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Spreading speed and profiles of solutions to a free boundary problem with Dirichlet boundary conditions

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A B S T R A C T

We discuss a free boundary problem for a reaction–diffusion equation with Dirichlet boundary conditions on both fixed and free boundaries of a one-dimensional interval. The problem was proposed by Du and Lin (2010) to model the spreading of an invasive or new species by putting Neumann boundary condition on the fixed boundary. Asymptotic properties of spreading solutions for such problems have been investigated in detail by Du and Lou (2015) and Du, Matsuzawa and Zhou (2014). The authors (2011) studied a free boundary problem with Dirichlet boundary condition. In this paper we will derive sharp asymptotic properties of spreading solutions to the free boundary problem in the Dirichlet case under general conditions on f. It will be shown that the spreading speed is asymptotically constant and determined by a semi-wave problem and that the solution converges to a semi-wave near the spreading front as $t \to \infty$ provided that the semi-wave problem has a unique solution.

1. Introduction

We discuss the following free boundary problem for a reaction–diffusion equation:

$$\text{(FBP)} \quad \begin{cases} u_t - du_{xx} = f(u), & t > 0,\ 0 < x < h(t), \\ u(t,0) = 0,\ u(t,h(t)) = 0, & t > 0, \\ h'(t) = -\mu u_x(t,h(t)), & t > 0, \\ h(0) = h_0,\ u(0,x) = u_0(x), & 0 \le x \le h_0, \end{cases}$$

where $\mu,\ h_0,\ d$ are given positive numbers, initial data (u_0, h_0) satisfies

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$$u_0 \in C[0, h_0] \cap C^2(0, h_0), \quad u_0(0) = u_0(h_0) = 0 \ \text{ and } \ u_0(x) > 0 \ \text{ in } \ (0, h_0) \tag{1.1}$$

and nonlinear function f meets

(A.1) $\quad f \in C^1[0, \infty),\ f(0) = f(1) = 0,\ f'(1) < 0,\ f(u) < 0$ for $u > 1$.

This type of a free boundary problem was proposed by Du–Lin [1] to model the spreading of an invasive or new species by putting Neumann condition on fixed boundary $x = 0$. See also Shigesada–Kawasaki [11] and Skellam [12] for the biological background of invasion models. After the appearance of [1], a lot of researchers have discussed such free boundary problems (see [2], [3], [4], [5], [6], [7], [8], [9] and the references therein). In particular, asymptotic properties of spreading solutions have been studied in great detail by Du–Lou [2] and Du–Matsuzawa–Zhou [3] in the case where both ends of the interval are moving boundaries determined by free boundary conditions of Stefan type.

In this paper we discuss the case where the habitat of a species is a one-dimensional interval and one end of the interval is a fixed boundary, whereas the other end is a moving boundary determined by Stefan condition $h'(t) = -\mu u_x(t, h(t))$. Homogeneous Dirichlet boundary conditions are imposed at the both ends of the interval. Biologically, this situation implies that the species cannot move across the fixed boundary and, therefore, it moves toward the moving boundary in order to get a new habitat. Denoting such an interval by $[0, h(t)]$ we consider the free boundary problem of the form (FBP). Under assumptions (1.1) and (A.1) we can show the following theorem on the existence and uniqueness of a global solution for (FBP).

Theorem 1. *Let f satisfy* (A.1). *Then for every initial data (u_0, h_0) satisfying* (1.1) *there exists a unique global solution (u, h) of* (FBP) *such that*

$$0 < u(t,x) \le C_1 \quad \text{and} \quad 0 < h'(t) \le \mu C_2 \quad \text{for} \ \ t > 0,\ 0 < x < h(t), \tag{1.2}$$

where C_1 and C_2 are positive constants depending only on $\|u_0\|_{C[0,h_0]}$ and $\|u_0\|_{C^1[0,h_0]}$, respectively. Moreover, it holds that

$$u_x(t,x) < 0 \tag{1.3}$$

for all (t,x) satisfying $t > 0$ and $\max\{h_0, h(t)/2\} \le x \le h(t)$.

One can refer to Kaneko–Yamada [6, Theorem 2.7] and Kaneko–Oeda–Yamada [5, Theorem 1.1] for the proof of Theorem 1 except for (1.3). For the proof of this decreasing property of u, see Lemma A.1 in Appendix.

Since the global existence of a unique solution to (FBP) has been established, the next stage is to study its asymptotic behavior as $t \to \infty$. It should be noted here that there are a pretty number of works for (FBP) and related problems. For instance, if f is of monostable type: f satisfies

$$\text{(A.1)} \quad \text{and} \quad f(u) > 0 \quad \text{for} \ \ u \in (0,1),$$

then it is well known that the spreading–vanishing dichotomy result holds true (see [6]). Hence any solution of (FBP) satisfies either spreading:

$$\lim_{t\to\infty} h(t) = \infty \quad \text{and} \quad \lim_{t\to\infty} u(t,x) = v^*(x) \ \text{ locally uniformly in } [0,\infty), \tag{1.4}$$

where $v^*(x)$ is a unique solution of

$$\text{(SP)}\qquad \begin{cases} dv_{xx}+f(v)=0, \quad v>0 \ \text{ for } \ 0<x<\infty, \\ v(0)=0, \quad \lim\limits_{x\to\infty} v(x)=1, \end{cases}$$

or vanishing:

$$\lim_{t\to\infty} h(t)<\infty \quad \text{and} \quad \lim_{t\to\infty} \|u(t,\cdot)\|_{C(0,h(t))}=0.$$

In the spreading case the local uniform convergence means the convergence in any compact subset of $[0,\infty)$. When spreading occurs, however, it holds that $\lim_{t\to\infty} h(t)=\infty$, and $[0,h(t)]\supset I$ for large t if I is a compact set. Hence the convergence property (1.4) does not give satisfactory information on the profile of the solution $u(t,x)$ near the spreading front $x=h(t)$.

Our main interest is to get better understanding on the large-time behavior of *spreading solution* (solution satisfying (1.4) as $t\to\infty$) (u,h) of (FBP) under general conditions on f. In addition to (A.1), we assume

(A.2) $\quad F(1)>0,\ F(1)>F(u^*)$ for any $u^*\in[0,1)$ satisfying $f(u^*)=0$ if it exists,

where $F(u)=\int_0^u f(s)\,ds$. Owing to (A.2), the phase plane analysis enables us to show that (SP) has a unique solution $v^*(x)$ (see Lemma 1). Then it is easy to prove that (FBP) admits a spreading solution satisfying (1.4) if h_0 and u_0 are suitably large (cf. [6]).

