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Rational dilation problems associated with constrained algebras ☆

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ABSTRACT

A set Ω is a spectral set for an operator T if the spectrum of T is contained in Ω, and von Neumann's inequality holds for T with respect to the algebra $R(\Omega)$ of rational functions with poles off of $\overline{\Omega}$. It is a complete spectral set if for all $r \in \mathbb{N}$, the same is true for $M_r(\mathbb{C}) \otimes R(\Omega)$. The rational dilation problem asks, if Ω is a spectral set for T, is it a complete spectral set for T? There are natural multivariable versions of this. There are a few cases where rational dilation is known to hold (eg, over the disk and bidisk), and some where it is known to fail, for example over the Neil parabola, a distinguished variety in the bidisk. The Neil parabola is naturally associated to a constrained subalgebra of the disk algebra $\mathbb{C} + z^2 A(\mathbb{D})$. Here it is shown that such a result is generic for a large class of varieties associated to constrained algebras. This is accomplished in part by finding a minimal set of test functions. In addition, an Agler–Pick interpolation theorem is given and it is proved that there exist Kaijser–Varopoulos style examples of non-contractive unital representations where the generators are contractions.

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1. Introduction

It was first recognized in the 1950s that there is a deep connection between the fact that over the unit disk $\mathbb{D}$ of the complex plane $\mathbb{C}$, von Neumann's inequality holds for any Hilbert space contraction operator, and that a contraction can be dilated to unitary operator (the Sz.-Nagy dilation theorem). A similar phenomenon is observed for a commuting pair of contractions, which according to Andô's theorem, dilate to a commuting pair of unitary operators.

More generally, an operator T in $\mathcal{B}(\mathcal{H})$, the bounded linear operators on a Hilbert space $\mathcal{H}$, is said to have a *rational dilation* (with respect to a compact set $\overline{\Omega}$) if there is a Hilbert space $\mathcal{K} \supset \mathcal{H}$ and a normal operator $N \in \mathcal{B}(\mathcal{K})$ with spectrum in the boundary of $\overline{\Omega}$ such that $f(T) = P_{\mathcal{H}} r(N)|_{\mathcal{H}}$ for all $f \in R(\overline{\Omega})$, the rational functions with poles off of $\overline{\Omega}$.

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There is a natural multivariable version of this.

Problem *(Rational dilation problem[1]).* Let Ω be a domain in $\mathbb{C}^n$ with compact closure and suppose that T is a commuting tuple of bounded operators on a Hilbert space $\mathcal{H}$ with spectrum contained in $\overline{\Omega}$. Furthermore, assume that for every $f \in R(\Omega)$, the set of rational functions with poles off of $\overline{\Omega}$, the von Neumann inequality holds; that is, $\|f(T)\| \leq \|f\|_\infty$, where $\|\cdot\|_\infty$ is the supremum norm over $\overline{\Omega}$. Does there exist a Hilbert space $\mathcal{K} \supset \mathcal{H}$ and commuting tuple of normal operators N on $\mathcal{K}$ with spectrum in the boundary of Ω such that $f(T) = P_{\mathcal{H}} r(N)|_{\mathcal{H}}$ for all $f \in R(\overline{\Omega})$? That is, does T have a *rational dilation* to N?

Here Arveson [6] is followed in defining the spectrum of a tuple T to be $\sigma(T) := \{\lambda \in \mathbb{C}^n : \text{ for } p : \mathbb{C}^n \to \mathbb{C} \text{ a polynomial, } p(\lambda) \in \sigma(p(T))\}$. He showed that this set is non-empty and compact, and that the spectral mapping theorem holds for all non-constant rational functions with poles off of $\sigma(T)$.

When the von Neumann inequality holds for an operator (or tuple of operators) T as in the statement of the rational dilation problem, $\overline{\Omega}$ is said to be a *spectral set* for T. It is not difficult to see that if T has a rational dilation, then $\overline{\Omega}$ is a spectral set for T; indeed, one also has the von Neumann inequality for $f \in R(\Omega) \otimes M_r(\mathbb{C})$, the matrix valued rational functions with poles off of $\overline{\Omega}$, for any finite r. Hence $\overline{\Omega}$ is a *complete spectral set* for T.

A nontrivial fact, also due to Arveson [6], is that T has a rational dilation if and only if $\overline{\Omega}$ is a complete spectral set for T. Thus the rational dilation problem can be reformulated as: *If $\overline{\Omega}$ is a spectral set for T, is it a complete spectral set for T?*

Given a set $X \subset \mathbb{C}^d$, a function $f : X \to \mathbb{C}$ is *analytic* if for every $x \in X$, there is an open neighborhood of x to which f extends analytically. Denote by $A(\Omega)$ the subalgebra of functions in $C(\overline{\Omega})$ which are analytic on Ω. At least over subsets of $\mathbb{C}$, there are various conditions which imply that $R(\Omega)$ is dense in $A(\Omega)$; for example, if Ω is finitely connected, then this is true. In this paper we concentrate on the setting where Ω is the intersection of a variety with $\overline{\mathbb{D}}^n$. Since the variety is the zero set of a polynomial, similar reasoning as in the one variable setting will dictate that $R(\Omega)$ is dense in $A(\Omega)$. How a bounded representation acts on $R(\Omega)$ is determined by its action on the generators, so such a representation extends continuously to $A(\Omega)$. This gives yet another formulation of the rational dilation problem over suitably nice Ω: *Is every contractive representation of $A(\Omega)$ completely contractive?*

An implication of the Sz.-Nagy dilation theorem is that contractive representations of $A(\mathbb{D})$ are completely contractive, and Andô's theorem allows us to draw the same conclusion for $A(\mathbb{D}^2)$. So for $\Omega = \mathbb{D}$ or $\mathbb{D}^2$, rational dilation holds. A more substantial argument is needed to prove that rational dilation holds for annuli [1] (but see also [17]), and intriguingly, there is a way of mapping an annulus to a distinguished variety of the bidisk [28] (that is, a variety $\mathcal{V}$ which intersects $\mathbb{D}^2$ and satisfies $\mathcal{V} \cap \partial\mathbb{D}^2 \subset \mathbb{T}^2$, which is the *distinguished*, or *Šilov boundary* of $\mathbb{D}^2$). Thus rational dilation holding for annuli is equivalent to it holding for a certain family of distinguished varieties in $\mathbb{D}^2$. It is natural to wonder if this is a legacy of rational dilation holding over $\mathbb{D}^2$, and so to speculate that perhaps rational dilation also holds for other distinguished varieties in $\mathbb{D}^2$.

Alas, this is too much to hope for. In [17], it was proved that rational dilation fails for the Neil parabola $\mathcal{N} = \{(z, w) \in \mathbb{D}^2 : z^2 = w^3\}$. The techniques are indirect. As with an annulus, one can associate $A(\mathcal{N})$ to another algebra. In this case, there is a complete isometry mapping $A(\mathcal{N})$ onto $A_{z^2}(\mathbb{D}) = \mathbb{C} + z^2 A(\mathbb{D})$, the subalgebra of $A(\mathbb{D})$, the functions of which have first derivative vanishing at 0. It is shown in [17] that this algebra has a contractive representation which is not 2-contractive, and so not completely contractive.

In this paper, we show that rational dilation fails without fail for algebras $A(\mathcal{V}_B)$ of functions which are analytic and continuous up to the boundary on distinguished varieties $\mathcal{V}_B$ of the N-disk associated to finite Blaschke products B with $N \geq 2$ zeros. We also prove that it fails on associated distinguished varieties of

[1] This problem is usually attributed to Halmos, and while this seems plausible, we have been unable to find a reference!

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the 2-disk, at least if B has two or more distinct zeros all of the same multiplicity. This enormously increases the set of examples where one can answer such questions.

The methods used were pioneered in [20] and [17], though they also have predecessors in [2], [18] and [19]. The first hurdle to be overcome is the construction of a minimal set of test functions for algebras of the form $\mathcal{A}_B = \mathbb{C}+B(z)A(\mathbb{D})$, as in [20]. Since it has N generators, this algebra is completely isometrically isomorphic to $A(\mathcal{V}_B)$, $\mathcal{V}_B$ a distinguished variety of the N-disk. It is also possible to consider the subalgebra $\mathcal{A}_B^0$ of $\mathcal{A}_B$ generated by the first two generators, B and zB, of $\mathcal{A}_B$. This is completely isometrically isomorphic to a subalgebra $\mathcal{A}(\mathcal{N}_B)$ on a distinguished variety of the bidisk. The algebras $\mathcal{A}_B$ were already studied from the dual viewpoint of families of kernels in [13], while we present here the first systematic study of the algebras $\mathcal{A}_B^0$.

For both $\mathcal{A}_B$ and $\mathcal{A}_B^0$ (with the condition on the zeros of B mentioned above), we construct an example of a contractive representation which is not completely contractive, yielding rational dilation results on the associated varieties. The strategy for doing this goes back to [18], though was undoubtedly familiar to Jim Agler even before this. One shows that there is a contractive representation which is not completely contractive. This is done by proving that certain matrix valued measures arising in the so-called Agler decomposition for matrix valued functions must diagonalize if rational dilation is to hold. Then it is a matter of finding a function for which this does not happen.

While it is well known that for $N > 2$, $A(\mathbb{D}^N)$ itself has contractive representations which are not completely contractive, it is not *a priori* the case that such a contractive representation of $A(\mathbb{D}^N)$ when restricted to a subalgebra is also not completely contractive. As a trivial example, consider $A(\mathbb{D}^2)$ in $A(\mathbb{D}^3)$. Likewise, simply knowing that a function algebra has a contractive representation which is not completely contractive does not necessarily imply the same is true for any algebra containing it. The Neil algebra as a subalgebra of $A(\mathbb{D}^2)$ is a case in point.

Various noteworthy observations are made in the course of the paper. For example, for both $\mathcal{A}_B$ and $\mathcal{A}_B^0$ minimal sets of test functions are constructed (for any B with two or more zeros), yielding optimal forms of Agler–Pick interpolation theorems. Kaiser–Varopoulos type examples of unital representations which are contractive on the generators of these algebras yet which fail to be contractive representations are also found. There is in addition a characterization of completely contractive representations along the line of the Sz.-Nagy dilation theorem, much like that proved by Broschinski for the Neil algebra [11].

The work is presented in the following order. Section 2 introduces the distinguished varieties associated to Blaschke products on which we will study the rational dilation problem, while Section 3 presents the rational dilation problem. Section 4 outlines the notion of test functions and their application to realization and interpolation problems. We show that there is no loss in generality in restricting to Blaschke products with at least one zero at 0. The Herglotz representation plays a central role, and there is a closed cone of positive measures which is fundamental. The extreme rays are connected with certain probability measures which, after a Cayley transform, yield a set of candidates for the test functions, as we see in Section 5. The next, and arguably most challenging step, is to show that the set of test functions found is in some sense minimal. This is addressed in Section 6, and then applied in Section 7 to give the Kaijser–Varopoulos style representation mentioned above and a Sz.-Nagy type dilation theorem. Finally, in Section 8 we tackle the rational dilation problem. The paper concludes with some remarks.

2. Distinguished varieties associated to Blaschke products

We begin by describing the distinguished varieties in the bidisk considered in this paper.

The following notation will be useful. For $x = (x_1, \dots, x_n) \in \mathbb{C}^n$, define $S_0(x) = 1$ and

$$S_k(x) = (-1)^k \sum_{1 \le x_1 \le \cdots \le x_n} x_{i_1} \cdots x_{i_n}, \quad k = 1, \dots, n,$$

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the kth (signed) symmetric sum of the elements of x. If $k > n$, define $S_k(x) = 0$. Then

$$\prod_{j=1}^{n}(z - x_j) = \sum_{k=0}^{n} S_k(x) z^{n-k} \qquad \text{and} \qquad \prod_{j=1}^{n}(1 - \overline{x_j} z) = \sum_{k=0}^{n} S_k(\overline{x}) z^k, \tag{1}$$

where $\overline{x} = (\overline{x_1}, \dots, \overline{x_n})$. Also define $S_0^{-i}(x) = -1$ and

$$S_k^{-i}(x) = S((x_1, \dots, x_{i-1}, -x_i, x_{i+1}, \dots, x_n)), \qquad k = 1, \dots, n.$$

Then $S_n^{-i}(x) = -S_n(x)$. For $x = \lambda \subset \mathbb{T}^n$,

$$S_k(\lambda) = S_n(\lambda) S_{n-k}(\overline{\lambda}) \qquad \text{and} \qquad S_k^{-i}(\lambda) = -S_n^{-i}(\lambda) S_{n-k}^{-i}(\overline{\lambda}).$$

Let

$$B(z) = \prod_{k=1}^{N} \frac{z - \alpha_k}{1 - \overline{\alpha_k} z}, \qquad z \in \overline{\mathbb{D}},$$

be a Blaschke product with at least two not necessarily distinct zeros. Define $x = B(z)$, $y = zB(z)$. Then the pair $(x, y) \in \overline{\mathbb{D}}^2$. Since

$$B(z)^{N+1} \prod_{k=1}^{N}(1 - \overline{\alpha_k} z) = B(z)^N \prod_{k=1}^{N}(z - \alpha_k), \tag{2}$$

the pair (x, y) satisfies the polynomial identity $P(x, y) = 0$, with

$$\begin{aligned} P(x, y) &= x \prod_{k=1}^{N}(x - \overline{\alpha_k} y) - \prod_{k=1}^{N}(y - \alpha_k x) \\ &= \sum_{k=0}^{N} \left(S_k(\overline{\alpha}) x^{N-k+1} y^k - S_k(\alpha) x^k y^{N-k} \right) \end{aligned}$$

Now suppose $(x, y) \in \mathbb{D}^2$ is any point satisfying $P(x, y) = 0$. If $x = 0$, then $y^N = 0$, and thus $y = 0$. So assume $x \neq 0$. Letting $z = y/x$, it follows that $x = B(z)$ and $y = zB(z)$, and so x and y have the form indicated above. Furthermore, since $|B(z)| \leq 1$ if and only if $|z| \leq 1$, $(x, y) \in \overline{\mathbb{D}}^2$ if and only if $z \in \overline{\mathbb{D}}$.

The locus described by $P(x, y) = 0$ defines a variety in $\mathbb{C}^2$. Furthermore, since $|x| = |B(z)| = 1$ implies that $|z| = 1$, $|x| = 1$ implies $|y| = 1$. Likewise, if $|y| = 1$, then the modulus of the Blaschke product $zB(z)$ is 1, and so once again $|z| = 1$, implying that $|x| = 1$. Hence this variety intersects the boundary of $\overline{\mathbb{D}}^2$ in $\mathbb{T}^2$, which is the Šilov or *distinguished boundary*; that is, $P(x, y) = 0$ defines a *distinguished variety*. Write $\mathcal{N}_B$ for $\{(x, y) \in \overline{\mathbb{D}}^2 : P(x, y) = 0\}$.

The variety $\mathcal{N}_B$ has a singularity solely at $(0, 0)$. If all of the zeros of B are distinct, then there is an $(N - 1)$-fold crossing at this point. At the other extreme, if all the zeros are the same, there is a cusp (for example, this is what happens with the Neil parabola, where $B(z) = z^2$). Intermediate cases give rise to mixtures of these.

The general theory of distinguished varieties of the bidisk as laid out by Agler and McCarthy in [4] (see also, [23]) shows that such varieties have a *determinantal representation*.

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Theorem 2.1 *([4]). Let $\mathcal{V}$ be a distinguished variety, defined as zero set of a polynomial $p \in \mathbb{C}[x,y]$ of minimal degree (m,n). Then, there is an $(m+n)\times(m+n)$ unitary matrix U, written as $U = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : \mathbb{C}^m \oplus \mathbb{C}^n \to \mathbb{C}^m \oplus \mathbb{C}^n$, such that*

(1) *A has no unimodular eigenvalues,*
(2) *$p(x,y)$ is a constant multiple of*

$$\det \begin{pmatrix} D - xI_n & xC \\ B & yA - I_m \end{pmatrix}, \qquad \text{and}$$

(3) *for the rational matrix valued inner function $\Psi(y) = D + yC\,(I_m - yA)^{-1}\,B$,*

$$\mathcal{V} = \left\{ (x,y) \in \overline{\mathbb{D}}^2 : \det(xI_n - \Psi(y)) = 0 \right\}.$$

Moreover, if Ψ is a matrix valued rational inner function on $\overline{\mathbb{D}}$, then

$$\left\{ (x,y) \in \overline{\mathbb{D}}^2 : \det(xI_n - \Psi(y)) = 0 \right\}$$

is a distinguished variety.

A straightforward calculation shows that a determinantal representation for $\mathcal{N}_B$ is obtained by choosing

$$\Psi(y) = \begin{pmatrix} 0 & -y & 0 & \cdots & 0 & 0 \\ 0 & \overline{\alpha_1}y & -y & 0 & \cdots & 0 \\ 0 & 0 & \overline{\alpha_2}y & -y & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \overline{\alpha_{N-1}}y & -y \\ (-1)^N & 0 & \cdots & \cdots & 0 & \overline{\alpha_N}y \end{pmatrix} T^{-1},$$

where

$$T = \begin{pmatrix} 1 & -\alpha_1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & -\alpha_2 & 0 & \cdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 & 1 & -\alpha_n \\ 0 & \cdots & \cdots & \cdots & 0 & 1 \end{pmatrix}.$$

By defining variables $x_j = z^{j-1}B$, $j = 1,\dots,N$, and again using (2), it is not hard to work out that there is an associated distinguished variety $\mathcal{V}_B$ in $\overline{\mathbb{D}}^N$. There will in general be multiple varieties which can be described with these variables from (2). They are obtained one from the other via the identities $x_j x_{N-j} = x_i x_{N-i}$.

Obviously, when N is sufficiently large, intermediate cases could be considered, associated to algebras on distinguished varieties in $\overline{\mathbb{D}}^n$, $2 < n < N$. The techniques needed to handle these are identical to those presented for $\mathcal{A}_B^0$ and $\mathcal{A}_B$.

Recall the notation $A(\mathcal{V}_B)$ for the algebra of analytic functions on $\mathcal{V}_B \cap \overline{D}^N$ which extend continuously to the boundary with the supremum norm, and $\mathcal{A}_B = \mathbb{C} + B\,A(\mathbb{D})$ the associated subalgebra of $A(\mathbb{D})$. If

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$B(z) = \prod_1^N \frac{z-\alpha_j}{1-\overline{\alpha_j}z}$, then $\mathcal{A}_B = \mathbb{C} + \prod_1^N (z-\alpha_j)\, A(\mathbb{D})$ since $\prod_1^N (1-\overline{\alpha_j}z) \in A(\mathbb{D})$. Thus $\mathcal{A}_B$ is generated by $\{Bz^j\}_{j=0}^{N-1}$. In connection with $A(\mathcal{N}_B)$, we are also interested in a subalgebra of $\mathcal{A}_B$ generated by B and zB, denoted by $\mathcal{A}_B^0$.

Theorem 2.2. *The algebra $A(\mathcal{V}_B)$ is completely isometrically isomorphic to the algebra $\mathcal{A}_B$. The algebra $A(\mathcal{N}_B)$ is completely isometrically isomorphic to the algebra $\mathcal{A}_B^0$, which consists of those functions in $\mathcal{A}_B$ for which the coefficients of terms of the form $z^i B^j$, $j = 0, \dots, N-2$ and $i = j+1, \dots N-1$, are 0. The algebra $\mathcal{A}_B^0$ is of codimension $N(N-1)/2$ in $A(\mathbb{D})$ and contains $\mathbb{C} + B^{N-1}A(\mathbb{D})$; so in particular, when $N = 2$, $\mathcal{A}_B^0 = \mathcal{A}_B$.*

Proof. The case of $A(\mathcal{N}_B)$ is only treated, the other being handled identically.

