\big|, \tag{1.2}$$

for $2 < \alpha < \frac{2K^2}{K^2-1}$, where

$$C_\alpha = \frac{10^6}{[(\alpha-1)(1-\nu)]^{1/\alpha}}, \quad \nu = 10^{4\alpha}\frac{\alpha-2}{\alpha-1}\left(24\pi^2K^2\right)^\alpha < 1.$$

The main technical problem of this estimate is evaluations of the quasiconformality coefficient K for quasidiscs. For this aim we use an equivalent description of quasidiscs in the terms of the Ahlfors's 3-point condition. This description of quasidiscs allows to obtain the estimates for the specific fractal type domains.

Theorem C. *Let $S_p \subset \mathbb{C}$, $1/4 \le p < 1/2$, be the Rohde snowflake. Then the spectrum of Neumann–Laplace operator in S_p is discrete, can be written in the form of a non-decreasing sequence*

$$0 = \mu_0(S_p) < \mu_1(S_p) \le \mu_2(S_p) \le \ldots \le \mu_n(S_p) \le \ldots,$$

and

$$\frac{1}{\mu_1(S_p)} \le \frac{C_\alpha^2 e^{4\left(1+e^{2\pi}(16/(1-2p))^5\right)^2}}{2^{40}\pi} \left(\frac{2\alpha-2}{\alpha-2}\right)^{\frac{2\alpha-2}{\alpha}} \times \exp\left\{\frac{\pi^2(2+\pi^4)^2 e^{4\left(1+e^{2\pi}(16/(1-2p))^5\right)^2}}{2^{41}\log 3}\right\} |S_p|\,,$$

for $2 < \alpha < \frac{2K^2}{K^2-1}$, where

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$$C_\alpha = \frac{10^6}{[(\alpha-1)(1-\nu)]^{1/\alpha}}, \quad \nu = 10^{4\alpha}\left(\frac{24\pi^2}{2^{40}}e^{4(1+e^{2\pi}C^5)^2}\right)^{\alpha}\frac{\alpha-2}{\alpha-1} < 1.$$

The quasiconformal coefficient K of the Rohde snowflake S_p satisfying the condition

$$K < 2^{-10}\exp\left\{\left(1+e^{2\pi}(16/(1-2p))^5\right)^2\right\}.$$

On the basis of Theorem A and Theorem C we can assert that eigenvalues of the Laplace operator depend on (quasi)conformal geometry of planar domains.

2. Sobolev spaces and Poincaré–Sobolev inequalities

Let $E \subset \mathbb{C}$ be a measurable set. For any $1 \le p < \infty$ we consider the Lebesgue space of locally integrable functions with the finite norm

$$\|f \mid L_p(E)\| := \left(\iint\limits_E |f(x,y)|^p \, dxdy\right)^{1/p} < \infty.$$

We define the Sobolev space $W_p^1(\Omega)$, $1 \le p < \infty$, as a Banach space of locally integrable weakly differentiable functions $f : \Omega \to \mathbb{R}$ equipped with the following norm:

$$\|f \mid W_p^1(\Omega)\| = \left(\iint\limits_\Omega |f(x,y)|^p \, dxdy\right)^{\frac{1}{p}} + \left(\iint\limits_\Omega |\nabla f(x,y)|^p \, dxdy\right)^{\frac{1}{p}}.$$

We also define the homogeneous seminormed Sobolev space $L_p^1(\Omega)$, $1 \le p < \infty$, of locally integrable weakly differentiable functions $f : \Omega \to \mathbb{R}$ equipped with the following seminorm:

$$\|f \mid L_p^1(\Omega)\| = \left(\iint\limits_\Omega |\nabla f(x,y)|^p \, dxdy\right)^{\frac{1}{p}}.$$

Recall that the embedding operator $i : L_p^1(\Omega) \to L_{1,\mathrm{loc}}(\Omega)$ is bounded.

If $f \in W_p^1(\mathbb{D})$, $1 \le p < \infty$, then for $0 \le \kappa = 1/p - 1/q < 1/2$ the Poincaré–Sobolev inequality

$$\left(\iint\limits_{\mathbb{D}} |f(z) - f_{\mathbb{D}}|^q \, dxdy\right)^{\frac{1}{q}} \le B_{q,p}(\mathbb{D}) \left(\iint\limits_{\mathbb{D}} |\nabla f(z)|^p \, dxdy\right)^{\frac{1}{p}}$$

holds (see, for example, [20,27]) with the constant

$$B_{q,p}(\mathbb{D}) \le \frac{2}{\pi^\kappa}\left(\frac{1-\kappa}{1/2-\kappa}\right)^{1-\kappa}.$$

This estimate is not applicable in the critical case $p = 1$ and $q = 2$. Now we obtain the upper estimate of the Poincaré constant in the Poincaré–Sobolev inequality for the Sobolev space $W_1^1(\mathbb{D})$ in this critical case. We use the following Gagliardo inequality for functions with compact support [18,33]:

$$\left(\iint\limits_\Omega \left|f(z)\right|^2 \, dxdy\right)^{\frac{1}{2}} \le \frac{1}{2\sqrt{\pi}} \iint\limits_\Omega |\nabla f(z)| \, dxdy, \tag{2.1}$$

where $\Omega \subset \mathbb{C}$ be a bounded Lipschitz domain.

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Theorem 2.1. *Let $f \in W_1^1(\Omega)$. Then for any $r > 0$ and any $z_0 \in \Omega : \operatorname{dist}(z_0, \partial\Omega) > 2r$, the following inequality*

$$\left(\iint\limits_{D(z_0,r)} |f(z) - f_{D(z_0,r)}|^2 \, dxdy \right)^{\frac{1}{2}} \leq \frac{3\sqrt{\pi^3}}{4} \iint\limits_{D(z_0,r)} |\nabla f(z)| \, dxdy \tag{2.2}$$

holds.

Proof. We prove this inequality in the case $f_{D(z_0,r)} = 0$. In this case the inequality (2.2) can be rewritten as

$$\left(\iint\limits_{D(z_0,r)} |f(z)|^2 \, dxdy \right)^{\frac{1}{2}} \leq \frac{3\sqrt{\pi^3}}{4} \iint\limits_{D(z_0,r)} |\nabla f(z)| \, dxdy. \tag{2.3}$$

The inequality (2.3) is invariant under translations and similarities and it is sufficient to prove one in for the disc $D(0,1)$.

Denote by $D(0,\delta)$ an open disc of the radius $\delta > 1$. Choose the cut function η in the form $\eta = \eta(|z|)$ such that

$$\eta(z) = \begin{cases} 1, & \text{if } |z| < 1, \\ 0, & \text{if } |z| > \delta \end{cases}$$

and linear for $z \in R_\delta$, where $R_\delta := D(0,\delta) \setminus \overline{D(0,1)}$. Then

$$|\nabla \eta(z)| = \begin{cases} \frac{1}{\delta - 1} & \text{if } z \in R_\delta, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\widetilde{f} : R_\delta \to \mathbb{R}$ be the extension function is defined by the rule

$$\widetilde{f}(z) = f\left(w(z)\right) \eta(z), \ w \in R_\delta,$$

where $w(z) = 1/\overline{z}$ is the inversion in the unit circle $S(0,1)$.

