



SUMS OF REGULAR SELFADJOINT OPERATORS IN HILBERT-C*-MODULES

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ABSTRACT. We introduce a notion of weak anticommutativity for a pair (S, T) of self-adjoint regular operators in a Hilbert C^* -module E . We prove that the sum $S + T$ of such pairs is self-adjoint and regular on the intersection of their domains. A similar result then holds for the sum $S^2 + T^2$ of the squares. We show that our definition is closely related to the Connes-Skandalis positivity criterion in KK -theory. As such we weaken a sufficient condition of Kucerovsky for representing the Kasparov product. Our proofs indicate that our conditions are close to optimal.

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1. INTRODUCTION

A well-known problem in functional analysis is to describe the domain and the spectral properties of the sum of two densely defined closed operators. In general nothing can be said as the intersection of the domains can be just $\{0\}$. The problem has a rich history and therefore in the next two sections we will summarize what is known in two quite different contexts. Thereafter we will describe the main theme of the paper.

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1.1. Banach space history of the problem. Given two densely defined unbounded operators A, B in some Banach space X with a joint ray, e.g. $i(0, \infty)$ or $(-\infty, 0)$, in the resolvent set. A basic problem is to give criteria which ensure the following to hold:

- (1) $\overline{A+B} - \lambda$ is invertible for λ in the said ray and large enough.
- (2) $A+B$ is a closed operator with domain $\mathcal{D}(A) \cap \mathcal{D}(B)$.

One of the first comprehensive papers on the problem [DPGr75] was motivated by evolution equations

$$\underbrace{-\partial_t^2 u}_A + \underbrace{\Lambda(t) u}_B + \lambda u = f,$$

with $\Lambda(t)$ being a family of partial differential operators parametrized by t .

The validity of (1) means that the equation $Ax+Bx+\lambda x = y$ is *weakly* solvable for λ large, that is given y there is a sequence $x_n \in \mathcal{D}(A) \cap \mathcal{D}(B)$ such that $x_n \rightarrow x$ and $(A+B+\lambda)x_n \rightarrow y$. (1) and (2) together mean that the equation $Ax+Bx+\lambda x = y$ is *strongly* solvable for λ large, that is given y there exists a solution $x \in \mathcal{D}(A) \cap \mathcal{D}(B)$.

One, and essentially the only approach to the problem in the Banach space context rests on the idea of viewing $A+B+\lambda$ as a (operator valued) function of B and writing the resolvent $(A+B+\lambda)^{-1}$ as the Dunford integral

$$P_\lambda := \frac{1}{2\pi i} \int_\Gamma (z + \lambda + A)^{-1} \cdot (z - B)^{-1} dz, \quad (1.1)$$

where Γ is a suitable contour encircling the spectrum of B . This approach works well only for *sectorial* operators with spectral angle $< \pi/2$. Eq. (1.1) equals the resolvent only if A and B are resolvent commuting and so it is not surprising that in the literature certain commutator conditions are formulated to ensure that Eq. (1.1) gives an appropriate approximation to the resolvent [DPGr75, DoVe87, LaTe87, Fuh93, MoPr97, KAWe01, PrSi07, Ro116].

1.2. KK-theory history of the problem. In the completely different context of KK-theory [Kas80] one encounters the problem of regular sums of operators when one tries to construct the notoriously complicated Kasparov product at the level of unbounded cycles [Mes14, BMvS16, MeRe16, KaLe12, KaLe13].

Here, the operators in question act on a Hilbert- (A, B) -bimodule E , which is a complete inner product module over the C^* -algebra B . For an unbounded B -linear operator T in E it makes sense to talk about self-adjointness and hence one might be tempted to believe that everything is as nice as in a Hilbert space. This, unfortunately (or fortunately), is not the case as the axiom of *regularity* does not come for free: analogously as in the Banach space context above an unbounded self-adjoint B -linear operator S in E is called *regular* if $S \pm \lambda$ has dense range for one and hence for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$. If $B = \mathbb{C}$ then regularity is equivalent to self-adjointness. In general, it is an additional feature, cf. [BAJu83, Wor91, Pie06, KaLe12].

An *unbounded Kasparov module* is a triple (A, E, D) consisting of a Hilbert (A, B) -bimodule E and a self-adjoint regular operator D , with compact resolvent, that

commutes with the dense subalgebra $\mathcal{A} \subset \mathcal{A}$ up to bounded operators. In the construction of the tensor product of two such modules (\mathcal{A}, X, S_X) and (\mathcal{B}, Y, T_Y) one encounters two problems.

The first one is the definition of the operator $T = 1 \otimes_{\nabla} T_Y$ on the module $E := X \otimes_{\mathcal{B}} Y$. Since T does not commute with B , one needs to incorporate extra data in the form of a *connection* ∇ . This is discussed in great generality in [MeRe16] and in this paper we will not be concerned with this construction.

Once a well-defined self-adjoint and regular connection operator T on E has been constructed from T_Y , the second problem that needs to be addressed is self-adjointness and regularity of the sum $D = S + T$, where $S = S_X \otimes 1$. The goal is then to formulate an appropriate smallness condition on the graded commutator $ST + TS$ such that $S + T$ is self-adjoint and regular on $\mathcal{D}(S) \cap \mathcal{D}(T)$.

The Banach space results mentioned in the previous paragraph do not (at least not a priori) apply to this situation as in general self-adjoint operators are sectorial with spectral angle $\pi/2$. Hence the sum of the spectral angles of S and T is π which is exactly the threshold for the validity of regularity results for sectorial operators, cf. Theorem 6.1 below. The methods in the Hilbert module case therefore resemble much more the methods known from Hilbert space theory.

1.3. The main results. Here we offer the following result which contains all previously known results in this context as special cases [MeS14, KaLe12, MeRe16].

Theorem 1.1. *Let S, T be self-adjoint and regular operators in the Hilbert-B-module E . Assume that*

- (1) *there are constants $C_0, C_1, C_2 > 0$ such that the form estimate*

$$\langle [S, T]x, [S, T]x \rangle \leq C_0 \cdot \langle x, x \rangle + C_1 \cdot \langle Sx, Sx \rangle + C_2 \cdot \langle Tx, Tx \rangle \quad (1.2)$$

holds for all $x \in \mathcal{F} := \mathcal{F}(S, T) = \{x \in \mathcal{D}(S) \cap \mathcal{D}(T) \mid Sx \in \mathcal{D}(T), Tx \in \mathcal{D}(S)\}$. This is an inequality in the C^ -algebra B .*

- (2) *There is a core $\mathcal{E} \subset \mathcal{D}(T)$ such that $(S + \lambda)^{-1}(\mathcal{E}) \subset \mathcal{F}(S, T)$ for $\lambda \in i\mathbb{R}$, $|\lambda| \geq \lambda_0$.*

Then $S + T$ is self-adjoint and regular on $\mathcal{D}(S) \cap \mathcal{D}(T)$. That is for $z \in \mathbb{C} \setminus \mathbb{R}$ and $y \in E$ the equation

$$Sx + Tx + z \cdot x = y$$

has a unique (strong) solution $x \in \mathcal{D}(S) \cap \mathcal{D}(T)$.

A more elaborate formulation can be found below in Theorem 2.6. Our main application of Theorem 2.6 is to the calculation of the Kasparov product of unbounded cycles in KK-theory.

Historically, the main tool for handling the Kasparov product has consisted of a guess-and-check procedure pioneered by Connes-Skandalis [CoSk84], and later refined by Kucerovsky [Kuc97]. This entails checking a set of three sufficient conditions to determine whether a cycle (\mathcal{A}, E, D) is the product of the cycles (\mathcal{A}, X, S_X) and (\mathcal{B}, Y, T_Y) . Although this avoids the aforementioned hard problems, it leaves one with the burden of coming up with a good guess for D in every particular instance, as well as proving that (\mathcal{A}, E, D) is a cycle.

In recent years, significant progress has been made on the constructive approach to finding D . In this setting, the first sufficient condition of Kucerovsky is satisfied whenever $D = S + T$ and T is a connection operator relative to T_Y . The second condition will be satisfied whenever $\mathcal{D}(S + T) \subset \mathcal{D}(S)$. In previous work the condition

$$\langle [S, T]x, [S, T]x \rangle \leq C(\langle x, x \rangle + \langle Sx, Sx \rangle),$$

was imposed to ensure self-adjointness of the sum $S + T$. This condition implies that

$$\langle (S + T)x, Sx \rangle + \langle Sx, (S + T)x \rangle \geq -\kappa \langle x, x \rangle,$$

for some $\kappa > 0$, which is the third sufficient condition appearing in [Kuc97, Theorem 13]. The form estimate (1.2) is in general not compatible with Kucerovsky's estimate. In Section 7 we prove that it is nonetheless sufficient to construct the Kasparov product.

Theorem 1.2. *Let (A, X, S_X) and (B, Y, T_Y) be unbounded Kasparov modules for (A, B) and (B, C) respectively and let $E := X \otimes_B Y$ and $S := S_X \otimes 1$. Suppose that $T : \mathcal{D}(T) \rightarrow E$ is an odd self-adjoint regular connection operator for T_Y such that*

- (i) *for all $a \in A$ we have $a : \mathcal{D}(T) \rightarrow \mathcal{D}(T)$ and $[T, a] \in \mathcal{L}(E)$;*
- (ii) *(S, T) is a weakly anticommuting pair.*

Then $(A, E, S + T)$ is an unbounded Kasparov module that represents the Kasparov product of (X, S_X) and (Y, T_Y) .

We note that the statement that the sum operator $D = S + T$ is a KK-cycle is part of this result. The proof consists of showing that weak anticommutation implies a weakened version of the sufficient conditions of Connes-Skandalis. In the constructive setting, this supersedes the result of Kucerovsky and covers a wider range of examples, provided that we construct our operator as a sum.

1.4. Outline. The paper is organized as follows: In Section 2 we first fix some notation and then introduce the decisive notion of a weakly anticommuting pair of self-adjoint regular operators. We put this definition into context and give a detailed comparison to previous such notions. Furthermore, by employing Clifford matrices we show how to switch back and forth between weakly anticommuting operators and weakly commuting operators. Thereafter we formulate our main Theorem on sums of self-adjoint regular operators followed by an outline of the structure of the proof.

The proof of the main Theorem on sums is spread over the technical Sections 3 and 4. In Section 5 we provide some applications. S^2 and T^2 are sectorial operators with spectral angle 0. So they cry for a Dore-Venni type Theorem. Although, they do not fulfill the prerequisites for any of the Dore-Venni type Theorems we know of we can nevertheless prove that $S^2 + T^2$ is self-adjoint and regular on $\mathcal{D}(S^2) \cap \mathcal{D}(T^2)$. As another application we show that the sum Theorem can be iterated to handle triples, and hence an arbitrary number of weakly anticommuting summands. This is motivated by the second author's program of constructing an appropriate category of KK-cycles.

In the survey type Section 6 we elaborate a bit more on the Banach space approach to sums of operators and outline an alternative approach to our main Theorem along the lines of the original Da Prato-Grisvard Dunford integral Eq. (1.1), the main result being Theorem 6.4.

The details on Theorem 1.2 can be found in Section 7. Finally, the appendix contains a few useful commutator identities which are needed in the proofs.

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2. WEAKLY ANTICOMMUTING OPERATORS AND SUMS

2.1. Notation. We assume familiarity with C*-algebras, Hilbert-C*-modules and unbounded and regular operators in Hilbert-C*-modules [LAN95, KAL12].

In the sequel E will always be a Hilbert-C*-module over the C*-algebra B . By $\mathcal{L}(E)$ we denote the C*-algebra of bounded adjointable module endomorphisms. S, T will denote self-adjoint regular operators in E . Domains of (semi)regular operators are denoted by $\mathcal{D}(\dots)$. Note that these are always dense submodules. Unless otherwise said, λ, μ denote resolvent parameters which are purely imaginary but bounded below by some $\lambda_0 > 0$; the specific value of λ_0 is irrelevant and may vary from statement to statement. In norm estimates C_1, C_2, \dots denote generic constants; they may also vary from statement to statement.

The basic problem we address is: if $ST + TS$ is “small” then $S + T$ should be self-adjoint and regular.

In the context of sectorial operators in certain Banach spaces this is a well studied problem with numerous publications, e.g. [DPGr75, DoVe87, PRSi07] and the references therein.

2.2. Weakly anticommuting operators. For a pair of operators S, T in a Hilbert-B-module E , we denote by $[S, T]$ the *anticommutator* $ST + TS$. This is in line with conventions regarding graded Hilbert-C*-modules and graded commutators in case the operators S and T are both odd for the grading. Note that if either S or T is not everywhere defined, then neither is the anticommutator $[S, T]$.

However, in order to work with commutators and anti-commutators at the same time, it will be convenient to also use the notation

$$[S, T]_{\pm} = ST \pm TS = \pm[T, S]_{\pm}.$$

Moreover, we will use the convention $[S, T] = [S, T]_{+}$. Some commutator identities are collected in the Appendix A.

Definition 2.1. Let S, T be self-adjoint and regular operators in the Hilbert-B-module E and set

$$\mathcal{F} := \mathcal{F}(S, T) = \{x \in \mathcal{D}(S) \cap \mathcal{D}(T) \mid Sx \in \mathcal{D}(T), Tx \in \mathcal{D}(S)\}. \quad (2.1)$$

The pair (S, T) is called weakly anticommuting if

- (1) there are constants $C_0, C_1, C_2 > 0$ such that for all $x \in \mathcal{F}$ the form estimate

$$\langle [S, T]x, [S, T]x \rangle \leq C_0 \cdot \langle x, x \rangle + C_1 \cdot \langle Sx, Sx \rangle + C_2 \cdot \langle Tx, Tx \rangle \quad (2.2)$$

holds in B .

- (2) There is a core $\mathcal{E} \subset \mathcal{D}(T)$ such that $(S + \lambda)^{-1}(\mathcal{E}) \subset \mathcal{F}(S, T)$ for $\lambda \in i\mathbb{R}$, $|\lambda| \geq \lambda_0$.

The pair is called weakly commuting if the estimate Eq. (2.2) holds with the commutator $ST - TS$ instead of the anticommutator $[S, T]$.

Remark 2.2. 1. This notion of weak (anti)commutativity is slightly more general than corresponding notions in [KALe12, Assumption 7.1], [MeRe16, Appendix A], [Les16, Sec. 3]. Cf. Section 2.3 below.

2. One should also compare weak (anti)commutativity to the commutator conditions appearing in earlier Banach space literature on sums of operators; see Sections 1.1 and 6.

3. By definition

$$\mathcal{F}(S, T) = (S + \lambda)^{-1}(\mathcal{D}(T)) \cap (T + \lambda)^{-1}(\mathcal{D}(S)).$$

If (2) holds with $\mathcal{E} = \mathcal{D}(T)$ then this implies the equality

$$\begin{aligned} \mathcal{F}(S, T) &= (S + \lambda)^{-1}(\mathcal{D}(T)) \\ &= \text{ran}((S + \lambda)^{-1} \cdot (T + \mu)^{-1}), \end{aligned} \quad \lambda, \mu \in i\mathbb{R}, |\lambda|, |\mu| \geq \lambda_0. \quad (2.3)$$

By Theorem 2.6 below indeed (2) does hold with $\mathcal{D}(T)$ instead of \mathcal{E} as well.

