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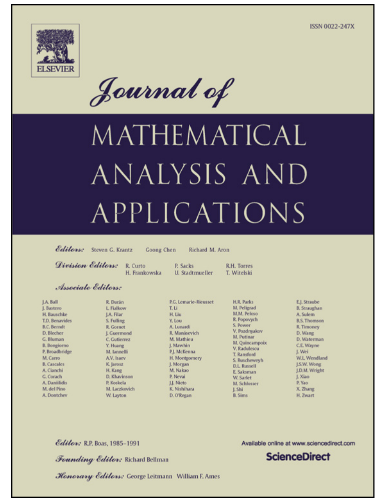

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On approximate roots of polynomials and systems of polynomials in ultrametric Banach algebras

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Abstract

We present methods that allow to estimate the distance between the approximate and exact zeros of some polynomial equations and systems of them (also infinite) in ultrametric Banach algebras. To make our results more useful, we consider that issue in a more general situation, i.e., for some functional equations of polynomial form; moreover, we do it almost everywhere (with respect to a given $\sigma-$ideal). As an auxiliary tool we prove an ultrametric version of a fixed point theorem in some function spaces (also almost everywhere). Our investigations have been motivated by several previous outcomes and the notion of Ulam stability.

Keywords: polynomial; system of polynomials; approximate root; ultrametric Banach algebra; nonlinear operator; approximate fixed point; almost everywhere; $\sigma-$ideal; Ulam stability.
2000 MSC: 39B82, 47H10.

1. Introduction and preliminaries

The roots of polynomials are very important in numerous investigations. Unfortunately, except some particular situations, we only can determine explicitly values that satisfy those equations approximately (accurately to a certain degree). Therefore it seems to be important to know how much those approximate solutions differ from the exact solutions to the equations. In this paper we study that problem in an ultrametric Banach algebra B, over a non-archimedean field $\mathbb{K}$.

Actually, we consider a more general issue, that is we investigate approximate solutions to the polynomial functional equation of the form

$$f(\mu(x)) + \sum_{j=0}^{m} a_j(x) f(\xi_j(x))^{p_j(x)} = 0, \tag{1}$$

for functions mapping a nonempty set X into B, where m is a positive integer, $\mu, \xi_0, \dots, \xi_m : X \to X$, $a_0, \dots, a_m : X \to B$ and $p_0, \dots, p_m : X \to \mathbb{N}_0$ (nonnegative integers) are given. Namely, we investigate functions $f : X \to B$ satisfying the inequality

$$\left\| f(\mu(x)) + \sum_{j=0}^{m} a_j(x) f(\xi_j(x))^{p_j(x)} \right\| \le \delta(x), \qquad x \in X, \tag{2}$$

with a given $\delta : X \to \mathbb{R}_+$ (nonnegative reals). Moreover, we study that inequality almost everywhere with respect to some $\sigma-$ideals in X, because it seems to be natural to assume that in some cases values of a function can be determined only outside some subsets of its domain (Corollary 5 supplies a very simple example of such situation).

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Note that if X has only one element x_0 and $p_0(x_0) = 0$, then, with $y := f(x_0)$, $\gamma = -a_0(x_0)$ (here we assume that, for each $b \in B$, b^0 is the neutral element in B) and $a_j := a_j(x_0)$, $p_j := p_j(x_0)$ for $j = 1, \dots, m$, the equation becomes the following classical polynomial equation in B

$$y + \sum_{j=1}^{m} a_j y^{p_j} = \gamma. \tag{3}$$

Therefore, in this way we also obtain results concerning approximate solutions to (3) in ultrametric Banach algebras.

Next, if $X = \{1, \dots, k\}$ with some $k \in \mathbb{N}$ (positive integers), $m \leq k$, $p_0(i) = 0$, $\mu(i) = i$ and $\xi_j(i) = j$ for $j = 1, \dots, m$ and $i = 1, \dots, k$, then (1) can be written as the following system of k polynomial equations (with k variables $y_1, \dots, y_k$)

$$y_i + \sum_{j=1}^{m} a_{ji} y_j^{p_{ji}} = \gamma_i, \qquad i = 1, \dots, k, \tag{4}$$

with $p_{ji} := p_j(i)$, $y_j := f(j)$, $a_{ji} := a_j(i)$, $\gamma_i := -a_0(i)$ for $j = 1, \dots, m$ and $i = 1, \dots, k$.

Note yet that, when for instance X is the set of integers or positive integers, $p_0(i) = 0$, $\mu(i) = i$ and $\xi_j(i) = j + i$ for $j = 1, \dots, m$ and $i \in X$, then (1) is the following system of infinitely many polynomial equations (with infinitely many variables y_i for $i \in X$) of the form

$$y_i + \sum_{j=1}^{m} a_{ji} y_{j+i}^{p_{ji}} = \gamma_i, \qquad i \in X, \tag{5}$$

where $p_{ji} := p_j(i)$, $a_{ji} := a_j(i)$, $y_i := f(i)$, $\gamma_i := -a_0(i)$ for $j = 1, \dots, m$ and $i \in X$.

Our results allow, in particular, to estimate the distance between the approximate and exact solutions of (3), (4) and (5) (see Corollaries 4-5, 9-12), which corresponds to the outcomes, e.g., in [8, 17, 19, 21, 23] motivated by the notion of Ulam stability (for more details and references on this notion see, e.g., [1, 3, 4, 12, 15, 16, 20]).

We apply a fixed point approach and therefore, except the answer to the above question, we also prove, as an auxiliary tool, an almost everywhere version of a fixed point theorem in some ultrametric function spaces, which can be applied in various similar studies in the ultrametric setting (cf., e.g., [1, 4, 5, 7, 6, 9]).

Now we recall some definitions and facts. Denote by $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{R}$ and $\mathbb{R}_+$ the sets of positive integers, nonnegative integers, reals and nonnegative real numbers, respectively.

An ultrametric space is a metric space (X, d) in which the metric satisfies the strong triangle inequality

$$d(x, y) \leq \max\{d(x, z), d(z, y)\}, \quad x, y, z \in X.$$

It is easily seen that a sequence (x_n) in an ultrametric space is Cauchy if and only if the sequence $d(x_{n+1}, x_n)$ converges to zero. A well-known example of an ultrametric space is $\mathbb{Q}_p$, the field of $p-$adic numbers equipped with the $p-$adic absolute value (for more information, examples, applications and properties of such spaces see for instance [22]). It has gained the interest of physicists because of connections to problems coming from quantum physics, $p-$adic strings and superstrings (cf., e.g., [18]).