Define the extension operator on Sobolev spaces

$$E : L_1^1(D(0,1)) \to L_1^1(D(0,\delta))$$

by the formula

$$(Ef)(z) = \begin{cases} f(z) & \text{if } z \in D(0,1), \\ \widetilde{f}(z) & \text{if } z \in R_\delta. \end{cases}$$

In order to estimate the norm $\|E\|$ of the extension operator $E : L_1^1(D(0,1)) \to L_1^1(D(0,\delta))$ we have

$$\|E \mid L_1^1(\mathbb{D}(0,\delta))\| = \iint\limits_{D(0,1)} |\nabla f(z)| \, dxdy + \iint\limits_{R_\delta} |\nabla \widetilde{f}(z)| \, dxdy.$$

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To estimate the second integral in the right side of this equality by elementary calculations we have

$$|\nabla \widetilde{f}(z)| = |\nabla \left(f(w(z))\eta(z)\right)| = |\nabla \left(f(w(z))\right) \cdot \eta(z) + f(w(z)) \cdot \nabla\eta(z)|$$
$$\leq |\nabla f(w(z))| + \frac{1}{\delta - 1} |f(w(z))|. \tag{2.4}$$

Hence

$$\iint\limits_{R_\delta} |\nabla \widetilde{f}(w(z))| \; dxdz \leq \iint\limits_{R_\delta} |\nabla f(w(z))| \; dxdy + \frac{1}{\delta - 1} \iint\limits_{R_\delta} |f(w(z))| \; dxdy.$$

First consider the integral

$$\begin{aligned}
\iint\limits_{R_\delta} |\nabla f(w(z))| \; dxdy &= \iint\limits_{R_\delta} |\nabla f|(w(z))|w'(z)|dxdy \\
&= \iint\limits_{R_\delta} |\nabla f|(w(z))|w'(z)||J(z,w)||J(z,w)|^{-1}dxdy \\
&\leq \sup_{z\in R_\delta} \frac{|w'(z)|}{|J(z,w)|} \iint\limits_{R_\delta} |\nabla f|(w(z))|J(z,w)| \; dxdy \\
&= \sup_{z\in R_\delta} \frac{|w'(z)|}{|J(z,w)|} \iint\limits_{D_\delta} |\nabla f|(w) \; dudv,
\end{aligned}$$

where $w : R_\delta \to D_\delta$, $w(z) = 1/\overline{z}$ and $D_\delta = \{w \in \mathbb{C} : 1/\delta < |z| < 1\}$. We calculate the norm of the derivative of the mapping w by the formula

$$|w'(z)| = |w_z| + |w_{\overline{z}}|$$

and the Jacobian of the mapping w by the formula

$$J(z,w) = |w_z|^2 - |w_{\overline{z}}|^2.$$

Here

$$w_z = \frac{1}{2}\left(\frac{\partial w}{\partial x} - i\frac{\partial w}{\partial y}\right) \quad \text{and} \quad w_{\overline{z}} = \frac{1}{2}\left(\frac{\partial w}{\partial x} + i\frac{\partial w}{\partial y}\right).$$

By elementary calculations

$$w_z = 0 \quad \text{and} \quad w_{\overline{z}} = -\frac{1}{\overline{z}^2}.$$

Hence

$$|w'(z)| = \frac{1}{|\overline{z}|^2} \quad \text{and} \quad |J(z,w)| = \frac{1}{|\overline{z}|^4}.$$

Finally we get

$$\iint\limits_{R_\delta} |\nabla f(w(z))| \; dxdy \leq \sup_{z\in R_\delta} |\overline{z}|^2 \iint\limits_{D_\delta} |\nabla f|(w) \; dudv \leq \delta^2 \iint\limits_{D(0,1)} |\nabla f|(w) \; dudv.$$

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In order to estimate the integral

$$\iint\limits_{R_\delta} |f\left(w(z)\right)| \; dxdy$$

we will use the change of variable formula.

We obtain

$$\begin{aligned}\iint\limits_{R_\delta} |f\left(w(z)\right)| \; dxdy &= \iint\limits_{R_\delta} |f\left(w(z)\right)||J(z,w)||J(z,w)|^{-1} \; dxdy \\ &\leq \sup_{z\in R_\delta} \frac{1}{|J(z,w)|} \iint\limits_{R_\delta} |f\left(w(z)\right)||J(z,w)| \; dxdy \\ &= \sup_{z\in R_\delta} \frac{1}{|J(z,w)|} \iint\limits_{D_\delta} |f(w)| \; dudv \leq \delta^4 \iint\limits_{D(0,1)} |f(w)| \; dudv.\end{aligned}$$

Using the following Poincaré–Sobolev inequality [1]

$$\iint\limits_{\Omega} |f(z)| \; dxdy \leq \frac{d}{2} \iint\limits_{\Omega} |\nabla f(z)| \; dxdy$$

where Ω be a convex domain with diameter d and $f \in W_1^1(\Omega)$, finally we obtain

$$\iint\limits_{R_\delta} |f\left(w(z)\right)| \; dxdy \leq \delta^4 \iint\limits_{D(0,1)} |\nabla f(w)| \; dudv.$$

Thus

$$\iint\limits_{R_\delta} |\nabla \widetilde{f}(z)| \; dxdy \leq \delta^2 \iint\limits_{D(0,1)} |\nabla f(w)| \; dudv + \frac{\delta^4}{\delta - 1} \iint\limits_{D(0,1)} |\nabla f(w)| \; dudv,$$

and consequently

$$\begin{aligned}\|E(f) \mid L_1^1(D(0,\delta))\| &\leq \left(1 + \delta^2 + \frac{\delta^4}{\delta - 1}\right) \iint\limits_{D(0,1)} |\nabla f(z)| \; dxdy \\ &= \left(1 + \delta^2 + \frac{\delta^4}{\delta - 1}\right) \|f \mid L_1^1(D(0,1))\|.\end{aligned}$$

Now, according to inequality (2.1) we obtain

$$\begin{aligned}\left(\iint\limits_{D(0,1)} |f(z)|^2 \; dxdy\right)^{\frac{1}{2}} &\leq \left(\iint\limits_{D(0,\delta)} |Ef(z)|^2 \; dxdy\right)^{\frac{1}{2}} \leq \frac{1}{2\sqrt{\pi}} \iint\limits_{D(0,\delta)} |\nabla Ef(z)| \; dxdy \\ &\leq \frac{\delta^4 + \delta^3 - \delta^2 + \delta - 1}{2\sqrt{\pi}(\delta - 1)} \iint\limits_{D(0,1)} |\nabla f(z)| \; dxdy \leq C(\delta) \iint\limits_{D(0,1)} |\nabla f(z)| \; dxdy.\end{aligned}$$

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Taking the optimal $\delta \approx 5/4$ we have

$$\left(\iint\limits_{D(0,1)} |f(z)|^2 \, dxdy \right)^{\frac{1}{2}} \leq \frac{3\sqrt{\pi^3}}{4} \iint\limits_{D(0,1)} |\nabla f(z)| \, dxdy. \quad \square$$

3. The doubling condition and the Hölder inequality

We recall necessary facts about conformal capacity. A well-ordered triple $(F_0, F_1; \Omega)$ of nonempty sets, where Ω is an open set in $\mathbb{C}$, and F_0, F_1 are closed subsets of Ω, is called a condenser on the complex plane $\mathbb{C}$.