As noted earlier, the algebra $\mathcal{Q}_B$ of rational functions with poles off of $\mathcal{N}_B$ is dense in $A(\mathcal{N}_B)$. Define a map $\rho : \mathcal{Q}_B \to \mathcal{A}_B$ by

$$\rho(p/q) = \frac{p(B, zB)}{q(B, zB)}, \qquad p, q \text{ polynomials},$$

and extending linearly. If it were the case that $q(B(\zeta), \zeta B(\zeta)) = 0$ for some $\zeta \in \overline{\mathbb{D}}$, then for $(x, y) = (B(\zeta), \zeta B(\zeta)) \in \mathcal{N}_B$, $q(x, y) = 0$, and so p/q cannot be in $\mathcal{A}(\mathcal{N}_B)$. Hence the image of ρ is in $A(\mathbb{D})$. Since the image is generated by B and zB, it equals $\mathcal{A}_B^0$. For $f \in A(\mathcal{N}_B)$, the maximum modulus principle holds for $\rho(f) = f(B, zB)$. Since $(x, y) \in \mathcal{N}_B \cap \mathbb{T}^2$ if and only if the associated z is in $\mathbb{T}$, f achieves its maximum modulus on $(x, y) \in \mathbb{T}^2 \cap \mathcal{N}_B$. Hence the map is isometric. The same reasoning shows that the map is a complete isometry, and so it extends to a completely isometric homomorphism from $A(\mathcal{N}_B)$ onto $\mathcal{A}_B^0$.

Now turn to the description of $\mathcal{A}_B^0$ in $\mathcal{A}_B$. Suppose for the time being that B has three or more zeros, and that there is some $f \in A(\mathcal{N}_B)$ such that $\rho(f) = z^2 B \in \mathcal{A}_B^0$. Then $\rho(xf) = z^2 B^2 = \rho(y^2)$. Since the map ρ is isometric, this implies that $xf = y^2$ in an open neighborhood U of $(0, 0)$. Fix any non-zero complex number t and let $C_t = \{(x, y) \in \mathbb{C}^2 : x = ty^2\}$. For y_0 small enough and non-zero, $(x_0, y_0) \neq (0, 0)$ is in $C_t \cap U$. Evaluating at (x_0, y_0) gives $f(x_0, y_0) = 1/t$. Hence f cannot be analytic, and so $z^2 B$ is not in $\mathcal{A}_B^0$.

The same argument shows that any term of the form $z^i B^j$, $j = 1, \dots, N-2$ and $i = j+1, \dots N-1$, while in $\mathcal{A}_B$, is not in $\mathcal{A}_B^0$. Obviously anything of this form where j is arbitrary and $i \leq j$ can be written as a product of powers of B and zB.

Now suppose B has $N \geq 2$ zeros and let $j = N-1$. Then

$$z^N B^{N-1} = \left(\prod_{j=1}^N (1 - \overline{\alpha_j} z) B - g \right) B^{N-1} = \left(\sum_{j=0}^N S_j(\overline{\alpha}) z^j B - g \right) B^{N-1},$$

where $\deg g \leq N-1$. All terms have the form $cz^i B^j$, c a constant and $i \leq j$, and hence are in $\mathcal{A}_B^0$. Also,

$$z^{N+k} B^{N-1} = z^k \left(\sum_0^N S_j(\overline{\alpha}) z^j B - g \right) B^{N-1},$$

so by an induction argument, all of these are in $\mathcal{A}_B^0$ as well. Hence, $\mathcal{A}_B^0 \supset B^{N-1} A(\mathbb{D})$. In particular, if B has only two zeros, B and zB generate the algebra $\mathcal{A}_B$, and in this case ρ is onto. □

3. The rational dilation problem and constrained algebras

Our goal is to study the *rational dilation problem* on $\mathcal{V} = \mathcal{V}_B$ (respectively, $\mathcal{N}_B$). Thus we consider n tuples of commuting operators $T = (T_1, \dots, T_n)$ ($n = N$ and $n = 2$, respectively) acting on a Hilbert space

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$\mathcal{H}$ having $\mathcal{V}$ as a *spectral set*. Recall that this means that the joint spectrum of T lies in $\mathcal{V}$ and for each $f \in \mathcal{Q}_B$, $\|f(T)\| \leq \|f\|$, where the left hand norm is the usual operator norm, and the right hand norm is the supremum norm of f on $\mathcal{V}$. This is a form of the von Neumann inequality, and as noted in the introduction, can be interpreted as stating that T induces a contractive unital representation of $\mathcal{Q}_B$, and hence $A(\mathcal{V})$.

The *rational dilation problem* then asks whether such a T dilates to a commuting tuple of normal operator $W = (W_1, \ldots, W_n)$ acting on some Hilbert space $\mathcal{K} \supset \mathcal{H}$ with spectrum contained in the distinguished boundary of $\mathcal{V} \subset \mathbb{T}^n$. By a *dilation*, we mean that $f(T) = P_{\mathcal{H}} f(W)|_{\mathcal{H}}$ for all $f \in \mathcal{Q}_B$. The tuple W is referred to as a *normal boundary dilation*. If W exists for all such T, rational dilation is said to hold, and otherwise, it fails.

For the tuple of normal operators W, not only is it the case that $\|f(W)\| \leq \|f\|$ for $f \in \mathcal{Q}_B$, but also for $f \in \mathcal{Q}_B \otimes M_r(\mathbb{C})$, $r \in \mathbb{N}$. Therefore if T has a normal boundary dilation, it is also true that $\|f(T)\| \leq \|f\|$ for $f \in \mathcal{Q}_B \otimes M_r(\mathbb{C})$. In other words, when rational dilation holds, contractive representations of $\mathcal{Q}_B$ (and hence $A(\mathcal{V})$) are completely contractive, and the converse also holds. Thus a strategy for showing that rational dilation fails on $A(\mathcal{V}_B)$ (respectively, $A(\mathcal{N}_B)$) is to find a contractive representation of $\mathcal{A}_B$ (respectively, $\mathcal{A}_B^0$) which is not completely contractive. This is the approach taken.

4. Test functions

Our method for solving the rational dilation problem requires finding a family of so-called "test functions" for the algebras $\mathcal{A}_B$ and $\mathcal{A}_B^0$. For other purposes (such as solving interpolation problems), it is useful for this family to be in some sense minimal. We give a brief synopsis the notion of test functions and their use in the solution of interpolation problems, and otherwise refer to [19] for further details. See also [3].

Let X be a set and $\Psi = \{\psi_\alpha\}$ a collection of complex valued functions on X. The elements of Ψ are called *test functions* if they satisfy two conditions:

- For any $x \in X$, $\sup_{\psi \in \Psi} |\psi(x)| < 1$, and
- The elements of Ψ separate the points of X.

Given a set of test functions Ψ, the set of *admissible kernels* $\mathcal{K}_\Psi$ consists of positive kernels k on $X \times X$ to $\mathbb{C}$ with the property that for each $\psi \in \Psi$, the kernel

$$((1 - \psi(x)\psi(y)^*)k(x,y)) \geq 0.$$

The admissible kernels allow us to define a function algebra $H^\infty(\mathcal{K}_\Psi)$ of those functions φ on X for which there is a $c \in \mathbb{R}^+$ such that for all $k \in \mathcal{K}_\Psi$,

$$((c1 - \varphi(x)\varphi(y)^*)k(x,y)) \geq 0.$$

The infimum over all such c defines a norm on $H^\infty(\mathcal{K}_\Psi)$ making it a Banach algebra. Obviously, the test functions are in the unit ball of this algebra. Because any positive kernel which is zero when $y \neq x$ is admissible, the norm of $H^\infty(\mathcal{K}_\Psi)$ will always be greater than or equal to the supremum norm, and so $H^\infty(\mathcal{K}_\Psi)$ is weakly closed (that is, closed under pointwise convergence).

A key result in the study of algebras generated through test functions is the realization theorem [19], which gives several equivalent characterizations of membership of the closed unit ball of the algebra $H^\infty(\mathcal{K}_\Psi)$. The relevant portion is stated here. Some notation: $C(\Psi)$ is the algebra of bounded continuous functions on Ψ, and $C(\Psi)^*$ is its continuous dual. Assume that Ψ is endowed with a suitable topology so that for all $x \in X$, the functions $E_x : \psi \in \Psi \mapsto \psi(x)$ are in $C(\Psi)$. In this case $E_x^* : \psi \in \Psi \mapsto \psi(x)^*$ is also in $C(\Psi)$.

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Theorem 4.1 *(Realization theorem). Let Ψ be a collection of test functions on a set X, $\mathcal{K}_\Psi$ the admissible kernels, and $H^\infty(\mathcal{K}_\Psi)$ the associated function algebra. For $\varphi : X \to \mathbb{C}$, the following are equivalent:*

(1) *$\varphi \in H^\infty(\mathcal{K}_\Psi)$ with $\|\varphi\| \leq 1$;*
(2) *There is a positive kernel $\Gamma : X \times X \to C(\Psi)^*$ such that for all $x, y \in X$,*

$$1 - \varphi(x)\varphi(y)^* = \Gamma(x,y)(1 - E_x E_y^*); \quad \text{and}$$

(3) *If π is any unital representation of $H^\infty(\mathcal{K}_\Psi)$ mapping the elements of Ψ to strict contractions (ie, norm strictly less than 1), then π is contractive.*

The proof of the realization theorem is the basis for the following interpolation theorem.

Theorem 4.2 *(Agler–Pick interpolation theorem). Let Ψ be a collection of test functions on a set X, $\mathcal{K}_\Psi$ the admissible kernels, and $H^\infty(\mathcal{K}_\Psi)$ the associated function algebra. Fix a finite set $F \subset X$. For $f : F \to \mathbb{C}$, the following are equivalent:*

(1) *There is a function $\varphi \in H^\infty(\mathcal{K}_\Psi)$ with $\|\varphi\| \leq 1$ such that $\varphi|_F = f$, and*
(2) *there is a positive kernel $\Gamma : F \times F \to C(\Psi)^*$ such that for $x, y \in F$,*

$$1 - \varphi(x)\varphi(y)^* = \Gamma(x,y)(1 - E_x E_y^*).$$

In summary, given a collection of test functions Ψ, first construct a set of admissible kernels $\mathcal{K}_\Psi$, and then from this a function algebra $H^\infty(\mathcal{K}_\Psi)$. In most situations though, an algebra $\mathcal{A}$ on a domain X is already at hand, and so for example, if one wishes to solve interpolation problems in $\mathcal{A}$, it is necessary to find a set of test functions Ψ generating $\mathcal{A}$. A trivial choice (disregarding possible degeneracies) is to let Ψ be the unit ball of $\mathcal{A}$. The ideal though is to choose Ψ to be as small as possible. Care is needed since it may be the case that removing finitely, or even countably many test functions still leaves a suitable set of test functions. Insisting that the set of test functions be (weakly) compact avoids this difficulty. Even then, the minimal set of test functions will only be defined up to automorphism. In any case, a compact family of test functions Ψ is said to be *minimal* for an algebra $\mathcal{A}(\mathcal{K}_\Psi)$ if there is no proper closed subset of Ψ such that the realization theorem holds for all functions in the unit ball of $\mathcal{A}(\mathcal{K}_\Psi)$.

Let us return our attention to the constrained algebras $\mathcal{A}_B$ and $\mathcal{A}_B^0$, and the construction of minimal sets of test functions Ψ_B and Ψ_B^0 for these, or rather, for the weak closure of these algebras, H_B^∞ and $H_B^{\infty,0}$. To simplify the work, it may be assumed that one of the zeros of B, written as α_0, equals 0. It turns out that this assumption in fact imposes no real restriction.

For suppose that B is a Blaschke product with zeros $\mathcal{Z}(B) = \{\alpha_0, \ldots, \alpha_n\}$ such that no $\alpha_j = 0$. Composing B with the Möbius map $m_{-\alpha_0} = (z + \alpha_0)/(1 + \overline{\alpha_0} z)$, to get a Blaschke product B' with zeros $\{\alpha'_j = m_{\alpha_0}(\alpha_j)\}_{j=0}^n$. Hence $\alpha'_0 = 0$. Obviously composing with m_{α_0} maps B' back to B. Since composition with $m_{\pm\alpha_0}$ leaves $H^\infty(\mathbb{D})$ invariant, $f \in H_B^\infty$ if and only if $f' = f \circ m_{-\alpha_0} \in H_{B'}^\infty$, and furthermore, $\|f\| = \|f'\|$.

Let $\Psi_{B'}$ be a family of test functions for $H_{B'}^\infty$, and define $\Psi = \{\psi' \circ m_{\alpha_0} : \psi' \in \Psi_{B'}\}$. Since m_{α_0} is an automorphism of the disk, $\Psi_{B'}$ maps injectively onto Ψ, and so it is possible to identify $C(\Psi_{B'})$ and $C(\Psi)$. For $x \in \mathbb{D}$, set $x' = m_{\alpha_0}(x)$. Then

$$E_x(\psi) = \psi(x) = \psi'(m_{\alpha_0}(x)) = \psi'(x') = E_{x'}(\psi').$$

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Let $\varphi \in H^\infty_B$ and set $\varphi' = \varphi \circ m_{-\alpha_0}$. Assume $\|\varphi'\| (= \|\varphi\|) = 1$. By the realization theorem and the assumption that $\Psi_{B'}$ is a family of test functions for $H^\infty_{B'}$, there is a positive kernel $\Gamma' : \mathbb{D} \times \mathbb{D} \to C(\Psi_{B'})^*$ such that for $x, y \in \mathbb{D}$, and $x' = m_{\alpha_0}(x)$, $y' = m_{\alpha_0}(y)$,

$$1 - \varphi(x)\varphi(y)^* = 1 - \varphi'(x')\varphi'(y')^* = \Gamma'(x', y')(1 - E_{x'}E^*_{y'}) = \Gamma(x, y)(1 - E_x E^*_y),$$

where $\Gamma(x, y) = \Gamma'(m_{\alpha_0}(x), m_{\alpha_0}(y))$ is a positive kernel from $\mathbb{D} \times \mathbb{D}$ to $C(\Psi)^*$. Conclude that H^∞_B is in the algebra $\mathcal{A}$ induced by the test functions Ψ and φ is in the unit ball of $\mathcal{A}$. Since the norm of φ in $\mathcal{A}$ is greater than or equal to the supremum norm (the norm in H^∞_B), the two norms must be equal. On the other hand, if φ is in the unit ball of $\mathcal{A}$, then for $\varphi' = \varphi \circ m_{-\alpha_0}$, the realization theorem implies that $\varphi' \in H^\infty_{B'}$, and so $\varphi \in H^\infty_B$. Thus $\mathcal{A} = H^\infty_B$ with the same norm, and the conclusion is that Ψ is a family of test functions for H^∞_B. By similar arguments, Ψ is minimal if and only if $\Psi_{B'}$ is minimal.

The same argument works when dealing with $\mathcal{A}^0_B$, so as needed, B will be replaced by B', where $0 \in \mathcal{Z}(B')$.

5. Herglotz representations and extreme rays

In this section, sets of test functions for the algebras $\mathcal{A}_B$ and $\mathcal{A}^0_B$ are determined. The strategy employed is as follows. Suppose that φ is in the unit ball of one of these algebras and that $\varphi(0) = 0$. A Cayley transform uniquely associates to this a function $f : \mathbb{D} \to \mathbb{H}$ with $f(0) = 1$, where $\mathbb{H}$ is the right half plane in $\mathbb{C}$. The function f has a Herglotz representation with respect to a unique probability measure μ on $\mathbb{T}$. The constraints of the algebra are encoded in the measure. The probability measures form a compact convex set, and μ can be represented as the integral with respect to a measure supported on the extreme points of this set. The set of inverse Cayley transforms of the functions which (modulo a unimodular constant) have Herglotz representations with respect to the extremal measures is then the candidate for the set of test functions.

If $\varphi \in H^\infty$ with $\varphi(0) = 0$ and norm at most 1, and $f = M \circ \varphi$ where $M(z) = \frac{1+z}{1-z}$ maps $\mathbb{D}$ to $\mathbb{H}$, then $\operatorname{Re} f \geq 0$ and $f(0) = 1$. The map M has inverse $M^{-1}(z) = \frac{1-z}{1+z}$, and hence there is a one to one correspondence between the set of functions in the unit ball of H^∞ which are zero at 0 and the set of holomorphic functions mapping $\mathbb{D}$ to $\mathbb{H}$ and the value 1 at 0.

By the Herglotz representation theorem, for any holomorphic $f : \mathbb{D} \to \mathbb{H}$ with $f(0) = 1$, there is a unique probability measure μ on $\mathbb{T}$ (usually referred to as the *Clark* or *Alexandrov–Clark measure*) such that

$$f(z) = \int_{\mathbb{T}} \frac{w + z}{w - z}\, d\mu(w),$$

and conversely, if μ is a probability measure on $\mathbb{T}$, then

$$f(z) := \int_{\mathbb{T}} \frac{w + z}{w - z}\, d\mu(w)$$

defines a holomorphic function on $\mathbb{D}$ to $\mathbb{H}$ with $f(0) = 1$.

The following can be cobbled together from other sources (see, for example, [12, Chapter 9]). We give a direct and elementary proof.

Lemma 5.1. *Let μ be a positive finite atomic measure on $\mathbb{T}$, $\mu = \{(\lambda_j, m_j)\}_{j=1}^n \subset \mathbb{T} \times \mathbb{R}_{>0}$, with f the function having Herglotz representation with this measure. Then $\varphi = M^{-1} \circ f$ is a unimodular constant multiple of a Blaschke product with n zeros, counting multiplicities, and $\varphi(0) \in \mathbb{R}$.*

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Conversely, given a Blaschke product φ with n zeros $\{\alpha_j\}$ counting multiplicities such that $\varphi(0) \in \mathbb{R}$, there is a positive finite atomic measure μ on $\mathbb{T}$ such that $f = M \circ \varphi(z)$ has a Herglotz representation with this measure. Furthermore, μ is a probability measure if and only if $\varphi(0) = 0$.

Proof. Let

$$f(z) = \int_{\mathbb{T}} \frac{w+z}{w-z}\, d\mu(w) = -\sum_{i=1}^{n} m_i \frac{z+\lambda_i}{z-\lambda_i},$$

a holomorphic function from $\mathbb{D}$ to $\mathbb{H}$. Set $m = \sum m_i$. Then

$$\begin{aligned} 1 \pm f(z) &= \frac{\frac{1}{m}\sum_i m_i \prod_j (z-\lambda_j) \mp \sum_i m_i (z+\lambda_i) \prod_{j\neq i}(z-\lambda_j)}{\prod_j (z-\lambda_j)} \\ &= \frac{\sum_{k=0}^{n} \left[\sum_{i=1}^{n} (\frac{m_i}{m} S_k(\lambda) \mp m_i S_k^{-i}(\lambda))\right] z^{n-k}}{\prod_j (z-\lambda_j)}, \end{aligned}$$

and

$$\varphi(z) := (M^{-1} \circ f)(z) = \frac{1-f(z)}{1+f(z)} = \frac{\sum_{k=0}^{n} \left[\sum_{i=1}^{n} (\frac{m_i}{m} S_k(\lambda) + m_i S_k^{-i}(\lambda))\right] z^{n-k}}{\sum_{k=0}^{n} \left[\sum_{i=1}^{n} (\frac{m_i}{m} S_k(\lambda) - m_i S_k^{-i}(\lambda))\right] z^{n-k}}$$

is a holomorphic map of the disk to itself.

Since the coefficient of z^n in the numerator of φ is $1+m > 0$, the numerator is a polynomial of degree n with complex roots $\alpha_1, \ldots, \alpha_n$. Express the numerator as $(1+m)\prod_j (z-\alpha_j)$. Then

$$(1+m) S_k(\alpha) = \sum_i (m_i S_k(\lambda) + m_i S_k^{-i}(\lambda)),$$

and so the denominator can be expressed as

$$\begin{aligned} \sum_{k=0}^{n} \left[\sum_{i=1}^{n} (\tfrac{m_i}{m} S_k(\lambda) - m_i S_k^{-i}(\lambda))\right] z^{n-k} &= S_n(\lambda) \sum_{k=0}^{n} \left[\sum_{i=1}^{n} (\tfrac{m_i}{m} \overline{S_{n-k}(\lambda)} + m_i \overline{S_{n-k}^{-i}(\lambda)})\right] z^{n-k} \\ = S_n(\lambda)(1+m) \sum_{k=0}^{n} \overline{S_k(\alpha)} z^k &= S_n(\lambda)(1+m) \prod_{j=1}^{n} (1-\overline{\alpha_j} z). \end{aligned}$$

Hence

$$\varphi(z) = \overline{S_n(\lambda)} \prod_{j=1}^{n} \frac{z-\alpha_j}{1-\overline{\alpha_j} z}.$$

Since $f(0) = \sum_i m_i \in \mathbb{R}$, the same is then true for $\varphi(0)$.