The notion of ultrametric valuation in a field is defined by the properties analogous to those of the $p-$adic absolute value. Namely, we say that a function $|\cdot| : \mathbb{K} \to \mathbb{R}_+$ is an ultrametric valuation in a field $\mathbb{K}$ provided it satisfies the following three conditions:

(i) $|a| = 0$ if and only if $a = 0$;

(ii) $|ab| = |a||b|$ for every $a, b \in \mathbb{K}$;

(iii) $|a + b| \leq \max\{|a|, |b|\}$ for every $a, b \in \mathbb{K}$.

If $|\cdot| : \mathbb{K} \to \mathbb{R}_+$ is an ultrametric valuation in a field $\mathbb{K}$, then we say that the pair $(\mathbb{K}, |\cdot|)$ is an ultrametric (or non-archimedean) field.

The $p-$adic absolute value is a classical example of an ultrametric valuation. But, for any field $\mathbb{K}$, there exists the trivial ultrametric valuation, which takes value 1 for all $a \in \mathbb{K} \setminus \{0\}$. Moreover, for each ultrametric field $(\mathbb{K}, |\cdot|)$, the function $d(a,b) := |a-b|$, $a, b \in \mathbb{K}$, is an ultrametric in $\mathbb{K}$ that is invariant (i.e., $d(a+c, b+c) = d(a,b)$ for every $a, b, c \in \mathbb{K}$).

Let Y be a linear space over an ultrametric field $(\mathbb{K}, |\cdot|)$. We say that a function $\|\cdot\| : Y \to \mathbb{R}_+$ is an ultrametric norm in Y provided the following three conditions are valid:

(i) $\|x\| = 0$ if and only if $x = 0$;

(ii) $\|ay\| = |a|\, \|y\|$ for $y \in Y$, $a \in \mathbb{K}$;

(iii) $\|x+y\| \le \max\{\|x\|, \|y\|\}$ for every $x, y \in Y$.

If $\|\cdot\| : Y \to \mathbb{R}_+$ is an ultrametric norm in Y, then the pair $(Y, \|\cdot\|)$ is said to be the ultrametric normed space and the function $d(x,y) := \|x-y\|$, $x, y \in Y$, is an invariant ultrametric in Y; we say that the ultrametric d is induced by the norm. A Banach ultrametric space is an ultrametric normed space in which the ultrametric induced by the norm is complete.

If Y is a commutative algebra over an ultrametric field, endowed with an ultrametric norm $\|\cdot\| : Y \to \mathbb{R}_+$ such that

$$\|xy\| \le \|x\|\, \|y\|, \qquad x, y \in Y,$$

then we say that Y is an ultrametric commutative algebra (with unit, if there exists an identity element in Y); if additionally $(Y, \|\cdot\|)$ is a Banach ultrametric space, then we say that Y is a Banach commutative algebra.

Let X be a nonempty set. A family $\mathcal{I} \subset 2^X$ is a $\sigma-$ideal in X if it contains the empty set, and subsets and countable unions of its elements.

Remark 1 Below we provide several natural examples of $\sigma-$ideals $\mathcal{I} \subset 2^X$.

(a) The trivial example is $\mathcal{I} = \{\emptyset\}$.

(b) If $D \subset X$ is nonempty, then 2^D (the family of all subsets of D) forms a $\sigma-$ideal; analogously $2^{X\setminus D}$ is a $\sigma-$ideal.

(c) If X is of cardinality greater than $\mathbb{N}$, then the family $\{A \subset X :\ A \text{ is at most countable}\}$ is a $\sigma-$ideal; moreover, if the cardinality of X is not of countable cofinality, then so is the set $\{A \subset X :\ \operatorname{card} A < \operatorname{card} X\}$.

(d) If X is either $\mathbb{R}^n$ or $\mathbb{C}^n$, then the family of all subsets of X that are of the Lebesgue measure zero forms a $\sigma-$ideal.

(e) If X is a topological space, then the family of all first category subsets of X is a $\sigma-$ideal.

(f) If X is an abelian Polish group, then $\sigma-$ideals are the family of all Haar zero subsets of X (see [11]), the family of all Christensen zero subsets of X (see [14]) and the family of all Haar meager subsets of X (see [13]).

We say that a property $p(x)$ holds $\mathcal{I}-$almost everywhere in X ($\mathcal{I}-$ (a.e.) in X, for short) if there exists a set $A \in \mathcal{I}$ such that $p(x)$ is valid for all $x \in X \setminus A$. Clearly, if $\mathcal{I} = \{\emptyset\}$, then every property $p(x)$, that holds $\mathcal{I}-$almost everywhere in X, actually holds for every $x \in X$.

Finally, given a nonempty set Y and $\mathcal{I} \subset 2^X$, we say that $g \in Y^X$ is an $\mathcal{I}-$unique function fulfilling some properties provided g fulfills the properties and $g(x) = h(x)$ $\mathcal{I}-$(a.e.) in X for every function $h \in Y^X$ satisfying those properties.

2. The main results

In this section we assume that $\mathcal{I}$ is a $\sigma-$ideal in a nonempty set X and B is an ultrametric Banach commutative algebra with the unit element denoted by e. For $r > 0$ we write

$$\mathcal{B}_r := \{u \in B^X :\ \|u(x)\| \le r \quad \mathcal{I}-\text{(a.e.) in } X\}.$$

Moreover, $m \in \mathbb{N}$, $a_0, \ldots, a_m \in B^X$, $p_0, \ldots, p_m : X \to \mathbb{N}_0$, $\mu, \xi_0, \ldots, \xi_m \in X^X$ and μ is bijective.

We consider approximate solutions to the polynomial functional equation (1), i.e., the equation

$$f(\mu(x)) + \sum_{j=0}^{m} a_j(x) f(\xi_j(x))^{p_j(x)} = 0, \tag{6}$$

where $f \in B^X$ is the unknown function and $f(x)^j = (f(x))^j$ for all $j \in \mathbb{N}_0$, $x \in X$ ($u^0 = e$ for each $u \in B$).