The value

$$\operatorname{cap}(E) = \operatorname{cap}(F_0, F_1; \Omega) = \inf \iint\limits_{\Omega} |\nabla v|^2 dxdy,$$

where the infimum is taken over all Lipschitz nonnegative functions $v : \overline{\Omega} \to \mathbb{R}$, such that $v = 0$ on F_0, and $v = 1$ on F_1, is called the conformal capacity of the condenser $E = (F_0, F_1; \Omega)$. If the set of admissible functions is empty, then $\operatorname{cap}(F_0, F_1; \Omega) = \infty$. For finite values of capacity $0 < \operatorname{cap}(F_0, F_1; \Omega) < +\infty$ there exists a unique continuous weakly differentiable function u_0 (an extremal function) such that:

$$\operatorname{cap}(F_0, F_1; \Omega) = \iint\limits_{\Omega} |\nabla u_0|^2 dxdy.$$

Quasiconformal mappings can be characterized in capacitary terms (see, for example [24]). Namely, a homeomorphism $\varphi : \Omega \to \Omega'$, $\Omega, \Omega' \subset \mathbb{C}$ is K-quasiconformal, if and only if

$$K^{-1} \operatorname{cap}(F_0, F_1; \Omega) \leq \operatorname{cap}(\varphi(F_0), \varphi(F_1); \Omega') \leq K \operatorname{cap}(F_0, F_1; \Omega)$$

for any condenser $E = (F_0, F_1; \Omega)$.

We will need the following estimate of the conformal capacity (see, for example [24,42]).

Denote $D(z_0, r)$ an open disc with center z_0 of radius r and $\overline{D(z_0, r)}$ its closure.

Lemma 3.1. *[24] Let $R > r > 0$. Then* $\operatorname{cap}\left(\overline{D(z_0, r)}, \mathbb{C} \setminus D(z_0, R); \mathbb{C}\right) = 2\pi(\log R/r)^{-1}$.

Consider capacity estimates for Teichmüller type condensers in $\overline{\mathbb{C}}$ (see, for example [42], Lemma 5.32).

Lemma 3.2. *Fix $0 < r < R$. Let F_0 and F_1 be continuums in $\mathbb{C}$ such that*

$$F_0 \cap S(0, \rho) \neq \emptyset \text{ and } F_1 \cap S(0, \rho) \neq \emptyset$$

for all $r < \rho < R$, where $S(0, \rho)$ is the circle of radius ρ. Then

$$\operatorname{cap}(F_0, F_1; D(0, R) \setminus \overline{D(0, r)}) \geq \frac{2}{\pi} \log \frac{R}{r}.$$

Denote by $R_T(t) = \overline{\mathbb{C}} \setminus \{[-1, 0] \cup [t, \infty]\}$, $t > 1$, the Teichmüller ring in $\overline{\mathbb{C}}$.

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Lemma 3.3. *Let $R_T(t)$ be the Teichmüller ring in $\overline{\mathbb{C}}$. Then*

$$\operatorname{cap} R_T(t) = \frac{2\pi}{\log \Phi(t)},$$

where Φ satisfies the conditions

$$t + 1 \le \Phi(t) < 32t,\ t > 1.$$

Using these capacity estimates we obtain estimates for a constant in the inverse Hölder inequality. We start from the weak Hölder inequality.

Lemma 3.4. *Let $\varphi : \Omega \to \Omega'$ be a K-quasiconformal mapping of planar domains $\Omega, \Omega' \subset \mathbb{C}$. Then for every disc $D(z_0, r)$ such that $D(z_0, 2r) \subset \Omega$ the weak inverse Hölder inequality for the first generalized derivatives of φ*

$$\left(\frac{1}{|D(z_0,r)|}\iint\limits_{D(z_0,r)} |\varphi'(z)|^2\ dxdy\right)^{\frac{1}{2}} \le \frac{24\pi^2 K}{|D(z_0,2r)|}\iint\limits_{D(z_0,2r)} |\varphi'(z)|\ dxdy \tag{3.1}$$

holds.

Proof. Derivatives of K-quasiconformal mappings satisfy the following inequality ([11], p. 274)

$$\left(\frac{1}{|D(z_0,r)|}\iint\limits_{D(z_0,r)} |\varphi'(z)|^2\ dxdy\right)^{\frac{1}{2}}$$
$$\le \frac{4K}{r}\left(\frac{1}{|D(z_0,2r)|}\iint\limits_{D(z_0,2r)} \left|\varphi(z) - \frac{1}{|D(z_0,2r)|}\iint\limits_{D(z_0,2r)} \varphi(z)\right|^2 dxdy\right)^{\frac{1}{2}}.$$

Applying to the right side of this inequality the Poincaré–Sobolev inequality (2.2), we obtain the required inequality (3.1). □

Directly by Lemma 3.4 and Theorem 4.2 from [11] we get the following weak Hölder inequality:

Theorem 3.5. *Let $\varphi : \Omega \to \Omega'$ be a K-quasiconformal mapping of planar domains $\Omega, \Omega' \subset \mathbb{C}$. Then for every disc $D(z_0, r)$ such that $D(z_0, 2r) \subset \Omega$ and for some $\sigma > 2$ the inequality*

$$\left(\frac{1}{|D(z_0,r)|}\iint\limits_{D(z_0,r)} |\varphi'(z)|^\sigma\ dxdy\right)^{\frac{1}{\sigma}} \le C_\sigma \left(\frac{1}{|D(z_0,2r)|}\iint\limits_{D(z_0,2r)} |\varphi'(z)|^2\ dxdy\right)^{\frac{1}{2}} \tag{3.2}$$

holds, where

$$C_\sigma = \frac{10^6}{[(\sigma-1)(1-\nu)]^{1/\sigma}}, \quad \nu = 10^{4\sigma}\frac{\sigma-2}{\sigma-1}\left(24\pi^2 K\right)^\sigma < 1.$$

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Given Theorem 3.5 for further estimates of the left side of the inequality (3.2) we use the doubling property of measures generated by Jacobians of quasiconformal mappings (see, for example, [29]). Recall that a Borel measure μ on a set Ω is doubling if there exist a constant $C_\mu \geq 1$ so that the inequality

$$\mu(D(z_0, 2r)) \leq C_\mu \cdot \mu(D(z_0, r)) < \infty$$

hold for all discs $D(z_0, r)$ in Ω.

In the following proposition we give an estimate of the constant C_μ in the measure doubling condition.

Proposition 3.6. *Let $\varphi : \mathbb{C} \to \mathbb{C}$ be a K-quasiconformal mapping. Then for any $z_0 \in \mathbb{C}$ and any $r > 0$ we have*

$$\iint\limits_{D(z_0,2r)} |J(z,\varphi)|\, dxdy \leq \exp\left\{\frac{K\pi^2(2+\pi^4)^2}{2\log 3}\right\} \iint\limits_{D(z_0,r)} |J(z,\varphi)|\, dxdy. \tag{3.3}$$

Proof. Because the inequality (3.3) is invariant under translations and similarities we can suppose that $z_0 = 0$ and radius $r = 1$. It is sufficient to show that

$$|\varphi(D(0,2))| \leq \exp\left\{\frac{K\pi^2(2+\pi^4)^2}{2\log 3}\right\} |\varphi(D(0,1))|. \tag{3.4}$$

Since φ is a quasiconformal mapping, $\varphi(0)$ is an interior point of the open connected set $U_0 = \varphi(D(0,1))$. Denote by $\lambda_0 = \operatorname{dist}(\varphi(0), \partial U_0)$ and by $\lambda_1 = \max\limits_{w\in\partial U_1} |\varphi(0) - w|$, $U_1 = \varphi(D(0,2))$.