4. It follows immediately from (2) of the definition that $\mathcal{F}(S, T)$ is a dense submodule of E . What is not immediately obvious but will be proved below in Cor. 3.6 is that $\mathcal{F}(S, T)$ is a core for S as well as for T . Theorem 2.6 below says even more. Namely, $\mathcal{F}(S, T)$ is a *joint* core for S and T in the sense that for $x \in \mathcal{D}(S) \cap \mathcal{D}(T)$ there is a sequence x_n such that $x_n \rightarrow x$, $Sx_n \rightarrow Sx$, $Tx_n \rightarrow Tx$. There are concrete formulas for the construction of x_n .

5. Self-adjointness implies that for $\lambda \in i\mathbb{R}$ the operator $S + \lambda$ is bounded below by $|\lambda|$. Hence the estimate Eq. (2.2) in the definition implies for $x \in \mathcal{F}(S, T)$ and $\lambda, \mu \in i\mathbb{R}$ with $|\lambda|, |\mu| \geq \lambda_0 > 0$

$$\begin{aligned} \|[S, T]x\| &\leq C \cdot (\|(S + \lambda)x\| + \|(T + \mu)x\|) \\ &\leq \frac{C}{|\mu|} \cdot \|(T + \mu)(S + \lambda)x\| + \frac{C}{|\lambda|} \cdot \|(S + \lambda)(T + \mu)x\|. \end{aligned} \quad (2.4)$$

2.3. Comparison to previous notions of weak (anti)commutativity. We show that weak *anticommutativity* in the sense of [MeRe16, Appendix A] implies weak *anticommutativity* in the sense of Def. 2.1. Similarly, [KaLe12, Assumption 7.1] implies weak *commutativity* in the sense of Def. 2.1. We denote by $\tau \in \{+, -\}$ a fixed choice of sign.

Proposition 2.3. *Suppose that for all $\lambda \in i\mathbb{R}$, $|\lambda| \geq \lambda_0 > 0$ large enough*

- (1) *there is a core $\mathcal{E} \subset \mathcal{D}(T)$ for T such that $(S + \lambda)^{-1}(\mathcal{E}) \subset \mathcal{D}(S) \cap \mathcal{D}(T)$,*
- (2) *$T(S + \lambda)^{-1}(\mathcal{E}) \subset \mathcal{D}(S)$,*
- (3) *$[S, T]_\tau(S + \lambda)^{-1}$ extends by continuity to a bounded (adjointable) map $E \rightarrow E$.*

Then S, T are weakly anticommuting (if $\tau = +$) resp. weakly commuting (if $\tau = -$) in the sense of Def. 2.1 with the constant C_2 in Eq. (2.2) being 0.

Remark 2.4. In (3) we put “adjointable” in parentheses because we do not have to assume this. Rather it follows because if $[S, T]_\tau(S + \lambda)^{-1}$ is a bounded module map $E \rightarrow E$ for $|\lambda| \geq \lambda_0$ then its adjoint is given by $(S + \bar{\lambda})^{-1}[T, S]_\tau$, which turns out to be bounded as well.

Proof. 1. Actually in this case it is easy to show a priori that the core \mathcal{E} can be replaced by $\mathcal{D}(T)$: let $x \in \mathcal{D}(T)$ and let $(x_n)_n \subset \mathcal{E}$ be a sequence with $x_n \rightarrow x$ and $Tx_n \rightarrow Tx$. Then by (1) we have $(S + \lambda)^{-1}x_n \in \mathcal{D}(T)$ and by continuity $(S + \lambda)^{-1}x_n \rightarrow (S + \lambda)^{-1}x$. By (2) we have $T(S + \lambda)^{-1}x_n \in \mathcal{D}(S)$ and by (3)

$$(S - \tau\lambda)T(S + \lambda)^{-1}x_n = (ST + \tau \cdot TS)(S + \lambda)^{-1}x_n - \tau \cdot Tx_n$$

converges as well. This shows that $T(S + \lambda)^{-1}x_n$ converges in $\mathcal{D}(S)$ and thus $(S + \lambda)^{-1}x \in \mathcal{D}(T)$ and $T(S + \lambda)^{-1}x \in \mathcal{D}(S)$. This proves that (1) and (2) hold for $\mathcal{D}(T)$ instead of \mathcal{E} .

To see that also (3) holds for $\mathcal{D}(T)$ instead of \mathcal{E} we need to show that the continuous extension of $((ST + \tau \cdot TS)(S + \lambda)^{-1})|_{\mathcal{E}}$ to E coincides with the now defined operator $((ST + \tau \cdot TS)(S + \lambda)^{-1})|_{\mathcal{D}(T)}$. We already know that $(T(S + \lambda)^{-1}x_n)_n$ converges in $\mathcal{D}(S)$ and from (3) we know that $((ST + \tau \cdot TS)(S + \lambda)^{-1}x_n)_n$ converges; thus also $(TS(S + \lambda)^{-1}x_n)_n$ converges. Summing up we have that $T(S + \lambda)^{-1}x \in \mathcal{D}(S)$, $S(T + \lambda)^{-1}x \in \mathcal{D}(T)$ and hence

$$(ST + \tau \cdot TS)(S + \lambda)^{-1}x = \lim_n (ST + \tau \cdot TS)(S + \lambda)^{-1}x_n$$

as claimed.

2. The first part of this proof shows that for $x \in \mathcal{D}(T)$ we have

$$(S + \lambda)^{-1}x \in \mathcal{D}(S) \cap \mathcal{D}(T), S(S + \lambda)^{-1}x \in \mathcal{D}(T), T(S + \lambda)^{-1}x \in \mathcal{D}(S),$$

thus $(S + \lambda)^{-1}(\mathcal{D}(T)) \subset \mathcal{F}(S, T)$, for $\lambda \in i\mathbb{R}$, $|\lambda| \geq \lambda_0$. In fact equality holds: namely, given $y \in \mathcal{F}(S, T)$ then $x := (S + \lambda)y \in \mathcal{D}(T)$ and hence $y = (S + \lambda)^{-1}x \in \mathcal{F}$.

On $\mathcal{F} = \text{ran}((S + \lambda)^{-1} \cdot (T + \mu)^{-1})$ we now have for a fixed $|\lambda| \geq \lambda_0$, using (3),

$$\begin{aligned} \langle [S, T]_\tau y, [S, T]_\tau y \rangle &= \langle [S, T]_\tau(S + \lambda)^{-1}(S + \lambda)y, [S, T]_\tau(S + \lambda)^{-1}(S + \lambda)y \rangle \\ &\leq C \cdot \langle (S + \lambda)y, (S + \lambda)y \rangle \leq C_0 \cdot \langle y, y \rangle + C_1 \cdot \langle Sy, Sy \rangle, \end{aligned} \quad (2.5)$$

and the result follows. \square

Remark 2.5. For $y \in \mathcal{F}(S, T)$, the form estimate

$$\langle [S, T]_\tau y, [S, T]_\tau y \rangle \leq C_1 \cdot \langle y, y \rangle + C_2 \cdot \langle Sy, Sy \rangle, \quad (2.6)$$

implies the norm estimate

$$\|[S, T]_\tau y\| \leq C_0 \cdot \|y\| + C_1 \cdot \|Sy\|, \quad (2.7)$$

but is in fact equivalent to it, provided that $(S + \lambda)^{-1} \mathcal{D}(T) \subset \mathcal{F}(S, T)$. This is seen by observing that Eq. (2.7) implies that

$$[S, T]_\tau (S + \lambda)^{-1} : (S + \lambda) \mathcal{F}(S, T) \rightarrow E,$$

extends to a bounded adjointable operator. Then by writing

$$[S, T]_\tau y = [S, T]_\tau (S + \lambda)^{-1} (S + \lambda) y,$$

and applying the standard form estimate $\langle Ry, Ry \rangle \leq \|R\|^2 \langle y, y \rangle$ for adjointable operators we obtain (2.6). Of course a similar equivalence holds when we exchange S and T . It should be noted that when both C_1 and C_2 are nonzero, we cannot a priori replace (2.2) by the corresponding norm estimate. For that we would need the regularity of the operator $D = S + T$, as the above argument works by using the operator $(1 + D^* D)^{-\frac{1}{2}}$.

2.4. Clifford algebras and (anti)commutators. We briefly explain how one can switch between commuting and anticommuting operators using Clifford algebra identities.

Let $\mathcal{Cl}(2)$ be the complex Clifford algebra on two unitary self-adjoint generators σ_i , $i = 1, 2$, satisfying the relations $\sigma_i \sigma_j + \sigma_j \sigma_i = 2\delta_{ij}$. In fact $\mathcal{Cl}(2) \simeq M(2, \mathbb{C})$ with generators given by ¹

$$\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 := \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \sigma_3 := i \sigma_1 \sigma_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Given operators S, T on the Hilbert-B-module E let $\hat{E} = E \otimes \mathbb{C}^2 = E \oplus E$ and consider the operators \hat{S} and \hat{T} on $E \oplus E$ given by

$$\mathcal{D}(\hat{S}) := \mathcal{D}(S) \oplus \mathcal{D}(S), \quad \hat{S} = S \otimes I = \begin{pmatrix} S & 0 \\ 0 & S \end{pmatrix}, \quad (2.8)$$

$$\mathcal{D}(\hat{T}) := \mathcal{D}(T) \oplus \mathcal{D}(T), \quad \hat{T} = T \otimes I = \begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix}. \quad (2.9)$$

The C^* -algebra $\mathcal{Cl}(2) \subset \mathcal{L}(\hat{E})$ is represented unitarily on \hat{E} . The submodules $\mathcal{D}(\hat{S})$ and $\mathcal{D}(\hat{T})$ are $\mathcal{Cl}(2)$ invariant, and the representation commutes with \hat{S}, \hat{T} , so that the operators

$$s_i := \hat{S} \sigma_i, \quad \mathcal{D}(s_i) := \mathcal{D}(\hat{S}), \quad t_j := \hat{T} \sigma_j, \quad \mathcal{D}(t_j) := \mathcal{D}(\hat{T}),$$

¹ σ_3 is the volume element of $\mathcal{Cl}(2)$, alternatively $\sigma_1, \sigma_2, \sigma_3$ generate one of the two irreducible representations of $\mathcal{Cl}(3) \simeq M(2, \mathbb{C}) \oplus M(2, \mathbb{C})$.

are all self-adjoint and regular in \hat{E} . It then holds that

$$\begin{aligned}\mathcal{F}(s_i, t_j) &:= \{x \in \mathcal{D}(s_i) \cap \mathcal{D}(t_j) \mid s_i x \in \mathcal{D}(t_j), t_j x \in \mathcal{D}(s_i)\} \\ &= \mathcal{F}(S, T) \oplus \mathcal{F}(S, T),\end{aligned}$$

and we have the following relations:

$$\begin{aligned}\hat{S}\sigma_i \cdot \hat{T}\sigma_j \pm \hat{T}\sigma_j \cdot \hat{S}\sigma_i &= (\hat{S}\hat{T} \mp \hat{T}\hat{S})\sigma_i\sigma_j, \quad i \neq j, \\ (\hat{S}\sigma_i)^2 &= \hat{S}^2, \quad (\hat{T}\sigma_i)^2 = \hat{T}^2,\end{aligned}\tag{2.10}$$

$$(\hat{S}\sigma_i + \lambda)^{-1} = (\hat{S}\sigma_i - \lambda)(\hat{S}^2 - \lambda^2)^{-1} = (\hat{S} - \lambda)^{-1}\sigma_i - (\lambda\sigma_i + \lambda)(\hat{S}^2 - \lambda^2)^{-1}.\tag{2.11}$$

It follows from Eq. (2.11) that for all i, j we have

$$(s_i + \lambda)^{-1}\mathcal{D}(t_j) \subset \mathcal{F}(s_i, t_j),$$

and then from Eq. (2.10) that for $i \neq j$, the pair (s_i, t_j) is weakly commuting whenever (S, T) is weakly anticommuting and vice versa. Out of a pair of weakly anticommuting operators we so obtain three pairs $(s_i, t_j), (i \neq j)$ of weakly commuting operators and similarly so for a pair of weakly commuting operators.

2.5. The Main Theorem.

Theorem 2.6. *Let S, T be weakly anticommuting operators in the Hilbert-B-module E . Then the operator $S + T$ is self-adjoint and regular on $\mathcal{D}(S) \cap \mathcal{D}(T)$. In more detail we have the following:*

(1) *There is a constant C such that for $x \in \mathcal{D}(S) \cap \mathcal{D}(T)$ we have*

$$\begin{aligned}C^{-1} \cdot (\langle x, x \rangle + \langle (S + T)x, (S + T)x \rangle) &\leq \langle x, x \rangle + \langle Sx, Sx \rangle + \langle Tx, Tx \rangle \\ &\leq C \cdot (\langle x, x \rangle + \langle (S + T)x, (S + T)x \rangle).\end{aligned}\tag{2.12}$$

(2) *For $\lambda, \mu \in i\mathbb{R}, |\lambda|, |\mu| \geq \lambda_0$ large enough we have*

$$(T + \mu)^{-1}(\mathcal{D}(S)) = \mathcal{F}(S, T) = (S + \lambda)^{-1}(\mathcal{D}(T))\tag{2.13}$$

and hence

$$\text{ran}((T + \mu)^{-1} \cdot (S + \lambda)^{-1}) = \mathcal{F}(S, T) = \text{ran}((S + \lambda)^{-1} \cdot (T + \mu)^{-1}).\tag{2.14}$$

(3) *For $\lambda_0 > 0$ large enough and $\lambda, \mu \in i\mathbb{R}, |\lambda| > |\mu| \geq \lambda_0, \mu/\lambda > 0$ the operator*

$$S + T + \frac{TS}{\lambda} + \mu : \mathcal{F} \rightarrow E$$

is bijective and its inverse $(S + T + \frac{TS}{\lambda} + \mu)^{-1}$ is a bounded adjointable operator. Moreover, for fixed μ and for all $x \in E$

$$\lim_{|\lambda| \rightarrow \infty} (S + T + \mu)(S + T + \frac{TS}{\lambda} + \mu)^{-1}x = x,\tag{2.15}$$

in norm and

$$\lim_{|\lambda| \rightarrow \infty} (S + T + \frac{TS}{\lambda} + \mu)^{-1} = (S + T + \mu)^{-1},$$

in operator norm.²

(4) $\mathcal{F}(S, T)$ is a core for S, T , and $S + T$.

Remark 2.7. Item (4) can be made more precise. For $x \in \mathcal{D}(S) \cap \mathcal{D}(T)$ it follows from (3) that

$$x_\lambda := (S + T + \frac{TS}{\lambda} + \mu)^{-1}(S + T + \mu)x$$

converges, as $|\lambda| \rightarrow \infty$, to x in the graph norm of $S + T$. By (1) this means that $x_\lambda \rightarrow x$, $Sx_\lambda \rightarrow Sx$, $Tx_\lambda \rightarrow Tx$.

Alternatively, the method of proof of (3) can be used to show the following slightly stronger convergence result: for $x \in E$ put

$$x_\lambda := \lambda^2 \cdot (T + \lambda)^{-1} \cdot (S + \lambda)^{-1}x \in \mathcal{F}(S, T), \quad (2.16)$$

for $\lambda \in i\mathbb{R}$, $|\lambda| > \lambda_0$. Then $\lim_{|\lambda| \rightarrow \infty} x_\lambda = x$. Furthermore, if $x \in \mathcal{D}(S)$ (resp. $\mathcal{D}(T)$) then also $\lim_{|\lambda| \rightarrow \infty} Sx_\lambda = Sx$ (resp. $\lim_{|\lambda| \rightarrow \infty} Tx_\lambda = Tx$).