In what follows, for a fixed $r > 0$, $\Lambda : \mathbb{R}_+^X \to \mathbb{R}_+^X$ is an operator given by

$$(\Lambda\eta)(x) := \max_{0 \le i \le m} \|a_i(\mu^{-1}(x))\| r^{p_i(\mu^{-1}(x))-1}\, \eta(\xi_i(\mu^{-1}(x))), \qquad \eta \in \mathbb{R}_+^X,\ x \in X. \tag{7}$$

We have the following theorem, which is in particular a counterpart of [8, Theorem 2] and [2, Theorem 3.4] for the ultrametric setting (the proof of it is provided at the end of this paper).

Theorem 2 *Assume that*

$$\xi_j^{-1}(I),\ \mu^{-1}(I),\ \mu(I) \in \mathcal{I}, \qquad I \in \mathcal{I},\ j \in \{0, 1, \ldots, m\}, \tag{8}$$

and $\delta \in \mathbb{R}_+^X$, $r > 0$, $f \in \mathcal{B}_r$ *satisfy the following three conditions*

$$\left\| f(\mu(x)) + \sum_{j=0}^{m} a_j(x) f(\xi_j(x))^{p_j(x)} \right\| \le \delta(x) \qquad \mathcal{I} - \text{(a.e.) in } X, \tag{9}$$

$$\max_{0 \le i \le m} \|a_i(x)\| r^{p_i(x)-1} \le 1 \qquad \mathcal{I} - \text{(a.e.) in } X, \tag{10}$$

$$\lim_{n \to \infty} \left(\Lambda^n \varepsilon\right)(x) = 0 \qquad \mathcal{I} - \text{(a.e.) in } X, \tag{11}$$

where $\varepsilon := \delta \circ \mu^{-1}$. *Then there is an* $\mathcal{I}-$*unique function* $g \in \mathcal{B}_r$ *such that*

$$g(\mu(x)) + \sum_{j=0}^{m} a_j(x) g(\xi_j(x))^{p_j(x)} = 0 \qquad \mathcal{I} - \text{(a.e.) in } X \tag{12}$$

and

$$\|g(x) - f(x)\| \le \sup_{n \ge 0} (\Lambda^n \varepsilon)(x) \qquad \mathcal{I} - \text{(a.e.) in } X. \tag{13}$$

Moreover, $g(x) := \lim_{n\to\infty} (\mathcal{T}^n f)(x)$ $\mathcal{I}-$*(a.e.) in* X, *for a certain operator* $\mathcal{T} : B^X \to B^X$ *satisfying for each* $u \in B^X$

$$(\mathcal{T}u)(x) = -\sum_{i=0}^{m} a_i(\mu^{-1}(x)) u(\xi_i(\mu^{-1}(x)))^{p_i(\mu^{-1}(x))} \qquad \mathcal{I} - \text{(a.e.) in } X. \tag{14}$$

Remark 3 Note that if we modify condition (10) in the following way

$$\max_{0 \le j \le m} \|a_j(x)\| r^{p_j(x)-1} \le d < 1 \qquad \mathcal{I} - \text{(a.e.) in } X, \tag{15}$$

then it implies (11) for every $\delta \in \mathbb{R}_+^X$ that is bounded (in view of (8)). So, Theorem 2 yields the following corollary.

Corollary 4 *Let* $\widehat{\delta} > 0$, $r > 0$ *and* (8) *be valid. If* $f \in \mathcal{B}_r$ *satisfies* (9) *with* $\delta(x) \equiv \widehat{\delta}$ *and* (15) *holds with some* $d > 0$, *then there is an* $\mathcal{I}$-*unique function* $g \in \mathcal{B}_r$ *satisfying* (12) *and such that*

$$\|g(x) - f(x)\| \le \widehat{\delta} \qquad \mathcal{I} - \text{(a.e.) in } X. \tag{16}$$

Before the next corollary, let us remind that, for any given $D \subset X$, the families $2^{X\setminus D}$ and 2^D are $\sigma-$ideals (see Remark 1 (b)); clearly, if in particular $D = X$, then

$$2^{X\setminus D} = 2^{\emptyset} = \{\emptyset\}.$$

Corollary 5 *Suppose that $X \in \{\mathbb{N}_0, \mathbb{Z}\}$, $D \subset X$ is nonempty, $r > 0$, $\widehat{\delta} > 0$, $d \in \mathbb{R}_+$, $p_{ji} \in \mathbb{N}_0$, $a_{ji} \in B$ for $j = 0, 1, \ldots, m$ and $i \in X$,*

$$\mu(X \setminus D) \subset X \setminus D, \qquad \xi_j(D) \subset D, \qquad j = 0, 1, \ldots, m, \tag{17}$$

$$\max_{0\le j\le m} \|a_{ji}\| r^{p_{ji}-1} \le d < 1, \qquad i \in D. \tag{18}$$

If a sequence $(z_i)_{i\in X}$ in $Y := \{b \in B : \|b\| \le r\}$ satisfies

$$\left\| z_{\mu(i)} + \sum_{j=0}^{m} a_{ji} z_{\xi_j(i)}^{p_{ji}} \right\| \le \widehat{\delta}, \qquad i \in D, \tag{19}$$

then there is a $2^{X\setminus D}$-unique sequence $(y_i)_{i\in X}$ in Y such that

$$\|z_i - y_i\| \le \widehat{\delta}, \qquad y_{\mu(i)} + \sum_{j=0}^{m} a_{ji} y_{\xi_j(i)}^{p_{ji}} = 0, \qquad i \in D. \tag{20}$$

Proof. We apply Corollary 4 with $\mathcal{I} = 2^{X\setminus D}$, $f(i) := z_i$, $a_j(i) := a_{ji}$ and $p_j(i) := p_{ji}$ for $j = 0, 1, \ldots, m$ and $i \in X$ (clearly, (17) implies (8)). $\square$

Remark 6 Corollary 5 shows that for a sequence $(z_i)_{i\in X}$ in Y we obtain a somewhat similar outcome as in Theorem 2 even if we replace the system of equations

$$z_{\mu(i)} + \sum_{j=0}^{m} a_{ji} z_{\xi_j(i)}^{p_{ji}} = 0, \qquad i \in X,$$

by a smaller system

$$z_{\mu(i)} + \sum_{j=0}^{m} a_{ji} z_{\xi_j(i)}^{p_{ji}} = 0, \qquad i \in D,$$

with any set $D \subset X$ such that (17) holds.