In the present paper, we will mainly investigate spreading solutions of (FBP); so that the solution (u,h) satisfies (1.4). Our purpose is to derive precise information on the asymptotic profiles of $(u(t,x),h(t))$ and, in particular, sharp estimates of $u(t,x)$ near the free boundary $x=h(t)$ when t is sufficiently large.

Here we should recall important works of Du–Lou [2] and Du–Matsuzawa–Zhou [3]. They discussed asymptotic behaviors of the global solution of the following free boundary problem:

$$\begin{cases} w_t-dw_{xx}=f(w), & t>0,\ 0<x<s(t), \\ w_x(t,0)=0,\ w(t,s(t))=0, & t>0, \\ s'(t)=-\mu w_x(t,s(t)), & t>0, \\ s(0)=s_0,\ w(0,x)=w_0(x), & 0\le x\le s_0. \end{cases} \tag{1.5}$$

In the above problem the Neumann boundary condition at $x=0$ takes the place of the Dirichlet boundary condition in (FBP). For (1.5), it was shown that the asymptotic profiles of global solutions are closely related with the following semi-wave problem:

$$\text{(SWP)}\qquad \begin{cases} dq''-cq'+f(q)=0,\ q(z)>0 \quad \text{for} \ \ z>0, \\ q(0)=0,\ \mu q'(0)=c,\ \lim\limits_{z\to\infty} q(z)=1. \end{cases}$$

Du–Lou [2] have discussed (1.5) when f satisfies the typical monostable, bistable or combustion condition in addition to (A.1). These conditions on f assure the existence of a unique solution $(q(z),c)=(q^*(z),c^*)$ of (SWP) (q^* is called *semi-wave*). Let $(w(t,x),s(t))$ be any spreading solution of (1.5) with initial data (w_0,s_0). Then it is known that the solution satisfies

$$\lim_{t\to\infty} \frac{s(t)}{t}=c^* \tag{1.6}$$

and $\lim_{t\to\infty} w(t,x)=1$ locally uniformly in $[0,\infty)$. Furthermore, a sharper estimate of the solution $(w(t,x),s(t))$ has been derived by Du–Matsuzawa–Zhou [3] with use of $(q^*(z),c^*)$ as follows: for any spreading solution $(w(t,x),s(t))$ of (1.5), there exists a constant $\hat{s}\in\mathbb{R}$ such that

$$\lim_{t\to\infty}\{s(t)-c^*t-\hat{s}\}=0, \quad \lim_{t\to\infty}s'(t)=c^* \tag{1.7}$$

and

$$\lim_{t\to\infty}\sup_{0\le x\le s(t)}|w(t,x)-q^*(s(t)-x)|=0. \tag{1.8}$$

See also the work of Liu–Lou [9], where similar asymptotic results were obtained for (1.5) by putting Robin boundary condition at $x=0$ in place of the Neumann condition.

In what follows, we will study whether any spreading solution $(u(t,x),h(t))$ of (FBP) satisfies analogous asymptotic estimates such as (1.6)–(1.8) or not. Furthermore we are intensely interested in what kind of influences are given by the Dirichlet boundary condition at $x=0$ on the asymptotic behavior of the solution. We finally put the following assumption on f:

(A.3) (SWP) has a unique solution $(q(z),c)=(q^*(z),c^*)$.

Functions that fulfill (A.3) include various types of nonlinearities such as monostable, bistable and combustion types that were discussed in e.g. [2], [3], [6] and [9]. It is known that *Holling's type III* nonlinearity also satisfies (A.3) (see also Kawai–Yamada [7]).

Our first result is given by the following theorem which corresponds to (1.6) for any spreading solution of (1.5).

Theorem 2. *Assume* (A.1), (A.2) *and* (A.3). *Let* (u,h) *be any solution of* (FBP) *satisfying* (1.4). *Then*

$$\lim_{t\to\infty}\frac{h(t)}{t}=c^*,$$

where c^* *is a positive number in* (A.3).

This result implies that $h(t)=c^*t+o(t)$ as $t\to\infty$. We can also give a sharper estimate for $h(t)$ although its proof requires a lengthy argument.

Theorem 3. *Assume* (A.1), (A.2) *and* (A.3). *Let* (u,h) *be any solution of* (FBP) *satisfying* (1.4) *and let* $(q^*(z),c^*)$ *be a unique solution of* (SWP). *Then there exists a constant* $\hat{H}\in\mathbb{R}$ *such that*

$$\lim_{t\to\infty}(h(t)-c^*t)=\hat{H} \quad \text{and} \quad \lim_{t\to\infty}h'(t)=c^*.$$

Moreover it holds that

$$\lim_{t\to\infty}\sup_{h(t)/2\le x\le h(t)}|u(t,x)-q^*(h(t)-x)|=0.$$

Remark 1. Theorem 3 implies that the semi-wave (q^*,c^*) is very important in the estimate of any spreading solution in the range $h(t)/2\le x\le h(t)$ for sufficiently large t. Therefore, the Dirichlet boundary condition at $x=0$ does not give any serious influence on asymptotic behaviors of $h(t)$ and $u(t,x)$ in a large region around the spreading front $x=h(t)$. This is the same as in the case where Robin boundary condition is imposed at the fixed boundary ([9]).

We can prove, in a different way from previous works, a key lemma (Lemma 2) for Theorem 3. Our method is to employ the representation of a stationary solution to get an estimate at a moving point $x=ct$ for $c>0$.

This paper is organized as follows: in Section 2 we give some properties on the stationary problem and the semi-wave problem and prepare a comparison principle; Section 3 is devoted to the proofs of the main theorems.

2. Preliminary results

2.1. Stationary problem (SP)

The existence of a positive solution to (SP) follows from Lemma 1.

Lemma 1. *Assume that f satisfies* (A.1) *and* (A.2)*. Then* (SP) *has a unique solution v^*.*

Proof. Let $v(x;p)$ be a solution of the following initial value problem

$$\begin{cases} dv_{xx} + f(v) = 0 \quad \text{for} \quad x \geq 0, \\ v(0) = 0, \; v_x(0) = p > 0. \end{cases} \tag{2.1}$$

Then it is easy to see that $v(x;p)$ satisfies

$$\frac{d}{2} v_x(x;p)^2 + F(v(x;p)) = \frac{dp^2}{2}, \tag{2.2}$$

where $F(v)$ is defined in (A.2). In vv_x-plane, $v(x;p)$ corresponds to an orbit starting from $(v, v_x) = (0, p)$. By (A.1) and (A.2), there exists a unique number $p^* > 0$ such that $d(p^*)^2/2 = F(1)$. This fact implies the existence of a unique orbit which connects $(0, p^*)$ and $(1, 0)$ in vv_x-plane. Clearly, $v(x;p^*)$ is a unique solution to (SP). □