The proof of Theorem 2.6 will be broken down as follows:

1. First we prove the form estimate Eq. (2.12), as a rather direct consequence of the form estimate Eq. (2.2). From Eq. (2.12) we derive that the operator $S + T$ is closed on $\mathcal{D}(S) \cap \mathcal{D}(T)$.

2. Next we will show (2) which shows, among other things, that a posteriori (2) of Def. 2.1 holds for $\mathcal{D}(T)$ instead of the core \mathcal{E} and that the roles of S, T , which a priori appear in Def. 2.1 (2) in an unsymmetric way, can be reversed. See Section 3.

3. For $\lambda \in i\mathbb{R}$ with $|\lambda| \geq \lambda_0$, the operators

$$A_\lambda := S + T + \frac{TS}{\lambda} : \mathcal{F} \rightarrow E,$$

are well defined. We prove a fundamental lower form bound for the operators $A_\lambda + \mu$. This allows to deduce λ -uniform norm bounds on the bounded adjointable inverse of these operators, and that for fixed μ the net $((A_\lambda + \mu)^{-1})_\lambda$ is operator norm Cauchy in λ .

4. Subsequently we show that the convergence in Eq. (2.15) holds true for all $x \in E$. From that we deduce directly that the operators $S + T + \mu$ have dense range, and therefore are essentially self-adjoint and regular on $\mathcal{F}(S, T)$. It is then readily established that the limit of the Cauchy net $((A_\lambda + \mu)^{-1})_\lambda$ is in fact $(S + T + \mu)^{-1}$, the resolvent of the sum operator.

5. In the weakly commuting case, we obtain that the operators

$$\begin{pmatrix} 0 & S + iT \\ S - iT & 0 \end{pmatrix}, \quad \begin{pmatrix} S & T \\ T & -S \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} S & iT \\ -iT & -S \end{pmatrix},$$

are self-adjoint and regular on $(\mathcal{D}(S) \cap \mathcal{D}(T))^{\oplus 2}$. In particular the operators $S \pm iT$ on $\mathcal{D}(S) \cap \mathcal{D}(T)$ are closed, regular and adjoints of each other.

²The limit $\lim_{|\lambda| \rightarrow \infty}$ is taken for the net of those λ with $|\lambda| > |\mu|$, $\mu/\lambda > 0$. In the sequel this is understood without repeatedly mentioning it.

3. DOMAIN CONSIDERATIONS: A CLOSER LOOK AT $\mathcal{F}(S, T)$

Estimates for inner products in Hilbert-C*-modules are a little more delicate since one is dealing with inequalities in C*-algebras. Recall that for $x, y \in E$ one has

$$\langle x, y \rangle + \langle y, x \rangle \leq \langle x, x \rangle + \langle y, y \rangle, \quad (3.1)$$

which follows immediately from expanding $0 \leq \langle x - y, x - y \rangle$. By replacing x by $\pm r \cdot x$ and y by $r^{-1} \cdot y$ we obtain for any $r > 0$ and $x, y \in E$

$$\pm (\langle x, y \rangle + \langle y, x \rangle) \leq r \cdot \langle x, x \rangle + \frac{1}{r} \cdot \langle y, y \rangle. \quad (3.2)$$

Note that this is an inequality from above and from below. This estimate will be used repeatedly in the sequel.

3.1. Graph norm estimate. We now show that the inequalities Eq. (2.12) hold true for $x \in \mathcal{F}(S, T) \subset \mathcal{D}(S) \cap \mathcal{D}(T)$.

Proposition 3.1. *Let S, T be weakly anticommuting operators in the Hilbert-C*-module E . There is a constant $C > 0$ such that for all $x \in \mathcal{F}(S, T)$ the form estimate*

$$\begin{aligned} C^{-1} \cdot (\langle x, x \rangle + \langle (S + T)x, (S + T)x \rangle) &\leq \langle x, x \rangle + \langle Sx, Sx \rangle + \langle Tx, Tx \rangle \\ &\leq C \cdot (\langle x, x \rangle + \langle (S + T)x, (S + T)x \rangle) \end{aligned}$$

holds true.

Proof. For any $x \in \mathcal{D}(S) \cap \mathcal{D}(T)$ we write

$$\begin{aligned} \langle (S + T)x, (S + T)x \rangle &= \langle Sx, Sx \rangle + \langle Tx, Tx \rangle + \langle Sx, Tx \rangle + \langle Tx, Sx \rangle \\ &\leq 2 \cdot (\langle Sx, Sx \rangle + \langle Tx, Tx \rangle). \end{aligned}$$

To prove the lower estimate, if we furthermore assume that

$$x \in \mathcal{F}(S, T) \subset \mathcal{D}(S) \cap \mathcal{D}(T)$$

we have for any $K > 0$ (cf. Eq. (3.2))

$$\pm (\langle Sx, Tx \rangle + \langle Tx, Sx \rangle) = \pm \langle [S, T]x, x \rangle \leq K^{-1} \cdot \langle [S, T]x, [S, T]x \rangle + K \cdot \langle x, x \rangle.$$

Thus for C as in Eq. (2.2) and any $0 < \varepsilon < 1$ we can choose $K > C$ such that $K^{-1}C < \varepsilon$. We then find

$$\begin{aligned} \langle [S, T]x, x \rangle &\geq -K^{-1} \cdot \langle [S, T]x, [S, T]x \rangle - K \cdot \langle x, x \rangle \\ &\geq -\varepsilon \cdot (\langle Sx, Sx \rangle + \langle Tx, Tx \rangle) - (K + \varepsilon) \cdot \langle x, x \rangle, \end{aligned}$$

and thus

$$\begin{aligned} \langle Sx, Sx \rangle + \langle Tx, Tx \rangle &\leq (1 + \varepsilon) \cdot (\langle Sx, Sx \rangle + \langle Tx, Tx \rangle) + \langle [S, T]x, x \rangle + (K + \varepsilon) \cdot \langle x, x \rangle \\ &= \langle (S + T)x, (S + T)x \rangle + \varepsilon \cdot (\langle Sx, Sx \rangle + \langle Tx, Tx \rangle) + (K + \varepsilon) \cdot \langle x, x \rangle, \end{aligned}$$

hence the estimate

$$(1 - \varepsilon) \cdot (\langle Sx, Sx \rangle + \langle Tx, Tx \rangle) \leq \langle (S + T)x, (S + T)x \rangle + (K + \varepsilon) \cdot \langle x, x \rangle.$$

With the constant

$$C := \max \left\{ \frac{1}{1-\varepsilon}, \frac{K+\varepsilon}{1-\varepsilon} \right\},$$

we find

$$\langle Sx, Sx \rangle + \langle Tx, Tx \rangle \leq C \cdot (\langle (S+T)x, (S+T)x \rangle + \langle x, x \rangle),$$

as desired. \square

3.2. Symmetry of the axioms for weak (anti)commutativity; proof of part (2) of the main Theorem 2.6. We first note that for arbitrary resolvent parameters λ, μ we have the equality of commutators

$$[S + \lambda, T + \mu]_- = [S, T]_-; \quad (3.3)$$

there is no simple analogue to this in the weakly anticommuting case.

In this section we establish part (2) of the main Theorem 2.6. For the proof we need two preparatory Lemmas.

Lemma 3.2. *Let S, T be weakly commuting operators. Then for λ_0 large enough we have for all $\lambda, \mu \in i\mathbb{R}$, $|\lambda|, |\mu| \geq \lambda_0$ and $x \in \mathcal{F}(S, T)$*

$$\|[S + \lambda, T + \mu]_- x\| = \|[S, T]_- x\| \leq C \cdot \left(\frac{1}{|\lambda|} + \frac{1}{|\mu|} \right) \cdot \|(S + \lambda)(T + \mu)x\|, \quad (3.4)$$

$$\|[S + \lambda, T + \mu]_- x\| = \|[S, T]_- x\| \leq C \cdot \left(\frac{1}{|\lambda|} + \frac{1}{|\mu|} \right) \cdot \|(T + \mu)(S + \lambda)x\|. \quad (3.5)$$

By increasing λ_0 we can arrange the constant $C \cdot \left(\frac{1}{|\lambda|} + \frac{1}{|\mu|} \right) \leq 2 \cdot C \cdot \frac{1}{\lambda_0}$ to be as small as we please.

Proof. By Eq. (3.3) and Eq. (2.4) we have

$$\begin{aligned} \|[S + \lambda, T + \mu]_- x\| &= \|[S, T]_- x\| \leq \frac{C_1}{|\mu|} \cdot \|(T + \mu)(S + \lambda)x\| + \frac{C_2}{|\lambda|} \cdot \|(S + \lambda)(T + \mu)x\| \\ &\leq \frac{C_1}{|\mu|} \cdot \|[T, S]_- x\| + \left(\frac{C_1}{|\mu|} + \frac{C_2}{|\lambda|} \right) \|(S + \lambda)(T + \mu)x\|. \end{aligned}$$

If λ_0 is large enough such that $\frac{C_1}{\lambda_0} < 1$ we obtain estimate Eq. (3.4). The proof of Eq. (3.5) is completely analogous. \square

Lemma 3.3. *Let A, B be closed densely defined and boundedly invertible operators in the Banach space X and suppose that a subspace*

$$\mathcal{E} \subset \mathcal{D}(A) \cap \mathcal{D}(B) \subset X$$

is a core for A . If there exists $0 < \varepsilon < 1/3$ such that for all $x \in \mathcal{E}$

$$\|Ax - Bx\| \leq \varepsilon \cdot (\|Ax\| + \|Bx\|), \quad (3.6)$$

then $\mathcal{D}(A) = \mathcal{D}(B)$ and the estimate Eq. (3.6) as well as

$$\begin{aligned} \|Ax - Bx\| &\leq \frac{2\varepsilon}{1-\varepsilon} \cdot \|Bx\|, \\ \|Ax - Bx\| &\leq \frac{2\varepsilon}{1-\varepsilon} \cdot \|Ax\| \end{aligned} \quad (3.7)$$

hold for all $x \in \mathcal{D}(A) = \mathcal{D}(B)$.

Proof. Clearly, for $x \in \mathcal{E}$ we infer from Eq. (3.6) that

$$\|Ax - Bx\| \leq 2\varepsilon \cdot \|Ax\| + \varepsilon \cdot \|Ax - Bx\|, \quad (3.8)$$

hence the first inequality in Eq. (3.7) for $x \in \mathcal{E}$; the second is seen by exchanging A and B . Thus for x in the dense subspace $A(\mathcal{E}) \subset X$ we have

$$\begin{aligned} \|BA^{-1}x - x\| &= \|B(A^{-1}x) - A(A^{-1}x)\| \\ &\leq 2\varepsilon \cdot \|x\| + \varepsilon \cdot \|BA^{-1}x - x\|, \end{aligned}$$

hence

$$\|BA^{-1}x - x\| \leq \frac{2\varepsilon}{1-\varepsilon} \cdot \|x\|; \quad \frac{2\varepsilon}{1-\varepsilon} < 1. \quad (3.9)$$

If $x \in X$ let $(x_n) \in A(\mathcal{E})$ be a sequence with $x_n \rightarrow x$. Then Eq. (3.9) and the closedness of B imply that $A^{-1}x \in \mathcal{D}(B)$ and Eq. (3.9) holds for x as well. Since $A^{-1}(X) = \mathcal{D}(A)$ it follows that $\mathcal{D}(A) \subset \mathcal{D}(B)$ and Eq. (3.6), (3.7) hold for all $x \in \mathcal{D}(A)$. From Eq. (3.9) it follows that BA^{-1} is bounded and invertible and thus B maps $\mathcal{D}(A)$ bijectively onto X . Since B is assumed to be invertible $\mathcal{D}(B) \rightarrow X$, it follows that $\mathcal{D}(B) = B^{-1}(X) = \mathcal{D}(A)$. \square

For future reference we note

Corollary 3.4. *If under the assumptions of Lemma 3.3, $\mathcal{X} \subset X$ is a dense subspace then $A^{-1}(\mathcal{X})$ is a core for A and for B .*

Proof. Since A is boundedly invertible, A^{-1} is a Banach space isomorphism from X onto $\mathcal{D}(A)$, the latter being equipped with the graph norm. Hence A^{-1} maps dense subspaces of X onto dense subspaces of $\mathcal{D}(A)$ and vice versa. Eq. (3.7) shows that the graph norms of A and B are equivalent, thus cores for A are cores for B and vice versa. \square

Proposition 3.5. *Let S, T be weakly (anti)commuting operators. Then (2) of Definition 2.1 holds with $\mathcal{E} = \mathcal{D}(T)$. That is $(S + \lambda)^{-1}(\mathcal{D}(T)) \subset \mathcal{F}(S, T)$ for $\lambda \in i\mathbb{R}, |\lambda| \geq \lambda_0$. Furthermore, we have for $\lambda, \mu \in i\mathbb{R}, |\lambda|, |\mu| \geq \lambda_0$*

$$\mathcal{F}(S, T) = \text{ran}((T + \mu)^{-1} \cdot (S + \lambda)^{-1}) = \text{ran}((S + \lambda)^{-1} \cdot (T + \mu)^{-1}),$$

that is, (2) of Theorem 2.6 holds.

Proof. The discussion in Section 2.4 shows that the statement for weakly commuting operators implies that for weakly anticommutating operators, so it suffices to consider only this case. Choosing λ_0 in Eq. (2.4) large enough we find for $x \in \mathcal{F}(S, T)$:

$$\begin{aligned} \|(S + \lambda) \cdot (T + \mu)x - (T + \mu) \cdot (S + \lambda)x\| &= \|(ST - TS)x\| \\ &\leq \varepsilon \cdot (\|(T + \mu)(S + \lambda)x\| + \|(S + \lambda)(T + \mu)x\|), \end{aligned}$$

with ε as small as we please, e.g. $< \frac{1}{3}$.

The result now follows from Lemma 3.3 with $A = (T + \mu)(S + \lambda)$, and $B = (S + \lambda)(T + \mu)$. Note that \mathcal{E} of Definition 2.1 is a core for T and hence $(S + \lambda)^{-1}(\mathcal{E}) \subset \mathcal{F}(S, T) \subset \mathcal{D}(B)$ is a core for A , cf. Cor. 3.4. \square

Corollary 3.6. *Let S, T be weakly (anti)commuting operators. Then $\mathcal{F}(S, T)$ is a core for S as well as a core for T .*

Proof. For $\lambda \in i\mathbb{R}, \lambda \neq 0$ the resolvent $(S + \lambda)^{-1}$ maps the dense subspace $\mathcal{D}(T)$ into a core for S and the resolvent $(T + \lambda)^{-1}$ maps the dense subspace $\mathcal{D}(S)$ into a core for T . By Prop. 3.5 and Definition 2.1 we have for $|\lambda|$ large enough $(S + \lambda)^{-1}(\mathcal{D}(T)) \subset \mathcal{F}(S, T)$ and $(T + \lambda)^{-1}(\mathcal{D}(S)) \subset \mathcal{F}(S, T)$, hence the claim. \square

4. APPROXIMATION OF THE RESOLVENT OF THE SUM $S + T$

During the whole section let S, T be a weakly anticommuting pair of self-adjoint regular operators in the Hilbert-B-module E .