Conversely, assume that $\varphi = cB$, where c is a unimodular constant, B is a Blaschke product with n zeros $\alpha_1, \ldots, \alpha_n$, counting multiplicities and $\varphi(0) \in \mathbb{R}$. Then

$$f(z) = \frac{1+\varphi(z)}{1-\varphi(z)} = \frac{\prod_j (1-\overline{\alpha_j} z) + c \prod_j (z-\alpha_j)}{\prod_j (1-\overline{\alpha_j} z) - c \prod_j (z-\alpha_j)} \tag{3}$$

is a holomorphic map from $\mathbb{D}$ to $\mathbb{H}$. Since $|S_n(\alpha)| < |c| = 1$, the leading coefficient in the denominator $C = \overline{S_n(\alpha)} - c$ is non-zero. Thus the denominator of f has n zeros in $\mathbb{C}\backslash\mathbb{D}$, $\lambda_1, \ldots, \lambda_n$. Write the denominator as $C\prod_j (z-\lambda_j)$.

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If the numerator and denominator of f have a common root w, then $\prod_j (w - \alpha_j) = 0$, implying $\lambda_k = \alpha_j \in \mathbb{D}$ for some k and j, which is a contradiction. The constant coefficient of the denominator equals $(1 - cS_n(\alpha))/C = c\overline{C}/C$, which has absolute value 1. Hence each $\lambda_j \in \mathbb{T}$.

Suppose that the denominator of f has a repeated root at some $\lambda \in \mathbb{T}$. Then the logarithmic derivative of φ,

$$\frac{\varphi'(z)}{\varphi(z)} = \sum_{k=1}^{n} \frac{1 - |\alpha_k|^2}{(1 - \overline{\alpha_k} z)(z - \alpha_k)} = \frac{-2f'(z)}{1 - f(z)^2},$$

is zero at λ. On the other hand, $\lambda \in \mathbb{T}$ implies

$$\frac{\varphi'(\lambda)}{\overline{\lambda}\varphi(\lambda)} = \sum_{k} \frac{1 - |\alpha_k|^2}{|\lambda - \alpha_k|^2} > 0, \tag{4}$$

giving a contradiction.

Consequently, since the denominator of f has n simple roots, f has a partial fraction decomposition

$$f(z) = -m - \sum_{k=1}^{n} m_k \frac{2\lambda_k}{z - \lambda_k}. \tag{5}$$

It remains to verify that each $m_k > 0$ and $m = \sum_k m_k$. This will then imply

$$f(z) = -\sum_{i=1}^{n} m_i \frac{z + \lambda_i}{z - \lambda_i},$$

meaning that f has a Herglotz representation with positive finite atomic measure $\mu = \{(\lambda_j, m_j)\}_{j=1}^{n}$ on $\mathbb{T} \times \mathbb{R}_{>0}$.

By (5), $\lim_{z \to \lambda_k} (z - \lambda_k) f(z) = -2\lambda_k m_k$. Also, since $\varphi(\lambda_k) = 1$,

$$\lim_{z \to \lambda_k} (z - \lambda_k) f(z) = \lim_{z \to \lambda_k} \frac{1 + \varphi(z)}{\frac{1 - \varphi(z)}{z - \lambda_k}} = \frac{-2}{\varphi'(\lambda_k)},$$

and so by (4)

$$m_k = \frac{1}{\lambda_k \varphi'(\lambda_k)} > 0.$$

The assumptions that $c \in \mathbb{T}$ and $\varphi(0) = cS_n(\alpha) \in \mathbb{R}$, along with (5) and (3), imply that

$$-m = \lim_{z \to \infty} f(z) = \frac{\overline{S_n(\alpha)} + c}{\overline{S_n(\alpha)} - c} = -\frac{1 + c\overline{S_n(\alpha)}}{1 - c\overline{S_n(\alpha)}} = -f(0) = m - 2\sum_k m_k.$$

Hence $m = \sum_k m_k$. Also, if $\alpha_j = 0$ for some j, then $m = 1$, and so μ is a probability measure. Conversely, if μ is a probability measure, then $c \prod_j \alpha_j = 0$, and so $\alpha_j = 0$ for some j. □

Recall the assumption that B' is a Blaschke product of degree bigger than 1 with a zero at $\alpha_0 = 0$ of multiplicity at least 1. Write t_j for the multiplicity of the zero α_j of B'.

If $f = M \circ \varphi$ where $\varphi \in H^\infty_{B'}$ with $\varphi(0) = 0$, then there are constraints imposed on the corresponding probability measure μ. For $j > 0$,

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$$1 = f(\alpha_j) = \int_{\mathbb{T}} \frac{w+\alpha_j}{w-\alpha_j}\, d\mu(w) = \int_{\mathbb{T}} \left[1 + \frac{2\alpha_j}{w-\alpha_j}\right] d\mu(w) = 1 + 2\alpha_j \int_{\mathbb{T}} \frac{1}{w-\alpha_j}\, d\mu(w),$$

and thus

$$\int_{\mathbb{T}} \frac{1}{w-\alpha_j}\, d\mu(w) = 0, \qquad j > 0.$$

By an induction argument,

$$f^{(k)}(z) = 2k! \int_{\mathbb{T}} \frac{w}{(w-z)^{k+1}}\, d\mu(w) = 2k! \left[\int_{\mathbb{T}} \frac{1}{(w-z)^k}\, d\mu(w) + \int_{\mathbb{T}} \frac{z}{(w-z)^{k+1}}\, d\mu(w)\right].$$

If the multiplicity t_j of the root α_j, is bigger than 1, then as neither M nor its derivatives have any zeros in $\mathbb{D}$, the Faà di Bruno formula implies that

$$f^{(k)}(\alpha_j) = 0, \qquad j > 0 \text{ and } k = 1, \dots t_j - 1.$$

Thus

$$0 = \int_{\mathbb{T}} \frac{1}{(w-\alpha_j)^k}\, d\mu(w), \qquad j > 0 \text{ and } k = 1, \dots t_j. \tag{6}$$

For $z = \alpha_0 = 0$, if $t_0 > 1$, then

$$0 = \int_{\mathbb{T}} \frac{1}{w^k}\, d\mu(w), \qquad k = 1, \dots t_0 - 1. \tag{7}$$

Consequently, the first $t_0 - 1$ moments of μ are zero, and other, more complex constraints are implied by the formulas involving the other roots.

Conversely, suppose that μ is a probability measure for which (6) and (7) hold. If, for example,

$$0 = \int_{\mathbb{T}} \frac{1}{w-\alpha_1}\, d\mu(w),$$

then

$$f(\alpha_1) = \int_{\mathbb{T}} \frac{w+\alpha_1}{w-\alpha_1}\, d\mu(w) = \int_{\mathbb{T}} \frac{w-\alpha_1}{w-\alpha_1}\, d\mu(w) + 2\alpha_1 \int_{\mathbb{T}} \frac{1}{w-\alpha_1}\, d\mu(w) = 1.$$

By the same reasoning, $f(\alpha_j) = 1$ for all j. Similar calculations show that $f^{(k)}(\alpha_j) = 0$ for $1 \le k \le t_j - 1$.

Denote the set of positive measures satisfying the constraints in (6) and (7) by $M^+_{B',\mathbb{R}}(\mathbb{T})$. This is a weak-$*$ closed, convex, locally compact set in the Banach space of finite Borel measures $M_{B',\mathbb{R}}(\mathbb{T}) = \overline{\bigvee M^+_{B',\mathbb{R}}(\mathbb{T})}$, and is additionally a cone since it is closed under sums, positive scalar multiples, and $M^+_{B',\mathbb{R}}(\mathbb{T}) \cap -M^+_{B',\mathbb{R}}(\mathbb{T}) = \{0\}$. Recall that in a convex set A in a vector space X, $E \subset A$ is an *extremal set* if whenever $a \in E$ and $a = tx + (1-t)y$ for $x, y \in A$ and $t \in (0,1)$, it follows that $x, y \in E$. One point extremal sets are *extreme points*, while extremal sets which are half lines are termed *extreme rays* (*extreme directions* in [2]). Here we follow the conventions laid out in Holmes [21].

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Write $M^{+,1}_{B',\mathbb{R}}(\mathbb{T})$ for the probability measures in $M^{+}_{B',\mathbb{R}}(\mathbb{T})$. This set is weak-$*$ closed, convex, and compact, and forms a *base* for $M^{+}_{B',\mathbb{R}}(\mathbb{T})$, in that any $\tilde{\mu} \in M^{+}_{B',\mathbb{R}}(\mathbb{T})$ is of the form $t\mu$ for some $\mu \in M^{+,1}_{B',\mathbb{R}}(\mathbb{T})$ and $t \geq 0$. By the Kreĭn–Mil'man theorem, $M^{+,1}_{B',\mathbb{R}}(\mathbb{T})$ is the closed convex hull of $\hat{\Theta}$, the set of its extreme points, and it is an elementary observation that $\mu \in M^{+,1}_{B',\mathbb{R}}(\mathbb{T})$ is an extreme point if and only if $\{t\mu : t \in \mathbb{R}^+\}$ is an extreme ray in $M^{+}_{B',\mathbb{R}}(\mathbb{T})$.

By the Choquet–Bishop–de Leeuw theorem [25], to any $\mu \in M^{+,1}_{B',\mathbb{R}}(\mathbb{T})$ there corresponds a ν on $\hat{\Theta}$ such that

$$\mu = \int_{\hat{\Theta}} \theta \, d\nu_\mu(\theta).$$

This is reminiscent of the Alexandrov disintegration theorem [12].

For $\mu \in \hat{\Theta}$, define

$$f_\mu(z) := \int_{\mathbb{T}} \frac{w+z}{w-z} \, d\mu(w),$$

an analytic function on $\mathbb{D}$ with positive real part and value 1 when $z = 0$. As in [20], this yields the so-called *Agler–Herglotz representation.*

Theorem 5.2 *(Agler–Herglotz representation associated to $\mathcal{A}_B$). Let f be analytic on $\mathbb{D}$ with positive real part, and suppose further that*

$$f(\alpha_0) = f(\alpha_j) = 1, \quad \text{and} \quad f^{(k)}(\alpha_j) = 0, \quad j = 1, \dots, n, \ 1 \leq k \leq t_j - 1. \tag{8}$$

Then there is a probability measure ν on the set of extreme points $\hat{\Theta}$ of $M^{+,1}_{B',\mathbb{R}}(\mathbb{T})$ such that

$$f(z) = \int_{\hat{\Theta}} f_\mu(z) \, d\nu(\mu). \tag{9}$$

Next turn to concretely characterizing the elements of $\hat{\Theta}$. This is done by first finding a dual characterization of the constraints in terms of the annihilator of H^∞_B. The following is in fact a special case of what is considered by Ball and Guerra-Huamán in [9]. Nevertheless, the special nature of the algebras considered here allow us to give much more specific information.

As usual, $L^2_{\mathbb{R}}(\mathbb{T})$ will stand for the Hilbert space of real valued square integrable functions on the unit circle. Also, $M_{\mathbb{R}}(\mathbb{T})$ stands for the space of finite regular real Borel measures on $\mathbb{T}$, which is the dual of $C_{\mathbb{R}}(\mathbb{T})$ with the norm topology, as well as being the weak-$*$ predual of this space. Every $\mu \in M_{\mathbb{R}}(\mathbb{T})$ is associated by means of a Poisson kernel to a harmonic function $\hat{\mu}$, which in this setting is the real part of a holomorphic function on $\mathbb{D}$. The space $L^2_{\mathbb{R}}(\mathbb{T})$ contains a subspace $L^2_{B',\mathbb{R}}(\mathbb{T})$ consisting of those functions which are the real parts of functions in $\mathcal{A}_{B'}$ restricted to $\mathbb{T}$.

Write $\alpha_0, \dots, \alpha_m$ for the distinct zeros of B' with respective multiplicities $t_0, \dots, t_m$, $N = \sum t_j$. Because each $\alpha_j \in \mathbb{D}$, a function in $\mathcal{A}_{B'}$ can be written as

$$\varphi(z) = c + \prod_0^N (z - \alpha_j)^{t_j} g(z),$$

for some $g \in A(\mathbb{D})$. It is a standard result that the complex annihilator ${\mathcal{A}_{B'}}^\perp$ is isometrically isomorphic to the dual of $A(\mathbb{D})/\mathcal{A}_{B'}$. The latter space is spanned by $z^k + \mathcal{A}_{B'}$, $k = 1, \dots, N-1$, and so has dimension $N-1$. Hence the dimension of ${\mathcal{A}_{B'}}^\perp$ is also $N-1$.

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The kernel functions

$$k_\alpha^{(i)}(z) = k^{(i)}(\alpha, z) := \frac{i! z^i}{(1-\overline{\alpha}z)^{i+1}} \tag{10}$$

have the property that $\left\langle \varphi, k_\alpha^{(i)} \right\rangle = \varphi^{(i)}(\alpha)$, the ith derivative of φ evaluated at α. So for $0 \le j \le m$, $1 \le i \le t_j - 1$ and $\varphi \in \mathcal{A}_{B'}$,

$$\left\langle \varphi, k_\alpha^{(i)} \right\rangle = 0.$$

This accounts for $-m + \sum t_m = N - m$ linearly independent functions in the annihilator. If $m > 0$, fix α_ℓ. Then for $j = 1, \dots, m$ and $j \neq \ell$,

$$\left\langle \varphi, k_{\alpha_\ell}^{(0)} - k_{\alpha_j}^{(0)} \right\rangle = c - c = 0.$$

These $m-1$ functions along with the previous $N-m$ functions then form a linearly independent set, and hence a basis for the complex annihilator of $\mathcal{A}_{B'}$. By the way, this argument works even when no $\alpha_i = 0$. Write $\{g_k\}$ for this set of functions.

These functions are connected to the constraints constructed above, since with the measure μ from the Herglotz representation, there will be h_j such that

$$0 = \langle \varphi, g_j \rangle = \int_{\mathbb{T}} h_j \, d\mu;$$

namely,

$$h_j = \left\langle \frac{1+\varphi}{1-\varphi}, g_j \right\rangle.$$

Lemma 5.3. *Let B be a Blaschke product with zeros $\alpha_0, \dots, \alpha_m$ with multiplicities $t_0, \dots, t_m$, and set $N = \sum t_j$. The annihilator $\mathcal{A}_B{}^\perp$ is an $N-1$ dimensional space, with basis made up of the functions $k^{(i)}(\alpha_j, \cdot)$, for all $0 \le j \le m$ such that $t_j > 1$ and $1 \le i \le t_j - 1$, as well as $k_{\alpha_0}^{(0)} - k_{\alpha_j}^{(0)}$ in case $m > 1$ and $j = 0, \dots, m$.*

If $\varphi \in \mathcal{A}_B$, then both $\operatorname{Re} h_k$ and $\operatorname{Im} h_k$ are orthogonal to μ in $L^2_{B',\mathbb{R}}(\mathbb{T})$. As explained in Section 4.1 of [9] (and generalizing similar results in [2]),

$$M_{B',\mathbb{R}}(\mathbb{T}) = L^2_{B',\mathbb{R}}(\mathbb{T})^\perp = \{\operatorname{Re} h_k, \ \operatorname{Im} h_k\}^\perp_{k=1,\dots,N-1},$$

and

$$C_{B',\mathbb{R}}(\mathbb{T})^\perp = \operatorname{span}\{\operatorname{Re} h_k \, ds, \ \operatorname{Im} h_k \, ds\}_{k=1,\dots,N-1},$$

a $(2N-2)$-dimensional space. Here ds represents arc-length measure on $\mathbb{T}$.

Theorem 5.4. *Let N be the number of zeros of B', counting multiplicities. If μ is an extreme point of $M^{+,1}_{B',\mathbb{R}}(\mathbb{T})$, then it is a probability measure on $\mathbb{T}$ supported at k points, where $N \le k \le 2N-1$.*

Proof. The idea of the proof for the upper bound is the same as for Theorem 5 of [20] and Lemma 3.5 of [2]. Since the codimension of $M_{B',\mathbb{R}}(\mathbb{T})$ in $M_{\mathbb{R}}(\mathbb{T})$ is $2N-2$, if a measure $\mu \ge 0$ is supported at $2N$ or more points, $\dim(M_{B',\mathbb{R}}(\mathbb{T}) \cap M_{\mathbb{R}}(\mathbb{T})) \ge 2$, and so this space contains a nonzero measure $\nu \ge 0$ which is linearly

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independent of μ. For small enough $\epsilon > 0$, $\mu \pm \epsilon\nu \geq 0$, and then since μ is a convex combination of these, it is not extremal.

Now consider the lower bound, and suppose μ is supported at $n < N$ points. By Lemma 5.1, μ is associated to a Blaschke product $\tilde{B} = \prod_1^n \frac{z-\beta_j}{1-\overline{\beta_j} z}$ with n zeros. Let $\alpha_0, \ldots, \alpha_m$ be the zeros of B with multiplicities $t_0, \ldots, t_m$. Since $\tilde{B} \in \mathcal{A}_B$, there is a constant c such that $\tilde{B}(\alpha_j) = c$ for all j. Then for each j, $\tilde{B}(z) - c$ is seen to have a zero of multiplicity t_j at α_j, and so $\tilde{B}(z) - c$ has at least N zeros. But

$$\tilde{B}(z) - c = \frac{\prod_1^n (z - \beta_j) - c \prod_1^n (1 - \overline{\beta_j} z)}{\prod_1^n (1 - \overline{\beta_j} z)},$$

and since the numerator is a polynomial of degree $n < N$, there is a contradiction. □

A similar construction can be carried out for $\mathcal{A}_B^0$. Since $\mathcal{A}_B^0 \subseteq \mathcal{A}_B$, for $\varphi \in \mathcal{A}_B^0$, $f = \frac{1+\varphi}{1-\varphi}$ will have a Herglotz representation with a measure μ satisfying the constraints in (6) and (7), as well as other, more complex constraints if $N > 2$.

Counting the functions in $A(\mathbb{D})$ of the form $z^i B^j$ as given in Theorem 2.2, the dimension of $A_B^{0\perp}$ is found to be $N(N-1)/2$. A total of $N-1$ of these are listed in Lemma 5.3. To recover the other $(N-1)(N-2)/2$ (where without loss of generality it is now assumed that $N > 2$), first observe that by Theorem 2.2, $\mathcal{A}_B^0 \supseteq \mathcal{A}_{B^{N-1}}$. So the remaining elements of a basis for $A_B^{0\perp}$ can be expressed in terms of the basis for $\mathcal{A}_{B^{N-1}}^{\perp}$ (as given by Lemma 5.3) minus those in $\mathcal{A}_B^{\perp}$ from the same lemma; that is, in terms of $k^{(i)}(\alpha_j, \cdot)$, $t_j \leq i \leq (N-1)t_j - 1$. There are a total of $N(N-2) - m$ such functions.