For the proof of the inequality (3.4) we estimate the ratio

$$\frac{|\varphi(D(0,2))|}{|\varphi(D(0,1))|} \leq \frac{\pi\lambda_1^2}{\pi\lambda_0^2} = \frac{\lambda_1^2}{\lambda_0^2},$$

because the disc $D(\varphi(0), \lambda_1)$ contains $\varphi(D(0,2))$ and the disc $D(\varphi(0), \lambda_0)$ is contained in $\varphi(D(0,1))$. For this aim we use the capacity estimates and quasi-invariance of conformal capacity under quasiconformal mappings and consider continuums F_0 and F_1 where F_0 be a line segment of length λ_0 joining $\varphi(0)$ to $w_0 \in \partial U_0$ and F_1 be a continuum in $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ joining a point w_1 in $\partial D(\varphi(0), \lambda_1) \cap U_1$ to ∞ in $\overline{\mathbb{C}} \setminus D(\varphi(0), \lambda_1)$.

Now we consider pre-images of continuums $\varphi^{-1}(F_0)$ and $\varphi^{-1}(F_1)$ in order to obtain a lower estimate of the capacity of the condenser $(\varphi^{-1}(F_0), \varphi^{-1}(F_1); D(0,\pi))$. Because capacity is invariant under rotations, without loss of generality we can suppose that $z_0 = \varphi^{-1}(w_0) = (1,0)$. Let $z_1 = \varphi^{-1}(w_1) = (2,\theta_0)$, where $(1,0)$ and $(2,\theta_0)$ are the polar coordinates.

Define the bi-Lipschitz mapping $\psi : \mathbb{C} \to \mathbb{C}$ which maps z_0 to z_0 and z_1 to $(2,0)$ by the rule

$$\psi(\rho,\theta) = (\rho, \theta - (\rho-1)\theta_0),$$

where (ρ,θ) are polar coordinates on the complex plane $\mathbb{C}$.

Now we estimate the Lipschitz coefficient L for the mapping ψ. For this aim we represent ψ as

$$\psi(x,y) = (\sqrt{x^2+y^2}\cos t, \sqrt{x^2+y^2}\sin t),$$

where

$$t = \arctan\frac{y}{x} - (\sqrt{x^2+y^2} - 1)\theta_0, \quad 0 \leq \theta_0 \leq \pi,$$

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and we use the following expressions

$$L = \|D\psi(x,y)\|_o \le \frac{\sqrt{2}}{2}\|D\psi(x,y)\|_e.$$

Here $D\psi(x,y)$ denotes the Jacobian matrix of the mapping $\psi = \psi(x,y)$, $\|\cdot\|_o$ and $\|\cdot\|_e$ denote the operator norm and the Euclidean norm respectively.

The Jacobian matrix has the form

$$D\psi(x,y) = \begin{pmatrix} \frac{x\cos t + y\sin t}{\sqrt{x^2+y^2}} + x\theta_0 \sin t & \frac{y\cos t - x\sin t}{\sqrt{x^2+y^2}} + y\theta_0 \sin t \\ -\frac{y\cos t - x\sin t}{\sqrt{x^2+y^2}} - x\theta_0 \cot t & \frac{x\cos t + y\sin t}{\sqrt{x^2+y^2}} - y\theta_0 \cot t \end{pmatrix}.$$

A straightforward calculation yields

$$\|D\psi(x,y)\|_e = \sqrt{2 + (x^2+y^2)\theta_0^2}.$$

Hence

$$L \le \frac{\sqrt{2}}{2}\|D\psi(x,y)\|_e = \sqrt{1 + \frac{(x^2+y^2)\theta_0^2}{2}} \le \sqrt{1 + \frac{\pi^4}{2}}.$$

Now we consider the ring $R = D(\frac{3}{2},\frac{3}{2}) \setminus D(\frac{3}{2},\frac{1}{2})$. Then by Lemma 3.2 using the monotonicity of the capacity and quasi-invariance under bi-Lipschtz mappings we obtain

$$\operatorname{cap}(\varphi^{-1}(F_0), \varphi^{-1}(F_1); D(0,\pi)) \ge \frac{1}{L^4}\operatorname{cap}(E_0, E_1; R) \ge \frac{8\log 3}{\pi(2+\pi^4)^2}, \tag{3.5}$$

where $E_0 = \psi(\varphi^{-1}(F_0)) \cap R$, $E_1 = \psi(\varphi^{-1}(F_1)) \cap R$.

On the other hand, taking into account the capacitary definition of the quasiconformal mapping and Lemma 3.1 we have

$$\begin{aligned} &\operatorname{cap}(\varphi^{-1}(F_0), \varphi^{-1}(F_1); D(0,\pi)) \le K \operatorname{cap}(F_0, F_1; \mathbb{C}) \\ &\le K \operatorname{cap}(D(\varphi(0),\lambda_0), \mathbb{C} \setminus D(\varphi(0),\lambda_1); \mathbb{C}) = \frac{2\pi K}{\log\frac{\lambda_1}{\lambda_0}}. \end{aligned} \tag{3.6}$$

Combining inequalities (3.5) and (3.6), we obtain

$$\frac{8\log 3}{\pi(2+\pi^4)^2} \le \frac{2\pi K}{\log\frac{\lambda_1}{\lambda_0}}.$$

By elementary calculations

$$\lambda_1 \le \exp\left\{\frac{K\pi^2(2+\pi^4)^2}{4\log 3}\right\}\lambda_0.$$

In order to obtain the required inequality it is necessary to perform the following straightforward calculations

$$\begin{aligned} |\varphi(D(0,2))| &\le \pi\lambda_1^2 \\ &\le \pi \exp\left\{\frac{K\pi^2(2+\pi^4)^2}{2\log 3}\right\}\lambda_0^2 \le \exp\left\{\frac{K\pi^2(2+\pi^4)^2}{2\log 3}\right\}|\varphi(D(0,1))|. \end{aligned}$$

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Thus

$$\iint\limits_{D(0,2)} |J(z,\varphi)|\,dxdy \le \exp\left\{\frac{K\pi^2(2+\pi^4)^2}{2\log 3}\right\} \iint\limits_{D(0,1)} |J(z,\varphi)|\,dxdy. \quad \square$$

For any planar K-quasiconformal homeomorphism $\varphi : \Omega \to \Omega'$ the following sharp result is known: $J(z,\varphi) \in L_{p,\mathrm{loc}}(\Omega)$ for any $1 \le p < \frac{K}{K-1}$ ([8,21]).