By Proposition 3.5, we have the equality of submodules

$$\mathcal{F}(S, T) = \text{ran}(S + \lambda)^{-1}(T + \mu)^{-1} = \text{ran}(T + \mu)^{-1}(S + \lambda)^{-1}$$

for $\lambda, \mu \in i\mathbb{R}, |\lambda|, |\mu| \geq \lambda_0$ large enough. The operators ST and TS are defined on $\mathcal{F}(S, T) \subset \mathcal{D}(S) \cap \mathcal{D}(T)$, which by Cor. 3.6 is a core for S as well as for T . We now consider the operator

$$A_\lambda : \mathcal{F}(S, T) \rightarrow E, \quad A_\lambda x := Sx + Tx + \frac{TSx}{\lambda},$$

which approximates $S + T$ strongly on $\mathcal{F}(S, T)$. In this section we show that this approximation holds in a much stronger sense.

4.1. The fundamental estimate.

Lemma 4.1. *For $\lambda, \mu \in i\mathbb{R}, |\lambda - \mu| < |\lambda|$, the operator $A_\lambda + \mu : \mathcal{F}(S, T) \rightarrow E$ is bijective and hence boundedly invertible.*

Proof. Recall that for a closable operator in a Banach space it is a consequence of the Closed Graph Theorem that being bijective is equivalent to being closed and boundedly invertible. We have

$$A_\lambda + \mu = \lambda^{-1}(T + \lambda)(S + \lambda) + \mu - \lambda =: B_\lambda + \mu - \lambda. \quad (4.1)$$

For $\lambda = \mu$ large enough this operator is boundedly invertible by Proposition 3.5. Moreover, since $\lambda, \mu \in i\mathbb{R}$ and S, T are self-adjoint, we have $\|(A_\lambda + \lambda)^{-1}\| \leq \frac{1}{|\lambda|}$. Thus

$$\|(A_\lambda + \mu)(A_\lambda + \lambda)^{-1} - \text{Id}\| = |\lambda - \mu| \cdot \|(A_\lambda + \lambda)^{-1}\| \leq \frac{|\lambda - \mu|}{|\lambda|} < 1,$$

hence the claim. \square

It is immediately clear that for $x \in \mathcal{F}(S, T)$ and $|\lambda| \rightarrow \infty$ we have the convergence

$$(A_\lambda + \mu)x \rightarrow (S + T + \mu)x.$$

We will show that this convergence holds in the resolvent sense. We now proceed to derive the form estimate that provides the cornerstone of our argument.

Lemma 4.2. For $\lambda \in i\mathbb{R} \setminus \{0\}$, $|\lambda| \geq \lambda_0$ large enough the operators $A_\lambda := S + T + \frac{TS}{\lambda}$ are closed on the domain

$$\mathcal{D}(A_\lambda) := \mathcal{F}(S, T)$$

and satisfy $\mathcal{D}(A_\lambda) = \mathcal{D}(A_\lambda^*)$. For $\lambda \in i\mathbb{R}$, $|\lambda| \geq \lambda_0$ and $\mu \in i\mathbb{R}$ with $|\lambda - \mu| < \lambda$, the operator $A_\lambda + \mu$ is boundedly invertible. Moreover, there exist $C, \mu_0 > 0$ such that for $\lambda, \mu \in i\mathbb{R}$, $|\lambda| > |\mu| \geq \mu_0$, $\lambda/\mu > 0$ we have

$$\|(A_\lambda + \mu)^{-1}\| \leq \frac{\sqrt{2}}{|\mu|}, \quad \|S(A_\lambda + \mu)^{-1}\| \leq \sqrt{2}, \quad \|T(A_\lambda + \mu)^{-1}\| \leq \sqrt{2}, \quad (4.2)$$

$$\|\frac{TS}{\lambda}(A_\lambda + \mu)^{-1}\| \leq 1, \quad \|[S, T](A_\lambda + \mu)^{-1}\| \leq C. \quad (4.3)$$

Proof. Recall that $A_\lambda + \lambda = \frac{1}{\lambda}(T + \lambda)(S + \lambda)$. Hence the claims about closedness, $\mathcal{D}(A_\lambda) = \mathcal{D}(A_\lambda^*)$ and bounded invertibility follow from Proposition 3.5 and Lemma 4.1. It remains to prove the claimed estimates.

Keep in mind that $\bar{\lambda} = -\lambda$ and similarly for μ , since these numbers are purely imaginary. Furthermore, we introduce the convenient abbreviation

$$\langle\langle y \rangle\rangle := \langle y, y \rangle \quad (4.4)$$

for $y \in E$.

For $x \in \mathcal{F}(S, T)$ we first expand

$$\begin{aligned} \langle\langle (A_\lambda + \mu)x \rangle\rangle &= \langle\langle Sx \rangle\rangle + \langle\langle Tx \rangle\rangle + \langle[S, T]x, x\rangle + \langle(S + T)x, (\frac{TS}{\lambda} + \mu)x\rangle \\ &\quad + \langle(\frac{TS}{\lambda} + \mu)x, (S + T)x\rangle + \langle\langle(\frac{TS}{\lambda} + \mu)x\rangle\rangle \\ &= \langle\langle Sx \rangle\rangle + \langle\langle Tx \rangle\rangle + \langle[S, T]x, x\rangle + \frac{1}{\lambda}(\langle Sx, TSx \rangle - \langle TSx, Sx \rangle) \\ &\quad + \langle Tx, \frac{TS}{\lambda}x \rangle + \langle \frac{TS}{\lambda}x, Tx \rangle + |\mu|^2 \langle\langle x \rangle\rangle + \langle\langle \frac{TS}{\lambda}x \rangle\rangle - \frac{\mu}{\lambda} \langle[S, T]x, x\rangle \\ &= \langle\langle Sx \rangle\rangle + \langle\langle Tx \rangle\rangle + |\mu|^2 \langle\langle x \rangle\rangle + \langle\langle \frac{TS}{\lambda}x \rangle\rangle \end{aligned} \quad (4.5)$$

$$+ (1 - \frac{\mu}{\lambda}) \langle[S, T]x, x\rangle \quad (4.6)$$

$$+ \langle Tx, \frac{[S, T]}{\lambda}x \rangle + \langle \frac{[S, T]}{\lambda}x, Tx \rangle. \quad (4.7)$$

The expressions in the lines Eq. (4.6) and Eq. (4.7) will be estimated separately from above and from below. Recall the estimates Eq. (3.1) and Eq. (3.2) as well as Eq. (2.2) which will be used repeatedly.

Expression Eq. (4.6). We have

$$\begin{aligned} \pm(1 - \frac{\mu}{\lambda}) \langle[S, T]x, x\rangle &\leq \frac{1}{2}(1 - \frac{\mu}{\lambda}) \left(\frac{1}{|\mu|} \langle\langle[S, T]x\rangle\rangle + |\mu| \langle\langle x \rangle\rangle \right) \\ &\leq \frac{C}{2|\mu|} \cdot \left(\langle\langle Sx \rangle\rangle + \langle\langle Tx \rangle\rangle + \langle\langle x \rangle\rangle \right) + \frac{|\mu|}{2} \langle\langle x \rangle\rangle. \end{aligned}$$

Expression Eq. (4.7).

$$\begin{aligned} \pm \left(\langle Tx, \frac{[S, T]}{\lambda} x \rangle + \langle \frac{[S, T]}{\lambda} x, Tx \rangle \right) &\leq \frac{1}{|\lambda|} \langle \langle Tx \rangle \rangle + |\lambda| \langle \langle \frac{[S, T]}{\lambda} x \rangle \rangle \\ &= \frac{1}{|\lambda|} \left(\langle \langle Tx \rangle \rangle + \langle \langle [S, T] x \rangle \rangle \right) \\ &\leq \frac{1}{|\lambda|} (C + 1) \langle \langle Tx \rangle \rangle + \frac{C}{|\lambda|} \langle \langle Sx \rangle \rangle + \frac{C}{|\lambda|} \langle \langle x \rangle \rangle. \end{aligned}$$

With these estimates we obtain

$$\begin{aligned} \langle \langle (A_\lambda + \mu)x \rangle \rangle &\geq \left(1 - \frac{C}{2|\mu|} - \frac{C}{|\lambda|}\right) \cdot \langle \langle Sx \rangle \rangle + \left(1 - \frac{C}{2|\mu|} - \frac{C+1}{|\lambda|}\right) \cdot \langle \langle Tx \rangle \rangle \\ &\quad + \left(|\mu|^2 - \frac{C}{2|\mu|} - \frac{C}{|\lambda|} - \frac{|\mu|}{2}\right) \cdot \langle \langle x \rangle \rangle + \langle \langle \frac{TS}{\lambda} x \rangle \rangle \end{aligned}$$

and since $|\mu| < |\lambda|, 0 < \frac{\mu}{\lambda} < 1$

$$\begin{aligned} \dots &\geq \left(1 - \frac{3/2C + 1}{|\mu|}\right) (\langle \langle Sx \rangle \rangle + \langle \langle Tx \rangle \rangle) \\ &\quad + \left(|\mu|^2 - \frac{3/2C}{|\mu|} - \frac{|\mu|}{2}\right) \cdot \langle \langle x \rangle \rangle + \langle \langle \frac{TS}{\lambda} x \rangle \rangle. \end{aligned}$$

Choosing μ_0 large enough we obtain for $|\mu| \geq \mu_0, |\lambda| > |\mu|, \frac{\mu}{\lambda} > 0$

$$\dots \geq \frac{1}{2} \langle \langle Sx \rangle \rangle + \frac{1}{2} \langle \langle Tx \rangle \rangle + \frac{|\mu|^2}{2} \langle \langle x \rangle \rangle + \langle \langle \frac{TS}{\lambda} x \rangle \rangle.$$

From this we infer for $x \in \mathcal{F}(S, T)$ the inequality $\|(A_\lambda + \mu)^{-1}x\|^2 \leq \frac{2}{|\mu|^2} \|x\|^2$, resp. the claimed $\|(A_\lambda + \mu)^{-1}\| \leq \frac{\sqrt{2}}{|\mu|}$.

Replacing x by $(A_\lambda + \mu)^{-1}x$ we find $\|S(A_\lambda + \mu)^{-1}\| \leq \sqrt{2}, \|T(A_\lambda + \mu)^{-1}\| \leq \sqrt{2}$, and $\|\frac{TS}{\lambda}(A_\lambda + \mu)^{-1}\| \leq 1$.

Applying the form estimate Eq. (2.2) and taking norms yields

$$\begin{aligned} \|[S, T](A_\lambda + \mu)^{-1}x\|^2 &\leq C \cdot \left(\|(A_\lambda + \mu)^{-1}x\|^2 + \|S(A_\lambda + \mu)^{-1}x\|^2 + \|T(A_\lambda + \mu)^{-1}x\|^2 \right) \leq C, \end{aligned}$$

and the proof is complete. \square

4.2. Self-adjointness and regularity.

Lemma 4.3. For $\mu_0 > 0$ large enough we have for $\lambda, \mu \in i\mathbb{R}, |\lambda| > |\mu| \geq \mu_0, \mu/\lambda > 0$

$$\begin{aligned} (A_\lambda + \mu)^{-1}(\mathcal{D}(S)) &\subset (S + \lambda)^{-1}(T + \lambda)^{-1}(\mathcal{D}(S)) \\ &\subset \mathcal{D}(S^2) \cap \mathcal{D}(STS) \cap \mathcal{D}(ST) \cap \mathcal{D}(TS). \end{aligned}$$

Proof. With $B_\lambda = A_\lambda + \lambda = \frac{1}{\lambda}(T + \lambda)(S + \lambda)$ as in Eq. (4.1) we have, cf. the proof of Lemma 4.1,

$$\begin{aligned} (A_\lambda + \mu)^{-1} &= B_\lambda^{-1} - (\mu - \lambda)B_\lambda^{-1}(A_\lambda + \mu)^{-1} \\ &= B_\lambda^{-1}(1 - (\mu - \lambda)(A_\lambda + \mu)^{-1}). \end{aligned} \tag{4.8}$$

The operator $1 - (\mu - \lambda)(A_\lambda + \mu)^{-1}$ preserves the domain of S , since

$$\text{ran}(A_\lambda + \mu)^{-1} = \mathcal{F}(S, T) \subset \mathcal{D}(S),$$

proving the first set inclusion. The second one follows from the identity

$$S(S + \lambda)^{-1} = 1 - \lambda(S + \lambda)^{-1},$$

the fact that $(T + \lambda)^{-1}$ preserves $\mathcal{D}(S)$ and the fact that

$$\text{ran}(T + \lambda)^{-1}(S + \lambda)^{-1} = \text{ran}(S + \lambda)^{-1}(T + \lambda)^{-1} \subset \mathcal{D}(ST) \cap \mathcal{D}(TS).$$

This proves the lemma. \square

Lemma 4.4. *Under the same assumptions as in Lemma 4.3 we have for $x \in \mathcal{D}(S)$ the equality*

$$\begin{aligned} S(S + T + \frac{TS}{\lambda} + \mu)^{-1}x + (S - T - \frac{ST}{\lambda} - \mu)^{-1}Sx \\ = (S - T - \frac{ST}{\lambda} - \mu)^{-1}[S, T](S + T + \frac{TS}{\lambda} + \mu)^{-1}x, \end{aligned}$$

and each of these elements is in $\mathcal{D}(T)$.

Proof. By the Lemma 4.3 we may write, on $\mathcal{D}(S)$,

$$\begin{aligned} S(S + T + \frac{TS}{\lambda} + \mu)^{-1} + (T - S - \frac{ST}{\lambda} - \mu)^{-1}S \\ = (T - S - \frac{ST}{\lambda} - \mu)^{-1} \cdot \left(S(S + T + \frac{TS}{\lambda} + \mu) + (T - S - \frac{ST}{\lambda} - \mu)S \right) \cdot (S + T + \frac{TS}{\lambda} + \mu)^{-1} \\ = (T - S - \frac{ST}{\lambda} - \mu)^{-1}[S, T](S + T + \frac{TS}{\lambda} + \mu)^{-1}, \end{aligned}$$

as claimed. The second summand on the left hand side maps into $\mathcal{D}(T)$, as does the right hand side, and thus the remaining term maps into $\mathcal{D}(T)$ as well. \square

Theorem 4.5. *The operator $S + T$ is self-adjoint and regular on $\mathcal{D}(S) \cap \mathcal{D}(T)$. More precisely, for $\mu \in i\mathbb{R}$ large enough the net $\left((S + T + \mu)(A_\lambda + \mu)^{-1} \right)_\lambda, \lambda \in i\mathbb{R}, |\lambda| > |\mu|, \mu/\lambda > 0$, converges strongly to 1 on E as $|\lambda| \rightarrow \infty$. The module*

$$\mathcal{F}(S, T) = \text{ran}(S + \lambda)^{-1}(T + \lambda)^{-1},$$

is a core for $S + T$. The net $\left((S + T + \frac{TS}{\lambda} + \mu)^{-1} \right)_\lambda, \lambda \in i\mathbb{R}, |\lambda| > |\mu|, \mu/\lambda > 0$, is norm convergent to the resolvent $(S + T + \mu)^{-1}$ of $S + T$ as $|\lambda| \rightarrow \infty$.