A general function in $\mathcal{A}_B^0$ has the form

$$\varphi(z) = \sum_{r=0}^{N-2} B(z)^r \sum_{s=0}^{r} a_{rs} z^s + B(z)^{N-1} g(z), \quad g \in A(\mathbb{D}),\ a_{rs} \in \mathbb{C} \text{ for all } r, s.$$

Let $v = (v_{\ell,j_\ell}) \in \mathbb{C}^{N(N-2)-m}$ be such that for all $\varphi \in \mathcal{A}_B^0$,

$$\begin{aligned} 0 &= \left\langle \varphi, \sum_{\ell=1}^{m} \sum_{j_\ell=t_\ell}^{(N-1)t_\ell - 1} v_{\ell,j_\ell} k_{\alpha_\ell}^{(j_\ell)} \right\rangle \\ &= \sum_{\ell=1}^{m} \sum_{j_\ell=t_\ell}^{(N-1)t_\ell-1} \overline{v_{\ell,j_\ell}} \sum_{r=0}^{N-2} \sum_{s=0}^{r} \sum_{q=0}^{j_\ell} a_{rs} \binom{j_\ell}{q} B^{r\,(j_\ell - q)}(\alpha_\ell) \alpha_\ell^{s\,(q)} \\ &= \sum_{r=0}^{N-2} \sum_{s=0}^{r} a_{rs} \left[\sum_{\ell=1}^{m} \sum_{j_\ell=t_\ell}^{(N-1)t_\ell-1} \left(\sum_{q=0}^{j_\ell} B^{r\,(j_\ell-q)}(\alpha_\ell) \alpha_\ell^{s\,(q)} \right) \overline{v_{\ell,j_\ell}} \right], \end{aligned}$$

with the shorthand notation $B^{r\,(j_\ell - q)}(\alpha_\ell) = \frac{d^{j_\ell - q}}{dz^{j_\ell - q}} B^r(z)|_{\alpha_\ell}$ and $\alpha_\ell^{s\,(q)} = \frac{d^q}{dz^q} z^s|_{\alpha_\ell}$. Define $N(N-1)/2$ vectors $b^{rs} = (b^{rs}_{\ell,j_\ell}) \in \mathbb{C}^{N(N-2)-m}$ by

$$\begin{aligned} b^{rs}_{\ell,j_\ell} &= \sum_{q=0}^{j_\ell} B^{r\,(j_\ell - q)}(\alpha_\ell) \alpha_\ell^{s\,(q)} \\ &= \begin{cases} \sum_{q=0}^{\min\{s, j_\ell - rq\}} \frac{s!}{(s-q)!} \binom{j_\ell}{q} B^{r\,(j_\ell - q)}(\alpha_\ell) \alpha_\ell^{s-q} & j_\ell - rq \geq 0, \\ 0 & \text{otherwise}, \end{cases} \end{aligned} \tag{11}$$

using the fact that $B^{r\,(n)}(\alpha_\ell) = 0$ if $n < rt_\ell$ for the last equality. Accordingly, there will be $(N-1)(N-2)/2$ nonzero, linearly independent vectors v such that $\langle b^{rs}, v \rangle = 0$.

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As a simple illustration of this, suppose $B(z) = z^N$, $N > 2$. Thus $m = \ell = 1$, $t_\ell = N$ and $N \le j_1 \le N(N-1)-1$. Nonzero entries in b^{rs} require $j_\ell = rN + t$ and $s = t$, so $rN \le j_\ell \le N(N-1)-1$. Clearly, if $r = 0$, then $s = 0$ and so $b^{00}_{1,j_\ell} = 0$ for all j_ℓ. On the other hand, for $r > 0$, it follows that since $s \le r \le N-2$, $j_\ell - rt_\ell \ge (N-1)r$. Thus the only nonzero term in the last sum in (11) occurs when $t = s$. Correspondingly, $j_\ell = rN + s$, in which case the vector b^{rs} has all entries equal to 0 except the $(1, rN+s)$ entry, which equals $s!\binom{rN+s}{s} = \frac{(rN+s)!}{(rN)!}$. It is then a straightforward exercise to choose the set of $(N-1)(N-2)/2$ linearly independent vectors v orthogonal to the vectors b^{rs}. For example, take $k_0^{(i)}(z) = i!z^i$ with $i \ne rN + s$, $1 \le r \le N-2$ and $0 \le s \le r$.

In fact this can be seen more directly, since a typical element of $\mathcal{A}_B^0$ has the form

$$\sum_{r=0}^{N-2}\sum_{s=0}^{r} a_{rs} z^{rN+s} + B(z)^{N-1} g(z), \quad g \in A(\mathbb{D}),\ a_{rs} \in \mathbb{C} \text{ for all } r, s.$$

The $N-1$ basis elements contributed from $\mathcal{A}_B^\perp$ have the form $k_0^{(i)}(z) = i!z^i$, where $1 \le i \le N-1$. Skip the Nth and $(N+1)$st $k_0^{(i)}$ since these are not orthogonal to $B(z) = z^N$ and $zB(z) = z^{N+1}$. However, $k_0^{(i)}$, $N+2 \le i \le 2N-1$, will give 0, and there are $N-2$ of these. Continue in this fashion, with the last basis element being $k_0^{(N(N-1)-1)}$, for a total of $N(N-1)/2$ basis vectors.

As it happens, not all of the functions in

$$\{cB^r z^s : 1 \le r \le N-2,\ 0 \le s \le r,\ cB^r(1) = 1\} \cup \{cB^{N-1} m_\alpha : \alpha \in \hat{\mathbb{D}},\ cB^{N-1}(1) m_\alpha(1) = 1\},$$

will be needed to form a set of test functions. Since for $r \ge s$,

$$\begin{aligned}
&1 - z^s B^r(z) B^{*r}(w) w^{*s} \\
&= 1 - B^{r-s}(z) B^{*r-s}(w) + B^{r-s}(z)(1 - z^s B^s(z) B^{*s}(w) w^{*s}) B^{*r-s}(w) \\
&= (1 - B(z)B^*(w)) + B(z)(1 - B(z)B^*(w))B^*(w) + \\
&\qquad \cdots + B(z)^{r-s-1}(1 - B(z)B^*(w))B^{*r-s-1}(w) + \\
&+ B(z)^{r-s}\left((1 - zB(z)B^*(w)w^*)B(w)^{*r-s} + zB(z)(1 - zB(z)B^*(w)w^*)B^*(w)w^* + \right. \\
&\qquad \left. \cdots + z^{s-1}B(z)^{s-1}(1 - zB(z)B^*(w)w^*)B^{*s-1}(w)w^{*s-1}\right) B^{*r-s}(w) \\
&= h_B(z)(1 - B(z)B^*(w))h_B^*(w) + h_{zB}(z)(1 - zB(z)B^*(w)w^*)h_{zB}^*(w),
\end{aligned}$$

the set

$$\{B, zB\} \cup \{cB^{N-1} m_\alpha : \alpha \in \hat{\mathbb{D}},\ cB^{N-1}(1) m_\alpha(1) = 1\}, \tag{12}$$

will suffice.

As before, equate the elements of the basis for ${\mathcal{A}_B}^{0\perp}$ with a set of constraints on a probability measure μ. Let $\mathcal{R}$ denote the collection of constraints. The elements of $\mathcal{R}$ will involve not only terms like those given in (6) and (7), but also linear combinations of such terms. In the example where $B(z) = z^N$, the constraints have the form given in (7), but now with $k \in \{0, \dots, N-1\} \cup \{N+2, \dots, 2N-1\} \cup \cdots \cup \{N(N-1)-1\}$.

A probability measure μ satisfying the constraints in $\mathcal{R}$ gives rise via the Herglotz representation to an analytic function f on $\mathbb{D}$ which has positive real part and equals 1 at 0. A Cayley transform of f then yields an element φ of the algebra $\mathcal{A}_B^0$ which is zero at 0. Conversely, every such element of the algebra gives rise to a probability measure satisfying these constraints. Write $M^{+,1}_{B',0,\mathbb{R}}(\mathbb{T})$ for the set of all such measures. This is a compact, convex set. Denote by $\hat{\Theta}^0$ the extreme points of this set of measures.

The proofs of the next three results mimic those given earlier in the context of $\mathcal{A}_B$, and so are omitted.

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Theorem 5.5 *(Agler–Herglotz representation associated to $\mathcal{A}_B^0$). Let f be analytic on $\mathbb{D}$ with positive real part, and suppose further that $\varphi = \frac{1-f}{1+f} \in \mathcal{A}_B$ with $\varphi(\alpha_j) = 0$, $1 \le j \le m$. Then there is a probability measure ν on the set of extreme points $\hat{\Theta}^0$ of $M^{+,1}_{B',0,\mathbb{R}}(\mathbb{T})$ such that*

$$f(z) = \int_{\hat{\Theta}^0} f_\mu(z)\, d\nu(\mu).$$

Lemma 5.6. *Let B be a Blaschke product with zeros $\alpha_0, \dots, \alpha_m$ with multiplicities $t_0, \dots, t_m$, and set $N = \sum t_j$. The annihilator ${\mathcal{A}_B}^{0\,\perp}$ is an $N(N-1)/2$ dimensional space containing $\mathcal{A}_B{}^\perp$ and contained in ${\mathcal{A}_{B^{N-1}}}^\perp$.*

Theorem 5.7. *Let N be the number of zeros of B', counting multiplicities. If μ is an extreme point of $M^{+,1}_{B',0,\mathbb{R}}(\mathbb{T})$, then μ is probability measure on $\mathbb{T}$ supported at k points, where $N \le k \le N(N-1)+1$.*

Now translate our results on measures to statements about functions in the unit balls of $\mathcal{A}_{B'}$ and $\mathcal{A}^0_{B'}$. Using a Cayley transform from the right half plane to the unit disk, for each $\mu \in M^{+,1}_{B',\mathbb{R}}(\mathbb{T})$ (respectively, $M^{+,1}_{B',0,\mathbb{R}}(\mathbb{T})$), define a map from $\mathbb{D}$ to itself

$$\psi_\mu := \frac{f_\mu - 1}{f_\mu + 1},$$

where f_μ is the function coming from the Herglotz representation corresponding to μ. Then

$$1 - \psi_\mu(z)\psi_\mu(w)^* = 2\frac{f_\mu(z) + f_\mu(w)^*}{(f_\mu(z)+1)(f_\mu(w)^*+1)}.$$

If μ is an extremal measure in either $M^{+,1}_{B',\mathbb{R}}(\mathbb{T})$ or $M^{+,1}_{B',0,\mathbb{R}}(\mathbb{T})$, by Theorem 5.4, it is a finitely supported atomic probability measure on $\mathbb{T}$. It then follows from Lemma 5.1 that for $\mathcal{A}_B$, $\psi_\mu = B'R_\mu$, where R_μ is a Blaschke product, with number of zeros between 0 and $N-1$ (where a Blaschke product with no zeros is taken to be the constant 1). For $\mathcal{A}^0_B$, R_μ will be a Blaschke product with the number of zeros is between 0 and $N(N-1)+1-N = (N-1)^2$. It is evident from the form of elements of $\mathcal{A}^0_B$ given in (12) that any Blaschke product corresponding to a measure with between N and $N(N-1)$ zeros is of the form B, zB or B^{N-1}. If it has $N(N-1)+1$ zeros, it is of the form $B^{N-1}m_\alpha$, where $m_\alpha = \frac{z-\alpha}{1-\overline{\alpha}z}$.

In both cases, the support of the measure μ corresponds to the set $\psi_\mu^{-1}(1)$. Ultimately, a subset of such functions will be used as test functions. By the realization theorem, any test function can be replaced by a unimodular constant times the test function. So for convenience, identify $\psi_\mu(z)$ with $\overline{\psi_\mu(1)}\psi_\mu(z)$. This amounts to ensuring that 1 is one of the support points for the measure μ. Let Θ (respectively, Θ^0) be the subset of measures in $\hat{\Theta}$ (respectively, $\hat{\Theta}^0$) having 1 as a support point, and write $\Psi_{B'}$ (respectively, $\Psi^0_{B'}$) for the collection $\{\psi_\mu\}_{\mu\in\Theta}$ (respectively, $\{\psi_\mu\}_{\mu\in\Theta^0}$). Then $\Psi_{B'}$ is a set of test functions for $\mathcal{A}_{B'}$ and $\Psi^0_{B'}$ is a set of test functions for $\mathcal{A}^0_{B'}$.

Suppose that φ is in the unit ball of $\mathcal{A}_{B'}$ with $\varphi(0) = 0$. Then $f = M \circ \varphi$ is a holomorphic function from $\mathbb{D}$ to $\mathbb{H}$ for which (8) holds. Also,

$$\varphi = \frac{f-1}{f+1}.$$

Hence

$$1 - \varphi(z)\varphi(w)^* = 2\frac{f(z) + f(w)^*}{(f(z)+1)(f(w)^*+1)}.$$

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As noted above, this in particular holds when $\varphi = \psi_\mu$ and $f = f_\mu$. Applying the Agler–Herglotz representation (Theorem 5.2), there is a probability measure ν on $\hat{\Theta}$ such that (9) holds. Thus

$$\begin{aligned} 1 - \varphi(z)\varphi(w)^* &= \frac{2}{(f(z)+1)(f(w)^*+1)} \int_{\hat{\Theta}} (f_\mu(z) + f_\mu(w)^*)\, d\nu(\mu) \\ &= \int_{\hat{\Theta}} H_\mu(z)(1 - \psi_\mu(z)\psi_\mu(w)^*)H_\mu(w)^*\, d\nu(\mu), \end{aligned} \tag{13}$$

$H_\mu = \frac{(f_\mu+1)}{f+1}$. It follows that

$$1 - \varphi(z)\varphi(w)^* = \Gamma(z,w)(1 - E(z)E(w)^*),$$

with $\Gamma : \mathbb{D} \times \mathbb{D} \to C(\Psi_{B'})$ the positive kernel given by

$$\Gamma(z,w)g = \int_{\hat{\Theta}} H_\mu(z) g(\psi_\mu) H_\mu(w)^*\, d\nu(\mu).$$

More generally, if $\varphi(0) = c \neq 0$, define

$$\tilde{\varphi}(z) = \frac{\varphi(z) - c}{1 - \overline{c}\varphi(z)}.$$

Then

$$1 - \tilde{\varphi}(z)\overline{\tilde{\varphi}(w)} = \frac{(1 - c\overline{c})(1 - \varphi(z)\overline{\varphi(w)})}{(1 - \overline{c}\varphi(z))(1 - c\overline{\varphi(w)})}.$$

Now define Γ as before, but with

$$H_\mu = \frac{\sqrt{1 - c\overline{c}}(1 - \overline{c}\varphi)(f_\mu + 1)}{\tilde{f} + 1},$$

where $\tilde{f} = M \circ \tilde{\varphi}$, and ν is chosen as the probability measure associated to $\tilde{\varphi}$ in the Agler–Herglotz theorem.

Once again, the same arguments work with $\mathcal{A}_B^0$ in place of $\mathcal{A}_B$. Combining Theorem 5.4 with Lemma 5.1 yields the following.

Corollary 5.8. *Let $\hat{\Theta}$ be the set of extreme measures in $M^{+,1}_{B',\mathbb{R}}(\mathbb{T})$. Then $\hat{\Theta}$ is a subset of the atomic probability measures supported at $N \leq k \leq 2N-1$ points in $\mathbb{T}$, where N is the number of zeros of B', counting multiplicity. Furthermore, the set*

$$\left\{ \psi_\mu = cBR_\mu : R_\mu \text{ a Blaschke product with between } 0 \text{ and } N-1 \text{ zeros, } c = \prod \frac{1 - \overline{\alpha_j}}{1 - \alpha_j} \right\}.$$

Corollary 5.9. *Let $\hat{\Theta}^0$ be the set of extreme measures in $M^{+,1}_{B',0,\mathbb{R}}(\mathbb{T})$. Let N be the number of zeros of B' counting multiplicities. Then $\hat{\Theta}^0$ is a subset of the atomic probability measures supported at $rN + s$ points in $\mathbb{T}$, $1 \leq r \leq N-2$, $0 \leq s \leq r$, which are associated to the Blaschke products $B^r z^s$, as well as those coming from B^{N-1} or $B^{N-1}m_\alpha$, m_α a Möbius map. Furthermore,*

$$\Psi^0_{B'} = \{cBz^s : s \in \{0,1\},\ cB(1) = 1\} \cup \left\{ cB^{N-1}m_\alpha : \alpha \in \hat{\mathbb{D}},\ cB^{N-1}(1)m_\alpha(1) = 1 \right\},$$

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is a collection of test functions for $\mathcal{A}^0_{B'}$, where $\hat{\mathbb{D}} = \mathbb{D} \cup \{\infty\}$ is the one point compactification of $\mathbb{D}$ and $m_\infty = 1$.

In Corollary 5.9, ${B'}^{N-1}$ and ${B'}^{N-1}z$ can obviously be removed from the set of test functions (or indeed, any countable subset of $\{{B'}^{N-1}m_\alpha\}$), but then the set would no longer be compact.

Applying a Möbius map if needed gives sets of test functions Ψ_B for $\mathcal{A}_B$ and Ψ^0_B for $\mathcal{A}^0_B$. The sets have exactly the same form, but with B' replaced by B. This is clear for $\mathcal{A}_B$. Let us check it for $\mathcal{A}^0_B$. Assume α is a zero of B, and let $B' = B \circ m_{-\alpha}$, which has a zero at 0. Set $x' = B'$ and $y' = B'z'$, where $z' = m_{-\alpha}$, and use these in defining P', the polynomial whose zero set is the variety for B'. The algebra $\mathcal{A}_{B'}$ has a set of test functions $\Psi^0_{B'}$ as described in Corollary 5.9. These are mapped completely isometrically isomorphically onto a set of test functions $\Psi_{\mathcal{N}(B')}$ for $A(\mathcal{N}_{B'})$. Set $x = x' \circ m_\alpha = B$ and $y = y' \circ m_\alpha = Bz$. This maps $\Psi_{\mathcal{N}_{B'}}$ to $\Psi_{\mathcal{N}_B}$, a set of test functions for $A(\mathcal{N}_B)$. Setting $z = y/x$ identifies these with a set of test functions of $\mathcal{A}^0_B$. Combining these gives a map from $\Psi^0_{B'}$ to Ψ^0_B which takes B' to B, $B'z'$ to Bz and $\{{B'}^{N-1}m_\alpha : \alpha \in \hat{\mathbb{D}}\}$ to $\{B^{N-1}m_\alpha : \alpha \in \hat{\mathbb{D}}\}$.

There is then a corresponding collection of admissible kernels $\mathcal{K}_{\Psi_B}$ (respectively, $\mathcal{K}_{\Psi^0_B}$), and function algebra $H^\infty(\mathcal{K}_{\Psi_B})$ (respectively, $H^\infty(\mathcal{K}_{\Psi^0_B})$).

Theorem 5.10. *The algebras $H^\infty(\mathcal{K}_{\Psi_B})$ and H^∞_B are isometrically isomorphic. Likewise, the algebras $H^\infty(\mathcal{K}_{\Psi^0_B})$ and ${H^\infty_B}^0$ are isometrically isomorphic.*

Proof. The difficult part of the proof has been done above. It is simply left to note that since any test function is in the unit ball of $H^\infty(\mathbb{D})$, the Szegő kernel k_s is admissible (for both $H^\infty(\mathcal{K}_{\Psi_B})$ and $H^\infty(\mathcal{K}_{\Psi^0_B})$). Hence for any function φ in the unit ball of either of these algebras, $((1 - \varphi(x)\varphi(y)^*)k_s(y,x))$ is a positive kernel, and so φ is in the unit ball of H^∞_B (respectively, ${H^\infty_B}^0$). □

6. Minimality of the set of test functions

At this point, Corollaries 5.8 and 5.9 give a fairly concrete description of a set of test functions, especially for $\mathcal{A}^0_{B'}$. However, in dealing with $\mathcal{A}_{B'}$, it is more useful for what follows to describe the test functions in terms of the placement of the zeros rather than the support points for the measure in the Herglotz representation. Obviously, in writing any test function as a Blaschke product, changing the order of the zeros does not change the function. There is also the point that the number of zeros of a test function for $\mathcal{A}_{B'}$ is between N and $2N-1$, where N is the number of zeros of B', so not all test functions will have the same number of zeros.