The next theorem shows some possible modifications that can be made in Theorem 2. It concerns a particular case of equation (6) with $p_0(x) = 0$ for $x \in X$ (then the form of ξ_0 does not matter), that is the equation

$$f(\mu(x)) + \sum_{j=1}^{m} a_j(x) f(\xi_j(x))^{p_j(x)} = \gamma(x),$$

with $\gamma(x) = -a_0(x)$ for $x \in X$. Thus we obtain another example of the situation when (11) holds. The theorem is an analogue of [8, Theorem 3] in the ultrametric case almost everywhere.

Theorem 7 *Suppose that $p_0(x) \equiv 0$, $\delta \in \mathbb{R}_+^X$, $r > 0$, (8) holds and, for every $i, j \in \{1, \ldots, m\}$, the following conditions are fulfilled $\mathcal{I}-$(a.e.) in X:*

$$\delta(\mu^{-1}(x)) \le \delta(x), \quad \delta(\xi_j(x)) \le \delta(x), \quad \|a_j(\mu^{-1}(x))\| \le \|a_j(x)\|, \quad \|a_j(\xi_i(x))\| \le \|a_j(x)\|, \tag{21}$$

$$r^{p_j(\mu^{-1}(x))} \le r^{p_j(x)}, \qquad r^{p_j(\xi_i(x))} \le r^{p_j(x)}. \tag{22}$$

If $f \in \mathcal{B}_r$ satisfies (9) and

$$\|a_0(x)\| \le r, \qquad \lambda(x) := \max_{1\le j\le m} \|a_j(x)\| r^{p_j(x)-1} < 1 \qquad \mathcal{I} - \text{(a.e.) in } X, \tag{23}$$

then there is an $\mathcal{I}-$unique function $g \in \mathcal{B}_r$ satisfying (12) and such that

$$\|g(x) - f(x)\| \le \delta(x) \qquad \mathcal{I} - \text{(a.e.) in } X. \tag{24}$$

The proof of Theorem 7 is provided in the last section.

Remark 8 The form of g in Theorem 7 can be easily deduced from the proof (see the last section). Namely, $g(x) := \lim_{n\to\infty}(\mathcal{T}^n f)(x)$ $\mathcal{I}-$(a.e.) in X, where the operator $\mathcal{T}$ satisfies (14) (cf. (49)) with $p_0(x) \equiv 0$.

If all functions a_j and p_j are constant (i.e., $a_j(x) \equiv a_j \in B$ and $p_j(x) \equiv p_j \in \mathbb{N}_0$ for $j = 0, 1 \dots, m$), then we obtain the following simplified version of Theorem 7.

Corollary 9 *Suppose that* $p_0 = 0$, $\widehat{\delta} > 0$, $r > 0$, (8) *holds and*

$$\|a_0\| \le r, \qquad \max_{1\le j\le m} \|a_j\| r^{p_j-1} < 1. \tag{25}$$

If $f \in \mathcal{B}_r$ *satisfies*

$$\left\| f(\mu(x)) + \sum_{j=0}^{m} a_j f(\xi_j(x))^{p_j} \right\| \le \widehat{\delta} \qquad \mathcal{I} - \text{(a.e.) in } X,$$

then there is an $\mathcal{I}-$*unique function* $g \in \mathcal{B}_r$ *satisfying*

$$g(\mu(x)) + \sum_{j=0}^{m} a_j g(\xi_j(x))^{p_j} = 0 \qquad \mathcal{I} - \text{(a.e.) in } X$$

and such that

$$\|g(x) - f(x)\| \le \widehat{\delta}, \qquad \mathcal{I} - \text{(a.e.) in } X. \tag{26}$$

Now, we present some further simple consequences of Theorem 7. The next corollary is to some extent a counterpart of [21, Theorem 18.1] and [8, Corollary 1].

Corollary 10 *Assume that* $\varepsilon > 0$, $r > 0$ *and* $c_0, \dots, c_m \in B$ *fulfill*

$$\|c_0\| \le r, \qquad \|c_1 - e\| < 1, \qquad \max_{2\le i\le m} \|c_i\| r^{i-1} < 1.$$

Then, for every $y \in B$ *satisfying*

$$\|y\| \le r, \qquad \|c_m y^m + \dots + c_1 y + c_0\| \le \varepsilon,$$

there exists a unique $y_0 \in B$ *such that*

$$\|y_0\| \le r, \qquad c_m y_0^m + \dots + c_1 y_0 + c_0 = 0, \qquad \|y - y_0\| \le \varepsilon.$$

Proof. We apply Theorem 7 for $X = \{x_0\}$ and $\mathcal{I} = \{\emptyset\}$, with $f(x_0) = y$, $a_1(x_0) = c_1 - e$, $a_i(x_0) = c_i$ for $i = 0$ and $i = 2, \dots, m$, and $p_i(x_0) = i$ for $i = 0, 1, \dots, m$; then, clearly, $\mu(x_0) = x_0$ and $\xi_i(x_0) = x_0$ for $i = 0, 1, \dots, m$, whence conditions (21)–(23) hold with $\delta(x_0) = \varepsilon$. □

Corollary 11 *Suppose that* $k \in \mathbb{N}$, $X = \{1, \dots, k\}$, $q_1, \dots, q_m \in \mathbb{N}_0$, $\overline{a}_j, \gamma_i \in B$ *for* $j = 1, \dots, m$ *and* $i = 1, \dots, k$, $\varepsilon > 0$, $r > 0$, *and*

$$\|\gamma_i\| \le r, \qquad \|\overline{a}_j\| < r^{1-q_j}, \qquad j = 1, \dots, m, \; i = 1, \dots, k. \tag{27}$$

If $z_1, \dots, z_k \in Y := \{b \in B : \|b\| \le r\}$ *satisfy*

$$\left\| z_{\mu(i)} + \sum_{j=1}^{m} \overline{a}_j z_{\xi_j(i)}^{q_j} - \gamma_i \right\| \le \varepsilon, \qquad i = 1, \dots, k, \tag{28}$$

then there are unique $y_1, \dots, y_k \in Y$ *such that*

$$y_{\mu(i)} + \sum_{j=1}^{m} \overline{a}_j y_{\xi_j(i)}^{q_j} = \gamma_i, \qquad i = 1, \dots, k, \tag{29}$$

$$\|z_j - y_j\| \le \varepsilon, \qquad j = 1, \dots, k. \tag{30}$$

Proof. We use Theorem 7 with $\mathcal{I} = \{\emptyset\}$, $a_0(i) = -\gamma_i$, $p_0(i) = 0$, $f(i) = z_i$, $a_j(i) = \overline{a}_j$, $p_j(i) = q_j$, and $\delta(i) = \varepsilon$ for $j = 1, \ldots, m$ and $i = 1, \ldots, k$ (then (21)–(23) hold). □