2.2. Semi-wave problem (SWP)

Here we will briefly explain how to solve (SWP). We follow the argument of Du and Lou [2] and consider the following problem instead of (SWP):

$$\begin{cases} dq'' - cq' + f(q) = 0, \quad q(z) > 0 \quad \text{for} \quad z > 0, \\ q(0) = 0, \quad \lim\limits_{z \to \infty} q(z) = 1 \end{cases} \tag{2.3}$$

for any given $c \geq 0$. It is effective to employ the phase plane method to approach (2.3). We introduce a new variable $p = q'$ and rewrite the first equation of (2.3) as follows:

$$\begin{cases} q' = p, \\ p' = \dfrac{1}{d}(cp - f(q)). \end{cases} \tag{2.4}$$

Note that a solution of (2.3) corresponds to a trajectory starting from a suitable point $(q, p) = (0, \omega)$ on p-axis and arriving at $(q, p) = (1, 0)$ in qp-plane. If such a trajectory $(q(z), p(z))$ satisfies $p(z) > 0$ for all $z \geq 0$, then one can express the trajectory as $p = p(q)$ for $q \in [0, 1)$. It satisfies

$$\begin{cases} \dfrac{dp}{dq} = \dfrac{1}{d}\left(c - \dfrac{f(q)}{p}\right) \quad \text{for} \quad 0 \leq q < 1, \\ p(0) = \omega, \quad p(1) = 0 \end{cases} \tag{2.5}$$

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with some $\omega > 0$. By virtue of $f'(1) < 0$, $(q, p) = (1, 0)$ is a saddle point for (2.5); therefore, there exist two trajectories around $(1, 0)$ for $q < 1$; one approaching $(1, 0)$ and the other leaving $(1, 0)$. We will take a trajectory approaching $(1, 0)$ from $q < 1$. Denote this trajectory by $p = p(q; c)$ for each $c \geq 0$. It is possible to show that $q(z; c)$ satisfies

$$\lim_{q \to 1} \frac{dp}{dq}(q; c) = \frac{c - \sqrt{c^2 - 4df'(1)}}{2d} < 0.$$

If $c = 0$, then assumption (A.2) assures that $p(q; 0)$ exists for all $q \in [0, 1)$ and $p(0; 0) > 0$. Therefore we see that, if $c > 0$ is small, then $p(q(z); c)$ exists for all $z \geq 0$ and $p(0; c) > 0$. Here we should recall the result of Du and Lou [2, Lemma 6.1], which asserts that $c \to p(q; c)$ is strictly decreasing as long as it exists. So define

$$c_0 = \sup\{\xi > 0 \mid p(q; c) > 0 \;\text{ for }\; q \in [0, 1) \;\text{ and }\; c \in [0, \xi]\}.$$

Then for each $c \in [0, c_0)$, there exists a unique trajectory connecting $(q, p) = (0, \omega)$ with $\omega > 0$ and $(q, p) = (1, 0)$. This trajectory corresponds to a solution $(q(z), p(z)) = (q_c(z), p_c(z))$ of (2.3) and satisfies

$$p(0; c) = q_c'(0) = \lim_{z \to 0} q_c'(z).$$

Therefore, if we want to look for a solution to (SWP), we only have to find c satisfying

$$\mu p(0; c) = c. \tag{2.6}$$

In (2.6) $\mu p(0; c)$ is strictly decreasing for $c \in [0, c_0]$ and the right-hand side is strictly increasing for $c \geq 0$. Thus the solution of (2.6) is unique if it exists. Our assumption (A.3) implies that (2.6) has a unique solution $c = c^*$; so that the solution q^* of (SWP) is given by

$$q^*(z) = q_{c^*}(z).$$

2.3. Comparison principle

In the study of asymptotic behaviors of global solutions to (FBP), the comparison theorem plays a very important role (for the proof see [6, Theorem 2.2]).

Theorem 4. *Assume* (A.1) *and let d, μ and T be positive numbers. Let $(\overline{u}, \overline{h})$ satisfy*

$$\begin{cases} \overline{u}_t \geq d\overline{u}_{xx} + f(\overline{u}), & (t, x) \in \overline{\Omega}, \\ \overline{u}(t, 0) \geq 0, \quad \overline{u}(t, \overline{h}(t)) = 0, & t \in (0, T], \\ \overline{h}'(t) \geq -\mu \overline{u}_x(t, \overline{h}(t)), & t \in (0, T], \end{cases} \tag{2.7}$$

where $\overline{\Omega} = \{(t, x) \in R^2 \mid 0 < x < \overline{h}(t), 0 < t \leq T\}$. Moreover, let $(\underline{u}, \underline{h})$ satisfy (2.7) with "$\geq$" and "$\overline{\Omega}$" replaced by "$\leq$" and "$\underline{\Omega} = \{(t, x) \in R^2 \mid 0 < x < \underline{h}(t), 0 < t \leq T\}$", respectively. If $\underline{h}(0) \leq \overline{h}(0)$ and $\underline{u}(0, x) \leq \overline{u}(0, x)$ in $[0, \underline{h}(0)]$, then it holds that

$$\underline{h}(t) \leq \overline{h}(t) \;\; \text{for} \;\; t \in [0, T], \quad \underline{u}(t, x) \leq \overline{u}(t, x) \;\; \text{for} \;\; (t, x) \in \underline{\Omega}.$$

Remark 2. When $(\overline{u}, \overline{h})$ satisfies (2.7), $\overline{h}(0) \geq h_0$ and $\overline{u}(0, x) \geq u_0(x)$ in $[0, h_0]$, such a pair $(\overline{u}, \overline{h})$ is called an upper solution of (FBP) for $0 \leq t \leq T$. A lower solution is defined when $(\underline{u}, \underline{h})$ in Theorem 4 meets $\underline{h}(0) \leq h_0$ and $\underline{u}(0, x) \leq u_0(x)$ in $[0, \underline{h}(0)]$.