Introduce the following order on elements of the one point compactification of the disk, $\hat{\mathbb{D}} = \mathbb{D} \cup \infty$, which take these considerations into account: $\zeta_1 \preceq \zeta_2$ in $\hat{\mathbb{D}}$ if either $|\zeta_1| < |\zeta_2|$ or $|\zeta_1| = |\zeta_2|$ and $\arg\zeta_1 \le \arg\zeta_2$. The point ∞ is the maximal element of $\hat{\mathbb{D}}$ with respect to this order, and 0 the minimal element.

This order can be used to describe the set of test functions for $\mathcal{A}_{B'}$. Let $\mathcal{Z}(B') = \{{\alpha'}_0 \le \cdots \le {\alpha'}_{N-1}\}$ be the (ordered) zeros of B'. So $B' = m_{{\alpha'}_0} \cdots m_{{\alpha'}_{N-1}}$, where $m_{{\alpha'}_j}$ is the Möbius map with zero ${\alpha'}_j$. If as an abuse of notation $m_\infty(z) = 1$, then a Blaschke product ${B'}_\alpha = B'R$ with between N and $2N-1$ zeros can be written as

$$ {B'}_\alpha(z) = \prod_{j=0}^{2N-2} m_{\alpha_j}, $$

where $\mathcal{Z}({B'}_\alpha) = \{0 = \alpha_0 \preceq \cdots \preceq \alpha_{2N-2}\}$, the ordered zeros of ${B'}_\alpha$ in $\hat{\mathbb{D}}$, contains $\mathcal{Z}(B')$.

The set

$$ \{c{B'}_\alpha : \mathcal{Z}({B'}_\alpha) \text{ an ordered } 2N-1 \text{ tuple containing the elements of } \mathcal{Z}(B') \text{ and } c = \overline{{B'}_\alpha(1)}\} $$

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contains $\Psi_{B'}$. On the other, by Corollary 5.8, any function in this set has a corresponding Herglotz representation with between N and $2N-1$ support points for the measure, the measure satisfies the constraints (6) and (7) since its zero set contains the zeros of B', and the constant c is chosen so that one of the support points is at 1. Therefore the opposite containment holds.

With this identification, view the measure in (13) as being on the set $\Psi_{B'}$ in place of the set of extremal measures $\hat{\Theta}$, so that

$$1-\varphi(z)\varphi(w)^* = \int\limits_{\Psi_{B'}} H_\psi(z)(1-\psi(z)\psi(w)^*)H_\psi(w)^*\,d\nu(\psi). \tag{14}$$

There is an obvious version of this for the algebra $\mathcal{A}^0_{B'}$, with $\Psi_{B'}$ replaced by $\Psi^0_{B'}$.

Theorem 6.1. *The set $\Psi_{B'}$ is a minimal set of test functions for the algebra $\mathcal{A}_{B'}$.*

Proof. The set $\Psi_{B'}$ is norm closed in $H^\infty(\mathbb{D})$. Endow it with the relative topology, as described in [19]. Suppose that some proper closed subset $\mathcal{C}$ of $\Psi_{B'}$ is a set of test functions for $\mathcal{A}_{B'}$. Then $\Psi_{B'}\backslash\mathcal{C}$ is relatively open, and some $\psi_0 = c_0\prod_{j=0}^{2N-2} m_{\tilde{\alpha}_j}$ is in this set, where $\mathcal{Z}(B') = \{0=\tilde{\alpha}_0, \tilde{\alpha}_{j_1}\ldots,\tilde{\alpha}_{j_{N-1}}\}$ are the zeros of B'. Since $\Psi_{B'}\backslash\mathcal{C}$ is relatively open, assume without loss of generality that no $\tilde{\alpha}_j=\infty$ and that any zero which is not a zero for B' is distinct from the zeros of B' and all such zeros are distinct from each other.

Let $\psi = c\prod_{k=0}^{2N-2} m_{\alpha_j}$ be in $\mathcal{C}$. For any α_k in $\mathcal{Z}(\psi)$ which occurs only once, set $k_{\alpha_j}(z) = 1/(1-\overline{\alpha_j}z)$, the Szegő kernel, where $k_\infty := 0$. More generally, if $\alpha\neq\infty$ is repeated, it is understood that the kernels $k^{(i)}_\alpha(z) = i!z^i/(1-\overline{\alpha}z)^{i+1}$ are used instead, where i runs from 0 to one less than the multiplicity of the root, though this is generally not written explicitly to avoid notational complexity. Define $k_{\tilde{\alpha}_j}$ in an identical manner.

To prove the theorem, argue by contradiction. To begin with, by the same reasoning to that found in the proof of Theorem 9 of [20], for $\psi\in\mathcal{C}$ and $1\leq\ell\leq 2N$, there exist functions $h_{\psi,\ell}\in L^2(\nu)$ such that (14) can be written as

$$1-\psi_0(z)\psi_0(w)^* = \int\limits_{\mathcal{C}}\sum_{\ell=1}^{2N} h_{\psi,\ell}(z)(1-\psi(z)\psi(w)^*)h_{\psi,\ell}(w)^*\,d\nu(\psi). \tag{15}$$

Furthermore, for $n=0,\ldots,2N-2$, there are constants c_{nj} such that

$$h_{\psi,\ell}k_{\alpha_n} = \sum_{j=0}^{2N-2} c_{nj}\,k_{\tilde{\alpha}_j}. \tag{16}$$

In particular, taking $n=0$ gives

$$h_{\psi,\ell} = \sum_{j=0}^{2N-2} c_{0j}\,k_{\tilde{\alpha}_j}.$$

The kernels extend to meromorphic functions on the Riemann sphere, as then does $h_{\psi,\ell}$. Plug this last identity back into (16), for $n>1$, to get

$$k_{\alpha_n}\sum_{j=0}^{2N-2} c_{0j}\,k_{\tilde{\alpha}_j} = \sum_{j=0}^{2N-2} c_{nj}\,k_{\tilde{\alpha}_j}. \tag{17}$$

Now use (17) to eliminate some of the terms and to eventually solve for $h_{\psi,\ell}$. If $\alpha_n\notin\mathcal{Z}(\psi_0)$, the left side of (16) has a pole at $1/\overline{\alpha_n}$, while the right side does not. In this case the only possibility is for $h_{\psi,\ell}=0$.

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Also, if α is a zero of multiplicity t_j in ψ and $\tilde{t}_j$ in ψ_0, then by (16) with k_{α_n} equal to $k_\alpha^{(t_j-1)}(z)$, it follows from (16) by counting pole multiplicities that $t_j \leq \tilde{t}_j$.

Since the number of zeros of $\psi_0 = 2N-1$ and is greater than or equal to the number of zeros of ψ, if the two have the same number of zeros, they are equal (up to multiplicative unimodular constant), which cannot happen. Hence ψ must have fewer than $2N-1$ zeros.

Consider $0 \neq \alpha_n \in \mathcal{Z}(B')$. Then $\alpha_n = \tilde{\alpha}_j$ for some j. If this is a zero of order 1 for ψ_0, then the right side of (17) has a pole of order at most 1 at $1/\overline{\tilde{\alpha}_j}$, while the left side has a pole of order 2 at this point if $c_{0j} \neq 0$. Hence $c_{0j} = 0$.

More generally, suppose that ψ_0 has a zero of order $m > 1$ at $\alpha_n \in \mathcal{Z}(B')$ (where now α_n may be 0). Let $\tilde{\alpha}_j = \cdots = \tilde{\alpha}_{j+m-1}$ be the m repeated zeros. If $\alpha_n \neq 0$, each $k_{\tilde{\alpha}_{j+i}}$, $0 \leq i \leq m-1$, has a pole of order between 1 and m, and so no term on the right side of (17) has a pole of order more than m at $1/\overline{\tilde{\alpha}_j}$. On the left side, if $k_{\alpha_n} = k_\alpha^{(m-1)}$ (which has a pole of order m) and if any of c_{0j} to $c_{0,j+m-1}$ are nonzero, the corresponding term has a pole of order bigger than m. Hence each of these coefficients must be zero.

Things are slightly different when $\alpha_n = 0$. In this case, $j = 0$ and each $k_{\tilde{\alpha}_i}$, $1 \leq i \leq m-1$, has a pole of order between 1 and $m-1$ at ∞ (note that $k_{\tilde{\alpha}_0} = 1$). So reasoning as before, no term on the right of (17) has a pole of order bigger than $m-1$ at ∞, while k_{α_n} has a pole of order $m-1$ there, the left side has a pole of order at least m at ∞ if any of $c_{0,1}$ to $c_{0,m-1}$ are nonzero. So all of these coefficients must also be zero.

Let $\alpha_n \in \mathcal{Z}(\psi)\backslash\mathcal{Z}(B') \subset \mathcal{Z}(\psi_0)\backslash\mathcal{Z}(B')$. By assumption all such zeros are of order 1. Once again, a pole count with (16) gives that the corresponding coefficient in $h_{\psi,\ell}$ is 0.

Combine these observations to conclude that

$$h_{\psi,\ell} = c_{00} + \sum_{\tilde{\alpha}_j \in \mathcal{Z}(\psi_0)\backslash\mathcal{Z}(\psi)} c_{0j} k_{\tilde{\alpha}_j} = g_{\psi,\ell} \prod_{\tilde{\alpha}_j \in \mathcal{Z}(\psi_0)\backslash\mathcal{Z}(\psi)} k_{\tilde{\alpha}_j}.$$

Recall that the elements of $\mathcal{Z}(\psi_0)\backslash\mathcal{Z}(B') \supset \mathcal{Z}(\psi_0)\backslash\mathcal{Z}(\psi)$ are distinct and none are repeats of elements of $\mathcal{Z}(B')$. Consequently,

$$g_{\psi,\ell}(z) = c_{00} \prod_{\tilde{\alpha}_j \in \mathcal{Z}(\psi_0)\backslash\mathcal{Z}(\psi)} (1 - \overline{\tilde{\alpha}_j} z) + \sum_{\tilde{\alpha}_j \in \mathcal{Z}(\psi_0)\backslash\mathcal{Z}(\psi)} c_{0j} \prod_{\tilde{\alpha}_n \in \mathcal{Z}(\psi_0)\backslash\mathcal{Z}(\psi),\, n \neq j} (1 - \overline{\tilde{\alpha}_n} z)$$

is a polynomial of degree at most $N-1$. So (17) becomes

$$g_{\psi,\ell} k_{\alpha_n} \prod_{\tilde{\alpha}_j \in \mathcal{Z}(\psi_0)\backslash\mathcal{Z}(\psi)} k_{\tilde{\alpha}_j} = \sum_{j=0}^{2N-2} c_{nj}\, k_{\tilde{\alpha}_j}. \tag{18}$$

Since by assumption B' has a zero at 0 of degree $t_0 \geq 1$, the right side of (18) has a pole at ∞ of order at most $t_0 - 1$ corresponding to $k_{\alpha_n} = k_0^{(t_0-1)}$. On the other hand, with this choice of k_{α_n}, the left side of (18) has a pole at ∞ of order $\deg g_{\psi,\ell} + t_0 - 1$. Hence $g_{\psi,\ell}$ is a constant, and so

$$h_{\psi,\ell} = g_{\psi,\ell} \frac{\prod_{\tilde{\alpha}_j \in \mathcal{Z}(\psi_0)\backslash\mathcal{Z}(B')} k_{\tilde{\alpha}_j}}{\prod_{\alpha_n \in \mathcal{Z}(\psi)\backslash\mathcal{Z}(B')} k_{\alpha_n}}, \qquad g_{\psi,\ell} \in \mathbb{C}. \tag{19}$$

Substitute the formula for $h_{\psi,\ell}$ from (19) into (15), multiply by $\prod_{\tilde{\alpha}_j \in \mathcal{Z}(\psi_0)\backslash\mathcal{Z}(B')} (1 - \overline{\tilde{\alpha}_j} z)(1 - \overline{\tilde{\alpha}_j} z)^*$ and use the identities in (1) to get

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$$\sum_{m,n=0}^{N-1} \left[z^m \overline{w}^n S_m(\overline{\tilde{\alpha}}) S_n(\overline{\tilde{\alpha}})^* - B'(z) B'(w)^* z^{N-1-m} \overline{w}^{N-1-n} S_m(\tilde{\alpha}) S_n(\tilde{\alpha})^* \right]$$
$$= \int_{\mathcal{C}} \sum_{\ell} \sum_{m,n=0}^{\deg \psi} |g_{\psi,\ell}|^2 \left(z^m \overline{w}^n S_m(\overline{\alpha_\psi}) S_n(\overline{\alpha_\psi})^* - B'(z) B'(w)^* z^{N-1-m} \overline{w}^{N-1-n} S_m(\alpha_\psi) S_n(\alpha_\psi)^* \right) d\nu(\psi). \tag{20}$$

As a shorthand notation, $\tilde{\alpha}$ stands for $\mathcal{Z}(\psi_0) \backslash \mathcal{Z}(B')$ and α_ψ for $\mathcal{Z}(\psi) \backslash \mathcal{Z}(B')$. Since there are only finitely many choices of ψ with $\mathcal{Z}(\psi) \subset \mathcal{Z}(\psi_0)$, the measure ν is finitely supported.

Consider the coefficient of $z^{N-1}\overline{w}^{N-1}$ in (20). On the left side, it is equal to $|S_{N-1}(\overline{\tilde{\alpha}})|^2 = \prod_{\tilde{\alpha}_j \in \mathcal{Z}(\psi_0)\backslash\mathcal{Z}(B')} |\alpha_j|^2 \neq 0$. On the other hand, since $\deg \psi < N-1$, the coefficient on the right side must be 0, giving a contradiction. Thus $\mathcal{C}$ cannot have been a set of test functions for the algebra $\mathcal{A}_{B'}$, and so $\Psi_{B'}$ is a minimal set of test functions. □

The minimality of the set of test functions for $\mathcal{A}^0_B$ follows, with some variation, the same sort of reasoning as in the proof of the last theorem.

Theorem 6.2. *The set $\Psi^0_{B'}$ is a minimal set of test functions for the algebra $\mathcal{A}^0_{B'}$.*

Proof. By Theorem 2.2, the case where B' has $N = 2$ zeros is covered by the last theorem, so assume from now on the $N > 2$. The elements B', zB' and the set $\{B'^{N-1} m_\alpha : \alpha \in \hat{\mathbb{D}}\}$ form distinct components of $\Psi^0_{B'}$. Hence removing an open subset of $\Psi^0_{B'}$ amounts to removing B', zB' or an open subset of $\{B'^{N-1} m_\alpha\}$, or a union of such sets. If the set removed contains a subset of $\{B'^{N-1} m_\alpha\}$, it is assumed that $\psi_0 = B'^{N-1} m_\alpha$ has been chosen with $\alpha \notin \mathcal{Z}(B')$.

As before,

$$1 - \psi_0(z)\psi_0(w)^* = \int_{\mathcal{C}} \sum_{\ell=1}^{L} h_{\psi,\ell}(z)(1 - \psi(z)\psi(w)^*) h_{\psi,\ell}(w)^* \, d\nu(\psi), \tag{21}$$

where now $L = N(N-1) + 2$. Denote zeros of ψ_0 by $\tilde{\alpha}_j$ and those of ψ by α_j. For $n = 0, \ldots, N(N-1)$, there are constants c_{nj} such that

$$h_{\psi,\ell} k_{\alpha_n} = \sum_{j=0}^{N(N-1)} c_{nj} \, k_{\tilde{\alpha}_j}. \tag{22}$$

If $\psi_0 = B'$ or zB', it is understood that the remaining α_js are ∞, and for these, $k_{\alpha_j} = 0$. When $n = 0$, (22) gives

$$h_{\psi,\ell} = \sum_{j=0}^{N(N-1)} c_{0j} \, k_{\tilde{\alpha}_j}. \tag{23}$$

Once again, all kernels and $h_{\psi,\ell}$ extend meromorphically to the Riemann sphere. Substituting this back into (21), for $n > 1$,

$$k_{\alpha_n} \sum_{j=0}^{N(N-1)} c_{0j} \, k_{\tilde{\alpha}_j} = \sum_{j=0}^{N(N-1)} c_{nj} \, k_{\tilde{\alpha}_j}.$$

Arguing as in the proof of Theorem 6.1, $\mathcal{Z}(\psi)$ is a proper subset of $\mathcal{Z}(\psi_0)$. In particular, $\psi_0 = B'$ is immediately ruled out. If $\psi_0 = zB'$,

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$$1 - zB'(z)B'(w)^*w^* = \sum_\ell h_{B',\ell}(z)(1 - B'(z)B'(w)^*)h_{B',\ell}(w)^*.$$

The set $\mathcal{Z}(B')$ consists of $\{\alpha_0, \dots \alpha_m\}$, where α_j has multiplicity t_j. Multiplying through by $\prod_j (1-\overline{\alpha_j}z)^{t_j}$ in (23),

$$g_{B',\ell}(z) = h_{B',\ell}(z)\prod_j (1-\overline{\alpha_j}z)^{t_j}$$

is a polynomial of degree less than $\deg B'$, and

$$\begin{aligned}&\prod_j (1-\overline{\alpha_j}z)^{t_j}\prod_j (1-\overline{\alpha_j}w)^{*\,t_j} - z\prod_j (\alpha_j - z)^{t_j}(\alpha_j - w)^{*\,t_j}w^* - g_{B',\ell}(z)g_{B',\ell}(w)^*\\ =&g_{B',\ell}(z)B'(z)B'(w)^*g_{B',\ell}(w)^*.\end{aligned}$$

If $g_{B',\ell} \neq 0$, there is a $w \in \mathbb{D}$ such that $B'(w)^*g_{B',\ell}(w)^* \neq 0$. Thus $g_{B',\ell}(z)B'(z)$ is a polynomial, and so the zeros of $g_{B',\ell}$ must cancel the poles of $B'(z)$. But since $\deg g_{B',\ell} < \deg B'$, which is impossible. Therefore $\psi_0 \neq zB'$.

Now assume that $\psi_0 = {B'}^{N-1}m_\alpha$, where $0 \neq \alpha \notin \mathcal{Z}(B')$. Since $|\mathcal{Z}(\psi)| < |\mathcal{Z}(\psi_0)|$, $\psi = B'$, zB' or ${B'}^{N-1}$. Since

$$\begin{aligned}&1 - {B'}^{N-1}(z){B'}^{*N-1}(w)\\ =&(1 - B'(z)B'(w)^*) + B'(z)(1 - B'(z)B'(w)^*)B'(w)^* + \cdots + B'(z)^{N-2}(1 - B'(z)B'(w)^*)B'(w)^{*\,N-2}\\ =&h(z)(1 - B'(z)B'(w)^*)h(w)^*,\end{aligned}$$

$h(z) = \begin{pmatrix}1 & B'(z) & \cdots & B'(z)^{N-2}\end{pmatrix}$, without loss of generality, $\psi \in \{B', zB'\}$. Hence

$$\begin{aligned}&1 - {B'}^{N-1}(z)m_\alpha(z)m_\alpha(w)^*{B'}^{N-1}(w^*)\\ =&\sum_\ell \left[h_{B',\ell}(z)(1 - B'(z)B'(w)^*)h_{B',\ell}(w)^* + h_{zB',\ell}(z)(1 - zB'(z)B'(w)^*w^*)h_{zB',\ell}(w)^*\right],\end{aligned} \tag{24}$$

where $h_{B',\ell}$ and $h_{zB',\ell}$ are as in (22).