Remark 4.6. In view of Prop. 3.1 the fact that $\mathcal{F}(S, T)$ is a core for $S + T$ implies that for $x \in \mathcal{D}(S) \cap \mathcal{D}(T)$ there exists a sequence $(x_n) \subset \mathcal{F}(S, T)$ such that $x_n \rightarrow x, Sx_n \rightarrow Sx$, and $Tx_n \rightarrow Tx$.

Proof. By definition we have

$$\mathcal{F}(S, T) \subset \mathcal{D}(S) \cap \mathcal{D}(T),$$

and by Proposition 3.1, the domain of the closure of $S + T$, restricted to $\mathcal{F}(S, T)$, is contained in $\mathcal{D}(S) \cap \mathcal{D}(T)$. We will show that

$$S + T + \mu : \mathcal{F}(S, T) \rightarrow E,$$

has dense range for $\mu \in i\mathbb{R}$ with $|\mu|$ sufficiently large. Since $S + T$ is symmetric and closed on $\mathcal{D}(S) \cap \mathcal{D}(T) \supset \mathcal{F}(S, T)$, this proves self-adjointness and regularity of $S + T$ on this domain, cf. [LAN95, Lemma 9.7]. Furthermore, that

$$(S + T + \mu)(\mathcal{F}(S, T))$$

is dense in E is equivalent to the fact that $\mathcal{F}(S, T)$ is a core for $S + T$.

By Lemmas 4.1 and 4.2, for $|\mu| \geq |\mu_0|$ large enough, the bounded adjointable operators $(A_\lambda + \mu)^{-1}$ map E into $\mathcal{F}(S, T)$. Furthermore, the net

$$\left((S + T + \mu)(A_\lambda + \mu)^{-1} \right)_{|\lambda| > |\mu|, \mu/\lambda > 0}$$

is uniformly bounded in norm and

$$(S + T + \mu)(A_\lambda + \mu)^{-1} = 1 - \frac{TS}{\lambda}(A_\lambda + \mu)^{-1},$$

so it suffices to show that $\frac{TS}{\lambda}(A_\lambda + \mu)^{-1}$ converges to 0 strongly on $\mathcal{D}(S)$. There, by virtue of Lemma 4.4, we can write

$$\begin{aligned} \frac{TS}{\lambda}(A_\lambda + \mu)^{-1}x &= \frac{-T}{\lambda}(S - T - \frac{ST}{\lambda} - \mu)^{-1}Sx \\ &\quad + \frac{T}{\lambda}(S - T - \frac{ST}{\lambda} - \mu)^{-1}[S, T](S + T + \frac{TS}{\lambda} + \mu)^{-1}x. \end{aligned}$$

Using the estimates (4.2) and (4.3), we see that both summands are $O(|\lambda|^{-1})$ in norm, and thus converge to 0. We conclude indeed that the operator $S + T + \mu : \mathcal{D}(S) \cap \mathcal{D}(T) \rightarrow E$ has dense range and it is therefore bijective as explained above. It follows that the resolvents $(S + T + \mu)^{-1}$ exist and that the submodule

$$\mathcal{F}(S, T) = \text{ran}(S + \lambda)^{-1}(T + \lambda)^{-1} \subset \mathcal{D}(S) \cap \mathcal{D}(T),$$

is a core for $S + T$. Now $S(S + T + \mu)^{-1}$ is a bounded adjointable operator so

$$\begin{aligned} \|(A_\lambda + \mu)^{-1} - (S + T + \mu)^{-1}\| &= \|(A_\lambda + \mu)^{-1} \frac{TS}{\lambda}(S + T + \mu)^{-1}\| \\ &\leq \frac{C}{|\lambda|} \|S(S + T + \mu)^{-1}\| \rightarrow 0, \end{aligned}$$

as $|\lambda| \rightarrow \infty$, which completes the proof. \square

5. APPLICATIONS

We present two applications of the main Theorem 2.6.

5.1. A Dore-Venni type theorem for $S^2 + T^2$.

Theorem 5.1. *Let S, T be weakly anticommuting operators in the Hilbert-B-module E . Then $S^2 + T^2$ is self-adjoint and regular on $\mathcal{D}(S^2) \cap \mathcal{D}(T^2)$. The latter equals $\mathcal{D}((S+T)^2)$ and is a subset of $\mathcal{F}(S, T)$.*

Remark 5.2. Ignoring domain questions for the moment one has

$$(S + T)^2 = S^2 + T^2 + [S, T].$$

A posteriori it indeed turns out that $[S, T]$ is relatively bounded w.r.t. $(S+T)^2$ with arbitrarily small relative bound. Hence once the domain claims are verified the Theorem is a consequence of the Hilbert-C*-module version of the Kato-Rellich Theorem [KALe12, Theorem 4.5].

Proof. By Theorem 2.6 and by Eq. (2.2) the closure of the operator $[S, T]$ has $\mathcal{D}(S) \cap \mathcal{D}(T)$ in its domain and $\overline{[S, T]}|_{\mathcal{D}(S) \cap \mathcal{D}(T)}$ is symmetric. Furthermore,

$$\mathcal{D}(S) \cap \mathcal{D}(T) \supset ((\mathcal{D}(S^2) \cap \mathcal{D}(T^2)) \cup \mathcal{D}((S+T)^2)).$$

In view of Theorem 2.6 (4) and Theorem 4.5 for $x \in \mathcal{D}(S) \cap \mathcal{D}(T)$ we choose an approximating sequence $(x_n) \subset \mathcal{F}(S, T)$ with $x_n \rightarrow x$, $Sx_n \rightarrow Sx$, and $Tx_n \rightarrow Tx$. By Eq. (2.2) then also $([S, T]x_n)$ converges to $\overline{[S, T]}x$ and hence

$$\begin{aligned} \langle \overline{[S, T]}x, \overline{[S, T]}x \rangle &\leq C_0 \langle x, x \rangle + C_1 \langle Sx, Sx \rangle + C_2 \langle Tx, Tx \rangle \\ &\leq C(\langle x, x \rangle + \langle (S+T)x, (S+T)x \rangle). \end{aligned} \quad (5.1)$$

Taking norms we find

$$\|\overline{[S, T]}x\| \leq C(\|x\| + \|(S+T)x\|).$$

Thus³ for any $\varepsilon > 0$ there exists a $C_\varepsilon > 0$ such that for $x \in \mathcal{D}((S+T)^2) \subset \mathcal{D}(S) \cap \mathcal{D}(T)$ one has

$$\|\overline{[S, T]}x\| \leq C_\varepsilon \cdot \|x\| + \varepsilon \cdot \|(S+T)^2x\|.$$

Thus on $\mathcal{D}((S+T)^2)$ the symmetric operator $\overline{[S, T]}$ is $(S+T)^2$ bounded with arbitrarily small bound. The Kato-Rellich Theorem for regular operators [KALe12, Theorem 4.5] now implies that $S^2 + T^2 = (S+T)^2 - [S, T]$ is self-adjoint and regular on $\mathcal{D}((S+T)^2)$.

Furthermore, this operator is obviously symmetric on the submodule

$$\mathcal{D}(S^2) \cap \mathcal{D}(T^2).$$

Therefore, if we can show that

$$\mathcal{D}((S+T)^2) \subset \mathcal{D}(S^2) \cap \mathcal{D}(T^2) \cap \mathcal{F}(S, T)$$

then we are done because a self-adjoint and regular operator does not have proper symmetric extensions.

³If A is a self-adjoint and regular operator in a Hilbert-C*-module it follows from the Spectral Theorem that for any $\varepsilon > 0$ there exists a C_ε such that for $x \in \mathcal{D}(A^2)$ one has $\|Ax\| \leq C_\varepsilon \cdot \|x\| + \varepsilon \cdot \|A^2x\|$.

To this end we use the Clifford matrices of Section 2.4. Then

$$\hat{S}_{\sigma_3} = \begin{pmatrix} S & 0 \\ 0 & -S \end{pmatrix}, \quad \hat{T} = \begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix}. \quad (5.2)$$

We have the relations

$$\begin{aligned} \hat{S}_{\sigma_3} \cdot \hat{T} \pm \hat{T} \cdot \hat{S}_{\sigma_3} &= (\hat{S}\hat{T} \pm \hat{T}\hat{S}) \cdot \sigma_3, \\ (\hat{S} \cdot \sigma_3 \pm \hat{T}) \cdot \hat{S}_{\sigma_1} + \hat{S}_{\sigma_1} \cdot (\hat{S} \cdot \sigma_3 \pm \hat{T}) &= \pm(\hat{S}\hat{T} + \hat{T}\hat{S}) \cdot \sigma_3, \end{aligned} \quad (5.3)$$

so the pair $(\hat{S}_{\sigma_3}, \hat{T})$ is also weakly anticommuting. Furthermore,

$$\hat{S}_{\sigma_3} + \hat{T} = \begin{pmatrix} S+T & 0 \\ 0 & T-S \end{pmatrix}, \quad (5.4)$$

and the pair $(\hat{S}_{\sigma_3} + \hat{T}, \hat{S}_{\sigma_1})$ is weakly anticommuting as well, cf. the last equation in Eq. (5.3). Namely, $\mathcal{F}(\hat{S}_{\sigma_1}, \hat{S}_{\sigma_3} + \hat{T})$ by definition equals the set

$$\{x \in \mathcal{D}(\hat{S}) \cap \mathcal{D}(\hat{T}) \mid \hat{S}x \in \mathcal{D}(\hat{S}) \cap \mathcal{D}(\hat{T}), \hat{T}x \in \mathcal{D}(\hat{S})\},$$

and thus

$$(\hat{S}_{\sigma_1} + \lambda)^{-1}(\mathcal{D}(\hat{S}) \cap \mathcal{D}(\hat{T})) \subset \mathcal{F}(\hat{S}_{\sigma_1}, \hat{S}_{\sigma_3} + \hat{T}).$$

By Theorem 2.6 (2) we thus also have the inclusion

$$(\hat{S}_{\sigma_3} + \hat{T} + \lambda)^{-1}(\mathcal{D}(\hat{S})) \subset \mathcal{F}(\hat{S}_{\sigma_1}, \hat{S}_{\sigma_3} + \hat{T}).$$

In view of the matrix representation Eq. (5.4) this implies

$$(S+T+\lambda)^{-1}(\mathcal{D}(S)) \subset \{x \in \mathcal{D}(S) \cap \mathcal{D}(T) \mid Sx \in \mathcal{D}(S) \cap \mathcal{D}(T), Tx \in \mathcal{D}(S)\}. \quad (5.5)$$

Now we are ready to prove the inclusion $\mathcal{D}((S+T)^2) \subset \mathcal{D}(S^2) \cap \mathcal{D}(T^2)$. Namely, let $x \in \mathcal{D}((S+T)^2)$. Then $(S+T+\lambda)x \in \mathcal{D}(S) \cap \mathcal{D}(T)$ and hence by Eq. (5.5) $x \in \mathcal{D}(S) \cap \mathcal{D}(T)$, $Sx \in \mathcal{D}(S) \cap \mathcal{D}(T)$, $Tx \in \mathcal{D}(S)$ and in particular $x \in \mathcal{F}(S, T)$. A posteriori $Tx = (S+T)x - Sx \in \mathcal{D}(T)$ as well. Thus we have shown that $x \in \mathcal{D}(S^2) \cap \mathcal{D}(T^2) \cap \mathcal{F}(S, T)$. \square

Corollary 5.3. *Let S, T be weakly commuting operators in the Hilbert-B-module E . Then $S^2 + T^2$ is self-adjoint and regular on $\mathcal{D}(S^2) \cap \mathcal{D}(T^2)$ and the latter equals*

$$\mathcal{D}((S+iT)^*(S+iT)) = \mathcal{D}((S-iT)^*(S-iT)).$$

Proof. Cf. Section 2.4, the operators

$$\hat{S}_{\sigma_1} = \begin{pmatrix} 0 & S \\ S & 0 \end{pmatrix}, \quad \hat{T}_{\sigma_2} = \begin{pmatrix} 0 & iT \\ -iT & 0 \end{pmatrix},$$

are weakly anticommuting. The previous Theorem gives that

$$(\hat{S}_{\sigma_1})^2 + (\hat{T}_{\sigma_2})^2 = \hat{S}^2 + \hat{T}^2,$$

is self-adjoint and regular on

$$\mathcal{D}(\hat{S}^2) \cap \mathcal{D}(\hat{T}^2) = \left(\mathcal{D}(S^2) \cap \mathcal{D}(T^2) \right)^{\oplus 2},$$

from which we conclude that $S^2 + T^2$ is self-adjoint and regular on $\mathcal{D}(S^2) \cap \mathcal{D}(T^2)$. The domain equality

$$\mathcal{D}((\hat{S}\sigma_1 + \hat{T}\sigma_2)^2) = \mathcal{D}((S - iT)^*(S - iT)) \oplus \mathcal{D}((S + iT)^*(S + iT)),$$

proves the remaining claim. \square

5.2. Iteration. In our early discussions on this project the following result appeared as a problem and seemed to be a major step in the second author's program to find a suitable category of unbounded KK-cycles. Therefore it was one of the driving forces to develop the sum theory of this paper. Although it became clear Theorem 5.4 does not resolve all remaining issues we think it is useful to record here for future reference.

Theorem 5.4. *Let $S_j, j = 1, 2, 3$, be self-adjoint and regular operators in the Hilbert-B-module E . Assume that $(S_1, S_2), (S_2, S_3), (S_1, S_3)$ are weakly anticommuting pairs. Then $S_1 + S_2$ and S_3 are weakly anticommuting and hence $S_1 + S_2 + S_3$ is self-adjoint and regular on $\mathcal{D}(S_1) \cap \mathcal{D}(S_2) \cap \mathcal{D}(S_3)$.*

Let us quickly explain why this is remarkable.⁴ According to the definition one needs to verify that $(S_1 + S_2 + \lambda)^{-1}(\mathcal{D}(S_3)) \subset \mathcal{F}(S_1 + S_2, S_3)$. A priori this is difficult since we have no concrete information about the resolvent of $S_1 + S_2$. However due to Theorem 2.6 the definition of weak anticommutativity is symmetric in the operators. Hence we know that also $(S_3 + \lambda)^{-1}(\mathcal{D}(S_j)) \subset \mathcal{F}(S_j, S_3)$ and, more importantly, that it suffices to show that $(S_3 + \lambda)^{-1}(\mathcal{D}(S_1) \cap \mathcal{D}(S_2)) \subset \mathcal{F}(S_1 + S_2, S_3)$. The latter turns out to be straightforward.

Proof. By Theorem 2.6 the operator $S_1 + S_2$ is self-adjoint and regular on $\mathcal{D}(S_1) \cap \mathcal{D}(S_2)$. By assumption we have

$$\begin{aligned} (S_3 + \lambda)^{-1}(\mathcal{D}(S_1)) &\subset \mathcal{F}(S_1, S_3), \\ (S_3 + \lambda)^{-1}(\mathcal{D}(S_2)) &\subset \mathcal{F}(S_2, S_3), \end{aligned}$$

thus

$$\begin{aligned} (S_3 + \lambda)^{-1}(\mathcal{D}(S_1 + S_2)) &\subset \mathcal{F}(S_1, S_3) \cap \mathcal{F}(S_2, S_3) \\ &= \{x \in \mathcal{D}(S_1) \cap \mathcal{D}(S_2) \cap \mathcal{D}(S_3) \mid \\ &\quad S_1x, S_2x \in \mathcal{D}(S_3), S_3x \in \mathcal{D}(S_1) \cap \mathcal{D}(S_2)\} \\ &\subset \mathcal{F}(S_1 + S_2, S_3). \end{aligned}$$

By the Remark 2.2 (3) and by Theorem 2.6 this implies

$$\mathcal{F}(S_1 + S_2, S_3) = (S_3 + \lambda)^{-1}(\mathcal{D}(S_1 + S_2)) = \mathcal{F}(S_1, S_3) \cap \mathcal{F}(S_2, S_3).$$

⁴It is more remarkable when one compares with the original definition of weak anticommutativity, cf. Sec. 2.3.