3. Proofs of main theorems

3.1. Proof of Theorem 2

Let (u_0, h_0) satisfy (1.1). We take a function $\hat{u}_0(x)$ satisfying $\hat{u}_0 \in C[0, h_0] \cap C^2(0, h_0)$,

$$\hat{u}_0'(0) = \hat{u}_0(h_0) = 0 \quad \text{and} \quad \hat{u}_0(x) \geq u_0(x),\ \hat{u}_0(x) > 0 \quad \text{in} \quad [0, h_0).$$

Let (w, s) be the solution of (1.5) with initial data (w_0, s_0) replaced by $(\hat{u}_0, h_0)$. By the strong maximum principle $w(t, x) > 0$ for $t > 0$ and $0 \leq x < s(t)$; so that $w(t, 0) > 0 = u(t, 0)$ for $t > 0$. Theorem 4 implies

$$s(t) > h(t) \quad \text{and} \quad w(t, x) > u(t, x) \quad \text{for} \quad t > 0, \quad 0 < x < h(t). \tag{3.1}$$

Recall that (u, h) is a spreading solution of (FBP). Hence (3.1) assures that (w, s) is also a spreading solution of (1.5), which satisfies

$$\lim_{t\to\infty} s(t) = \infty \quad \text{and} \quad \lim_{t\to\infty} w(t, x) = 1 \quad \text{locally uniformly in} \quad [0, \infty).$$

When f satisfies (A.1), (A.2) and (A.3), the argument used in the proof of [2, Theorem 1.10] is valid to derive (1.6). Thus we see

$$\limsup_{t\to\infty} \frac{h(t)}{t} \leq \lim_{t\to\infty} \frac{s(t)}{t} = c^*, \tag{3.2}$$

where c^* is the positive constant in (A.3).

In what follows, we will show

$$\liminf_{t\to\infty} \frac{h(t)}{t} \geq c^*. \tag{3.3}$$

Take any $c \in (0, c^*)$ such that c is sufficiently close to c^*. We now consider the following problem to construct a lower solution to (FBP):

$$\begin{cases} dq'' - cq' + f(q) = 0, \\ q(0) = 0, q'(0) = c^*/\mu. \end{cases} \tag{3.4}$$

Let $q = q_c(z)$ be the solution of (3.4). In qp-plane of §2.2, this solution corresponds to a trajectory of (2.4) starting from $(q, p) = (0, c^*/\mu)$ on p-axis. By the phase plane analysis, we can show that this trajectory hits q-axis at $q = Q_c \in (0, 1)$ because c is smaller than c^* and sufficiently close to c^*. This fact implies that there exists a unique number $z_c > 0$ such that

$$q_c'(z) > 0 \quad \text{for} \quad z \in [0, z_c), \quad q_c'(z_c) = 0 \quad \text{and} \quad q_c(z_c) = Q_c.$$

We may consider that Q_c is very close to 1 by making c sufficiently close to c^*. We now use the notation in §2.1. In vv_x-plane, we choose a trajectory $(v(x; p), v_x(x; p))$ corresponding to (2.1) such that the trajectory passes through $(v, v_x) = (Q_c, 0)$. Since $v(x; p)$ must satisfy (2.2), we get

$$p = p_c := \sqrt{\frac{2F(Q_c)}{d}}.$$

Let $v = v_c(x)$ be the solution of (2.1) with $v_x(0) = p_c$. Then there exists a unique number $x_c > 0$ such that

$$v_c'(x) > 0 \quad \text{for} \quad x \in (0, x_c), \quad v_c'(x_c) = 0 \quad \text{and} \quad v_c(x_c) = Q_c.$$

We are ready to construct a lower solution to (FBP). Define $\underline{u}(t,x)$ and $\underline{h}(t)$ by

$$\underline{h}(t) = ct + x_c + z_c, \quad \underline{u}(t,x) = \begin{cases} q_c(\underline{h}(t) - x), & \underline{h}(t) - z_c \le x \le \underline{h}(t), \\ Q_c, & x_c < x < \underline{h}(t) - z_c, \\ v_c(x), & 0 \le x \le x_c. \end{cases}$$

We will show that $(\underline{u}(t,x), \underline{h}(t))$ is a lower solution to (FBP). For $x \in (0, x_c) \cup (\underline{h}(t) - z_c, \underline{h}(t))$, $\underline{u}(t,x)$ satisfies

$$\underline{u}_t - d\underline{u}_{xx} = f(\underline{u}).$$

It holds for $x \in (x_c, \underline{h}(t) - z_c)$ that

$$\underline{u}_t - d\underline{u}_{xx} = 0 \le f(\underline{u}) = f(Q_c)$$

because, when $c(< c^*)$ is close to c^*, Q_c is close to 1 and $f(Q_c) > 0$ by (A.1). Moreover, it is easy to show that

$$\underline{h}'(t) = c \le c^* = -\mu \underline{u}_x(t, \underline{h}(t)) = \mu q_c'(0) \quad \text{for } t > 0,$$
$$\underline{u}(t,0) = 0, \ \underline{u}(t, \underline{h}(t)) = 0 \quad \text{for } t > 0.$$

Here observe that the solution v^* of (SP) satisfies

$$v^*(x) > v_c(x) \quad \text{for} \quad x \in (0, x_c)$$

by the phase plane analysis. Therefore, since $\lim_{t\to\infty} u(t,x) = v^*(x)$ locally uniformly in $[0,\infty)$, there exists a constant $T_0 = T_0(c)$ such that

$$h(T_0) \ge \underline{h}(0) = x_c + z_c, \quad u(T_0, x) \ge \underline{u}(0,x) \quad \text{for } 0 \le x \le \underline{h}(0).$$

Then Theorem 4 assures

$$\underline{h}(t) \le h(t + T_0) \quad \text{for } t > 0, \quad \underline{u}(t,x) \le u(t + T_0, x) \quad \text{for } t > 0 \ \text{ and } \ 0 < x < \underline{h}(t). \tag{3.5}$$

Hence

$$\lim_{t\to\infty} \frac{h(t)}{t} \ge \lim_{t\to\infty} \frac{\underline{h}(t - T_0)}{t} = \lim_{t\to\infty} \frac{c(t - T_0) + x_c + z_c}{t} = c.$$

Since c is an arbitrary number such that $c^* - c > 0$ is sufficiently small, we get (3.3). Thus the proof is complete. □

Remark 3. In Liu–Lou [8], the same result as Theorem 2 was given without proof for (FBP) with the Dirichlet condition replaced by Robin condition when f is the typical monostable or combustion type.

3.2. Proof of Theorem 3

Since $f'(1) < 0$, there exist $\rho > 0$ and $\eta > 0$ such that

$$f(u) \ge \rho(1 - u) \quad \text{for} \quad 1 - \eta < u < 1. \tag{3.6}$$

We will prove Theorem 3 by using a series of lemmas.

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Lemma 2. *Let* (u, h) *be any solution of* (FBP) *satisfying* (1.4). *Let* c *be any positive number such that* $c^* - c > 0$ *is sufficiently small. Then there exist positive numbers* δ, M *and* T *depending on* c *such that*

$$u(t, ct) \geq 1 - Me^{-\delta t} \quad \text{for} \quad t \geq T. \tag{3.7}$$

Proof. Define $\underline{u}$ and Q_c as in the proof of Theorem 2. For any $c \in (0, c^*)$ such that c is sufficiently close to c^*, it follows from (3.5) that

$$u(t + T_0, x) \geq \underline{u}(t, x) = Q_c \quad \text{for} \quad t \geq 0, \quad x_c \leq x \leq x_c + ct. \tag{3.8}$$

We make c sufficiently close to c^* (if necessary) so that $Q_c \geq 1 - \eta$. Take any $0 < \alpha < 1$ and choose a sufficiently large T_1 such that $\alpha c T_1 > x_c$.