As before, write the distinct elements of $\mathcal{Z}(B')$ as $\{0 = \alpha_0, \dots \alpha_m\}$, where α_j has multiplicity t_j. Then $\mathcal{Z}(\psi_0)$ consists of $\{\alpha_0, \dots \alpha_m, \alpha\}$, where α_j has multiplicity $(N-1)t_j$ and α has multiplicity 1. By pole counting, the coefficient of the kernel $k_{\alpha_j}^{((N-2)t_j+\ell-1)}$, $1 \leq \ell \leq t_j$ in (23) is 0. Moreover, when $\alpha_j = 0$ and $\psi = zB'$, the coefficient of $k_0^{((N-2)t_0-1)}$ also equals 0. Therefore,

$$h_{\psi,\ell} = g_{\psi,\ell}(1-\overline{\alpha}z)\prod_j (1-\overline{\alpha_j}z)^{-(N-2)t_j},$$

where $g_{\psi,\ell} = \sum_s g_{\psi,\ell,s}z^s = c\prod_j (z - \beta_j)$ is a polynomial of degree at most $N(N-2)-1$ when $\psi = B'$ and $N(N-2)-2$ when $\psi = zB'$. Write Z for the set of zeros of ${B'}^{N-2}m_\alpha$, counting multiplicities, and $\beta_{B',\ell}$, $\beta_{zB',\ell}$ for the set of roots of $g_{B',\ell}$ and $g_{zB',\ell}$. Multiplying through in equation (24) by $(1-\overline{\alpha}z)\prod_j (1-\overline{\alpha_j}z)^{(N-2)t_j}\prod_j (1-\overline{\alpha_j}w)^{*\,(N-2)t_j}(1-\overline{\alpha}w)^*$ and using (1),

$$
\begin{aligned}
&\sum_{m,n=0}^{N(N-2)+1} \left[z^m \overline{w}^n S_m(\overline{Z}) S_n(\overline{Z})^* - B'(z)B'(w)^* z^{N(N-2)+1-m} \overline{w}^{N(N-2)+1-n} S_m(Z) S_n(Z)^* \right] \\
&= \sum_{\ell} \sum_{m,n=0}^{N(N-2)} |g_{B',\ell,0}|^2 \left[z^m \overline{w}^n S_{N(N-2)-m}(\beta_{B',\ell}) S_{N(N-2)-n}(\beta_{B',\ell}) \right. \\
&\qquad \left. - B'(z)B'(w)^* z^{N(N-2)-m} \overline{w}^{N(N-2)-n} S_m(\beta_{B',\ell}) S_n(\beta_{B',\ell}) \right] \\
&\quad + \sum_{\ell} \sum_{m,n=0}^{N(N-2)-1} |g_{zB',\ell,0}|^2 \left[z^m \overline{w}^n S_{N(N-2)-1-m}(\beta_{zB',\ell}) S_{N(N-2)-1-n}(\beta_{zB',\ell}) \right. \\
&\qquad \left. - B'(z)B'(w)^* z^{N(N-2)-m} \overline{w}^{N(N-2)-n} S_m(\beta_{zB',\ell}) S_n(\beta_{zB',\ell}) \right].
\end{aligned} \tag{25}
$$

On the left, the coefficient of $B'(z)B'(w)^* z^{N(N-2)+1} \overline{w}^{N(N-2)+1}$ in (25) is $S_0(Z)S_0(Z)^* = 1$, while on the right, the coefficient is 0, yielding a contradiction. Hence $\Psi^0_{B'}$ is a minimal set of test functions. □

Recalling the discussion preceding Theorem 5.10, the following corollary is seen to hold.

Corollary 6.3. *The set*

$$\Psi_B = \{cB_\alpha : \alpha \text{ an ordered } 2N-1 \text{ tuple containing the elements of } \mathcal{Z}(B) \text{ and } c = \overline{B_\alpha(1)}\}$$

is a minimal set of test functions for $\mathcal{A}_B$, and the set

$$\Psi^0_B = \{B\} \cup \{zB\} \cup \{cBm_\alpha : \alpha \in \hat{\mathbb{D}} \text{ and } c = \overline{B(1)m_\alpha(1)}\},$$

is a minimal set of test functions for $\mathcal{A}^0_B$.

This leads us to a refinement of the realization theorem, Theorem 4.1.

Theorem 6.4. *Let $\Psi = \Psi_B$ (respectively, Ψ^0_B) be the minimal set of test functions for the algebra $\mathcal{A}_B$ (respectively, $\mathcal{A}^0_B$). For $\varphi : \mathbb{D} \to \mathbb{C}$, the following are equivalent:*

(1) *$\varphi \in \mathcal{A}_B$ (respectively, $\mathcal{A}^0_B$) with $\|\varphi\| \le 1$;*
(2) *There is a positive measure μ from $\mathbb{D} \times \mathbb{D}$ to $C(\Psi_B)^*$ (respectively, $C(\Psi^0_B)^*$) and $H_\psi \in H^2(\mathbb{D})$ such that for all $z, w \in \mathbb{D}$,*

$$1 - \varphi(z)\varphi(w)^* = \int_{\Psi} H_\psi(z)(1 - \psi(z)\psi(w)^*) H_\psi(w)^* \, d\mu_{z,w}(\psi).$$

Given a finite set $S \subset \mathbb{D}$, $n = |S|$, write $\mathcal{C}_{1,S}$ for the set of matrices in $M_n(\mathbb{C})$ of the form

$$\left(\int_{\mathcal{C}} (1 - \psi(z)\psi(w)^*) \, d\mu_{z,w}(\psi) \right)_{z,w \in S},$$

$\mathcal{C} = \Psi_B$ (respectively, Ψ^0_B), and $(\mu_{z,w})$ an $M_n(\mathbb{C})$-valued positive Borel measure on Ψ_B. This set is a norm closed cone, contains all positive matrices, and is also closed under conjugation (see [19]). The realization theorem then can be restated as saying that φ is in the unit ball of $\mathcal{A}_B$ (respectively, $\mathcal{A}^0_B$) if and only if for all finite sets $S \subset \mathbb{D}$, the matrix $(1 - \varphi(z)\varphi(w)^*)_{z,w \in S} \in \mathcal{C}_{1,S}$.

As usual, there is also an Agler–Pick interpolation theorem [17] (but see also [8], [13], [22] and [27]).

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Theorem 6.5. *Let Ψ_B (respectively, Ψ_B^0) be the minimal set of test functions for the algebra $\mathcal{A}_B$ (respectively, $\mathcal{A}_B^0$). Let F, a finite subset of $\mathbb{D}$, $|F| = n$, and $\xi : F \to \mathbb{D}$ be given.*

(1) *There exists φ in $\mathcal{A}_B$ (respectively, $\mathcal{A}_B^0$) satisfying $\|\varphi\| \leq 1$ and $\varphi|_F = \xi$;*
(2) *for each k in $\mathcal{K}_{\Psi_B}$ (respectively, $\mathcal{K}_{\Psi_B^0}$), the kernel defined by*

$$F \times F \ni (z, w) \mapsto (1 - \xi(z)\xi(w)^*)k(z, w)$$

is positive.

7. Completely contractive representations and dilations

As Arveson showed, there is an intimate connection between completely contractive representations and dilations. For the disk algebra $A(\mathbb{D})$ and the bidisk algebra $A(\mathbb{D}^2)$, the Sz.-Nagy dilation theorem and Andô's theorem tell us that any representation of one of these algebras which sends the generators to contractions is automatically completely contractive.

For a constrained algebra $\mathcal{A}_B$, there is a similar characterization of those representations which are completely contractive. This was first observed by Broschinski [11] for the Neil algebra $\mathcal{A}_{z^2}$.

Theorem 7.1. *A unital representation $\pi : \mathcal{A}_B \to \mathcal{B}(\mathcal{H})$, $\mathcal{H}$ a Hilbert space, is completely contractive if and only if there is a unitary operator U acting on a Hilbert space $\mathcal{K} \supset \mathcal{H}$ such that for $j \in \mathbb{N}$,*

$$\pi(z^j B) = P_{\mathcal{H}} U^j B(U)|_{\mathcal{H}}.$$

Proof. Suppose that π is a map of the given form. By linearity, π extends to functions of the form pB, p a polynomial. By the spectral theorem for normal operators, the representation is bounded, and so extends to a representation of $\mathcal{A}_B$. Since B is inner, the spectrum of $x(U) := B(U)$ and $y(U) := UB(U)$ define normal operators with spectrum on the boundary of $\mathcal{N}_B$, and so by Arveson's theorem, π is completely contractive.

Conversely, if π is completely contractive, it induces a completely positive map on the operator space $\mathcal{A}_B + \mathcal{A}_B^*$ by $\pi(f + g^*) = \pi(f) + \pi(g)^*$. An application of the Arveson extension theorem extends π to a completely positive map on $C(\mathbb{T})$. The Stinespring dilation theorem then yields a dilation of this to a representation ρ with the property that $\rho(z) = U$, which is unitary. □

By using Theorem 2.2, the same argument gives a dilation theorem for the algebra $\mathcal{A}_B^0$.

Theorem 7.2. *A unital representation $\pi : \mathcal{A}_B^0 \to \mathcal{B}(\mathcal{H})$, $\mathcal{H}$ a Hilbert space, is completely contractive if and only if there is a unitary operator U acting on a Hilbert space $\mathcal{K} \supset \mathcal{H}$ such that for $1 \leq i \leq N - 2$ and $1 \leq j \leq i$, and for $i = N - 1$ and $j \in \mathbb{N}$,*

$$\pi(z^j B^i) = P_{\mathcal{H}} U^j B(U)^i|_{\mathcal{H}}.$$

As in [17], it happens that even though there is a contraction $T := P_{\mathcal{H}} U|_{\mathcal{H}}$, for neither algebra is it necessarily the case that $\pi(B) = B(T)$ and $\pi(zB) = TB(T)$.

Proposition 7.3. *For both $\mathcal{A}_B$ and $\mathcal{A}_B^0$, there is a completely contractive representation π in $\mathcal{B}(\mathcal{H})$ for which there is no operator $T \in \mathcal{B}(\mathcal{H})$ such that $\pi(B) = B(T)$ and $\pi(zB) = TB(T)$.*

Proof. Consider $\mathcal{A}_B$ to begin with. Let $\alpha_0, \ldots, \alpha_m$ be the zeros of B with multiplicities $t_0, \ldots, t_m$, respectively, and recall the functions $g_1, \ldots, g_{N-1}$ defined in terms of the kernel functions $k_{\alpha_j}^{(i)}$ (given in (10)) in

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the paragraphs preceding Theorem 5.4. By definition, $k_{\alpha_j}^{(i)}$ is divisible by z^i (and no higher power of z) and a simple calculation shows that likewise, the functions $k_{\alpha_\ell}^{(0)} - k_{\alpha_j}^{(0)}$, $j \neq \ell$ are divisible by z but no higher power of z. Each g_j is in $H^2(\mathbb{D})$, the functions in $L^2(\mathbb{T})$ (with normalized Lebesgue measure) where the coefficients of z^j are zero when $j < 0$.

Define $\mathcal{H} \subset H^2(\mathbb{D})$ to be the orthogonal complement of the span of g, where either $g = k_{\alpha_j}^{(1)}$ for some j or $g = k_{\alpha_\ell}^{(0)} - k_{\alpha_j}^{(0)}$. The degree of B is at least 2, so there is always one such g in the set of complex annihilators of $\mathcal{A}_B$. Since $\operatorname{ran} B$ is orthogonal to the span of g, $\mathcal{H}$ is invariant under multiplication by both B and zB.

Let U be the bilateral shift on $L^2(\mathbb{T})$, which is unitary. Then $\mathcal{H}$ is invariant under both $B(U)$ and $UB(U)$. Furthermore, $U^*h \in H^2(\mathbb{D})$ and z does not divide U^*h. Since each g_j is divisible by z, this implies that U^*h is not in the annihilator of $\mathcal{A}_B$.

Suppose that there exists $T \in \mathcal{B}(\mathcal{H})$ such that $\pi(B) = B(T) = B(U)|_{\mathcal{H}}$ and $\pi(zB) = TB(T) = UB(U)|_{\mathcal{H}}$. As B is inner, both $\pi(B)$ and $\pi(zB)$ are isometries. The quotient space $\hat{\mathcal{H}} = H^2(\mathbb{D})/\bigvee g$ is isometrically isomorphic to $\mathcal{H}$. Let q be the quotient map. Since $\mathcal{H}$ is invariant under U, T passes to a contraction operator $\hat{T}$ on the quotient space and $\hat{T}^j B(\hat{T})$ are isometries, $j = 0, 1$. Also, there is an isometry $V : \hat{\mathcal{H}} \to L^2(\mathbb{T})$ such that $\hat{\pi}(z^j B) := \hat{T}^j B(\hat{T}) = V^* U^j B(U) V$, $j \in \mathbb{N} \cup \{0\}$. Hence by Theorem 7.1, $\hat{\pi}$ defines a completely contractive representation of $\mathcal{A}_B$ into $\mathcal{B}(\hat{\mathcal{H}})$.

Since $U(U^*g) = g$, $\hat{T}q(U^*g) = 0$. As noted, the map $\hat{T}$ is isometric, and so it follows that $q(U^*g) = 0$. But since U^*g is not in the annihilator of $\mathcal{A}_B \supset \bigvee g$, $q(U^*g)$ cannot be 0, giving a contradiction.

The representation $\hat{\pi}$ of $\mathcal{A}_B$ constructed above restricts to a completely contractive representation of $\mathcal{A}_B^0$. Since there is no operator $\hat{T}$ such that $\hat{T}^j B(\hat{T})$, $j = 0, 1$, and these latter are in $\mathcal{A}_B^0$, the claim holds for $\mathcal{A}_B^0$ as well. □

Along the lines of the example due to Kaijser and Varopoulos on the tridisk [29] (see also [17]), it will be shown that for both $\mathcal{A}_B$ and $\mathcal{A}_B^0$ there are representations sending the generators to contractions which are not contractive.

Theorem 7.4. *There is a non-contractive unital representation π of $\mathcal{A}_B$ (respectively, $\mathcal{A}_B^0$) which maps the generators to contractions.*

Proof. Only the algebra $\mathcal{A}_B^0$ is considered, since the argument for $\mathcal{A}_B$ is identical.

Let $\mathcal{C}_0 = \{B, zB\}$, the generators of $\mathcal{A}_B^0$. By Theorem 6.4, there is a finite set F and a function φ in the unit ball of $\mathcal{A}_B^0$ such that $1 - \varphi(z)\varphi(w)^* = \int_{\Psi_B^0} (1 - \psi(z)\psi(w)^*)\, d\mu_{z,w}(\psi)$, $z, w \in F$, but such that there is no finite positive Borel measure $(\mu_{xy}^0)_{x,y\in F}$ with the property that $1 - \varphi(z)\varphi(w)^* = \int_{\mathcal{C}_0} (1 - \psi(z)\psi(w)^*)\, d\mu_{z,w}^0(\psi)$ for all $z, w \in F$. Consequently, there is a linear functional strictly separating the closed cone

$$\left\{ \left(\int_{\mathcal{C}_0} (1 - \psi(z)\psi(w)^*)\, d\mu_{z,w}^0(\psi) \right)_{z,w\in F} : \mu_{z,w}^0 \text{ a finite positive Borel measure} \right\}$$

from $(1 - \varphi(z)\varphi(w)^*)_{z,w\in F}$. By a standard GNS construction, this results in a unital representation π of $\mathcal{A}_B^0$ for which $\pi(B)$ and $\pi(zB)$ are contractions, yet $\pi(\varphi)$ is not contractive. □

Since $\mathcal{A}(\mathcal{N}_B)$ and $\mathcal{A}_B^0$ are (completely) isometrically isomorphic, this of course means that there is also a non-contractive unital representation of $\mathcal{A}(\mathcal{N}_B)$ which is contractive on generators.

8. Contractive, but not completely contractive, representations of $\mathcal{A}_B$ and $\mathcal{A}_B^0$

In this section, it is proved that for any B with two or more zeros, there exist contractive representations of $\mathcal{A}_B$ which are not completely contractive. Likewise, if B has two or more zeros all of the same multiplicity,

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there exist contractive representations of $\mathcal{A}_B^0$ which are not completely contractive. Indeed, in all cases there is such a representation which is not 2-contractive. The minimal set of test functions is generically referred to as Ψ.

Similarly to Section 6, for a finite set $S \subset \mathbb{D}$, define $\mathcal{C}_{2,S}$ as the set of matrices in $M_{2|S|}(\mathbb{C})$ of the form

$$\left(\int_{\Psi} (1 - \psi(z)\psi(w)^*)\, d\mu_{z,w}(\psi) \right)_{z,w\in S},$$

$(\mu_{z,w})$ a positive Borel measure on Ψ with entries in $M_2(\mathbb{C})$. As with $C_{1,S}$, this is a norm closed cone, contains all positive matrices, and is closed under conjugation (see [17]).

Given a finite set $S \subset \mathbb{D}$, let I_S be the ideal of functions in $\mathcal{A} = \mathcal{A}_B$ or $\mathcal{A}_B^0$ vanishing on S. The quotient map $q : \mathcal{A} \to \mathcal{A}/I_S$ is completely contractive. Assuming the set S and a function Φ in the unit ball of $M_2 \otimes \mathcal{A}$ can be chosen so that $(I_2 - \Phi(z)\Phi(w)^*)_{z,w\in S} \notin \mathcal{C}_{2,S}$ a cone separation argument and GNS construction implies that there is a representation $\tau : \mathcal{A}/I_S \to \mathcal{B}(\mathcal{H})$ with the property that $\pi = q \circ \tau$ is contractive but not 2-contractive (and hence not completely contractive) (see [17], Proposition 3.5).

Following [17], let

$$R(z) = \begin{pmatrix} m_{p_1} & 0 \\ 0 & 1 \end{pmatrix} U \begin{pmatrix} 1 & 0 \\ 0 & m_{p_2} \end{pmatrix}, \tag{26}$$

where U a unitary matrix in $M_2(\mathbb{C})$ with non-zero off diagonal entries, concretely chosen as

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Define $\Phi = B^n(z)R(z)$ with $n = 1$ when $\Phi \in M_2 \otimes \mathcal{A}_B$ and $n = N - 1$ when $\Phi \in M_2 \otimes \mathcal{A}_B^0$. In both cases, $\|\Phi\| \leq 1$.

From here on, $p_1, p_2 \notin \mathcal{Z}(B) \cup \{0\}$ are taken to be distinct points, and S is a set of $2N^2 - 3N + 5$ points in $\mathbb{D}$ containing p_1, p_2 and the zeros in $\mathcal{Z}(B)$ (including repeated roots). In this case, $S' := S \backslash \{\mathcal{Z}(B) \cup \{p_1, p_2\}\}$ consists of $2(N-1)^2 + 1$ distinct points, and we assume that these are chosen so that any polynomial which is zero on S' has degree greater than $2(N-1)^2$. Define

$$\Delta_{\Phi,S} = (1 - \Phi(z)\Phi(w)^*)_{z,w\in S}.$$

There are several results from [17] which will be needed in what follows. Some include small variations on what is found there. Where the proofs are essentially unaltered, they are left out.

Since the zeros of B are $\{\alpha_j\}_0^m$ with multiplicities $\{t_j\}_0^n$, $\sum t_j = N$, it follows that B^{n-1} has the same zeros, but with multiplicities $\{(n-1)t_j\}$ summing to $(n-1)N$. There are then $(n-1)N$ linearly independent kernels $\{k_{\alpha_0}^{(s)} : 0 \leq s \leq (n-1)t_0 - 1\} \cup \cdots \cup \{k_{\alpha_m}^{(s)} : 0 \leq s \leq (n-1)t_m - 1\}$. As a shorthand, we write $\{\tilde{k}_i\}_{i=1}^{(n-1)N}$ for these kernels. When $n = 1$, this is taken to be the empty set.

Lemma 8.1 *([17, Lemma 4.3]). There exist linearly independent vectors $v_1, v_2 \in \mathbb{C}^2$ along with $2(n-1)N+2 = 2n^2$ functions $a_j : S \to \mathbb{C}^2$ in the span of*

$$E = \{k_{p_1}v_1, \tilde{k}_1 v_1, \ldots, \tilde{k}_{(n-1)N}v_1\} \cup \{k_{p_2}v_2, \tilde{k}_1 v_2, \ldots, \tilde{k}_{(n-1)N}v_2\}$$

(with $E = \{k_{p_1}v_1, k_{p_2}v_2\}$ when $n = 1$) such that

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$$\frac{I_2 - B^{n-1}(z)R(z)R(w)^*B^{n-1}(w)^*}{1-zw^*} = \sum_{j=1}^{2n^2} a_j(z)a_j(w)^*.$$

With the algebra $\mathcal{A}_B$ the number of terms will be 2, while for $\mathcal{A}_B^0$ it will be $2(N-1)^2$.