On $\mathcal{F}(S_1 + S_2, S_3)$ we thus have, using the abbreviation Eq. (4.4),

$$\begin{aligned} \langle \langle [S_1 + S_2, S_3]x \rangle \rangle &= \langle \langle [S_1, S_3]x \rangle \rangle + \langle \langle [S_2, S_3]x \rangle \rangle \\ &= \langle [S_1, S_3]x, [S_2, S_3]x \rangle + \langle [S_2, S_3]x, [S_1, S_3]x \rangle \\ &\leq 2(\langle \langle [S_1, S_3]x \rangle \rangle + \langle \langle [S_2, S_3]x \rangle \rangle) \\ &\leq C'_0 \cdot \langle x, x \rangle + C'_1 \cdot \langle S_1 x, S_1 x \rangle + C'_2 \langle S_2 x, S_2 x \rangle + C'_3 \cdot \langle S_3 x, S_3 x \rangle \\ &\leq C_0 \cdot \langle x, x \rangle + C_1 \cdot \langle (S_1 + S_2)x, (S_1 + S_2)x \rangle + C_2 \cdot \langle S_3 x, S_3 x \rangle, \end{aligned}$$

where we used Eq. (3.1) and the form estimate for the weakly anticommuting pairs $(S_1, S_3), (S_2, S_3)$. Hence $S_1 + S_2, S_3$ is a weakly anticommuting pair of operators. \square

6. COMPARISON TO BANACH SPACE RESULTS FOR SUMS OF OPERATORS

There exists quite some literature on the problem of closedness and regularity of sums of sectorial operators in Banach spaces [DPGr75, DoVe87, LaTe87, Fuh93, MoPr97, KAWe01, PrSio7, Roi16]. One might wonder whether and how our results compare to these. After all a Hilbert- C^* -module is a Banach space and our operators S, T, S^2, T^2 are *sectorial* operators. It is the purpose of this section to put this into context.

Let X be a Banach space and let A be a densely defined operator in X . One should think of A as being S^2 or T^2 above. The operator A is called *sectorial* in the open sector $\Sigma_\theta := \{z \in \mathbb{C} \mid z \neq 0, |\arg z| < \theta\}$ if $\ker A = \{0\}$, $\text{ran } A$ is dense, $A + \lambda$ is invertible for $\lambda \in \Sigma_\theta$ and

$$M_\theta := \sup_{\lambda \in \Sigma_\theta} |\lambda| \cdot \|(A + \lambda)^{-1}\| < \infty.$$

The *spectral angle* of A is defined by

$$\phi_A := \inf\{\phi > 0 \mid A \text{ is sectorial in } \Sigma_{\pi-\phi}\}.$$

Clearly, if S is a self-adjoint and regular operator in the Hilbert- C^* -module E then S^2 is sectorial with spectral angle 0, while for any $\delta > 0$ the operator $iS + \delta$ is sectorial with spectral angle $\pi/2$.

Now given two sectorial operators A, B in a Banach space, the analogue of the regularity problem in Hilbert- C^* -modules splits into the subproblems to decide whether $A + B$ is closed on $\mathcal{D}(A) \cap \mathcal{D}(B)$ and whether $A + B + \lambda$ has dense range for $\lambda > 0$ large (resp. $\overline{A + B}$ is sectorial). This should be compared to Theorem 2.6 (1), (3) and to Theorem 5.1.

One of the seminal results on this is the following:

Theorem 6.1 (Da Prato and Grisvard [DPGr75], cf. also [PrSio7, Sec. 3]). *Let A, B be sectorial operators in a Banach space X with spectral angles ϕ_A, ϕ_B . Assume that $\psi_A > \phi_A, \psi_B > \phi_B, \psi_A + \psi_B < \pi, (A + \lambda)^{-1}(\mathcal{D}(B)) \subset \mathcal{D}(B)$, and*

$$\|(B(A + \lambda)^{-1} - (A + \lambda)^{-1}B)(\mu + B)^{-1}\| \leq \frac{c}{(1 + |\lambda|)^\alpha |\mu|^\beta}, \quad (6.1)$$

for $\lambda \in \Sigma_{\pi-\psi_A}, \mu \in \Sigma_{\pi-\psi_B}$ and fixed numbers $\alpha, \beta > 0, \beta < 1, \alpha + \beta > 1$.

Then $A + B$ is sectorial with spectral angle $\leq \max(\psi_A, \psi_B)$.

In [LATe87] Eq. (6.1) was replaced by a more flexible but also more involved estimate, cf. [PRSi07, Sec. 3].

Proving closedness of $A + B$ on $\mathcal{D}(A) \cap \mathcal{D}(B)$ requires additional assumptions on the Banach space (class \mathcal{HT}) and that A, B admit bounded imaginary powers. The seminal result is that of Dore and Venni [DoVe87] for resolvent commuting operators. This was later improved by Monnieux and Prüss [MoPr97] for non-commuting operators satisfying the above mentioned Labbas-Terreni [LATe87] commutator condition, and by Prüss and Simonett [PRSi07] for pairs satisfying the Da Prato-Grisvard commutator condition Eq. (6.1).

For our operators in a Hilbert-C*-module E bounded imaginary powers are a non-issue. That is, for a self-adjoint and regular operator S in E it follows trivially from the Spectral Theorem that S has bounded imaginary powers, i.e. that $\{S^{it} \mid -1 \leq t \leq 1\}$ is bounded in $\mathcal{L}(E)$.

The class \mathcal{HT} condition seems to be “orthogonal” to Hilbert-C*-module theory: recall that a Banach space X is called of class \mathcal{HT} if the Hilbert transform

$$Hf(t) := \lim_{\varepsilon \searrow 0} \int_{|s| \geq \varepsilon} f(t-s) \frac{ds}{s}, \quad f \in C_c^\infty(\mathbb{R}, X),$$

extends by continuity to a bounded linear operator $L^2(\mathbb{R}, X) \rightarrow L^2(\mathbb{R}, X)$. Here, $L^2(\mathbb{R}, X)$ is the completion of $C_c^\infty(\mathbb{R}, X)$ with respect to the norm

$$\|f\| := \left(\int_{\mathbb{R}} \|f(x)\|_X^2 dx \right)^{1/2}.$$

We asked experts on the aforementioned Banach space theory but the following problem could not be clarified:

Problem 6.2. *Let E be a Hilbert-B-module. Decide whether E is of class \mathcal{HT} or not.*

The problem here is that in general $L^2(\mathbb{R}, E)$ is not a Hilbert-B-module, but is strictly smaller than the external tensor product $L^2(\mathbb{R}) \hat{\otimes}_{\mathbb{C}} E$. The latter is the completion of $C_c^\infty(\mathbb{R}, E)$ with respect to the inner product

$$\langle f, g \rangle = \int_{\mathbb{R}} \langle f(x), g(x) \rangle_E dx \in B$$

resp. the induced norm

$$\|f\| = \left\| \int_{\mathbb{R}} \langle f(x), g(x) \rangle_E dx \right\|_B^{1/2}.$$

Clearly, since the Hilbert transform $H_0 : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ is bounded the Hilbert transform on $L^2(\mathbb{R}) \hat{\otimes}_{\mathbb{C}} E$ is nothing but $H_0 \hat{\otimes} \text{id}$, which is bounded as well. This leads to

Problem 6.3. *Is it true that for Hilbert-C*-modules the proof of the above mentioned Dore-Venni type results go through by exploiting instead of the \mathcal{HT} condition the (obviously true) condition of boundedness of the Hilbert transform on $L^2(\mathbb{R}) \hat{\otimes}_{\mathbb{C}} E$.*

Finally, we outline an alternative approach to the main Theorem 2.6 using the method due to Da Prato-Grisvard. While their results have been generalized and

refined, all subsequent publications essentially employ the basic pattern which can already be found in [DPGr75]. Namely, given sectorial operators A, B then view $A+B+\lambda$ as a (operator valued) function of B . Then the resolvent $(A+B+\lambda)^{-1}$ should be given by the Dunford integral

$$P_\lambda := \frac{1}{2\pi i} \int_\Gamma (z + \lambda + A)^{-1} \cdot (z - B)^{-1} dz, \quad (6.2)$$

where Γ is a contour of the form $(\infty, r)e^{i\theta} \cup re^{i[\theta, \pi-\theta]} \cup (r, \infty)e^{-i\theta}$ with $\psi_B < \theta < \min(\psi, \pi - \psi_A)$.

If A and B are resolvent commuting, then P_λ equals the resolvent $(A+B+\lambda)^{-1}$. In all other cases P_λ is only an approximation to the resolvent and the main part of the work is to formulate commutator conditions on A and B ensuring that P_λ maps a sufficiently large space into $\mathcal{D}(A) \cap \mathcal{D}(B)$.

Turning to a pair of weakly anticommuting operators S, T we cannot apply the pattern outlined above since our commutator conditions Def. 2.1 concern commutators of S and T but not of S^2, T^2 . Therefore, it is unrealistic to prove commutator estimates on S^2, T^2 à la Da Prato-Grisvard resp. Labbas-Terreni.

However, we can slightly modify Eq. (6.2) to obtain a resolvent approximation of $S + T + i\lambda$ instead of $S^2 + T^2 + \lambda$.⁵

Namely, from the estimate Eq. (2.4) one infers

$$\|[S, T](S + i\lambda)^{-1}(T + i\mu)^{-1}\| \leq C\left(\frac{1}{|\lambda|} + \frac{1}{|\mu|}\right)$$

hence, cf. Theorem 6.1,

$$\|[T^2, (S^2 + \lambda)^{-1}](T^2 + \mu)^{-1}\| \leq \frac{C}{|\lambda|} \left(\frac{1}{\sqrt{|\lambda|}} + \frac{1}{\sqrt{|\mu|}} \right).$$

However, we may not expect the stronger domain inclusion

$$(S^2 + \lambda)^{-1}(\mathcal{D}(T^2)) \subset \mathcal{D}(T^2)$$

to hold.

Nevertheless, without further assumptions, these estimates and the axioms of weak anticommutativity allow to prove

Theorem 6.4. *Let*

$$P_\lambda := \frac{1}{2\pi i} \int_\Gamma (z + \lambda + S^2)^{-1}(S + T - i\lambda)(z - T^2)^{-1} dz.$$

Then for $y \in \mathcal{D}(S) \cap \mathcal{D}(T)$ and λ large we have $P_\lambda y \in \mathcal{D}(S) \cap \mathcal{D}(T)$ and

$$(S + T + i\lambda)P_\lambda y = (I + R_\lambda)y,$$

with $\|R_\lambda\| < 1$, hence $\text{ran}(S + T + i\lambda)$ dense.

So in principle the original idea of Da Prato-Grisvard together with the axioms of weak anticommutativity lead to yet another proof of the closedness and regularity statements in Theorem 2.6. Further details are omitted and hence left to the reader.

⁵In this section λ, μ denote real parameters.

7. THE KASPAROV PRODUCT OF UNBOUNDED MODULES

In this section we describe how Theorem 4.5 can be applied in the constructive approach to the Kasparov product. For background on unbounded KK-theory we refer to [BAJ83, BLA98, CON94, KUC97, KUC00, MES14, MER16, KALE13].

7.1. Weakly anticommuting operators and the Kasparov product. Kasparov's KK-theory [KAS80] is a powerful tool in operator K-theory [BLA98]. It associates to a pair of separable C*-algebras (A, B) an abelian group $KK_0(A, B)$. The main feature of KK-theory is the existence of an associative bilinear product

$$KK_0(A, B) \times KK_0(B, C) \rightarrow KK_0(A, C), \quad (7.1)$$

defined for all separable C*-algebras A, B and C .

A $\mathbb{Z}/2$ -grading on a Hilbert C*-module E is a self-adjoint operator $\gamma \in \mathcal{L}(E)$ such that $\gamma^2 = 1$. An operator $F \in \mathcal{L}(E)$ is *even* if $F\gamma = \gamma F$ and *odd* if $\gamma F = -F\gamma$.

Elements of the group $KK_0(A, B)$ are given by the following data:

Definition 7.1. [KAS80] Let (A, B) be a pair of separable C*-algebras. A Kasparov module for (A, B) is a pair (E, F) where

- (i) E is a $\mathbb{Z}/2$ -graded Hilbert C*-module over B together with a *-homomorphism $A \rightarrow \mathcal{L}(E)$;
- (ii) $F \in \mathcal{L}(E)$ is an odd operator such that $a(1 - F^2)$, $a(F - F^*)$ and $[F, a]$ are elements of $\mathcal{K}(E)$.

Here, $[\cdot, \cdot]$ denotes the *graded* commutator which on homogeneous elements x, y of parity $\partial x, \partial y$ is defined by $[x, y] = xy - (-1)^{\partial x \cdot \partial y} yx$. That is $[a, b] = [a, b]_+$ if a, b are both odd and $[a, b] = [a, b]_-$ if one of them is even. For the commutators introduced in Section 2.2 we will therefore always write $[\cdot, \cdot]_+$ resp. $[\cdot, \cdot]_-$ to make the sign of the second summand explicit.

For an (A, B) Hilbert bimodule E we will refer to the C*-algebra

$$\{K \in \mathcal{L}(E) \mid \forall a \in A \quad aK, Ka \in \mathcal{K}(E)\}$$

as the C*-algebra of *A*-locally compact operators on E . For self-adjoint elements $Q, R \in \mathcal{L}(E)$ we say that $Q \leq R$ *modulo A-locally compact operators* if there exists a locally compact operator K such that $Q \leq R + K$.

In [COSK84, Theorem A.5] Connes-Skandalis provided sufficient conditions that determine the product (7.1).

Theorem 7.2. Let (X, F_X) and (Y, F_Y) be Kasparov modules for (A, B) and (B, C) respectively. Suppose that $(X \otimes_B Y, F)$ is a (A, C) Kasparov module such that

- (i) for all $x \in X$ the operator $y \mapsto \gamma(x) \otimes F_Y y - F(x \otimes y)$ is in $\mathcal{K}(Y, X \otimes_B Y)$;
- (ii) there is $0 \leq \kappa < 2$ such that for all $a \in A$ the operator inequality

$$a^*[F_X \otimes 1, F]a \geq -\kappa a^*a,$$

holds modulo *A*-compact operators.

Then $(X \otimes_B Y, F)$ represents the Kasparov product of (X, F_X) and (Y, F_Y) .

Notice that condition (ii) is weaker than what is stated in [CoSk84, Theorem A.5] (see [Kuc97, Definition 4] and [BLa98, Definition 18.4.1]). This weakening will be of vital importance for our main theorem.

In order to describe the external product in KK-theory in a constructive way, Baaj-Julg introduced the following refinement of Kasparov modules.