Proposition 3.7. *For any conformal homeomorphism $\varphi : \mathbb{D} \to \Omega$ of the unit disc $\mathbb{D}$ onto a K-quasidisc Ω derivatives $\varphi' \in L_p(\mathbb{D})$ for any $1 \le p < \frac{2K^2}{K^2-1} \subset \mathrm{HI}(\Omega)$.*

Proof. Any conformal homeomorphism $\varphi : \mathbb{D} \to \Omega$ can be extended to a K^2-quasiconformal homeomorphism $\widetilde{\varphi}$ of the whole plane to the whole plane by reflection. Hence $\widetilde{\varphi}'$ belongs to the class $L_{p,\mathrm{loc}}(\mathbb{C})$ for any $1 \le p < \frac{2K^2}{K^2-1}$ ([8,21]). Therefore φ' belongs to the class $L_p(\mathbb{D})$. $\square$

On the base of the weak Hölder inequality and the doubling condition we obtain integral estimates of complex derivatives of conformal mapping $\varphi : \mathbb{D} \to \Omega$ of the unit disc onto a K-quasidisc Ω:

Theorem B. *Let $\Omega \subset \mathbb{R}^2$ be a K-quasidisc and $\varphi : \mathbb{D} \to \Omega$ be a conformal mapping. Suppose that $2 < \alpha < \frac{2K^2}{K^2-1}$. Then*

$$\left(\iint\limits_{D(0,1)} |J(z,\varphi)|^{\frac{\alpha}{2}}\,dxdy\right)^{\frac{2}{\alpha}} \le \frac{C_\alpha^2 K^2 \pi^{\frac{2}{\alpha}-1}}{4} \exp\left\{\frac{K^2\pi^2(2+\pi^4)^2}{2\log 3}\right\} \cdot |\Omega|, \tag{3.7}$$

where

$$C_\alpha = \frac{10^6}{[(\alpha-1)(1-\nu)]^{1/\alpha}}, \quad \nu = 10^{4\alpha}\frac{\alpha-2}{\alpha-1}(24\pi^2K^2)^\alpha < 1.$$

Proof. Since $\varphi : \mathbb{D} \to \Omega$ is a conformal mapping, then by the inequality (3.2), using the equality $J(z,\varphi) = |\varphi'(z)|^2$, we have

$$\left(\iint\limits_{D(0,1)} |J(z,\varphi)|^{\frac{\alpha}{2}}\,dxdy\right)^{\frac{2}{\alpha}} = |D(0,1)|^{\frac{2}{\alpha}} \left(\frac{1}{|D(0,1)|}\iint\limits_{D(0,1)} |\varphi'(z)|^\alpha\,dxdy\right)^{\frac{1}{\alpha}\cdot 2}$$

$$\le C_\alpha^2 |D(0,1)|^{\frac{2}{\alpha}} \left(\frac{1}{|D(0,2)|}\iint\limits_{D(0,2)} |\varphi'(z)|^2\,dxdy\right) = \frac{C_\alpha^2 \pi^{\frac{2}{\alpha}-1}}{4} \iint\limits_{D(0,2)} |\varphi'(z)|^2\,dxdy.$$

In the disc $D(0,2)$ an extension of the conformal mapping φ is a K^2-quasiconformal homeomorphism. Hence taking into account the inequality

$$|\varphi'(z)|^2 \le K^2 |J(z,\varphi)| \quad \text{for almost all } z \in D(0,2)$$

by the inequality (3.3) we get

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$$\frac{C_\alpha^2\pi^{\frac{2}{\alpha}-1}}{4}\iint\limits_{D(0,2)}|\varphi'(z)|^2\,dxdy\le\frac{C_\alpha^2K^2\pi^{\frac{2}{\alpha}-1}}{4}\iint\limits_{D(0,2)}|J(z,\varphi)|\,dxdy$$
$$\le\frac{C_\alpha^2K^2\pi^{\frac{2}{\alpha}-1}}{4}\exp\left\{\frac{K^2\pi^2(2+\pi^4)^2}{2\log 3}\right\}\iint\limits_{D(0,1)}|J(z,\varphi)|\,dxdy.$$

Thus, considering that

$$\iint\limits_{D(0,1)}|J(z,\varphi)|\,dxdy=|\Omega|$$

we have

$$\left(\iint\limits_{D(0,1)}|J(z,\varphi)|^{\frac{\alpha}{2}}\,dxdy\right)^{\frac{2}{\alpha}}\le\frac{C_\alpha^2K^2\pi^{\frac{2}{\alpha}-1}}{4}\exp\left\{\frac{K^2\pi^2(2+\pi^4)^2}{2\log 3}\right\}\cdot|\Omega|.\quad\square$$

4. Estimates of the hyperbolic α-dilatation and the first non-trivial eigenvalue of the Neumann–Laplace operator

In the work [27] were obtained the following estimates of the first non-trivial eigenvalue of the Neumann–Laplace operator in quasidiscs:

Proposition 4.1. *Suppose a conformal homeomorphism $\varphi:\mathbb{D}\to\Omega$ maps the unit disc $\mathbb{D}$ onto a K-quasidisc Ω. Then*

$$1/\mu_1(\Omega)\le\frac{4}{\sqrt[\alpha]{\pi^2}}\left(\frac{2\alpha-2}{\alpha-2}\right)^{\frac{2\alpha-2}{\alpha}}Q(\alpha,\Omega)^{2/\alpha}$$

for any $2<\alpha<\frac{2K^2}{K^2-1}$.

By Proposition 4.1 and Theorem B we obtain the lower estimates of the first non-trivial eigenvalue of the Neumann–Laplace operator in K-quasidisc in terms of quasiconformal geometry of domains:

Theorem A. *Let $\Omega\subset\mathbb{C}$ be a K-quasidisc. Then the spectrum of Neumann–Laplace operator in Ω is discrete, can be written in the form of a non-decreasing sequence*

$$0=\mu_0(\Omega)<\mu_1(\Omega)\le\mu_2(\Omega)\le\ldots\le\mu_n(\Omega)\le\ldots,$$

and

$$\frac{1}{\mu_1(\Omega)}\le\frac{K^2C_\alpha^2}{\pi}\left(\frac{2\alpha-2}{\alpha-2}\right)^{\frac{2\alpha-2}{\alpha}}\exp\left\{\frac{K^2\pi^2(2+\pi^4)^2}{2\log 3}\right\}\cdot\big|\Omega\big|,\tag{4.1}$$

for $2<\alpha<\frac{2K^2}{K^2-1}$, where

$$C_\alpha=\frac{10^6}{[(\alpha-1)(1-\nu)]^{1/\alpha}},\quad\nu=10^{4\alpha}\frac{\alpha-2}{\alpha-1}(24\pi^2K^2)^\alpha<1.$$

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Proof. By Proposition 4.1 for any $2 < \alpha < \frac{2K^2}{K^2-1}$ we have

$$1/\mu_1(\Omega) \le \frac{4}{\sqrt[\alpha]{\pi^2}} \left(\frac{2\alpha-2}{\alpha-2}\right)^{\frac{2\alpha-2}{\alpha}} Q(\alpha,\Omega)^{2/\alpha}. \tag{4.2}$$

By Theorem B

$$\begin{aligned} Q(\alpha,\Omega)^{\frac{2}{\alpha}} &:= \left(\iint\limits_{D(0,1)} |J(z,\varphi)|^{\frac{\alpha}{2}}\, dxdy\right)^{\frac{2}{\alpha}} \\ &\le \frac{C_\alpha^2 K^2 \pi^{\frac{2}{\alpha}-1}}{4} \exp\left\{\frac{K^2\pi^2(2+\pi^4)^2}{2\log 3}\right\} \cdot |\Omega|. \end{aligned} \tag{4.3}$$

Combining inequalities (4.2) and (4.3) and performing a straightforward calculations we obtain the required inequality. □

As an application of Theorem A, we obtain the lower estimates of the first non-trivial eigenvalue on the Neumann eigenvalue problem for the Laplace operator in the star-shaped and spiral-shaped domains.