Corollary 12 *Suppose that $X \in \{\mathbb{N}_0, \mathbb{Z}\}$, $r > 0$, $\delta_i \geq 0$, $q_j \in \mathbb{N}_0$, $\widehat{a}_j, \gamma_i \in B$ for $j = 1, \ldots, m$ and $i \in X$, and*

$$\max_{1\leq k\leq m} \|\widehat{a}_k\| r^{q_k-1} < 1, \qquad \delta_{\xi_j(i)} \leq \delta_i, \qquad \|\gamma_i\| \leq r, \qquad j = 1, \ldots, m,\ i \in X. \tag{31}$$

If a sequence $(z_i)_{i\in X}$ in $Y := \{b \in B : \|b\| \leq r\}$ satisfies

$$\Big\| z_i + \sum_{j=1}^{m} \widehat{a}_j z_{\xi_j(i)}^{q_j} - \gamma_i \Big\| \leq \delta_i, \qquad i \in X, \tag{32}$$

then there is a unique sequence $(y_i)_{i\in X}$ in Y such that

$$\|z_i - y_i\| \leq \delta_i, \qquad y_i + \sum_{j=1}^{m} \widehat{a}_j y_{\xi_j(i)}^{q_j} = \gamma_i, \qquad i \in X. \tag{33}$$

Proof. Let $a_0(i) := -\gamma_i$, $p_0(i) := 0$, $\delta(i) := \delta_i$, $f(i) := z_i$, $\mu(i) := i$, $a_j(i) := \widehat{a}_j$ and $p_j(i) := q_j$ for $j = 1, \ldots, m$ and $i \in X$. Then it is easy to check that conditions (21)–(23) are valid. Hence, it is enough to apply Theorem 7 with $\mathcal{I} = \{\emptyset\}$. □

Remark 13 The values of y_0 and y_i in Corollaries 10–12 can be described analogously as in Remark 8, with a suitable operator $\mathcal{T}$.

3. Auxiliary fixed point theorem

For the proofs of Theorems 2 and 7 we need an auxiliary fixed point result in some function spaces (in the ultrametric settings). We prove it in a bit more general form than it is necessary in the proof, because it corresponds to recent results in, e.g., [4, 5, 10] and complements the main theorem in [2] (proved for the case of classical complete metric spaces).

In this part (Y, d) denotes a complete ultrametric space, X is a nonempty set and $\mathcal{I}$ stands for a $\sigma-$ideal in X, unless explicitly stated otherwise.

Let $\Lambda : \mathbb{R}_+^X \to \mathbb{R}_+^X$. We say that Λ has the property (C0) if, for each sequence $(\delta_n)_{n\in\mathbb{N}}$ of elements of $\mathbb{R}_+^X$,

$$\lim_{n\to\infty} \delta_n(x) = 0 \quad \mathcal{I} - \text{(a.e.) in } X \implies \lim_{n\to\infty} (\Lambda\delta_n)(x) = 0 \quad \mathcal{I} - \text{(a.e.) in } X. \tag{C0}$$

Next, Λ is said to be $\mathcal{I}-$nondecreasing if, for every $\delta_1, \delta_2 \in \mathbb{R}_+^X$,

$$\delta_1(x) \leq \delta_2(x) \quad \mathcal{I} - \text{(a.e.) in } X \implies (\Lambda\delta_1)(x) \leq (\Lambda\delta_2)(x) \quad \mathcal{I} - \text{(a.e.) in } X.$$

Finally, an operator $\mathcal{T} : Y^X \to Y^X$ is called $\Lambda-$contractive $\mathcal{I}-$(a.e.) if for any $u, v \in Y^X$ and $\delta \in \mathbb{R}_+^X$

$$d(u(x), v(x)) \leq \delta(x) \quad \mathcal{I} - \text{(a.e.) in } X \implies d((\mathcal{T}u)(x), (\mathcal{T}v)(x)) \leq (\Lambda\delta)(x) \quad \mathcal{I} - \text{(a.e.) in } X.$$

In the sequel, given $u, v \in Y^X$, the function $|u, v| : X \to \mathbb{R}$ is always defined by $|u, v|(x) := d(u(x), v(x))$ for $x \in X$. We have the following fixed point theorem in the ultrametric case.

Theorem 14 *Assume that $\Lambda : \mathbb{R}_+^X \to \mathbb{R}_+^X$ has the property* (C0) *and let $\mathcal{T} : Y^X \to Y^X$ be $\Lambda-$contractive $\mathcal{I}-$(a.e.). Suppose that there are functions $\varepsilon \in \mathbb{R}_+^X$ and $f \in Y^X$ such that*

$$d\big((\mathcal{T}f)(x), f(x)\big) \leq \varepsilon(x) \qquad \mathcal{I} - (a.e.) \text{ in } X \tag{34}$$

and

$$\lim_{n\to\infty}(\Lambda^n\varepsilon)(x) = 0 \qquad \mathcal{I}-(a.e.)\ in\ X. \tag{35}$$

Then the limit

$$g(x) := \lim_{n\to\infty}(\mathcal{T}^n f)(x) \tag{36}$$

exists $\mathcal{I}-$*(a.e.) in* X.

Moreover, the following two statements are valid.

(i) Any function $g \in Y^X$*, satisfying* (36) $\mathcal{I}-$*(a.e.) in* X*, fulfils the condition*

$$(\mathcal{T}g)(x) = g(x) \qquad \mathcal{I}-(a.e.)\ in\ X \tag{37}$$

and is an $\mathcal{I}-$*unique function, from* Y^X*, such that*

$$d((\mathcal{T}^n f)(x), g(x)) \le \sup_{j\ge n}(\Lambda^j\varepsilon)(x) =: \sigma_n(x) \qquad \mathcal{I}-(a.e.)\ in\ X \tag{38}$$

for every $n \in \mathbb{N}_0$.