For any fixed $T \geq T_1$ consider the following initial boundary value problem:

$$\begin{cases} \Psi_t = d\Psi_{xx} + \rho(1 - \Psi), & t > 0, \ \alpha cT < x < cT, \\ \Psi(t, \alpha cT) = \Psi(t, cT) = Q_c, & t > 0, \\ \Psi(0, x) = Q_c, & \alpha cT \leq x \leq cT. \end{cases} \tag{3.9}$$

Here it should be noted from (3.8) that

$$u(t + T + T_0, x) \geq \underline{u}(t + T, x) = Q_c \quad \text{for} \ \ t \geq 0, \ \alpha cT \leq x \leq cT.$$

Hence

$$u(t + T + T_0, x) \geq Q_c = \Psi(t, x) \quad \text{for} \ \ (t, x) \in \Gamma_{\alpha, c, T},$$

where

$$\Gamma_{\alpha,c,T} = \{(0, x) \mid \alpha cT \leq x \leq cT\} \cup \{(t, \alpha cT) \mid t \geq 0\} \cup \{(t, cT) \mid t \geq 0\}.$$

Since $Q_c < 1$, the standard comparison theorem for (3.9) (cf. Protter–Weinberger [10], Smoller [13]) gives

$$1 - \eta \leq Q_c \leq \Psi(t, x) \leq 1 \quad \text{for} \ \ t \geq 0, \ \alpha cT \leq x \leq cT.$$

Moreover it follows from (3.6) that

$$f(\Psi(t, x)) \geq \rho(1 - \Psi(t, x)) \quad \text{for} \ \ t \geq 0, \ \alpha cT \leq x \leq cT.$$

Therefore, applying the comparison theorem to (3.9) again, one can see

$$u(t + T + T_0, x) \geq \Psi(t, x) \quad \text{for} \ \ t \geq 0, \ \alpha cT \leq x \leq cT. \tag{3.10}$$

As the second step we will give an estimate for Ψ from below. For this purpose, it should be noted that Ψ satisfies

$$\lim_{t \to \infty} \Psi(t, x) = \zeta(x) \quad \text{uniformly for} \ \ x \in [\alpha cT, cT],$$

where ζ is a solution of

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$$\begin{cases} d\zeta_{xx} + \rho(1-\zeta) = 0, & \alpha cT < x < cT, \\ \zeta(\alpha cT) = \zeta(cT) = Q_c. \end{cases}$$

Here ζ is expressed as

$$\zeta(x) = 1 - \frac{1-Q_c}{1+A^{1-\alpha}}\{A^{-\alpha}e^{\sqrt{\rho/d}x} + Ae^{-\sqrt{\rho/d}x}\} \in (Q_c, 1) \tag{3.11}$$

with $A = e^{\sqrt{\rho/d}cT}$. If we define $w(t,x) = \zeta(x) - \Psi(t,x)$, then w meets

$$\begin{cases} w_t = dw_{xx} - \rho w, & t > 0,\ \alpha cT < x < cT, \\ w(t,\alpha cT) = w(t,cT) = 0, & t > 0, \\ w(0,x) = \zeta(x) - Q_c > 0, & \alpha cT \le x \le cT. \end{cases}$$

It is easy to see

$$0 < w(t,x) = \zeta(x) - \Psi(t,x) \le (1-Q_c)e^{-\rho t} \quad \text{for}\ \ t \ge 0,\ \alpha cT < x < cT.$$

Hence it follows from (3.10) that

$$u(t+T+T_0, x) \ge \Psi(t,x) \ge \zeta(x) - (1-Q_c)e^{-\rho t} \quad \text{for}\ \ t > 0,\ \alpha cT \le x \le cT. \tag{3.12}$$

Setting $x = (1+\alpha)cT/2$ in (3.11), we deduce from $A^{-\alpha}e^{\sqrt{\rho/d}x} + Ae^{-\sqrt{\rho/d}x} = 2A^{(1-\alpha)/2}$ that

$$\begin{aligned} \zeta\Big(\frac{1+\alpha}{2}cT\Big) &= 1 - \frac{2(1-Q_c)}{1+A^{1-\alpha}}A^{\frac{1-\alpha}{2}} \\ &\ge 1 - 2(1-Q_c)e^{-(1-\alpha)\sqrt{\rho/d}cT/2}. \end{aligned}$$

This estimate together with (3.12) implies

$$\begin{aligned} u\Big(t+T, \frac{1+\alpha}{2}cT\Big) &\ge \zeta\Big(\frac{1+\alpha}{2}cT\Big) - (1-Q_c)e^{-\rho(t-T_0)} \\ &\ge 1 - 2(1-Q_c)e^{-(1-\alpha)\sqrt{\rho/d}cT/2} - (1-Q_c)e^{-\rho(t-T_0)} \end{aligned}$$

for $t \ge T_0$. We set $t = \varepsilon T$ for any $\varepsilon > 0$ in the above estimate (take T large enough to satisfy $\varepsilon T \ge T_0$); so that

$$\begin{aligned} u\left((1+\varepsilon)T, \frac{1+\alpha}{2}cT\right) &\ge 1 - (1-Q_c)\left\{2e^{-(1-\alpha)\sqrt{\rho/d}cT/2} + e^{\rho T_0}e^{-\rho\varepsilon T}\right\} \\ &\ge 1 - \frac{M}{2}(e^{-(1-\alpha)\sqrt{\rho/d}cT/2} + e^{-\rho\varepsilon T}) \end{aligned} \tag{3.13}$$

with $M = 2(1-Q_c)\max\{2, e^{\rho T_0}\}$. We again set $t = (1+\varepsilon)T$ in (3.13) with T satisfying $T \ge T_1$ and $\varepsilon T \ge T_0$; then we get

$$u\left(t, \frac{(1+\alpha)c}{2(1+\varepsilon)}t\right) \ge 1 - \frac{M}{2}(e^{-(1-\alpha)\sqrt{\rho/d}ct/\{2(1+\varepsilon)\}} + e^{-\rho\varepsilon t/(1+\varepsilon)}) \tag{3.14}$$

for $t \ge \max\{(1+\varepsilon)T_1, \{(1+\varepsilon)/\varepsilon\}T_0\}$.