A simply connected domain Ω^* is β-star-shaped (with respect to $z_0 = 0$) if the function $\varphi(z)$, $\varphi(0) = 0$, conformally maps a unit disc $\mathbb{D}$ onto Ω^* and satisfies the condition [17]:

$$\left|\arg \frac{z\varphi'(z)}{\varphi(z)}\right| \le \beta\pi/2, \quad 0 \le \beta < 1, \quad |z| < 1.$$

A simply connected domain Ω_s is β-spiral-shaped (with respect to $z_0 = 0$) if the function $\varphi(z)$, $\varphi(0) = 0$, conformally maps a unit disc $\mathbb{D}$ onto Ω_s and satisfies the condition [37,38]:

$$\left|\arg e^{i\delta} \frac{z\varphi'(z)}{\varphi(z)}\right| \le \beta\pi/2, \quad 0 \le \beta < 1, \quad |\delta| < \beta\pi/2, \quad |z| < 1.$$

In [17] and [37,38], respectively, it is shown that boundaries of domains Ω^* and Ω_s are a K-quasicircles with $K = \cot^2(1-\beta)\pi/4$.

Setting $\Omega = \Omega^*$ or $\Omega = \Omega_s$ Theorem A implies

$$\frac{1}{\mu_1(\Omega)} \le \frac{C_\alpha^2 \cot^4(1-\beta)\frac{\pi}{4}}{\pi} \left(\frac{2\alpha-2}{\alpha-2}\right)^{\frac{2\alpha-2}{\alpha}} \exp\left\{\frac{\pi^2(2+\pi^4)^2 \cot^4(1-\beta)\frac{\pi}{4}}{2\log 3}\right\} \cdot |\Omega|,$$

where

$$C_\alpha = \frac{10^6}{[(\alpha-1)(1-\nu)]^{1/\alpha}}, \quad \nu = 10^{4\alpha}(24\pi^2 \cot^4(1-\beta)\pi/4)^\alpha \frac{\alpha-2}{\alpha-1} < 1.$$

5. Quasiconformal reflection

Let Γ be a Jordan curve on the Riemann sphere, and denote its complementary components by Ω, Ω^*. Suppose that there exists a sense-reversing quasiconformal mapping ψ of the sphere onto itself which maps Ω on Ω^* and keeps every point on Γ fixed. Such mappings are called quasiconformal reflections.

Denote by H the upper half-plane and by H^* the lower half-plane of the complex plane $\mathbb{C}$. Consider a conformal mapping φ of H on Ω and a conformal mapping φ_* of H^* to Ω^*. Then a mapping $\varphi_*^{-1} \circ \psi \circ \varphi$ is

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a quasiconformal mapping of H onto H^* which induces a monotone mapping $h = \varphi_*^{-1} \circ \varphi$ of the real axis on itself [4].

In [4] L. Ahlfors and A. Beurling derived a necessary and sufficient condition for a boundary mapping h to be restriction of a quasiconformal mapping of H on itself (or on its reflection H^*). Without loss of generality it may be assumed that $h(\infty) = \infty$. Then h admits a quasiconformal extension if and only if it satisfies an M-condition, namely an inequality

$$\frac{1}{M} \leq \frac{h(x+t) - h(x)}{h(x) - h(x-t)} \leq M \tag{5.1}$$

which is to be fulfilled for all real x, t, $t > 0$, and with a constant $M \neq 0, \infty$. More precisely, if h has a K-quasiconformal extension, then (5.1) holds with a $M = M(K) < e^{\pi K}/16$, and if (5.1) holds, then h has a K-quasiconformal extension such that $K = K(M) < M^2$.

In [2] L. Ahlfors proved that a Jordan curve Γ admits a K-quasiconformal reflection if and only if Γ satisfies the Ahlfors's 3-point condition. In the following theorem, using the Ahlfors scheme, we calculate the upper estimates for the coefficient of quasiconformality K for quasicircles given by the Ahlfors's 3-point condition.

Theorem 5.1. *Let a Jordan curve Γ satisfies the Ahlfors's 3-point condition: there exists a constant C such that*

$$|\zeta_3 - \zeta_1| \leq C|\zeta_2 - \zeta_1| \tag{5.2}$$

for any three point on Γ where ζ_3 is between ζ_1 and ζ_2. Then Γ admits a K-quasiconformal reflection where K depends only on C and

$$K < \frac{1}{2^{10}} \exp\left\{\left(1 + e^{2\pi} C^5\right)^2\right\}.$$

Proof. We will use the notations

$$\alpha_1 = \arc \zeta_1\zeta_3, \quad \alpha_2 = \arc \zeta_3\zeta_2, \quad \beta_1 = \arc \zeta_2\infty, \quad \beta_2 = \arc \zeta_1\infty.$$

Thus

$$\operatorname{cap}(\alpha_1, \beta_1; \Omega) \cdot \operatorname{cap}(\alpha_2, \beta_2; \Omega) = 1 \text{ and } \operatorname{cap}(\alpha_1, \beta_1; \Omega^*) \cdot \operatorname{cap}(\alpha_2, \beta_2; \Omega^*) = 1.$$

Through the conformal mapping of Ω, let ζ_1, ζ_3, ζ_2 correspond to $x - t$, x, $x + t$. This means that

$$\operatorname{cap}(\alpha_1, \beta_1; \Omega) = \operatorname{cap}(\alpha_2, \beta_2; \Omega) = 1.$$

Through the conformal mapping of Ω^*, ζ_1, ζ_3, ζ_2 correspond to $h(x - t)$, $h(x)$, $h(x + t)$.

In [2] Ahlfors proved that

$$\operatorname{cap}(\alpha_1, \beta_1; \Omega^*) \leq \pi \left(1 + e^{2\pi} C^5\right)^2. \tag{5.3}$$

Now we obtain the lower estimate for the capacity $\operatorname{cap}(\alpha_1, \beta_1; \Omega^*)$ using the capacity estimates for the Teichmüller condenser: denote by

$$y = (h(x+t) - h(x))/(h(x) - h(x-t)),$$

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then by Lemma 3.3

$$\operatorname{cap} R_T(y) = \frac{2\pi}{\log \Phi(y)} > \frac{2\pi}{\log 32y}. \tag{5.4}$$

Combining inequalities (5.3) and (5.4), we obtain

$$\frac{2\pi}{\log 32y} < \operatorname{cap} R_T(y) = \operatorname{cap}(\alpha_1, \beta_1; \Omega^*) \le \pi\left(1 + e^{2\pi} C^5\right)^2$$

or

$$\frac{h(x+t) - h(x)}{h(x) - h(x-t)} \le \frac{1}{32} \exp\left\{\frac{\left(1 + e^{2\pi} C^5\right)^2}{2}\right\}.$$

Hence h satisfies an M-condition, i.e.

$$\frac{1}{M(C)} \le \frac{h(x+t) - h(x)}{h(x) - h(x-t)} \le M(C)$$

where

$$M(C) = \frac{1}{32} \exp\left\{\frac{\left(1 + e^{2\pi} C^5\right)^2}{2}\right\}.$$

So, [4] there exist a K-quasiconformal reflection such that $K < M^2(C)$.