(ii) If for σ_0*, defined in* (38)*, we have*

$$\lim_{n\to\infty}(\Lambda^n\sigma_0)(x) = 0 \qquad \mathcal{I}-(a.e.)\ in\ X, \tag{39}$$

then any function $g \in Y^X$*, satisfying* (37) *and such that*

$$d(f(x), g(x)) \le \sigma_0(x) \qquad \mathcal{I}-(a.e.)\ in\ X, \tag{40}$$

is $\mathcal{I}-$*unique.*

Proof. First we show by induction that, for every $n \in \mathbb{N}_0$, there is $A_n \in \mathcal{I}$ such that

$$d\big((\mathcal{T}^{n+1}f)(x), (\mathcal{T}^n f)(x)\big) \le (\Lambda^n\varepsilon)(x), \qquad x \in X \setminus A_n. \tag{41}$$

Clearly, by (34), the case $n = 0$ is trivial. Now fix $n \in \mathbb{N}_0$ and suppose that (41) is valid. Since $\mathcal{T}$ is $\Lambda-$contractive $\mathcal{I}-$(a.e.), according to the inductive assumption, we get

$$d\big((\mathcal{T}^{n+2}f)(x), (\mathcal{T}^{n+1}f)(x)\big) \le \Lambda(\Lambda^n\varepsilon)(x) = (\Lambda^{n+1}\varepsilon)(x)$$

for all $x \in X \setminus A_{n+1}$, with some $A_{n+1} \in \mathcal{I}$. This completes the proof of (41).

Now, using (41), for every $n, j \in \mathbb{N}_0$, $n > j$, we have

$$d\big((\mathcal{T}^n f)(x), (\mathcal{T}^j f)(x)\big) \le \max_{j\le i\le n-1}\left(\Lambda^i\varepsilon\right)(x), \qquad x \in X \setminus \left(\bigcup_{i=j}^{n-1} A_i\right). \tag{42}$$

Since the equality in (35) holds for $x \in X \setminus C$ with some $C \in \mathcal{I}$, it follows from the above estimate that, for every $x \in X \setminus (C \cup \bigcup_{i=0}^{\infty} A_i)$, $\big((\mathcal{T}^n f)(x)\big)_{n\in\mathbb{N}}$ is a Cauchy sequence and, as (Y, d) is complete, the limit $g(x)$ given by (36) exists for all $x \in X \setminus D$, where

$$D := C \cup \bigcup_{i=0}^{\infty} A_i \in \mathcal{I}.$$

Taking $n \to \infty$ in (42), we obtain that the inequality in (38) holds for all $x \in X \setminus D$.

Next, we prove (i). So, take $g \in Y^X$ satisfying (36) $\mathcal{I}-$(a.e.) in X. Then, using (C0) with $\delta_n = |\mathcal{T}^n f, g|$, according to (36), we have $\lim_{n\to\infty}(\Lambda|\mathcal{T}^n f, g|)(x) = 0$ for $x \in X \setminus E$, with some $E \in \mathcal{I}$. As $\mathcal{T}$ is $\Lambda-$contractive $\mathcal{I}-$(a.e), this means that

$$d((\mathcal{T}^{n+1}f)(x), (\mathcal{T}g)(x)) \le (\Lambda|\mathcal{T}^n f, g|)(x), \qquad n \in \mathbb{N}_0,\ x \in X \setminus F,$$

with some $F \in \mathcal{I}$, whence it follows that

$$\lim_{n\to\infty} d((\mathcal{T}^{n+1}f)(x), (\mathcal{T}g)(x)) = 0, \qquad x \in X \setminus (E \cup F).$$

Hence, in view of (36), $g(x) = (\mathcal{T}g)(x)$ for $x \in X \setminus (E \cup F \cup D)$.

Suppose that also $g_1 \in Y^X$ satisfies (38) (with g replaced by g_1). Then there is a set $G \in \mathcal{I}$ such that

$$d((\mathcal{T}^n f)(x), g(x)) \le \sup_{i\ge n}(\Lambda^i\varepsilon)(x), \qquad x \in X \setminus G, n \in \mathbb{N},$$

$$d((\mathcal{T}^n f)(x), g_1(x)) \le \sup_{i\ge n}(\Lambda^i\varepsilon)(x), \qquad x \in X \setminus G, n \in \mathbb{N}.$$

This means that

$$\begin{aligned} d(g(x), g_1(x)) &\le \max\{d((\mathcal{T}^n f)(x), g(x)), d((\mathcal{T}^n f)(x), g_1(x))\} \\ &\le \sup_{i\ge n}(\Lambda^i\varepsilon)(x), \qquad x \in X \setminus G, n \in \mathbb{N}. \end{aligned}$$

Hence, with $n \to \infty$, (in view of (35)) we obtain $d(g_1(x), g(x)) = 0$ for $x \in X \setminus (G \cup C)$. Thus $g = g_1$ $\mathcal{I}-$(a.e.) in X.

Finally, assume that (39) holds and $g_1, g_2 \in Y^X$ are such that

$$(\mathcal{T}g_i)(x) = g_i(x), \qquad x \in X \setminus I_i,\ i = 1, 2, \tag{43}$$

$$d(f(x), g_i(x)) \le \sigma_0(x), \qquad x \in X \setminus I_i,\ i = 1, 2. \tag{44}$$

with some $I_1, I_2 \in \mathcal{I}$. Since $\mathcal{T}$ is $\Lambda-$contractive $\mathcal{I}-$(a.e.), for each $n \in \mathbb{N}$ we have

$$d(g_1(x), g_2(x)) = d((\mathcal{T}^n g_1)(x), (\mathcal{T}^n g_2)(x)) \le (\Lambda^n\sigma_0)(x), \qquad x \in X \setminus J_n, \tag{45}$$

with some $J_n \in \mathcal{I}$. Hence letting $n \to \infty$, by (39), we have $g_1 = g_2$ $\mathcal{I}-$(a.e.) in X, which ends the proof. $\square$

Note that Theorem 14 can be reformulated in the following weaker, but much simpler form.

Corollary 15 *Assume that $\Lambda : \mathbb{R}_+^X \to \mathbb{R}_+^X$ has the property* (C0) *and let $\mathcal{T} : Y^X \to Y^X$ be $\Lambda-$contractive $\mathcal{I}-$(a.e.). Suppose that there are functions $\varepsilon \in \mathbb{R}_+^X$ and $f \in Y^X$ such that* (34) *and* (35) *are valid. Then there exists an $\mathcal{I}-$unique function $g \in Y^X$ satisfying* (37) *and* (38) *for every $n \in \mathbb{N}_0$. Moreover,*

$$g(x) := \lim_{n\to\infty}(\mathcal{T}^n f)(x) \qquad \mathcal{I} - \text{(a.e.) in } X. \tag{46}$$

Proof. Clearly, (36) means that there is a set $D \in \mathcal{I}$ such that, for each $x \in X \setminus D$, the limit $\lim_{n\to\infty}(\mathcal{T}^n f)(x)$ exists in X. So, it is enough to define g by $g(x) = 0$ for $x \in D$ and

$$g(x) := \lim_{n\to\infty}(\mathcal{T}^n f)(x), \qquad x \in X \setminus D.$$

$\square$

4. Proofs of Theorems 2 and 7

First we present a proof for Theorem 2.