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Let $c \in (0, c^*)$ be any number such that c is sufficiently close to c^*. Then one can choose $\alpha \in (0,1),\ \varepsilon > 0$ and $\tilde{c} \in (c, c^*)$ satisfying

$$c = \frac{(1+\alpha)\tilde{c}}{2(1+\varepsilon)}$$

(Indeed both ε and $1-\alpha$ must be sufficiently small to achieve $\tilde{c} < c^*$.) Then (3.14) is valid with c replaced by $\tilde{c}$. Therefore, putting

$$\delta = \min\left\{\frac{(1-\alpha)\tilde{c}}{2(1+\varepsilon)}\sqrt{\frac{\rho}{d}}\ ,\ \frac{\rho\varepsilon}{1+\varepsilon}\right\}$$

we deduce

$$u(t, ct) \geq 1 - Me^{-\delta t} \quad \text{for} \quad t \geq \max\left\{(1+\varepsilon)T_1, \frac{1+\varepsilon}{\varepsilon}T_0\right\}.$$

Thus the proof is complete. □

Remark 4. In Lemma 2 it can be seen from its proof that one can choose δ satisfying $\delta \in (0, \rho)$ such that $u(t, ct) \geq 1 - Me^{-\delta t}$ for large t, where ρ is a positive constant in (3.6).

We can also extend Lemma 2 to the following lemma.

Lemma 3. *Let (u, h) be any solution of* (FBP) *satisfying* (1.4). *Let $c \in (0, c^*)$ be any number such that c is sufficiently close to c^*. Then there exists a number $T > 0$ such that*

$$u(t,x) \geq 1 - Me^{-\delta t} \qquad \textit{for} \quad t \geq T \quad \textit{and} \quad \frac{h(t)}{2} \leq x \leq ct,$$

where M and δ are the positive constants given in Lemma 2.

Proof. Since c is sufficiently close to c^* (e.g. $c > 2c^*/3$) and $\lim_{t\to\infty} h(t)/t = c^*$ by Theorem 2, we see $h_0 \leq h(t)/2 \leq ct$ for $t > T$ with some large $T > 0$; and $\{(t,x) \in \mathbb{R}^2 \mid t \geq T,\ h(t)/2 \leq x \leq ct\}$ is not an empty set. By Theorem 1 (or Lemma A.1 in Appendix) we find $u_x(t,x) < 0$ for $t > 0,\ \max\{h_0, h(t)/2\} \leq x \leq h(t)$. Hence

$$u(t,x) \geq u(t,ct) \quad \text{for} \quad t > T,\ h(t)/2 \leq x \leq ct.$$

Note here that $u(t,ct) \geq 1 - Me^{-\delta t}$ for $t \geq T$ by Lemma 2 (we choose T large if necessary). Thus we get the conclusion. □

We will establish the uniform boundedness of $|h(t) - c^*t|$ to study the asymptotic behavior of $h(t) - c^*t$ as $t \to \infty$.

Lemma 4. *Let (u, h) be any solution of* (FBP) *satisfying* (1.4). *Then there exist two numbers H_1 and H_2 in $\mathbb{R}$ such that*

$$H_1 \leq h(t) - c^*t \leq H_2$$

for all $t > 0$.

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Proof. Let (w, s) be the solution of (1.5) with initial data $(w(0, \cdot), s(0)) = (\hat{u_0}, h_0)$ as in the proof of Theorem 2. By the result of Du–Matsuzawa–Zhou [3, Proposition 3.1] to the Neumann problem, there exists a positive constant $\hat{H}$ such that

$$|s(t) - c^* t| \le \hat{H} \qquad \text{for all} \ \ t \ge 0.$$

Therefore, making use of (3.1), we see

$$h(t) - c^* t < s(t) - c^* t < \hat{H} \qquad \text{for all} \ \ t \ge 0. \tag{3.15}$$

In what follows, we will show a lower bound of $h(t) - c^* t$. Take $c \in (0, c^*)$ such that c is sufficiently close to c^*. Then it follows from Lemma 2 that (3.7) holds true for some positive constants δ, M and T. We next choose $T^* \ge T$ such that $Me^{-\delta T^*} \le \eta/2$, where η is a positive number appearing in (3.6). Let $(q^*(z), c^*)$ be the solution of (SWP). Since $q^*(z)$ is a strictly increasing function in z satisfying $\lim_{z\to\infty} q^*(z) = 1$, there exists a positive number z^* satisfying $q^*(z^*) = 1/2$. Recalling $u(T^*, cT^*) \ge 1 - Me^{-\delta T^*}$ by (3.7), we can choose $x^* \in (0, z^*)$ such that

$$u(T^*, cT^* + x) \ge \frac{1 - Me^{-\delta T^*}}{2} \qquad \text{for all} \ \ x \in [0, x^*].$$

After these preparations we define

$$\begin{cases} \underline{h}(t) = c^*(t - T^*) + \sigma M(e^{-\delta t} - e^{-\delta T^*}) + cT^* + x^*, & t \ge T^*, \\ \underline{u}(t, x) = (1 - Me^{-\delta t}) q^*(\underline{h}(t) - x), & t \ge T^*, \ ct \le x \le \underline{h}(t), \end{cases} \tag{3.16}$$

where $\sigma > 0$ is a parameter to be determined later. Note $ct < \underline{h}(t)$ for $t \ge T^*$ when T^* is sufficiently large. Since u satisfies (3.7) and $0 < q^*(z) < 1$ for $z > 0$, it is easy to see

$$\underline{u}(t, ct) = (1 - Me^{-\delta t}) q^*(\underline{h}(t) - ct) \le 1 - Me^{-\delta t} \le u(t, ct)$$

and $\underline{u}(t, \underline{h}(t)) = 0$ for $t \ge T^*$. We also see that

$$\begin{aligned} \underline{h}'(t) - (-\mu \underline{u}_x(t, \underline{h}(t))) &= (c^* - \sigma\delta Me^{-\delta t}) - \mu(1 - Me^{-\delta t})(q^*)'(0) \\ &= (c^* - \sigma\delta) Me^{-\delta t} \\ &\le 0 \end{aligned}$$

if σ satisfies $\sigma\delta \ge c^*$. Moreover, it is possible to follow the arguments in the work of Du–Matsuzawa–Zhou [3, Lemma 3.3] to show

$$\underline{u}_t - d\underline{u}_{xx} - f(\underline{u}) \le 0 \qquad \text{for} \ \ t \ge T^*, \ \frac{h(t)}{2} \le x \le \underline{h}(t).$$

Since $\lim_{t\to\infty} h(t)/t = c^*$, one can deduce

$$\underline{h}(T^*) = cT^* + x^* \le h(T^*)$$

by making T^* large (if necessary). Finally, if $cT^* \le x \le cT^* + x^*$, then

$$\underline{u}(T^*, x) = (1 - Me^{-\delta T^*}) q^*(\underline{h}(T^*) - x) \le (1 - Me^{-\delta T^*}) q^*(x^*) < \frac{1 - Me^{-\delta T^*}}{2} \le u(T^*, x)$$

(use $q^*(x^*) < q^*(z^*) = 1/2$).