Finally we get

$$K < \left(\frac{1}{32} \exp\left\{\frac{\left(1 + e^{2\pi} C^5\right)^2}{2}\right\}\right)^2 = \frac{1}{2^{10}} \exp\left\{\left(1 + e^{2\pi} C^5\right)^2\right\}. \quad \square \tag{5.5}$$

Combining Theorem A and Theorem 5.1 we obtain

Corollary 5.2. *Let a domain $\Omega \subset \mathbb{R}^2$ be bounded by a Jordan curve Γ that satisfies the Ahlfors's 3-point condition. Then*

$$\frac{1}{\mu_1(\Omega)} \le \frac{C_\alpha^2 e^{2\left(1+e^{2\pi}C^5\right)^2}}{2^{20}\pi} \left(\frac{2\alpha - 2}{\alpha - 2}\right)^{\frac{2\alpha-2}{\alpha}} \exp\left\{\frac{\pi^2(2+\pi^4)^2 e^{2\left(1+e^{2\pi}C^5\right)^2}}{2^{21}\log 3}\right\} \cdot |\Omega|,$$

for $2 < \alpha < \frac{2K^2}{K^2-1}$, where

$$C_\alpha = \frac{10^6}{[(\alpha-1)(1-\nu)]^{1/\alpha}}, \quad \nu = 10^{4\alpha} \left(\frac{24\pi^2}{2^{20}} e^{2\left(1+e^{2\pi}C^5\right)^2}\right)^\alpha \frac{\alpha - 2}{\alpha - 1} < 1.$$

6. Examples in fractal type domains

Rohde snowflake. In [36] S. Rohde constructed a collection S of snowflake type planar curves with the intriguing property that each planar quasicircle is bi-Lipschitz equivalent to some curve in S. We use the formulation of Rohde snowflakes and the picture from [31].

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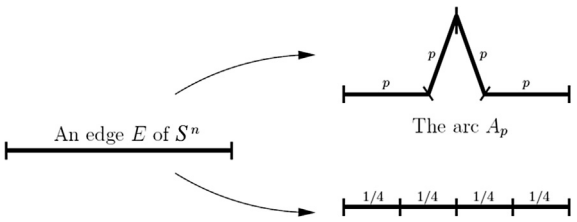

Fig. 6.1. Construction of a Rohde-snowflake.

Rohde's catalog is

$$S := \bigcup_{1/4\le p<1/2} S_p$$

where p is a snowflake parameter. Each curve in S_p is built in a manner reminiscent of the construction of the von Koch snowflake. Thus, each $S \in S_p$ is the limit of a sequence S^n of polygons where S^{n+1} is obtained from S^n by using the replacement rule illustrated in Fig. 6.1: for each of the 4^n edges E of S^n we have two choices, either we replace E with the four line segments obtained by dividing E into four arcs of equal diameter, or we replace E by a similarity copy of the polygonal arc A_p pictured at the top right of Fig. 6.1. In both cases E is replaced by four new segments, each of these with diameter $(1/4)\mathrm{diam}(E)$ in the first case or with diameter $p\,\mathrm{diam}(E)$ in the second case. The second type of replacement is done so that the "tip" of the replacement arc points into the exterior of S^n. This iterative process starts with S^1 being the unit square, and the snowflake parameter, thus the polygon arc A_p, is fixed throughout the construction.

The sequence S^n of polygons converges, in the Hausdorff metric, to a planar quasicircle S that we call a *Rohde snowflake* constructed with snowflake parameter p. Then S_p is the collection of all Rohde snowflakes that can be constructed with snowflake parameter p.

In [31] established that each Rohde snowflake S in S_p is C-bounded turning with

$$C = C(p) = \frac{16}{1-2p}, \quad 1/4 \le p < 1/2.$$

A planar curve Γ satisfies the C-bounded turning, $C \ge 1$, if for each pair of points x, y, on Γ, the smaller diameter subarc $\Gamma[x,y]$ of Γ that joins x, y satisfies

$$\mathrm{diam}(\Gamma[x,y]) \le C|x-y|. \tag{6.1}$$

The C-bounded turning condition (6.1) is equivalent the Ahlfors's 3-point condition (5.2) with the same constant C [24].

The following theorem gives the lower estimates of the first non-trivial eigenvalue of the Neumann–Laplace operator in domains type a Rohde snowflakes:

Theorem C. *Let $S_p \subset \mathbb{R}^2$, $1/4 \le p < 1/2$, be the Rohde snowflake. Then the spectrum of Neumann–Laplace operator in S_p is discrete, can be written in the form of a non-decreasing sequence*

$$0 = \mu_0(S_p) < \mu_1(S_p) \le \mu_2(S_p) \le \ldots \le \mu_n(S_p) \le \ldots,$$

and

$$\frac{1}{\mu_1(S_p)} \le \frac{C_\alpha^2 e^{4\left(1+e^{2\pi}(16/(1-2p))^5\right)^2}}{2^{40}\pi} \left(\frac{2\alpha-2}{\alpha-2}\right)^{\frac{2\alpha-2}{\alpha}} \times \exp\left\{\frac{\pi^2(2+\pi^4)^2 e^{4\left(1+e^{2\pi}(16/(1-2p))^5\right)^2}}{2^{41}\log 3}\right\} |S_p|,$$

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for $2 < \alpha < \frac{2K^2}{K^2-1}$, *where*

$$C_\alpha = \frac{10^6}{[(\alpha-1)(1-\nu)]^{1/\alpha}}, \quad \nu = 10^{4\alpha}\left(\frac{24\pi^2}{2^{40}}e^{4(1+e^{2\pi}C^5)^2}\right)^{\alpha}\frac{\alpha-2}{\alpha-1} < 1.$$

The proof of Theorem C immediately follows from the Corollary 5.2, given that any L-bi-Lipschitz planar homeomorphism is K-quasiconformal with $K = L^2$.

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Appendix A

Firstly we discuss a new notion of the hyperbolic α-dilatation $Q(\alpha,\Omega)$ and its convergence hyperbolic interval $\mathrm{HI}(\Omega)$ in connection with known results. Of course, quantities $Q(\alpha,\Omega)$ and $\mathrm{HI}(\Omega)$ can be defined also in non bounded domains.

Recall definitions:

$$Q(\alpha,\Omega) := \iint\limits_{\mathbb{D}} |\varphi'(z)|^\alpha \, dxdy = \iint\limits_{\Omega} \left|\left(\varphi^{-1}\right)'(w)\right|^{2-\alpha} dudv$$

where $\varphi : \mathbb{D} \to \Omega$ is a Riemann conformal homeomorphism;

$$\mathrm{HI}(\Omega) := \{\alpha \in \mathbb{R} : Q(\alpha,\Omega) < \infty\}.$$

Let us remark that by the definition

$$Q(2,\Omega) = |\Omega|,$$

i.e. is finite for any domain of finite measure. We don't know any simple geometric interpretation of $Q(\alpha,\Omega)$ for a number α other than 2.

In these new terms of **the Inverse Brennan's conjecture** [13] states: Let $\Omega \subset \mathbb{C}$ be a simply connected planar domain with nonempty boundary, and $\varphi : \mathbb{D} \to \Omega$ be a conformal homeomorphism. Then

$$\mathrm{HI}(\Omega) = (-2, 2/3).$$

The upper bound $\alpha = 2/3$ is proved, the lower bound is only conjectured.

By [26] for bounded domains it is proved that

$$\mathrm{HI}(\Omega) = (-2, 2].$$

The upper bound $\alpha = 2$ is proved, the lower bound is only conjectured.

Note that for smooth bounded domains [13]

$$\mathrm{HI}(\Omega) = (-\infty, \infty).$$

In [27] we demonstrated that for any K-quasidisc $1 \le \alpha < \frac{2K^2}{K^2-1} \subset \mathrm{HI}(\Omega)$. It can be reformulated in terms of the Ahlfors condition.

Conjecture. *Interval* $\mathrm{HI}(\Omega)$ *is defined by the hyperbolic geometry of* Ω.