Proof. Observe that Λ (given by (7)) can be written in the form

$$(\Lambda\eta)(x) := \max_{0\le i\le m} L_i(x)\eta(h_i(x)), \qquad \eta \in \mathbb{R}_+^X,\ x \in X, \tag{47}$$

with $L_i(x) := \|a_i(\mu^{-1}(x))\| r^{p_i(\mu^{-1}(x))-1}$ and $h_i := \xi_i \circ \mu^{-1}$ for $i = 0, 1, \ldots, m$. So, it is easily seen that Λ is $\mathcal{I}-$nondecreasing (also for $\mathcal{I} = \{\emptyset\}$).

We show that Λ satisfies (C0). So, take a sequence $(\delta_n)_{n\in\mathbb{N}}$ of elements of $\mathbb{R}_+^X$ such that

$$\lim_{n\to\infty} \delta_n(x) = 0, \qquad x \in X \setminus K,$$

for some $K \in \mathcal{I}$. Then

$$\lim_{n\to\infty} \delta_n(h_i(x)) = 0, \qquad i = 0, 1, \ldots, m, \;\; x \in X \setminus h_i^{-1}(K),$$

and, according to (8), $h_i^{-1}(K) \in \mathcal{I}$ for $i = 0, 1, \ldots, m$. Consequently,

$$\lim_{n\to\infty} (\Lambda\delta_n)(x) = \lim_{n\to\infty} \max_{0\le i\le m} L_i(x)\delta_n(h_i(x)) = 0, \qquad x \in X \setminus D,$$

where

$$D := \bigcup_{i=0}^{m} h_i^{-1}(K) \in \mathcal{I}.$$

This means that (C0) is valid.

Next, we show that (11) yields (39), where $\sigma_0(x) := \sup_{j\ge 0} (\Lambda^j\varepsilon)(x)$ for $x \in X$, with $\varepsilon := \delta\circ\mu^{-1}$. To this end note that (11) implies the condition

$$\lim_{n\to\infty} \sigma_n(x) = 0 \qquad \mathcal{I} - \text{(a.e.) in } X, \tag{48}$$

where $\sigma_n(x) := \sup_{j\ge n} (\Lambda^j\varepsilon)(x)$ for $x \in X$ and $n \in \mathbb{N}_0$. Further,

$$\begin{aligned}(\Lambda\sigma_0)(x) &= \max_{0\le i\le m} L_i(x) \sup_{j\ge 0}(\Lambda^j\varepsilon)(h_i(x)) = \sup_{j\ge 0} \max_{0\le i\le m} L_i(x)(\Lambda^j\varepsilon)(h_i(x))\\ &= \sup_{j\ge 0}(\Lambda^{j+1}\varepsilon)(x) = \sigma_1(x), \qquad x \in X,\end{aligned}$$

and by induction we obtain analogously that

$$(\Lambda^n\sigma_0)(x) = \sigma_n(x), \qquad n \in \mathbb{N}_0, \;\; x \in X.$$

Hence, in view of (48), $\lim_{n\to\infty}(\Lambda^n\sigma_0)(x) = 0 \;\; \mathcal{I}-\text{(a.e.) in } X$.

Clearly $Y := \{b \in B : \|b\| \le r\}$ is a complete ultrametric space with the ultrametric d, given by

$$d(y_1, y_2) = \|y_1 - y_2\|, \qquad y_1, y_2 \in Y.$$

Let $I \in \mathcal{I}$ be such that

$$\|f(x)\| \le r, \qquad x \in X \setminus I,$$

and the inequalities in (9) and (10), and the equality in (11) hold for $x \in X \setminus I$. Define the operator $\mathcal{T} : B^X \to B^X$ as follows

$$(\mathcal{T}u)(x) := \begin{cases} -\sum_{i=0}^{m} a_i(\mu^{-1}(x))\; u(\xi_i(\mu^{-1}(x)))^{p_i(\mu^{-1}(x))}, & x \in X \setminus \mu(I);\\ 0, & x \in \mu(I),\end{cases} \qquad u \in B^X. \tag{49}$$

Observe that, by (10) and (49),

$$\begin{aligned}\|(\mathcal{T}u)(x)\| &= \Big\| \sum_{i=0}^{m} a_i(\mu^{-1}(x))\; u(\xi_i(\mu^{-1}(x)))^{p_i(\mu^{-1}(x))} \Big\|\\ &\le \max_{0\le i\le m} \|a_i(\mu^{-1}(x))\|\; r^{p_i(\mu^{-1}(x))} \le r\end{aligned}$$

for every $u \in Y^X$ and $x \in X \setminus \mu(I)$ and $\|(\mathcal{T}u)(x)\| = 0$ for $x \in \mu(I)$, whence $\mathcal{T}(Y^X) \subset Y^X$.

Next, since $\|u(x)\| \le r$ for $u \in Y^X$ and $x \in X$, we have

$$\begin{aligned}\|u(x)^n - v(x)^n\| &\le \|u(x) - v(x)\| \Big\| \sum_{j=0}^{n-1} u(x)^j v(x)^{n-j-1} \Big\|\\ &\le \|u(x) - v(x)\| r^{n-1}, \qquad u, v \in Y^X, \; x \in X, \; n \in \mathbb{N},\end{aligned} \tag{50}$$

and consequently, using (49) and (50), we obtain

$$\begin{aligned}
\|(\mathcal{T}u)(x)-(\mathcal{T}v)(x)\| &= \Big\| \sum_{i=0}^{m} a_i(\mu^{-1}(x)) \, \big(u(\xi_i(\mu^{-1}(x)))^{p_i(\mu^{-1}(x))} - v(\xi_i(\mu^{-1}(x)))^{p_i(\mu^{-1}(x))} \big) \Big\| \\
&\le \max_{0\le i\le m} \|a_i(\mu^{-1}(x))\| \, \big\| u(\xi_i(\mu^{-1}(x))) - v(\xi_i(\mu^{-1}(x))) \big\| \, r^{p_i(\mu^{-1}(x))-1} \\
&= \max_{0\le i\le m} \|a_i(\mu^{-1}(x))\| \, r^{p_i(\mu^{-1}(x))-1} \, d(u(\xi_i(\mu^{-1}(x)), v(\xi_i(\mu^{-1}(x)))
\end{aligned}$$