We are ready to apply the comparison principle between (u, h) and $(\underline{u}, \underline{h})$. Hence it is possible to prove

$$\underline{h}(t) \leq h(t), \quad \underline{u}(t,x) \leq u(t,x) \quad \text{for} \quad t \geq T^*, \ \frac{h(t)}{2} \leq x \leq \underline{h}(t) \tag{3.17}$$

in the same way as Theorem 2. Since $\underline{h}(t) - c^*t$ is uniformly bounded for $t \geq T^*$, (3.17) implies that $h(t) - c^*t$ is uniformly bounded from below for all $t \geq 0$. □

Lemma 5. *Let (u, h) be any solution of* (FBP) *satisfying* (1.4). *Then there exists a constant $H^* \in \mathbb{R}$ such that*

$$\lim_{t\to\infty} (h(t) - c^*t) = H^* \quad \text{and} \quad \lim_{t\to\infty} h'(t) = c^*.$$

Proof. Let $H(t) := h(t) - c^*t$. We first show, for any $k \in \mathbb{R}$, $H(t) - k$ changes its sign at most finitely many times. Its proof can be carried out essentially in the same way as the work of Liu–Lou [9, Lemma 6.2] (see also [4, Proposition 3.6]) by using the zero number argument. Since $H(t)$ is globally bounded by Lemma 4, there exist a sequence $\{t_n\}$ and a constant H^* in $\mathbb{R}$ such that $\lim_{n\to\infty} H(t_n) = H^*$. Let $\tilde{H} \in \mathbb{R}$ be any accumulating point of $H(t)$ as $t \to \infty$, that is, $\lim_{n\to\infty} H(t'_n) = \tilde{H}$ for some sequence $\{t'_n\} \subset \mathbb{R}$. If $H^* \neq \tilde{H}$, then $H(t) - k$ for $\min\{H^*, \tilde{H}\} < k < \max\{H^*, \tilde{H}\}$ changes its sign infinitely many times because $\lim_{n\to\infty} H(t_n) = H^*$ and $\lim_{n\to\infty} H(t'_n) = \tilde{H}$. This is a contradiction. Therefore $\tilde{H} = H^*$; and $\lim_{t\to\infty} H(t) = \lim_{t\to\infty}(h(t) - c^*t) = H^*$.

We next prove $\lim_{t\to\infty} h'(t) = c^*$. Otherwise it holds for a sequence $\{t_n\}$ with $\lim_{n\to\infty} t_n = \infty$ such that $\liminf_{n\to\infty} h'(t_n) \neq c^*$. Let $H_n(t) = H(t + t_n)$. Then it is easy to see in the same way as [4, Lemma 3.11] that, up to a subsequence $\{\tilde{t}_n\} \subset \{t_n\}$ with $\lim_{n\to\infty} \tilde{t}_n = \infty$,

$$\lim_{n\to\infty} H(\tilde{t}_n + \cdot) = H^* \quad \text{in} \quad C^{1+\alpha}_{loc}(\mathbb{R})$$

for any $\alpha \in (0, 1/2)$. Hence $\lim_{n\to\infty} H'(\tilde{t}_n) = 0$, and $\lim_{n\to\infty} h'(\tilde{t}_n) = c^*$. This is a contradiction, and thus we obtain the conclusion. □

Proof of Theorem 3. Since we have shown Lemma 5, we only have to derive a certain kind of sharp estimate for $u(t, x)$. It follows from (3.17) that

$$(1 - Me^{-\delta t})q^*(\underline{h}(t) - x) \leq u(t,x) \quad \text{for} \quad t \geq T^*, \ \frac{h(t)}{2} \leq x \leq \underline{h}(t), \tag{3.18}$$

where $\underline{h}(t)$ is defined in Lemma 4. On the other hand, making use of (3.1) and [3, Lemma 3.2], we can deduce

$$u(t,x) \leq (1 + M'e^{-\delta' t})q^*(\overline{h}(t) - x) \quad \text{for} \quad t \geq T^*, \ \frac{h(t)}{2} \leq x \leq h(t) \tag{3.19}$$

with some positive constants M' and δ' (we again take T^* large if necessary). In (3.19)

$$\overline{h}(t) = c^*(t - T^*) - M'\sigma'(e^{-\delta' t} - e^{-\delta' T^*}) + C^*$$

with some suitable positive numbers σ' and C^*. Since u satisfies (3.18) and (3.19), it is sufficient to repeat the argument developed in the proof of [3, Theorem 1.2] (see also [4]) in order to show

$$\lim_{t\to\infty} \sup_{h(t)/2 \leq x \leq h(t)} |u(t,x) - q^*(h(t) - x)| = 0.$$

The proof is complete. □

Remark 5. Let (u, h) be any solution to (FBP) satisfying (1.4). Under the same assumptions as Theorem 3, it may hold that

$$\lim_{t\to\infty} \sup_{0\le x\le h(t)} |u(t,x) - \min\{v^*(x), q^*(h(t)-x)\}| = 0,$$

where $q^*(z)$ is the function in (A.3). We will discuss this matter elsewhere.

4. Concluding remarks

We can apply methods and ideas in the preceding sections to more general nonlinearity f. In [7], free boundary problem (1.5) has been discussed when f satisfies the following conditions:

$$\text{(A.4)} \quad \begin{cases} f \ \text{ is of class } \ C^1[0,\infty) \ \text{ and satisfies } \ f(0) = f(u_1^*) = f(u_2^*) = f(u_3^*) = 0, \\ f'(0) > 0, f'(u_1^*) < 0, f'(u_2^*) > 0, f'(u_3^*) < 0, \displaystyle\int_{u_1^*}^{u_3^*} f(u)du > 0 \\ \text{and } \ f(u) \neq 0 \ \text{ for } \ u \neq 0, u_1^*, u_2^*, u_3^*. \end{cases}$$

Under (A.4) it was shown that (1.5) has two types of spreading solutions $(w(t,x), s(t))$:

(1) small spreading; $\lim_{t\to\infty} s(t) = \infty$ and $\lim_{t\to\infty} w(t,x) = u_1^*$ uniformly in any compact set of $[0,\infty)$,
(2) big spreading; $\lim_{t\to\infty} s(t) = \infty$ and $\lim_{t\to\infty} w(t,x) = u_3^*$ uniformly in any compact set of $[0,\infty)$.