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References

[1] G. Acosta, R.G. Durán, An optimal Poincaré inequality in L^1 for convex domains, Proc. Amer. Math. Soc. 132 (1) (2004) 195–202.
[2] L. Ahlfors, Quasiconformal reflections, Acta Math. 109 (1963) 291–301.
[3] L. Ahlfors, Lectures on Quasiconformal Mappings, D. Van Nostrand Co., Inc., Toronto, Ont., New York, London, 1966.
[4] L. Ahlfors, A. Beurling, The boundary correspondence under quasiconformal mapping, Acta Math. 56 (1956) 125–142.
[5] M.S. Ashbaugh, Isoperimetric and universal inequalities for eigenvalues, in: Spectral Theory and Geometry, Edinburgh, 1998, in: London Math. Soc. Lecture Note Ser., vol. 273, Cambridge Univ. Press, Cambridge, 1999, pp. 95–139.
[6] M.S. Ashbaugh, R.D. Benguria, Universal bounds for the low eigenvalues of Neumann Laplacians in n-dimensions, SIAM J. Math. Anal. 24 (1993) 557–570.
[7] M.S. Ashbaugh, H.A. Levine, Inequalities for the Dirichlet and Neumann eigenvalues of the Laplacian for domains on spheres, in: Journées "Équations aux Dérivées Partielles", Saint-Jean-de-Monts, 1997, École Polytech., Palaiseau, 1997, Exp. No. I, 15 pp.
[8] K. Astala, Area distortion of quasiconformal mappings, Acta Math. 173 (1994) 37–60.
[9] A.F. Beardon, D. Minda, The hyperbolic metric and geometric function theory, in: Quasiconformal Mappings and Their Applications, Narosa, New Delhi, 2007, pp. 9–56.
[10] L. Bieberbach, Über die Koeffizienten derjenigen Polenzreihen, welche eine schlichte Abbildung des Einheitskreises vermitteln, S.-B. Preuss. Akad. Wiss. (1916) 940–955.
[11] B. Bojarski, T. Iwaniec, Analytical foundation of the theory of quasiconformal mappings in $\mathbb{R}^n$, Ann. Acad. Sci. Fenn. Ser. A I Math. 8 (1983) 257–324.
[12] B. Brandolini, F. Chiacchio, E.B. Dryden, J.J. Langford, Sharp Poincaré inequalities in a class of non-covex sets, J. Spectr. Theory, accepted, arXiv:1608.01236.
[13] J. Brennan, The integrability of the derivative in conformal mapping, J. Lond. Math. Soc. (2) 18 (2) (1978) 261–272.
[14] V.I. Burenkov, V. Gol'dshtein, A. Ukhlov, Conformal spectral stability for the Dirichlet–Laplace operator, Math. Nachr. 288 (2015) 1822–1833.
[15] C. Enache, G.A. Philippin, On some isoperimetric inequalities involving eigenvalues of symmetric free membranes, ZAMM Z. Angew. Math. Mech. 95 (2015) 424–430.
[16] L. Esposito, C. Nitsch, C. Trombetti, Best constants in Poincaré inequalities for convex domains, J. Convex Anal. 20 (2013) 253–264.
[17] M. Fait, Y. Krzyz, Y. Zygmunt, Explicit quasiconformal extension for some classes of univalent functions, Comment. Math. Helv. 51 (2) (1976) 279–285.
[18] H. Federer, W.H. Fleming, Normal and integral currents, Ann. of Math. 72 (1960) 458–520.
[19] V. Ferone, C. Nitsch, C. Trombetti, A remark on optimal weighted Poincaré inequalities for convex domains, Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl. 23 (2012) 467–475.
[20] D. Gilbarg, N.S. Trudinger, Elliptic Partial Differential Equations of Second Order, Springer-Verlag, Berlin, Heidelberg, New York, 1977.
[21] V.M. Gol'dshtein, The degree of summability of generalized derivatives of quasiconformal homeomorphisms, Sibirsk. Mat. Zh. 22 (6) (1981) 22–40 (in Russian). English transl.: Sib. Math. J. 22 (6) (1981) 821–836.
[22] V. Gol'dshtein, L. Gurov, Applications of change of variables operators for exact embedding theorems, Integral Equations Operator Theory 19 (1994) 1–24.
[23] V. Gol'dshtein, L. Gurov, A. Romanov, Homeomorphisms that induce monomorphisms of Sobolev spaces, Israel J. Math. 91 (1995) 31–60.
[24] V.M. Gol'dshtein, Yu.G. Reshetnyak, Quasiconformal Mappings and Sobolev Spaces, Kluwer Academic Publisher, Dordrecht, Boston, London, 1990.
[25] V. Gol'dshtein, A. Ukhlov, Weighted Sobolev spaces and embedding theorems, Trans. Amer. Math. Soc. 361 (2009) 3829–3850.
[26] V. Gol'dshtein, A. Ukhlov, Sobolev homeomorphisms and Brennan's conjecture, Comput. Methods Funct. Theory 14 (2014) 247–256.
[27] V. Gol'dshtein, A. Ukhlov, On the first eigenvalues of free vibrating membranes in conformal regular domains, Arch. Ration. Mech. Anal. 221 (2) (2016) 893–915.
[28] H. Hedenmalm, S. Shimorin, Weighted Bergman spaces and the integral means spectrum of conformal mappings, Duke Math. J. 127 (2) (2005) 341–393.
[29] J. Heinonen, P. Koskela, Weighted Sobolev and Poincaré inequalities and quasiregular mappings of polynomial type, Math. Scand. 77 (1995) 251–271.
[30] S. Hencl, P. Koskela, T. Neiminen, Dimension gap under conformal mappings, Adv. Math. 230 (2012) 1423–1441.
[31] D.A. Herron, D. Meyer, Quasicircles and bounded turning circles modulo bi-Lipschitz maps, Rev. Mat. Iberoam. 28 (3) (2012) 603–630.
[32] R.S. Laugesen, C. Morpurgo, Extremals of eigenvalues of Laplacians under conformal mapping, J. Funct. Anal. 155 (1998) 64–108.
[33] V. Maz'ya, Sobolev Spaces: with Applications to Elliptic Partial Differential Equations, Springer, Berlin, Heidelberg, 2010.
[34] L.E. Payne, H.F. Weinberger, An optimal Poincaré inequality for convex domains, Arch. Ration. Mech. Anal. 5 (1960) 286–292.
[35] G. Pólya, On the eigenvalues of vibrating membranes, Proc. Lond. Math. Soc. 11 (1961) 419–433.
[36] S. Rohde, Quasicircles modulo bi-Lipschitz maps, Rev. Mat. Iberoam. 17 (2001) 643–659.
[37] M.A. Sevodin, Univalence conditions in spiral domains, Tr. Semin. Kraev. Zad. 23 (1986) 193–200 (in Russian).

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[38] T. Sugawa, Quasiconformal extension of strongly spirallike functions, Comput. Methods Funct. Theory 12 (1) (2012) 19–30.
[39] G. Szegö, Inequalities for certain eigenvalues of a membrane of given area, J. Ration. Mech. Anal. 3 (1954) 343–356.
[40] A. Ukhlov, On mappings, which induce embeddings of Sobolev spaces, Sib. Math. J. 34 (1993) 185–192.
[41] D. Valtorta, Sharp estimate on the first eigenvalue of the p-Laplacian, Nonlinear Anal. 75 (2012) 4974–4994.
[42] M. Vuorinen, Conformal Geometry and Quasiregular Mappings, Lecture Notes in Math., vol. 1319, Springer-Verlag, Berlin, Heidelberg, New York, 2006.
[43] H.F. Weinberger, An isoperimetric inequality for the n-dimensional free membrane problem, Arch. Ration. Mech. Anal. 5 (1956) 633–636.

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