for $u,v \in Y^X$ and $x \in X \setminus \mu(I)$. Thus we have proved that

$$d((\mathcal{T}u)(x),(\mathcal{T}v)(x)) \le (\Lambda|u,v|)(x), \qquad x \in X, \tag{51}$$

for every $u,v \in Y^X$. Since Λ is $\mathcal{I}$-nondecreasing, this means that $\mathcal{T}$, restricted to Y^X, is $\Lambda-$contractive $\mathcal{I}-$(a.e.).

Set

$$A := I \cup \mu(I) \in \mathcal{I}$$

and define $\widetilde{f} \in Y^X$ as follows

$$\widetilde{f}(x) := \begin{cases} f(x), & x \in X \setminus I; \\ 0, & x \in I. \end{cases}$$

Then, by (9),

$$d\big((\mathcal{T}\widetilde{f})(x), \widetilde{f}(x)\big) = \Big\| \widetilde{f}(x) - (\mathcal{T}\widetilde{f})(x) \Big\| \le \delta(\mu^{-1}(x)) = \varepsilon(x), \qquad x \in X \setminus A.$$

Moreover, in view of (11), (35) is valid. So, we can apply Theorem 14 for $\widetilde{f}$ and thereby deduce that the limit (36) with f replaced by $\widetilde{f}$ exists for $x \in X \setminus D$, with some $D \in \mathcal{I}$. Write $g(x)=0$ for $x \in D$. In view of Theorem 14, it is easily seen that so defined $g \in Y^X$ is an $\mathcal{I}-$unique function satisfying (37) and (40). Clearly, (12) and (37) are equivalent and (40) is just (13). □

Now, we prove Theorem 7.

Proof. First, note that (23) implies (10). Let Λ be given by

$$(\Lambda\eta)(x) := \max_{1\le i\le m} \|a_i(\mu^{-1}(x))\| r^{p_i(\mu^{-1}(x))-1} \, \eta(\xi_i(\mu^{-1}(x))), \qquad \eta \in \mathbb{R}_+^X, \ x \in X. \tag{52}$$

According to (21) and (22), $\lambda(\xi_j(\mu^{-1}(x))) \le \lambda(x)$ $\mathcal{I}$-(a.e.) in X for $j=1,\dots,m$. Therefore, if for some $n \in \mathbb{N}$,

$$(\Lambda^n\delta)(x) \le \lambda(x)^n \delta(x), \qquad \mathcal{I}-\text{(a.e.) in } X, \tag{53}$$

then (since Λ is $\mathcal{I}-$nondecreasing), again by (21) and (22),

$$\begin{aligned}
(\Lambda^{n+1}\delta)(x) &= \Lambda(\Lambda^n\delta)(x) \\
&\le \max_{1\le i\le m} \|a_i(\mu^{-1}(x))\| r^{p_i(\mu^{-1}(x))-1} \, \lambda(\xi_i(\mu^{-1}(x)))^n \delta(\xi_i(\mu^{-1}(x))) \\
&\le \max_{1\le i\le m} \|a_i(x)\| r^{p_i(x)-1} \, \lambda(x)^n \delta(x) = \lambda(x)^{n+1}\delta(x)
\end{aligned}$$

$\mathcal{I}-$(a.e.) in X, which yields that (53) holds for all $n \in \mathbb{N}$. Hence, in view of (23),

$$\sigma_0(x) := \sup_{j\ge 0} (\Lambda^j\delta)(x) = \delta(x) \qquad \mathcal{I}-\text{(a.e.) in } X,$$

and therefore

$$\lim_{n\to\infty} (\Lambda^n\sigma_0)(x) = \lim_{n\to\infty} (\Lambda^n\delta)(x) = 0 \qquad \mathcal{I}-\text{(a.e.) in } X.$$

Now, it is enough to define $\mathcal{T}$ by (49) (with $p_0(x) \equiv 0$) and argue analogously as in the proof of Theorem 2. Indeed, assuming that the inequalities in (23) are fulfilled for $x \in X \setminus I$, with some $I \in \mathcal{I}$, we have

$$\begin{aligned}\|(\mathcal{T}u)(x)\| &= \Big\| \sum_{i=0}^{m} a_i(\mu^{-1}(x))\, u(\xi_i(\mu^{-1}(x)))^{p_i(\mu^{-1}(x))} \Big\| \\ &\leq \max\{a_0(\mu^{-1}(x)),\ \lambda(\mu^{-1}(x)) \cdot r\} \leq r, \qquad x \in X \setminus \mu(I),\end{aligned}$$

and $\|(\mathcal{T}u)(x)\| = 0 \leq r$ for $x \in \mu(I)$; consequently $\mathcal{T}(Y^X) \subset Y^X$. Next, according to (50),

$$\begin{aligned}\|(\mathcal{T}u)(x) - (\mathcal{T}v)(x)\| &\leq \max_{1\leq i\leq m} \|a_i(\mu^{-1}(x))\| \, \big\|u(\xi_i(\mu^{-1}(x))) - v(\xi_i(\mu^{-1}(x)))\big\| \, r^{p_i(\mu^{-1}(x))-1} \\ &= (\Lambda|u,v|)(x)\end{aligned}$$

for $u, v \in Y^X$ and $x \in X$. This means that $\mathcal{T}$ restricted to Y^X is Λ-contractive $\mathcal{I}-$(a.e.).

Defining the function $\widetilde{f} \in Y^X$ as in the previous proof, we get

$$d\big((\mathcal{T}\widetilde{f})(x), \widetilde{f}(x)\big) = \Big\|\widetilde{f}(x) - (\mathcal{T}\widetilde{f})(x)\Big\| \leq \delta(\mu^{-1}(x)) \leq \delta(x) \qquad \mathcal{I} - \text{(a.e.) in } X.$$

Finally, we use Theorem 14 with $\varepsilon = \delta$ and f replaced by $\widetilde{f}$. This completes the proof. □

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