For $\zeta \in \mathbb{D}$,

$$k_\zeta(z) = \frac{\sqrt{1-|\zeta|^2}}{1-z\overline{\zeta}},$$

denotes the normalized Szegő kernel, with $k_\infty = 0$. Then for all $\zeta \in \hat{\mathbb{D}}$ (recall that $m_\infty = 1$),

$$\frac{1 - m_\zeta(z)m_\zeta(w)^*}{1-zw^*} = k_\zeta(z)k_\zeta(w)^*.$$

More generally, if G is a Blaschke product with zero set $\mathcal{Z}(G) = \{\zeta_0 = \infty, \zeta_1, \dots, \zeta_\ell\}$ (including multiplicities),

$$\begin{aligned}\frac{1-G(z)G(w)^*}{1-zw^*} &= \sum_{j=1}^{\ell} \left(\prod_{i=0}^{j-1} m_{\zeta_i}\right) \frac{1-m_{\zeta_j}(z)m_{\zeta_j}(w)^*}{1-zw^*} \left(\prod_{i=1}^{j-1} m_{\zeta_i}\right) \\ &= \sum_{j=1}^{\ell} \left(\prod_{i=0}^{j-1} m_{\zeta_i}\right) k_{\zeta_j}(z)k_{\zeta_j}(w)^* \left(\prod_{i=1}^{j-1} m_{\zeta_i}\right) \\ &= K_\zeta(z)K_\zeta(w)^*,\end{aligned}$$

where $K_\zeta = \begin{pmatrix} k_{\zeta_1} & k_{\zeta_2}m_{\zeta_1} & \cdots & k_{\zeta_\ell}\prod_{i=1}^{\ell-1} m_{\zeta_i}\end{pmatrix}$.

Apply this to $B^{n-1}R_\lambda$, where $R_\lambda = \prod_{j=1}^{N-n} m_{\lambda_j}$ with $\lambda = (\lambda_j)_{j=0}^{N-n} \in \hat{\mathbb{D}}^{N-n}$, to obtain

$$\frac{1-\psi_\lambda(z)\psi_\lambda(w)^*}{1-zw^*} = \frac{1-B(z)B(w)^*}{1-zw^*} + B(z)K_\lambda(z)K_\lambda(w)^*B(w)^*. \tag{27}$$

When $n = 1$ (for $\mathcal{A}_B$), $K_\lambda = \begin{pmatrix} k_{\lambda_1} & \cdots & k_{\lambda_{N-1}}\prod_{i=1}^{N-2} m_{\lambda_i}\end{pmatrix}$, where any term with $\lambda_i = \infty$ is 0, and when $n = N-1$ (for $\mathcal{A}_B^0$), $K_\lambda = \begin{pmatrix} k_{\alpha_1} & \cdots & k_\lambda \prod_{i=1}^{N} m_{\lambda_i}^{N-2}\end{pmatrix}$, and only the last term involves $\lambda \in \mathbb{D}$.

Suppose that $\Delta_{\Phi,S} \in \mathcal{C}_{2,S}$. Applying (27) and Lemma 8.1, there exist linearly independent vectors $v_1, v_2 \in \mathbb{C}^2$ and functions $a_j : S \to \mathbb{C}^2$ in the span of E such that

$$\begin{aligned}\frac{I_2 - \Phi(z)\Phi(w)^*}{1-zw^*} &= \frac{1-B(z)B(w)^*}{1-zw^*}I_2 + B(z)\frac{I_2 - B^{n-1}(z)R(z)R(w)^*B^{n-1}(w)^*}{1-zw^*}B(w)^* \\ &= \frac{1-B(z)B(w)^*}{1-zw^*}I_2 + B(z)\left(\sum_1^{2n^2} a_j(z)a_j(w)^*\right)B(w)^* \\ &= \frac{1-B(z)B(w)^*}{1-zw^*}\mu_{z,w}(\Psi) + B(z)B(w)^*\mu_{zw}(\{zB\}) \\ &\quad + B(z)B(w)^* \int_{\Psi\setminus\{B,zB\}} \frac{1-B^{n-1}(z)R_\lambda(z)R_\lambda(w)^*B^{n-1}(w)^*}{1-zw^*}\, d\mu_{z,w}(\psi_\lambda) \\ &= \frac{1-B(z)B(w)^*}{1-zw^*}\mu_{z,w}(\Psi) + B(z)B(w)^*\mu_{zw}(\{zB\}) \\ &\quad + B(z)B(w)^* \int_{\Psi\setminus\{B,zB\}} K_\lambda(z)K_\lambda(w)^*\, d\mu_{z,w}(\psi_\lambda).\end{aligned} \tag{28}$$

Define positive (ie, positive semidefinite) kernels A, D and $\tilde{D}$ on $S \times S$ by

$$A(z,w) = \mu_{z,w}(\Psi)$$
$$D(z,w) = B(z)\left(\sum_1^{2n^2} a_j(z)a_j(w)^*\right)B(w)^*$$
$$\tilde{D}(z,w) = B(z)\left(\mu_{zw}(\{zB\}) + \int_{\Psi\setminus\{B,zB\}} K_\lambda(z)K_\lambda(w)^*\, d\mu_{z,w}(\psi_\lambda)\right)B(w)^*.$$

Then

$$D(z,w) - \tilde{D}(z,w) = \frac{1 - B(z)B(w)^*}{1 - zw^*}\left(A(z,w) - I_2\right).$$

Lemma 8.2 *(See also [17, Lemma 5.2]). Assume that $\Delta_{\Phi,S} \in C_{2,S}$. With the above notation,*

(i) *The $M_2(\mathbb{C})$ valued kernel $A - [I] := (A(z,w) - I_2)_{z,w\in S}$ is positive;*
(ii) *The $M_2(\mathbb{C})$ valued kernel $D - \tilde{D}$ is positive with rank at most $2(n-1)N + 2$;*
(iii) *The range of $\tilde{D}$ lies in the range of D, which is in $E_B = BE$; and*
(iv) *For $z, w \in S' = S\setminus\{\mathcal{Z}(B) \cup \{p_1, p_2\}\}$, there are at most $2n^2$ functions $r_j : S' \to \mathbb{C}^2$ such that for $i = 1, \dots N$, $r_j m_{\alpha_0} \cdots m_{\alpha_{i-1}} k_{\alpha_i} \in E$ ($m_{\alpha_0} \cdots m_{\alpha_{i-1}} = 1$ if $i = 0$) and*

$$A(z,w) = I_2 + \sum_j r_j(z) r_j(w)^*.$$

Furthermore, if $r_j(z) \neq 0$ for some $z \in S'$ then there are at most $2n^2 - 1$ points in S' where r_j is zero.

Proof. Recall that $\mathcal{Z} = \mathcal{Z}(B) \subseteq \mathcal{Z}(\psi)$ for all $\psi \in \Psi$. Hence for $\alpha \in \mathcal{Z}(B)$ and $w \in S$,

$$I_2 = I_2 - \Phi(\alpha)\Phi(w)^* = \int_\Psi (1 - \psi(\alpha)\psi(w)^*)\, d\mu_{\alpha,w}(\psi_\lambda) = \int_\Psi d\mu_{\alpha,w}(\psi_\lambda) = A(\alpha, w).$$

Fix α and factor $(A(z,w))_{z,w\neq\alpha} = LL^*$. By positivity of A, there is a contraction G such that H, the column matrix of $2N-3$ copies of I_2, can be factored as $H = LG$. Hence $LL^* \geq LGG^*L^* = HH^*$, a $(2N-3)\times(2N-3)$ matrix with all entries equal to I_2. Hence

$$A \geq \begin{pmatrix} HH^* & H \\ H^* & I_2 \end{pmatrix} = [I_2].$$

This takes care of (i).

The kernel $\left(\frac{1-B(z)B(w)^*}{1-zw^*}\right) = K_{\mathcal{Z}}(z)K_{\mathcal{Z}}(w)^* \geq 0$. Since the Schur product of positive matrices is positive,

$$D - \tilde{D} = \left(\frac{1 - B(z)B(w)^*}{1 - zw^*}\right)(A - [I_2]) \geq 0.$$

Thus $\operatorname{ran}\tilde{D} \subset \operatorname{ran} D \subset BE$, proving (ii) and (iii).

Now turn to (iv). By the proof of (i), the rank-nullity theorem, and since $\left(\frac{1-B(z)B(w)^*}{1-zw^*}\right)_{z,w\in S'} > 0$, the rank of $(A(z,w) - I_2)_{z,w\in S'}$ is at most $2n^2$, and so $(A - I_2)(z,w) = \sum_1^{2n^2} r_j(z)r_j(w)^*$, where $r_j : S' \to \mathbb{C}^2$. Thus

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$$\left(K_{\mathcal{Z}}(z)\sum_{1}^{2n^2} r_j(z)r_j(w)^*K_{\mathcal{Z}}(w)^*\right) \le (D(z,w)) = \left(B(z)\left(\sum_{1}^{2n^2} a_j(z)a_j(w)^*\right)B(w)^*\right) \tag{29}$$

The left side of (29) is the sum of positive matrices of the form

$$\left(\left(r_j(z)m_{\alpha_0}(z)\cdots m_{\alpha_{i-1}}(z)k_{\alpha_i}(z)k_{\alpha_i}(w)^*m_{\alpha_{i-1}}(w)^*\cdots m_{\alpha_1}(w)^*r_j(w)^*\right)\right),$$

$1 \le j \le 2n^2$, $1 \le i \le N$, where $m_{\alpha_1}\cdots m_{\alpha_{i-1}} = 1$ if $i = 0$. Consequently,

$$r_j m_{\alpha_1}\cdots m_{\alpha_{i-1}}k_{\alpha_i} \in BE. \tag{30}$$

It is worth noting for later use that the order of the elements in $\mathcal{Z}$ does not effect the r_js.

Write

$$r_j m_{\alpha_0}\cdots m_{\alpha_{i-1}}k_{\alpha_i} = w_{1ji}v_1 + w_{2ji}v_2, \tag{31}$$

where

$$w_{1ji} = B\left(c_{ji0}k_{p_1} + \sum_{\ell} c_{ji\ell}\tilde{k}_\ell\right), \tag{32}$$

with a similar expression for w_{2ji}. (The sums are absent if $N = 2$, and the notation introduced just before Lemma 8.1 is used.) If $r_j(z) \neq 0$ for some $z \in S'$, then either $w_{1ji}(z) \neq 0$ or $w_{2ji}(z) \neq 0$. Assume it is w_{1ji}, since the other case is identical. Clearing the denominators of the term in parentheses in (32), we have a non-trivial polynomial of degree less than $2n^2 - 1$. By the assumption we have made regarding S' and since B is non-zero on S', if this polynomial is 0 at more than $2n^2 - 1$ points, then it is identically zero on $\mathbb{D}$, and consequently, $w_{1ji} = 0$ there, giving a contradiction. The functions m_{α_i} and k_{α_i} are nonzero on S', so $r_j(z) = 0$ on a set of cardinality at most $2n^2 - 1$ if $r_j \neq 0$. □

Lemma 8.3 *(See [17, Lemma 5.3]). Assume $n = 1$ (corresponding to the algebra $\mathcal{A}_B$), or $n = N - 1 > 1$ (corresponding to the algebra $\mathcal{A}_B^0$), in which case it is assumed that B has more than one distinct zero and all zeros of B have the same multiplicity. If $\Delta_{\Phi,S} \in C_{2,S}$, then $A = [I_2]$.*

Proof. If $n = 1$ (corresponding to the algebra $\mathcal{A}_B$), then as noted following (32), $w_{1ji} = c_{ji0}k_{p_1}B$. By Lemma 8.2, if $r_j \neq 0$, $r_j(z) \neq 0$ on a subset of S' of cardinality at least $2(N-1)^2 + 1$. Since $r_j k_{\alpha_1} = c_{j10}k_{p_1}B = c_{ji0}k_{p_1}k_{\alpha_1}B$, and $B(z) \neq 0$ if $z \in S'$, $c_{j10}k_{p_1} = c_{ji0}k_{p_1}k_{\alpha_1}$ on S' and hence meromorphically on $\hat{\mathbb{C}}$. Since $p_1 \neq \alpha_1$, pole counting gives $c_{j10} = c_{ji0} = 0$. If α_1 is a zero of B of multiplicity $t_1 > 1$, a similar argument with $r_j m_{\alpha_1}^\ell k_{\alpha_1}$, with $1 \le \ell \le t_1$ can instead be used. Consequently $r_j = 0$ and so $A = [I_2]$.

Next turn to $\mathcal{A}_B^0$. Assume $n = N - 1 > 1$, that B has more than one distinct zero and all zeros of B have the same multiplicity. Fix j and write $r = r_j$ and $\{\alpha_i\}_{i=0}^m$ for the zeros of B, with multiplicities t. Applying a Möbius map if necessary, there is no loss in generality in assuming that there is an i such that $\alpha_i = 0$.

As was pointed out after (30), the order chosen for the zeros does not effect r. So given a permutation σ of the numbers $\{0,\dots,m\}$, it follows that (31) and (32),

$$r m_{\alpha_{\sigma^{-1}(0)}}\cdots m_{\alpha_{\sigma^{-1}(i-1)}}k_{\alpha_{\sigma^{-1}(j)}} = w_{1j}^\sigma v_1 + w_{2j}^\sigma v_2,$$

where

$$w_{1j}^{\sigma} = B\left(c_{j0}^{\sigma}k_{p_1} + \sum_{\ell=0}^{m}\sum_{s=0}^{(n-1)t-1} c_{j\ell s}^{\sigma}k_{\ell}^{(s)}\right),$$

with a similar expression for w_{2j}^{σ}.

Suppose the ith Möbius map m_i has been applied so that $\alpha_i = 0$. Choose a permutation σ so that $\sigma^{-1}(i) = 0$ and for some $1 \le i' \le m$, $\sigma^{-1}(i') = 1$. Write $p = m_{\alpha_0}(p_1)$, $\beta_s = m_{\alpha_0}(\alpha_s)$, $\kappa_p = k_p$, $\kappa_s^{(\ell)} = k_{\beta_s}^{(\ell)}$, $m_s = m_{m_{\alpha_0}(\alpha_s)}$, and B' for B after the application of m_i. Then w_{1j}^{σ} becomes

$$W_{1j}^{\sigma} = B'\left(c_{j0}^{\sigma}\kappa_p + \sum_{\ell=0}^{m}\sum_{s=0}^{(n-1)t-1} c_{j\ell s}^{\sigma}\kappa_{\ell}^{(s)}\right), \tag{33}$$

and

$$W_{1j}^{\sigma} = W_{10}^{\sigma} m_{\sigma^{-1}(1)} \cdots m_{\sigma^{-1}(j-1)} k_{\sigma^{-1}(j)}. \tag{34}$$

As long as $\sigma^{-1}(i) = 0$, the coefficients in W_{10}^{σ} do not otherwise depend on σ.

By (34) with $j = 1$ and $\sigma^{-1}(1) = i'$, unless the coefficient $c_{j\ell,(n-1)t-1}^{\sigma} = 0$, the right side has a pole of higher order than the left at $1/\overline{\alpha}_{i'}$. Allowing i' to run over all possible choices, the result is that

$$W_{11}^{\sigma} = B'\left(c_{10}^{\sigma}\kappa_p + c_{10,(n-1)t-1}^{\sigma}\kappa_0^{((n-1)t-1)} + \sum_{\ell=0}^{m}\sum_{s=0}^{(n-1)t-2} c_{1\ell s}^{\sigma}\kappa_{\ell}^{(s)}\right).$$

Under the inverse Möbius map, this becomes

$$rk_{\alpha_i} = B\left(c_{i0}k_{p_1} + c_{ii,(n-1)t-1}k_{\alpha_i}^{((n-1)t-1)} + \sum_{\ell=0}^{m}\sum_{s=0}^{(n-1)t-2} c_{i\ell s}k_{\ell}^{(s)}\right).$$

Multiply this equation by $k_{\alpha_{i'}}$, $i' \ne i$, and similarly, multiply the equation for $rk_{\alpha_{i'}}$ by k_{α_i}. Take the difference. Then since B is non-zero on S', the term

$$c_{ii,(n-1)t-1}k_{\alpha_i}^{((n-1)t-1)}k_{\alpha_{i'}} - c_{i'i',(n-1)t-1}k_{\alpha_{i'}}^{((n-1)t-1)}k_{\alpha_i} = 0,$$

as it otherwise is the only term with a pole of order $(n-1)t-1$ at ∞. Thus

$$c_{ii,(n-1)t-1}k_{\alpha_i}^{((n-1)t-2)} = c_{i'i',(n-1)t-1}k_{\alpha_{i'}}^{((n-1)t-2)},$$

and since i' was arbitrary, linear independence of the kernels then implies that $c_{ii,(n-1)t-1} = 0$ for all i. So for all σ, (33) becomes

$$W_{1j}^{\sigma} = B'\left(c_{j0}^{\sigma}\kappa_p + \sum_{\ell=0}^{m}\sum_{s=0}^{(n-1)t-2} c_{j\ell s}^{\sigma}\kappa_{\ell}^{(s)}\right), \qquad j = 0, 1.$$

Now repeating the same argument sufficiently many times, this last equation reduces to

$$W_{1j}^{\sigma} = B'\left(c_{j0}^{\sigma}\kappa_p + \sum_{\ell=0}^{m}\kappa_{\ell}\right), \qquad j = 0, 1.$$

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Hence

$$rk_{\alpha_i} = B\left(c_{i0}k_{p_1} + \sum_{\ell=0}^{m} c_{i\ell}k_{\alpha_\ell}\right), \qquad i = 1, \dots m.$$

If $r(z) = 0$ for $m+1$ or more choices of $z \in S'$, then since $B(z) \neq 0$ on S' and by linear independence of the kernels on the right, all coefficients are zero, meaning that $r = 0$.

So assume this is not the case. For $i' \neq i$, $k_{\alpha_{i'}} r k_{\alpha_i} = k_{\alpha_i} r k_{\alpha_{i'}}$. Since S' is sufficiently large, this must hold for all $z \in \mathbb{D}$, and then extends meromorphically to $\hat{\mathbb{C}}$. But then $k_{\alpha_{i'}} r k_{\alpha_i}$ has a pole at $1/\overline{\alpha}_{i'}$ of order one larger than $k_{\alpha_i} r k_{\alpha_{i'}}$ does unless $c_{ii'} = 0$. Since i' was arbitrary, $c_{ii'} = 0$ for all $i' \neq i$. Hence for all i

$$rk_{\alpha_i} = B\left(c_{i0}k_{p_1} + c_{ii}k_{\alpha_i}\right),$$

and so

$$r(1 - \overline{p}_1 z) = B\left(c_{i0}(1 - \overline{\alpha}_i z) + c_{ii}(1 - \overline{p}_1 z)\right), \qquad i = 1, \dots, m.$$

It follows from the assumption $m > 1$ that

$$c_{i0}(1 - \overline{\alpha}_i z) + c_{ii}(1 - \overline{p}_1 z) = c_{i'0}(1 - \overline{\alpha}_{i'} z) + c_{i'i'}(1 - \overline{p}_1 z), \qquad \alpha_{i'} \neq \alpha_i.$$

Evaluating at three non-collinear points in $S \backslash \mathcal{Z}(B)$ gives $c_{i0} = c_{ii} = 0$ for all i, which yields a contradiction. As a consequence, r must be 0. The proof of the lemma now follows for $\mathcal{A}_B^0$ as well. □

For $\mathcal{A}_B^0$, the last lemma covers, among other things, the setting where all zeros of B are distinct. In fact the result holds more broadly, such as when B has three zeros, two of which are the same and the third distinct, though the proof becomes more complicated. Despite our best efforts, it appears that the case when B has $N > 2$ identical zeros cannot be done in this way, at least with the choice made of the function Φ. We do not have a succinct characterization of all the possible choices of roots of B for which the lemma holds with this choice of Φ.