Definition 7.3 ([BAJu83]). *Let (A, B) be a pair of separable C^* -algebras. An unbounded Kasparov module for (A, B) is a triple (A, E, D) where*

- (i) E is a $\mathbb{Z}/2$ -graded Hilbert C^* -module over B together with a $*$ -homomorphism $A \rightarrow \mathcal{L}(E)$;
- (ii) $D : \mathcal{D}(D) \rightarrow E$ is a self-adjoint regular operator such that $\alpha(D \pm i)^{-1} \in \mathcal{K}(E)$ for all $\alpha \in A$;
- (iii) $\mathcal{A} \subset A$ is a norm dense $*$ -subalgebra such that $\alpha : \mathcal{D}(D) \rightarrow \mathcal{D}(D)$ and $[D, \alpha]$ extends to an element in $\mathcal{L}(E)$ for all $\alpha \in \mathcal{A}$.

A continuous function $\chi : \mathbb{R} \rightarrow [-1, 1]$ is called a *normalizing function* if

$$\chi(-x) = -\chi(x) \quad \text{and} \quad \lim_{x \rightarrow \pm\infty} \chi(x) = \pm 1.$$

If (E, D) is an unbounded Kasparov module then $(E, \chi(D))$ is a Kasparov module ([BAJu83]) whose class does not depend on the choice of χ . Notice that the difference of any two normalizing functions χ_1, χ_2 is an element of $C_0(\mathbb{R})$, which is generated by $(x \pm i)^{-1}$. Since $(D \pm i)^{-1}$ are locally compact, so is $\chi_1(D) - \chi_2(D)$ and the two functions give homotopic Kasparov modules, cf. [HiRooo, Sec. 10.6].

Theorem 7.4. *Let (A, X, S_X) and (B, Y, T_Y) be unbounded Kasparov modules for (A, B) and (B, C) respectively and let $E := X \otimes_B Y$ and $S := S_X \otimes 1$. Suppose that $T : \mathcal{D}(T) \rightarrow E$ is an odd self-adjoint regular operator such that*

- (i) *there is a dense B -submodule $\mathcal{X} \subset \mathcal{D}(S) \subset X$ for which the algebraic tensor product $\mathcal{X} \otimes_B^{\text{alg}} \mathcal{D}(T_Y)$ is a core for T and for all homogenous elements $x \in \mathcal{X}$ and all $y \in \mathcal{D}(T_Y)$ the operator*

$$y \mapsto \gamma(x) \otimes T_Y y - T(x \otimes y)$$

defines an element of $\mathcal{L}(Y, E)$;

- (ii) *for all $\alpha \in A$ we have $\alpha : \mathcal{D}(T) \rightarrow \mathcal{D}(T)$ and $[T, \alpha] \in \mathcal{L}(E)$;*
- (iii) *(S, T) is a weakly anticommuting pair.*

Then $(A, E, S+T)$ is an unbounded Kasparov module that represents the Kasparov product of (X, S_X) and (Y, T_Y) .

Proof. The fact that $(E, S+T)$ is an unbounded Kasparov module follows quite easily: the sum $S+T$ is self-adjoint and regular by condition (iii), and has bounded commutators with \mathcal{A} by condition (ii). By condition (i) and [MeRe16, Lemma 4.3] we have $\alpha(S + \lambda)^{-1}(T + \mu)^{-1} \in \mathcal{K}(E)$ for all $\alpha \in A$. In particular

$$\alpha B_\lambda^{-1} = \frac{\alpha}{\lambda} (S + \lambda)^{-1} (T + \lambda)^{-1} \in \mathcal{K}(E),$$

with B_λ as in Lemma 4.1. By Eq. (4.8) we have

$$(A_\lambda + \mu)^{-1} = B_\lambda^{-1} - (\mu - \lambda) B_\lambda^{-1} (A_\lambda + \mu)^{-1},$$

and it follows that $a(A_\lambda + \mu)^{-1} \in \mathcal{K}(E)$. By Theorem 4.5

$$a(S + T + \mu)^{-1} = \lim_{\lambda \rightarrow \infty} a(A_\lambda + \mu)^{-1},$$

is a norm limit and we conclude that $a(S + T + \mu)^{-1} \in \mathcal{K}(E)$.

To show that $(E, S + T)$ represents the Kasparov product, consider the normalizing functions

$$\chi(x) := \frac{2}{\pi} \arctan(x), \quad b(x) := x(1 + x^2)^{-1/2}.$$

We will prove that $\chi(D)$ satisfies the conditions of Theorem 7.2. By [Kuc97, Proposition 14] the operators $b(D)$ and $b(T_Y)$ satisfy condition (i) of Theorem 7.2. Since $\chi(D) - b(D)$ is A locally compact on E and $b(T_Y) - \chi(T_Y)$ is B locally compact on Y , $\chi(D)$ and $\chi(T_Y)$ satisfy condition (i) as well.

The fact that after a suitable homotopy, $\chi(D)$ and $\chi(S)$, satisfy condition (ii) follows from Proposition 7.12 in Section 7.3. \square

Remark 7.5. Theorem 7.4 should be compared to [Kuc97, Theorem 13]. There, fewer assumptions are imposed on the form of the product operator, in particular it need not arise as a sum. The case where

$$\langle [S, T]x, [S, T]x \rangle \leq C(\langle x, x \rangle + \langle Sx, Sx \rangle),$$

is covered by the latter result. This assumption was in place in [KALe13, MeRe16]. However, as soon as there is a nontrivial relative bound to T as well, condition (iii) of [Kuc97, Theorem 13] may not be satisfied. An example of such a situation is given in [BoMe18].

Remark 7.6. The construction of operators T satisfying hypotheses (i) and (ii) of Theorem 7.4 is the subject of the of the papers [KALe13, MeS14, MeRe16]. Indeed in [MeRe16] it was shown that up to equivalence, such a T can always be constructed. In geometric situations, an operator T with the required properties can often be written down explicitly, see for example [BMvS16, KavS16].

7.2. A form estimate for the absolute value of the sum. We denote by S and T a weakly anti-commuting pair of operators on the Hilbert C^* -module E , and by $D := S + T$ their sum operator, which is self-adjoint and regular. Our goal is to obtain a form estimate for the anticommutator $[S, T]$ relative to the positive operator $|D|$ defined through functional calculus. As we wish to work on the domain of D we consider the extension $\overline{[S, T]}$, as in the proof of Theorem 5.1.

Lemma 7.7. *For $0 \leq \operatorname{Re} z \leq 1$ the operator $P_z := (1 + |D|)^{-z} \overline{[S, T]} (1 + |D|)^{z-1}$ is bounded on $\mathcal{D}(S) \cap \mathcal{D}(T)$, extends to an adjointable operator and $\|P_z\| \leq \|P_0\|$.*

Proof. The operator $1 + |D| : \mathcal{D}(D) \rightarrow E$ is boundedly invertible and by Eq. (5.1) the operator $\overline{[S, T]} : \mathcal{D}(D) \rightarrow E$ is bounded, when $\mathcal{D}(D)$ is equipped with the graph norm of D . Hence

$$P_0 := \overline{[S, T]} (1 + |D|)^{-1} : E \rightarrow E$$

is bounded and consequently the densely defined operator

$$(1 + |D|)^{-1} \overline{[S, T]} : \mathcal{D}(S) \cap \mathcal{D}(T) \rightarrow E$$

is bounded as well and its closure P_1 equals the adjoint of P_0 .

We now adapt the interpolation argument of [Les05, Appendix A] to the case of Hilbert C^* -modules. For $\operatorname{Re} z > 0$ the operators $(1 + |D|)^{-z}$ preserve $\mathcal{D}(D)$. For $x, y \in \mathcal{D}(D)$ the function

$$f_{x,y} : z \mapsto \langle (1 + |D|)^{-z} \overline{[S, T]} (1 + |D|)^{-1+z} x, y \rangle = \langle P_z x, y \rangle,$$

is weakly holomorphic on the strip $0 \leq \operatorname{Re} z \leq 1$. Since

$$\begin{aligned} \langle P_z x, P_z x \rangle &\leq \langle x, x \rangle + \langle D(1 + |D|)^{-1+z} x, D(1 + |D|)^{-1+z} x \rangle \\ &\leq \langle x, x \rangle + \langle Dx, Dx \rangle, \end{aligned}$$

we infer that $\|P_z x\| \leq \|x\| + \|Dx\|$ and

$$\|f_{x,y}(z)\| \leq \|P_z x\| \|y\| \leq (\|x\| + \|Dx\|) \|y\|$$

so $f_{x,y}$ is a bounded function. Now let $\varphi : B \rightarrow \mathbb{C}$ be a state. The function $z \mapsto \varphi \circ f_{x,y}(z)$ is bounded and holomorphic in the strip $0 \leq \operatorname{Re} z \leq 1$. By the Phragmén-Lindelöf Theorem (aka Hadamard 3-line Theorem in this case) the function is bounded by its suprema on the boundary $\operatorname{Re} z \in \{0, 1\}$. For such z it holds that $\|P_z\| = \|P_0\| = \|P_1\| = \|P_0^*\|$. So we obtain that

$$\begin{aligned} |\varphi(\langle P_z x, y \rangle)| &\leq \sup_{\operatorname{Re} w \in \{0, 1\}} |\varphi(\langle P_w x, y \rangle)| \\ &\leq \sup_{\operatorname{Re} w \in \{0, 1\}} \|\langle P_w x, y \rangle\| \leq \|P_0\| \|x\| \|y\|. \end{aligned}$$

Since this holds for all states φ it follows that $\|\langle P_z x, y \rangle\| \leq \|P_0\| \|x\| \|y\|$ and hence $\|P_z\| \leq \|P_0\|$. \square

By a *rescaling* of a weakly anticommuting pair (S, T) we mean a weakly anticommuting pair of the form $(\lambda S, \lambda T)$ for some $\lambda > 0$.

Proposition 7.8. *Let (S, T) be a weakly anticommuting pair and $D = S + T$ their sum. For all $x \in \mathcal{D}(S) \cap \mathcal{D}(T)$ the form estimate*

$$\langle \overline{[S, T]} x, x \rangle \leq C(\langle x, x \rangle + \langle |D|x, x \rangle), \quad (7.2)$$

holds true, with C a constant independent of x . Consequently, for all $\mu > 0$ we have the operator estimate

$$\pm(1 + \mu^2 D^2)^{-1} [S, T] (1 + \mu^2 D^2)^{-1} \leq C(1 + |D|)(1 + \mu^2 D^2)^{-2}. \quad (7.3)$$

After a suitable rescaling of the pair (S, T) we can achieve that $C < \varepsilon$ for any $\varepsilon > 0$.

Proof. The operator $P_{1/2}$ is self-adjoint whence

$$\begin{aligned} \langle [S, T] x, x \rangle &= \langle P_{1/2} (1 + |D|)^{1/2} x, P_{1/2} (1 + |D|)^{1/2} x \rangle \\ &\leq \|P_0\| \langle (1 + |D|)^{1/2} x, (1 + |D|)^{1/2} x \rangle \\ &= \|P_0\| (\langle x, x \rangle + \langle |D|x, x \rangle), \end{aligned}$$

which proves the form estimate (7.2) with $C = \|P_0\|$. The operator estimate (7.3) now follows in a straightforward manner. Replacing S, T by $\lambda S, \lambda T$ for $0 < \lambda < 1$ we obtain

$$\langle [\lambda S, \lambda T]x, x \rangle = \lambda^2 \langle [S, T]x, x \rangle \leq \lambda^2 C (\langle x, x \rangle + \langle |D|x, x \rangle) \leq \lambda C (\langle x, x \rangle + \langle |\lambda D|x, x \rangle).$$

Thus, by taking λ sufficiently small we may assume that C is as small as we like. \square

7.3. Proof of the positivity condition. We use the integral representation of the function $\arctan = \tan^{-1}$

$$\arctan(x) = \int_0^x \frac{1}{1+t^2} dt = \int_0^1 \frac{x}{1+\mu^2 x^2} d\mu.$$

For any self-adjoint regular operator D , the bounded adjointable operator $\chi(D) := \frac{2}{\pi} \arctan(D)$ then has the representation

$$\chi(D) = \frac{2}{\pi} \int_0^1 D(1+\mu^2 D^2)^{-1} d\mu,$$

as a *strongly* convergent integral (cf. [Kuc97, Lemma 8]).

We now consider a weakly anticommuting pair of operators (S, T) in a Hilbert C^* -module E . Recall from Section 2.4 that the Clifford algebra $\mathbb{C}\ell(2)$ is represented unitarily on $E \oplus E$ and that the $\mathbb{C}\ell(2)$ action commutes with \hat{S}, \hat{T} ((2.8), (2.9)) and that the action preserves their domains. Denote by $\omega := \sigma_3 = i\sigma_1\sigma_2$ the volume element of $\mathbb{C}\ell(2)$ and we let $\hat{D}_\pm := \hat{S} \pm \omega \hat{T}$ and $\hat{D} := \hat{D}_+$, cf. Eq. (5.2)–(5.4). Since ω commutes with \hat{S}, \hat{T} we have that the pair $(\hat{S}, \pm \omega \hat{T})$ is weakly anticommuting as well and that $\hat{S} \pm \omega \hat{T}$ is self-adjoint and regular with domain $\mathcal{D}(\hat{S}) \cap \mathcal{D}(\hat{T})$. Recall also that

$$\hat{S} + \omega \hat{T} = \begin{pmatrix} S + T & 0 \\ 0 & S - T \end{pmatrix}.$$

So for the time being we may w.l.o.g. omit the hat decorator and assume that S, T are $\mathbb{C}\ell(2)$ invariant.

Lemma 7.9. *For $\mu > 0$ the operator*

$$\begin{aligned} K_\mu &:= (1 + \mu^2 D_-^2)^{-1} - (1 + \mu^2 D_+^2)^{-1} \\ &= 2(1 + \mu^2 D_+^2)^{-1} \mu^2 [S, T] (1 + \mu^2 D_-^2)^{-1} \end{aligned} \tag{7.4}$$

$$= 2(1 + \mu^2 D_-^2)^{-1} \mu^2 [S, T] (1 + \mu^2 D_+^2)^{-1} \tag{7.5}$$

is A locally compact, as are the operators $D_\pm K_\mu$. Moreover

$$\sup\{\|K_\mu\|, \|D_\pm K_\mu\| : \mu \in (0, \infty)\} < \infty,$$

and thus integrate to A -locally compact operators over any finite interval $(0, x]$.