When we consider (FBP) under (A.4), we can prove that (FBP) also admits two types of spreading solutions such as

(i) small spreading; $\lim_{t\to\infty} h(t) = \infty$ and $\lim_{t\to\infty} u(t,x) = v_1^*(x)$ uniformly in any compact set of $[0,\infty)$,
(ii) big spreading; $\lim_{t\to\infty} h(t) = \infty$ and $\lim_{t\to\infty} u(t,x) = v_3^*(x)$ uniformly in any compact set of $[0,\infty)$.

Here v_i^* $(i = 1, 3)$ is a solution of the following stationary problem:

$$\begin{cases} dv_{xx} + f(v) = 0, \quad v(x) > 0 \quad \text{for } \ x > 0, \\ v(0) = 0, \quad \lim_{x\to\infty} v(x) = u_i^* \quad (i = 1, 3). \end{cases} \tag{4.1}$$

The existence and uniqueness of the solution v_i^* satisfying (4.1) can be shown with use of the phase-plane analysis under (A.4). We can give some sufficient conditions for the occurrence of such small spreading and big spreading solutions. This result will be discussed elsewhere.

The arguments developed in the preceding sections are valid with obvious modifications for the study of asymptotic profiles of spreading solutions under (A.4). The corresponding semi-wave problem is given by

$$\begin{cases} dq'' - cq' + f(q) = 0, \quad q(z) > 0 \quad \text{for } \ z > 0, \\ q(0) = 0, \ \mu q'(0) = c, \quad \lim_{z\to\infty} q(z) = u_i^*, \quad (i = 1, 3). \end{cases} \tag{4.2}$$

For (4.2) with $i = 1$, it was proved in [7] that there exists a unique solution $(q(z), c) = (q_S(z), c_S)$. Therefore, we can show the following estimate similarly to Theorem 3.

Theorem 5. *Under* (A.4)*, let* (u, h) *be any small spreading solution of* (FBP) *satisfying* $\sup_{0\le x\le h(T)} u(T,x) < u_2^*$ *with some* $T > 0$*. Then there exists a constant* $H_S \in \mathbb{R}$ *such that*

$$\lim_{t\to\infty}(h(t) - c_S t) = H_S \quad \textit{and} \quad \lim_{t\to\infty} h'(t) = c_S.$$

Moreover, it holds that

$$\lim_{t\to\infty} \sup_{h(t)/2\le x\le h(t)} |u(t,x) - q_S(h(t) - x)| = 0.$$

The solvability of (4.2) with $i = 3$ is delicate and there are two cases:

(Case A) There exists a unique solution $(q(z), c) = (q_B(z), c_B)$ of (4.2) with $i = 3$ for every $\mu > 0$.
(Case B) There exists a positive constant μ^* such that (4.2) with $i = 3$ admits a unique solution $(q(z), c) = (q_B(z), c_B)$ for every $\mu \in (0, \mu^*)$ and no solution for every $\mu \in [\mu^*, \infty)$.

For the proof of this fact we can refer to [7, Theorem 4.1]. If (4.2) with $i = 3$ has a unique solution, then it is possible to prove the following theorem similarly to Theorem 5.

Theorem 6. *Under* (A.4)*, let* (u, h) *be any big spreading solution of* (FBP) *and assume that* (4.2) *with* $i = 3$ *has a unique solution* $(q_B(z), c_B)$*. Then there exists a constant* $H_B \in \mathbb{R}$ *such that*

$$\lim_{t\to\infty}(h(t) - c_B t) = H_B \quad \textit{and} \quad \lim_{t\to\infty} h'(t) = c_B.$$

Moreover, it holds that

$$\lim_{t\to\infty} \sup_{h(t)/2\le x\le h(t)} |u(t,x) - q_B(h(t) - x)| = 0.$$

Remark 6. In Case B, (4.2) with $i = 3$ has no solution if $\mu \ge \mu^*$. Under this condition, if we repeat the arguments in the proof of Theorem 2, we are able to show that any big spreading solution satisfies

$$\lim_{t\to\infty} \frac{h(t)}{t} = c_S$$

(see the corresponding result in [7, Theorem 4.2]). We infer that such a big spreading solution $u(t,x)$ will be approximated by $q_S(h(t) - x)$ near $x = h(t)$. However, we have not succeeded yet in deriving a sharp estimate on such $u(t,x)$ for large t.

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Appendix A

Lemma A.1. *Let* (u, h) *be any solution of* (FBP)*. Then it holds that*

$$u_x(t,x) < 0$$

for all (t,x) *satisfying* $t > 0$ *and* $\max\{h_0, h(t)/2\} \le x \le h(t)$*.*

Proof. We will prove this lemma by the reflection method as in [2, Lemma 2.8]. Let ℓ be any number satisfying

$$h_0 < \ell < \lim_{t\to\infty} h(t).$$

Since $h(t)$ is strictly increasing by (1.2), there exists a unique number $T_\ell > 0$ such that $h(T_\ell) = \ell$. We define a set Ω_ℓ by

$$\Omega_\ell = \{(t,x) \mid T_\ell \le t < T_\ell^* \ \text{ and } \ 2\ell - h(t) \le x \le \ell\}$$

if there exists T_ℓ^* such that $h(T_\ell^*) = 2\ell$, or

$$\Omega_\ell = \{(t,x) \mid T_\ell \le t \ \text{ and } \ 2\ell - h(t) \le x \le \ell\}$$

if $h(t) < 2\ell$ for all $t > 0$. Note that $\Omega_\ell \subset \{(t,x) \mid t \ge 0 \ \text{ and } \ 0 \le x \le h(t)\}$; so that $(t,x) \in \Omega_\ell$ implies $h(t)/2 \le \ell \le h(t)$. Consider $w(t,x)$ defined by

$$w(t,x) = u(t,x) - u(t, 2\ell - x) \qquad \text{for } \ (t,x) \in \Omega_\ell.$$

Clearly, w satisfies

$$w_t = dw_{xx} + a(t,x)w \qquad \text{for } \ (t,x) \in \Omega_\ell,$$

where $a(t,x)$ is a continuous function given by

$$a(t,x) = \int_0^1 f'(\theta u(t,x) + (1-\theta)u(t, 2\ell - x))d\theta.$$

On the lateral boundary of Ω_ℓ, $w(t,\ell) = 0$ and

$$w(t, 2\ell - h(t)) = u(t, 2\ell - h(t)) - u(t, h(t)) = u(t, 2\ell - h(t)) > 0.$$

Therefore, it follows from the strong maximum principle (see [10] or [13]) that $w(t,x) > 0$ in $\mathrm{Int}\Omega_\ell$. This fact together with Hopf's boundary lemma implies $w_x(t,\ell) = 2u_x(t,\ell) < 0$. Since ℓ is any number satisfying $\max\{h_0, h(t)/2\} \le \ell \le h(t)$ for each $t > 0$, one can obtain the conclusion. $\square$

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