With minor notational changes, the proof of the following is essentially that of [17, Lemma 5.5]. It recalls (28) and uses the fact, proved in Lemma 4.2 of [17], that for positive $M_2(\mathbb{C})$ valued measures $\mu_{z,w}$ with the property that $\mu_{z,w}(\Psi) = I_2$ for all $z, w \in S$, there is a measure μ independent of z and w such that $\mu_{z,w} = \mu$ for all z, w.

Lemma 8.4 *([17, Lemma 5.5]). For the algebra $\mathcal{A}_B$, if $\Delta_{\Phi,S} \in C_{2,S}$, then there exists an $M_2(\mathbb{C})$ valued measure μ on Ψ such that $\mu(\Psi) = I_2$ and for all $z, w \in S \backslash \mathcal{Z}(B)$,*

$$\frac{I_2 - R(z)R(w)^*}{1 - zw^*} = \int_{\Psi \backslash \{B\}} K_\lambda(z) K_\lambda(w)^* \, d\mu(\psi_\lambda).$$

There is a similar lemma for $\mathcal{A}_B^0$.

Lemma 8.5. *Assume B has two or more distinct zeros and all zeros of B have the same multiplicity. For the algebra $\mathcal{A}_B^0$, if $\Delta_{\Phi,S} \in C_{2,S}$, then there exists an $M_2(\mathbb{C})$ valued measure μ on Ψ such that $\mu(\Psi) = I_2$ and for all $z, w \in S \backslash \mathcal{Z}(B)$,*

$$\frac{I_2 - B^{N-2}(z)R(z)R(w)^*B^{N-2}(w)^*}{1 - zw^*} = \mu(\{zB\}) + \int_{\Psi \backslash \{B, Bz\}} K_\lambda(z) K_\lambda(w)^* \, d\mu(\psi_\lambda).$$

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Here λ lies in $\mathbb{D}\backslash\mathcal{Z}(B)$.

For ν a 2×2 matrix valued measure and $\gamma\in\mathbb{C}^2$, define the scalar measure $\nu_\gamma(\omega)=\gamma^*\nu(\omega)\gamma$. In case ν is a positive measure, ν_γ is also positive.

Lemma 8.6 *(See also [17, Lemma 4.5]). Suppose ν is an $M_2(\mathbb{C})$-valued positive measure on $\Psi\backslash\{B\}$. For each $\gamma\in\mathbb{C}^2$ the measure ν_γ is a nonnegative linear combination of at most n point masses if and only if there exist (possibly not distinct) points $\zeta_1,\dots,\zeta_n\in\hat{\mathbb{D}}^{N-1}\backslash\{\infty^{N-1}\}$ and positive semidefinite matrices $Q_1,\dots,Q_n$ such that*

$$\nu=\sum_{j=1}^{n}\delta_{\zeta_j}Q_j,$$

where for each j, δ_{ζ_j} is a scalar unit point measure on $\Psi\backslash\{B\}$ supported at ψ_{ζ_j}.

Proof. One direction is obvious, so for the converse, assume that every ν_γ is a nonnegative linear combination of at most n point masses. Let ν be a $M_2(\mathbb{C})$-valued measure on $\Psi\backslash\{B\}$, written as (ν_{ij}) with respect to the standard basis $\{e_1,e_2\}$. Obviously, for $i=1,2$, $\nu_{ii}=e_i^*\nu e_i$ is a positive measure and $\nu_{ji}=\nu_{ij}^*$. If $\nu_{ii}(\Omega)=0$ for a Borel subset $\Omega\subset\Psi\backslash\{B\}$, then $\nu_{ij}(\Omega)=0$. Hence ν_{ij} is absolutely continuous with respect to both ν_{11} and ν_{22}, and has its support contained in the intersection of the supports of these measures.

Let $n_{ij}=|\operatorname{supp}\nu_{ij}|$. For $\gamma=\begin{pmatrix}\gamma_1 & \gamma_2\end{pmatrix}^t$,

$$\nu_\gamma=|\gamma_1|^2\sum_{\ell=1}^{n_{11}}c_\ell^{1,1}\delta_{\eta_{1,1},\ell}+|\gamma_2|^2\sum_{\ell=1}^{n_{22}}c_\ell^{2,2}\delta_{\eta_{2,2},\ell}+2\sum_{\ell=1}^{n_{12}}\operatorname{Re}\left(\gamma_1\gamma_2^*c_\ell^{1,2}\right)\delta_{\eta_{1,2},\ell}.$$

Assuming $\gamma_1,\gamma_2=|\gamma_2|e^{i\theta}$ are nonzero, the set $C=\left\{\gamma_1|\gamma_2|e^{-i\theta}c_\ell^{1,2}:\theta\in[0,2\pi)\right\}$ is a circle, and there are at most two values of θ where $2\operatorname{Re}\left(\gamma_1\gamma_2^*c_\ell^{1,2}\right)=-|\gamma_1|^2c_{\ell_1}^{1,1}-|\gamma_2|^2c_{\ell_2}^{2,2}$, with $\delta_{\eta_{1,2},\ell}=\delta_{\eta_{1,1},\ell_1}=\delta_{\eta_{2,2},\ell_2}$. Ranging over all ℓ, there are at most a finite number of such θ. Choosing θ avoiding these points, it follows that $\operatorname{supp}\nu_\gamma=\operatorname{supp}\nu_{11}\cup\operatorname{supp}\nu_{22}$. By assumption, at most n of these points can be distinct, and hence ν has the form claimed. □

Lemma 8.7 *(See also [17, Lemma 5.6]). If $\Delta_{\Phi,S}\in C_{2,S}$ and the algebra is $\mathcal{A}_B$, then the measure μ has the form $\mu=\delta_1P_1+\delta_2P_2+\delta_{12}P_{12}+\delta_\infty P_\infty$, where each P_* is a 2×2 positive matrix, $P_\infty+P_1+P_2+P_{12}=I_2$, each δ_* is a scalar unit point measure on Ψ, with δ_∞, δ_1, δ_2 and δ_{12} supported at B, Bm_{p_1}, Bm_{p_2} and $Bm_{p_1}m_{p_2}$ (in case $Bm_{p_1}m_{p_2}\in\Psi$), respectively. If the algebra is $\mathcal{A}_B^0$ and B has two or more distinct zeros, all with the same multiplicity, then $\mu=\delta_1P_1+\delta_2P_2$.*

For the algebra $\mathcal{A}_B$ with $N=2$, the $Bm_{p_1}m_{p_2}$ term will not be present.

Proof. Write μ_0 for the restriction of μ in Lemma 8.4 to $\Psi\backslash\{B\}$. For $\gamma\in\mathbb{C}^2$, define the scalar valued Borel measure $\nu_\gamma:=\gamma^*\mu_0\gamma$ on $\Psi\backslash\{B\}$. Then by Lemmas 8.4, 8.1 and 8.2 there are $2n^2$ functions $r_j:S\to\mathbb{C}^2$ with ranges contained in $E=\{k_{p_1}v_1,k_{p_2}v_2\}$ if the algebra is $\mathcal{A}_B$ $(n=1)$, or $E=\{k_{p_1}v_1,\tilde{k}_1v_1,\dots,\tilde{k}_{(n-1)N}v_1\}\cup\{k_{p_2}v_2,\tilde{k}_1v_2,\dots,\tilde{k}_{(n-1)N}v_2\}$ in case of $\mathcal{A}_B^0$ $(n=N-1)$, such that

$$\gamma^*\left(\sum_{j=1}^{2n^2}r_j(z)r_j(w)^*\right)\gamma=\int_{\Psi\backslash\{B\}}K_\lambda(z)K_\lambda(w)^*\,d\nu_\gamma(\psi_\lambda).$$

Fix a set of $(n-1)^2+3$ non-zero points $X=\{z_j\}\subset S\backslash\mathcal{Z}(B)$. Using the notation introduced just before Lemma 8.1, set $K=\{\tilde{k}_j\}\cup\{k_{p_1},k_{p_2}\}$ for $\mathcal{A}_B^0$ and $K=\{k_{p_1},k_{p_2}\}$ for $\mathcal{A}_B$. Define a codimension 1

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subspace $V := \bigvee_{k\in K}(k(z))_{z\in X}$ in $\mathbb{C}^{(n-1)^2+3}$, and let $c = (\overline{c(z)})$ be a unit vector orthogonal to V. Suppose that one of the entries of c is zero. Take it to be $c(z_{(n-1)^2+3})$, reordering X if necessary. Then the vector $(\overline{c(z_j)})_1^{(n-1)^2+2} \in \mathbb{C}^{(n-1)^2+2}$ is orthogonal to $\bigvee_{k\in K}(k(z_j))_1^{(n-1)^2+2}$. Since the latter spans $\mathbb{C}^{(n-1)^2+2}$, $c = 0$, giving a contradiction. So no $c(z_j)$ is 0.

Any $\gamma \in \mathbb{C}^2$ is in the span of the dual basis $\{w_1, w_2\}$ to $\{v_1, v_2\}$, and so for any γ,

$$\sum_{z,w\in X} c(z)\gamma^* \left(\sum_{j=1}^{2n^2} r_j(z)r_j(w)^*\right)\gamma c(w)^* = 0.$$

Thus

$$0 = \int_{\Psi\setminus\{B\}} \left|\sum_{z\in X} \psi_\lambda(z)c(z)\right|^2 d\nu_\gamma(\psi_\lambda).$$

For $\mathcal{A}_B^0$ with $N > 2$, this has the form

$$0 = \mu(\{zB\})\|c\|^2 + \int_{\Psi\setminus\{B,zB\}} \left|\sum_{z\in X} K_\lambda(z)c(z)\right|^2 d\nu_\gamma(\psi_\lambda),$$

an immediate consequence of which is that $\mu(\{zB\}) = 0$. Here, $K_\lambda = K_{B^{N-3}m_\lambda}$. Otherwise, for $\mathcal{A}_B$,

$$0 = \int_{\Psi\setminus\{B\}} \left|\sum_{z\in X} K_\lambda(z)c(z)\right|^2 d\nu_\gamma(\psi_\lambda),$$

where now if $\psi_\lambda = BG$, G a Blaschke product, $K_\lambda = K_G$.

For both algebras, it follows that

$$\sum_{z\in X} K_\lambda(z)c(z) = 0 \qquad \nu_\gamma - a.e.$$

Let $\mathcal{Z}(B) = \{\alpha_1, \dots, \alpha_N\}$. Recall that for $\mathcal{A}_B$, $\lambda \in \hat{\mathbb{D}}^{N-1}$ and K_λ is the vector $(\prod_{s=1}^{i-1} m_{\lambda_s} k_{\lambda_i})_{i=1}^N$, with the product absent when $i = 1$. For $\mathcal{A}_B^0$, $\lambda \in \mathbb{D}$ and $K_\lambda = \begin{pmatrix} k_{\alpha_1} & \cdots & B^{N-2}k_\lambda \end{pmatrix}$. In both cases, for ν_γ almost every ψ_λ, for every i and for $\lambda = \lambda_1$,

$$\sum_{z\in X} B^{n-1}(z)k_\lambda \in V.$$

Thus for $z \in X$,

$$B^{n-1}(z)k_\lambda(z) = \sum_{k\in K} c_k k(z).$$

Clearing the denominators, this gives for $i = 0$,

$$\begin{aligned} 0 = &\prod (z-\alpha_\ell)^{n-1}(1-\overline{p}_1 z)(1-\overline{p}_2 z) \\ &- (1-\overline{\lambda}_1 z)\left(a(z)(1-\overline{p}_1 z)(1-\overline{p}_2 z) + (c_1(1-\overline{p}_1 z) + c_2(1-\overline{p}_2 z))\prod(1-\overline{\alpha}_\ell z)^{n-1}\right), \end{aligned}$$

where a is a polynomial of degree at most $(n-1)^2$ and c_1, c_2 are constants. Therefore we have a polynomial of degree at most $(n-1)^2+2$ which is zero at $(n-1)^2+3$ distinct points, and so must be identically zero. Thus the equation holds for all $z \in \mathbb{C}$.

Rewrite this as

$$\begin{aligned}&\left(\prod (z-\alpha_\ell)^{n-1} - a(z)(1-\overline{\lambda}_1 z)\right)(1-\overline{p}_1 z)(1-\overline{p}_2 z)\\ &= \prod (1-\overline{\alpha}_\ell z)^{n-1}(c_1(1-\overline{p}_1 z)+c_2(1-\overline{p}_2 z))(1-\overline{\lambda}_1 z).\end{aligned}$$

The left side has zeros at $1/\overline{p}_1$ and $1/\overline{p}_2$, so the right must as well, implying that for a constant $\tilde{c}$,

$$(c_1(1-\overline{p}_1 z)+c_2(1-\overline{p}_2 z))(1-\overline{\lambda}_1 z) = \tilde{c}(1-\overline{p}_1 z)(1-\overline{p}_2 z).$$

If $\tilde{c}=0$, then $c_1=c_2=0$. This forces $\prod (z-\alpha_\ell)^{n-1} = a(z)(1-\overline{\lambda}_1 z)$, which is impossible. So $\tilde{c}\neq 0$, and $\lambda = p_1$ and $c_2 = 0$, or $\lambda = p_2$ and $c_1 = 0$. In the case of $\mathcal{A}_B^0$, there is nothing further to prove.

For $\mathcal{A}_B$, assume $\lambda_1 = p_1$, and suppose that

$$m_{p_1} k_{\lambda_2} = c_1 k_{p_1} + c_2 k_{p_2}.$$

Clearing the denominators gives a quadratic polynomial which is zero at the three points of X, and hence in all of $\mathbb{C}$. Therefore this equation holds meromorphically in $\hat{\mathbb{C}}$. If $c_2 = 0$, then $\lambda_2 = \infty$, which is the case already considered. Likewise it may be assumed that $c_1 \neq 0$. Thus both sides have poles at $1/\overline{p}_1$ and $1/\overline{p}_2$; that is, $\lambda_2 = p_2$. The case where $\lambda_1 = p_2$ is identical, and then $\lambda_2 = \infty$ or p_1.

Next assume that B has 3 or more zeros and

$$m_{p_1} m_{p_2} k_{\lambda_3} = c_1 k_{p_1} + c_2 k_{p_2}.$$

Clearing denominators again gives a quadratic polynomial which is zero at the three points of X, and hence in all of $\mathbb{C}$. The equation thus holds meromorphically in $\hat{\mathbb{C}}$. Examining the poles, it is clear that the only possibility is for $\lambda_3 = \infty$, and so this reduces to the last case considered.

Consequently, for $\mathcal{A}_B$ there are at most three distinct possibilities, where exactly one $\lambda_j = p_1$ and the rest are ∞, where exactly one $\lambda_j = p_2$ and the rest are ∞, and where one $\lambda_j = p_1$ another equals p_2 and the rest are ∞. □

Theorem 8.8. *Suppose that $\Phi = B(z)R(z) \in M_2 \otimes \mathcal{A}_B$ and $\Phi = B(z)^{N-1}R(z) \in M_2 \otimes \mathcal{A}_B^0$, with R chosen as in (26), and B having two or more distinct zeros all with the same multiplicity in the latter case. Then $\Delta_{\Phi,S} \notin C_{2,S}$. Consequently, there are contractive unital representations of $\mathcal{A}_B$ and $\mathcal{A}_B^0$ which are not 2-contractive, and hence not completely contractive.*

Proof. From Lemma 8.7, for $\mathcal{A}_B$,

$$R(z)R(w)^* = m_{p_1}(z)m_{p_1}(w)^* P_1 + m_{p_2}(z)m_{p_2}(w)^* P_2 + m_{p_1}(z)m_{p_2}(z)m_{p_2}(w)^* m_{p_1}(w)^* P_{12} + P_\infty$$

Hence

$$R(p_1)R(p_2)^* = \frac{1}{2}\begin{pmatrix} 0 & 0 \\ m_{p_1}(p_2)^* & 1 \end{pmatrix} = P_\infty \geq 0.$$

Thus $p_2 = p_1$, a contradiction.

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For $\mathcal{A}_B^0$, by Lemma 8.7, $B^{N-1}(z)R(z)R(w)^*B^{N-1}(w)^* = m_{p_1}(z)m_{p_1}(w)^*P_1 + m_{p_2}(z)m_{p_2}(w)^*P_2$. This time, evaluating at $z = p_1$, $w = p_2$ gives

$$0 = \frac{1}{2}B^{N-2}(p_1)B^{N-2}(p_2)^* \begin{pmatrix} 0 & 0 \\ m_{p_1}(p_2)^* & 1 \end{pmatrix},$$

which is only possible if p_1 or $p_2 \in \mathcal{Z}(B)$, contrary to assumption.

Thus $\Delta_{\Phi,S} \notin C_{2,S}$. □

Corollary 8.9. *For any finite Blaschke product B with two or more zeros, rational dilation fails over the distinguished variety $\mathcal{V}_B \subseteq \overline{\mathbb{D}}^N$. If B has two or more distinct zeros all of the same multiplicity, rational dilation fails over the distinguished variety $\mathcal{N}_B \subseteq \overline{\mathbb{D}}^2$.*

9. Conclusion

As mentioned at the end of Section 2, for $2 \leq n \leq N$, and a Blaschke product B with N zeros, one can consider the algebra generated by $B, Bz, \ldots, Bz^n$. This will be completely isometrically isomorphic to an algebra of holomorphic functions on a distinguished variety in $\overline{\mathbb{D}}^n$, and arguments similar to those given here can be used to study such algebras and the rational dilation problem over the attendant varieties.

We speculate that rational dilation fails on $\mathcal{A}(\mathcal{N}_B)$ even without the restrictions imposed here. There are a few cases we have been able to verify (for example, B with three zeros, two of which are the same). However, what on the surface should be the easiest case (B has three or more zeros which are all the same) resists our approach, at least with the Φ used here.

For both $\mathcal{A}(\mathcal{N}_B)$ and $\mathcal{A}(\mathcal{V}_B)$ (as well as the other algebras mentioned), there is thus a hierarchy of unital representations. The nicest, though smallest class, are those which are completely contractive. Next are the contractive representations, a class that in a small coterie of examples (eg, simply connected planar sets with smooth boundaries, the bidisk, annuli, the symmetrized bidisk [5]) agrees with the completely contractive representations. In the context of this paper, it is however strictly larger, since there are examples of contractive representations which are not 2-contractive, much as with the tri-disk [24]. It is natural to wonder if there are 2-contractive representations which are not 3-contractive, and so on. Possibly some of the ideas presented here, along with the work of Ball and Guerra Huamán [9,10], could enable the construction of a minimal set of test functions for $M_2(\mathbb{C}) \otimes \mathcal{A}(\mathcal{N}_B)$ and $M_2(\mathbb{C}) \otimes \mathcal{A}(\mathcal{V}_B)$, which would be a first step in analyzing this question.

Function algebras on distinguished varieties are intimately connected to function algebras on multiply connected domains [28,30] (see also [17] and [23]). Perhaps the techniques employed here will enable the extension of the results in [26] to general multiply connected domains.

Finally, there is the class of bounded unital representations which send the algebra generators to contractions. As in the case of the tridisk [29], these are seen to comprise a strictly larger class of representations than that of the contractive representations of $\mathcal{A}(\mathcal{V})$ for the varieties $\mathcal{V}$ we considered here. By comparison, for the disk, bidisk and symmetrized bidisk, such representations are automatically contractive, and so as noted, completely contractive. This is not universal though. Consider an annulus $\mathbb{A}$ (assumed without loss of generality to be centered at 0 with outer radius 1 and inner radius r). Any minimal set of test functions over this set is infinite, so there are representations sending the generators z and r/z of $A(\mathbb{A})$ to contractions which are not contractive representations. On the other hand, there is a uniform bound for the norm growth in this case, since the von Neumann inequality holds up to multiplication of the function norm by $K > 1$ (see [7,14], as well as [15] and [16]). This relation between spectral and complete K-spectral sets is another area worth exploring in $\mathcal{A}(\mathcal{V})$.

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