Proof. Since $\mathcal{D}(D_\pm^2) = \mathcal{D}(S^2) \cap \mathcal{D}(T^2) \subset \mathcal{F}(S, T)$ by Theorem 5.1, formula (7.4) follows by direct calculation and (7.5) by taking adjoints. Using (7.4) for D_+ and (7.5) for D_- it follows that $D_\pm K_\mu$ is locally compact. Because of the presence of the factor μ^2 in this equation, multiplication by D_\pm still yields a family of operators that is uniformly bounded in μ . \square

From now on we write D for D_+ . Consider the bounded adjointable operators $\chi(D)$ and $\chi(S)$

$$\frac{4}{\pi^2} [\chi(D), \chi(S)] = \int_0^1 \int_0^1 (1 + \lambda^2 S^2)^{-1} S D (1 + \mu^2 D^2)^{-1} + (1 + \mu^2 D^2)^{-1} D S (1 + \lambda^2 S^2)^{-1} d\lambda d\mu. \quad (7.6)$$

We will show that, for any $\kappa > 0$, a suitable rescaling of the operators D and S gives that $[\chi(D), \chi(S)] \geq -\kappa$, modulo A -locally compact operators. We therefore discard the multiplicative factor $\frac{4}{\pi^2}$. We apply the identity

$$(1 + \lambda^2 S^2)(1 + \lambda^2 S^2)^{-1} = (1 + \lambda^2 S^2)^{-1}(1 + \lambda^2 S^2) = 1,$$

and multiply the first summand of (7.6) from the right and second summand from the left. The integrand can thus be written as the sum of the operator

$$(1 + \lambda^2 S^2)^{-1} S D (1 + \mu^2 D^2)^{-1} (1 + \lambda^2 S^2) (1 + \lambda^2 S^2)^{-1}, \quad (7.7)$$

and its adjoint. Expanding $D = S + \omega T$ in

$$S D (1 + \mu^2 D^2)^{-1} (1 + \lambda^2 S^2) = D (1 + \mu^2 D^2)^{-1} \lambda^2 S^2 + S (1 + \mu^2 D^2)^{-1} D$$

gives us a sum of four terms

$$S^2 (1 + \mu^2 D_+^2)^{-1} \lambda^2 S^2 + S (1 + \mu^2 D^2)^{-1} S \quad (7.8)$$

$$+ S (1 + \mu^2 D^2)^{-1} \omega T + \lambda S \cdot \omega T (1 + \mu^2 D_+^2)^{-1} S \cdot \lambda S. \quad (7.9)$$

The summands (7.8) are nonnegative and can thus be discarded. By adding the adjoints of (7.9) and multiply by $(1 + \lambda^2 S^2)^{-1}$ from the left and from the right (cf. (7.7)), we need to address the integral of the sum of operators

$$P_\lambda \cdot R_\mu \cdot P_\lambda + Q_\lambda \cdot R_\mu \cdot Q_\lambda,$$

where

$$R_\mu := \omega T (1 + \mu^2 D^2)^{-1} S + S (1 + \mu^2 D^2)^{-1} \omega T \quad (7.10)$$

$$P_\lambda := (1 + \lambda^2 S^2)^{-1}, \quad Q_\lambda := \lambda S (1 + \lambda^2 S^2)^{-1}, \quad (7.11)$$

so that up to positive operators (7.6) can be written

$$\int_0^1 P_\lambda \left(\int_0^1 R_\mu d\mu \right) P_\lambda d\lambda + \int_0^1 Q_\lambda \left(\int_0^1 R_\mu d\mu \right) Q_\lambda d\lambda. \quad (7.12)$$

We will prove that for any $\varepsilon > 0$ there is a rescaling of the pair (S, T) such that the integral $\int_0^1 R_\mu d\mu \geq -\varepsilon$ modulo A -locally compact operators. By [MeRe16, Lemma 4.3] $P_\lambda K P_\lambda$ and $Q_\lambda K Q_\lambda$ are A -locally compact whenever K is. Since $\|P_\lambda\| \leq 1$ and $\|Q_\lambda\| \leq 1$ this allows us to estimate (7.12) from below as well. Note that since we are integrating over $[0, 1]$, perturbing the integrand by a function $f(\mu)$ with values in the A -locally compact operators that is *uniformly bounded in μ* yields an A -locally compact perturbation after integration.

Our first goal is to find another expression for R_μ . We first consider the algebraic identity Eq. (A.5) and show that it holds with $a = \sigma_2 S$ and $b = D = S + \omega T$.

Lemma 7.10. *The self-adjoint regular operators S , T and $D_{\pm} = S \pm \omega T$ satisfy the identities*

$$[(1 + D^2)^{-1}, \sigma_j S]_- = (1 + D^2)^{-1} [\sigma_j S, D]_- D (1 + D^2)^{-1} + D (1 + D^2)^{-1} [\sigma_j S, D]_- (1 + D^2)^{-1}, \quad (7.13)$$

$$[(1 + D^2)^{-1}, \sigma_j T]_- = (1 + D^2)^{-1} [\sigma_j T, D]_+ D (1 + D^2)^{-1} - D (1 + D^2)^{-1} [\sigma_j T, D]_+ (1 + D^2)^{-1}, \quad (7.14)$$

on $\mathcal{D}(S) \cap \mathcal{D}(T)$ for $j = 1, 2$.

Proof. Recall from the proof of Theorem 5.1 that the pair $(D, \sigma_j S)$ is weakly commuting for $j = 1, 2$ and that the pair $(D, \sigma_j T)$ is weakly anticommuting for $j = 1, 2$. The commutator identities in Lemma A.1 in Appendix A apply including domains with $b = D_{\pm}$, $a = \sigma_j S$ or $a = \sigma_j T$ and $\lambda \in i\mathbb{R} \setminus \{0\}$. We prove Eq. (7.13) using the resolvent identities

$$(1 + D^2)^{-1} = (D + i)^{-1} (D - i)^{-1} = (D - i)^{-1} (D + i)^{-1}.$$

The Leibniz rules of Lemma A.1 give the identities

$$[(1 + D^2)^{-1}, \sigma_j S]_- = (1 + D^2)^{-1} [\sigma_j S, D]_- (D \pm i)^{-1} + (D \mp i)^{-1} [\sigma_j S, D]_- (1 + D^2)^{-1},$$

$$[(1 + D^2)^{-1}, \sigma_j T]_- = (1 + D^2)^{-1} [\sigma_j T, D]_+ (D \pm i)^{-1} - (D \mp i)^{-1} [\sigma_j T, D]_+ (1 + D^2)^{-1},$$

on $\mathcal{D}(S) \cap \mathcal{D}(T)$. Averaging these equalities for $\pm i$ and using that

$$(D + i)^{-1} + (D - i)^{-1} = 2D(1 + D^2)^{-1},$$

then give us Eq. (7.13) and Eq. (7.14) on $\mathcal{D}(S) \cap \mathcal{D}(T)$. \square

Lemma 7.11. *Recalling the notation $D_{\pm} = S \pm \omega T$ we have for $\mu > 0$ the equality of operators*

$$\begin{aligned} \omega R_{\mu} &= (1 + \mu^2 D_+^2)^{-1} [S, T] (1 + \mu^2 D_-^2)^{-1} \\ &\quad + \mu D_- (1 + \mu^2 D_+^2)^{-1} [\overline{S}, \overline{T}] (1 + \mu^2 D_-^2)^{-1} \mu D_- \end{aligned} \quad (7.15)$$

This amounts to an equality

$$\begin{aligned} \omega R_{\mu} &= (1 + \mu^2 D_-^2)^{-1} [S, T] (1 + \mu^2 D_-^2)^{-1} \\ &\quad + \mu D_- (1 + \mu^2 D_-^2)^{-1} [\overline{S}, \overline{T}] (1 + \mu^2 D_-^2)^{-1} \mu D_- \end{aligned} \quad (7.16)$$

modulo an A -locally compact perturbation that is uniformly bounded in μ .

Proof. Note that by definition Eq. (7.10) $R_{\mu} = R(\mu, S, T)$ is a rational function of μ and the (non-commuting) variables S, T , and we have the relation $R(1, \mu S, \mu T) = \mu^2 R(\mu, S, T)$. The same is true for the right hand side of Eq. (7.15). Hence it suffices to prove the claim for $\mu = 1$. It then follows in general by replacing S, T, D by $\mu S, \mu T, \mu D$ resp.

We have $\omega R := \omega R_1 = S(1 + D^2)^{-1}T + T(1 + D^2)^{-1}S$. For the first summand we calculate on $\mathcal{D}(S) \cap \mathcal{D}(T)$ using the commutator identity Eq. (7.14) and the Leibniz rule

$$\begin{aligned} S(1 + D^2)^{-1}T &= S(1 + D^2)^{-1}\sigma_2 T \sigma_2 \\ &= \sigma_2 S T (1 + D^2)^{-1}\sigma_2 + S[(1 + D^2)^{-1}, \sigma_2 T]_- \sigma_2 \\ &= \sigma_2 S T (1 + D^2)^{-1}\sigma_2 - S D (1 + D^2)^{-1}[\sigma_2 T, D]_+ (1 + D^2)^{-1}\sigma_2 \\ &\quad + S(1 + D^2)^{-1}[\sigma_2 T, D]_+ D (1 + D^2)^{-1}\sigma_2 \end{aligned}$$

and for the second summand using Eq. (7.13)

$$\begin{aligned} T(1 + D^2)^{-1}S &= T(1 + D^2)^{-1}\sigma_2 S \sigma_2 \\ &= \sigma_2 T S (1 + D^2)^{-1}\sigma_2 + T[(1 + D^2)^{-1}, \sigma_2 S]_- \sigma_2 \\ &= \sigma_2 T S (1 + D^2)^{-1}\sigma_2 + T D (1 + D^2)^{-1}[\sigma_j S, D]_- (1 + D^2)^{-1} \\ &\quad + T(1 + D^2)^{-1}[\sigma_j S, D]_- D (1 + D^2)^{-1}\sigma_2, \end{aligned}$$

Adding up and using the identities

$$\begin{aligned} D\omega &= \omega D = \omega S + T, \\ D\sigma_j &= \sigma_j D_-, \quad j = 1, 2 \\ [\sigma_j S, D]_- &= -\omega \sigma_j [S, T]_+, \quad j = 1, 2 \\ [\sigma_j T, D]_+ &= \sigma_j [S, T]_+, \quad j = 1, 2 \end{aligned}$$

we find

$$\begin{aligned} \omega R &= \sigma_2 [S, T]_+ (1 + D^2)^{-1}\sigma_2 \\ &\quad - D^2 (1 + D^2)^{-1}\sigma_2 [S, T]_+ (1 + D^2)^{-1}\sigma_2 \\ &\quad + D_- (1 + D^2)^{-1}\sigma_2 [S, T]_+ D (1 + D^2)^{-1}\sigma_2. \end{aligned}$$

Noting that $1 - D^2(1 + D^2)^{-1} = (1 + D^2)^{-1}$ allows to combine the first and the second summand. Moving the first σ_2 to the far right replaces D by D_- in between. Altogether this gives

$$\omega R = (1 + D^2)^{-1}[S, T]_+ (1 + D_-^2)^{-1} + D_- (1 + D^2)^{-1}[S, T]_+ (1 + D_-^2)^{-1}$$

whence the first claim.

By Lemma 7.4 we can replace $(1 + \mu^2 D_+^2)^{-1}$ by $(1 + \mu^2 D_-^2)^{-1}$ in both summands of R_μ at the cost of an error term that is uniformly bounded in μ . The latter expression thus equals

$$(1 + \mu^2 D_-^2)^{-1}[S, T](1 + \mu^2 D_-^2)^{-1} + \mu D_- (1 + \mu^2 D_-^2)^{-1}[S, T](1 + \mu^2 D_-^2)^{-1}\mu D_-,$$

modulo A -locally compact perturbations that are uniformly bounded in μ . \square

We arrive at the following Proposition, which completes the proof of Theorem 7.4.

Proposition 7.12. *Let E be a Hilbert C^* -module over B and let $A \rightarrow \mathcal{L}(E)$ be a $*$ -homomorphism. Furthermore, let (S, T) be a weakly anticommuting pair such that $D_{\pm} = S \pm T$ has A -locally compact resolvent. Then for every $\kappa > 0$, (S, T) can be rescaled so that for $\chi(x) := \frac{2}{\pi} \arctan(x)$ the operators $\chi(S)$ and $\chi(D_+)$ satisfy the operator estimate*

$$[\chi(S), \chi(D)] \geq -\kappa,$$

up to an A -locally compact perturbation.

Proof. Let $\varepsilon > 0$ and rescale (S, T) so that the operator estimate (7.3) holds true. We apply Lemma 7.11 to \hat{S}, \hat{T} . Then the upper left corner of the corresponding \hat{R}_{μ} gives up to A -locally compact perturbations which are uniformly bounded in μ :

$$\begin{aligned} R_{\mu} &= (1 + \mu^2 D_-^2)^{-1} [S, T] (1 + \mu^2 D_-^2)^{-1} + \mu D_- (1 + \mu^2 D_-^2)^{-1} [S, T] (1 + \mu^2 D_-^2)^{-1} \mu D_- \\ &\geq -\varepsilon (1 + |D_-|) ((1 + \mu^2 D_-^2)^{-2} + (1 + \mu^2 D_-^2)^{-1}) \\ &\geq -2\varepsilon (1 + |D_-|) (1 + \mu^2 D_-^2)^{-1} \\ &= -2\varepsilon |D_-| (1 + \mu^2 D_-^2)^{-1}. \end{aligned}$$

Cf. Equation (7.12) we have, modulo A -locally compact perturbations that

$$\begin{aligned} \frac{4}{\pi^2} [\chi(D_+), \chi(S)] &= \int_0^1 P_{\lambda} \left(\int_0^1 R_{\mu} d\mu \right) P_{\lambda} d\lambda + \int_0^1 Q_{\lambda} \left(\int_0^1 R_{\mu} d\mu \right) Q_{\lambda} d\lambda \\ &\geq -4\varepsilon \int_0^1 |D_-| (1 + \mu^2 D_-^2)^{-1} d\mu \\ &\geq -2\pi\varepsilon, \end{aligned}$$

since

$$\pm \arctan(|D_-|) = \pm \int_0^1 |D_-| (1 + \mu^2 D_-^2)^{-1} d\mu \leq \frac{\pi}{2}.$$

Thus, choosing $\varepsilon = \frac{2\kappa}{\pi^3}$ and rescaling (S, T) according to Proposition 7.8 yields that $[\chi(D_+), \chi(S)] \geq -\kappa$ modulo A -locally compact perturbations. \square

APPENDIX A. COMMUTATOR IDENTITIES

We collect here some useful identities for (graded) commutators. In the sequel a, b, c, \dots denote elements in a unital \mathbb{C} -algebra. This section is concerned only with algebraic identities. When applying to unbounded operators the equality of domains needs to be checked separately.

Recall from Sections 2.2, 2.3

$$[a, b]_{\tau} := a \cdot b + \tau b \cdot a, \quad \tau \in \{+, -\}. \quad (\text{A.1})$$

Lemma A.1. *For $\sigma, \tau \in \{+, -\}$ one has the Leibniz rules*

$$[a, b \cdot c]_{\tau} = [a, b]_{\sigma} \cdot c - \sigma b \cdot [a, c]_{-\sigma\tau}, \quad (\text{A.2})$$

$$[a \cdot b, c]_{\tau} = a \cdot [b, c]_{\sigma} - \sigma [a, c]_{-\sigma\tau} \cdot b. \quad (\text{A.3})$$

This follows immediately by expanding the left and right hand sides.

Lemma A.2. *Assume that for $\lambda \in \mathbb{C}$ the element $\pm\lambda + b$ resp. $\lambda + b^2$ is invertible; λ is an abbreviation for $\lambda \cdot 1$. Then*

$$(\lambda + b)^{-1}a = a(\lambda - \tau b)^{-1} - (\lambda + b)^{-1}[b, a]_{\tau}(\lambda - \tau b)^{-1}, \quad (\text{A.4})$$

$$\begin{aligned} [(\lambda + b^2)^{-1}, a]_{-} &= (\lambda + b^2)^{-1}b[a, b]_{-}(\lambda + b^2)^{-1} \\ &\quad + (\lambda + b^2)^{-1}[a, b]_{-}b(\lambda + b^2)^{-1}, \end{aligned} \quad (\text{A.5})$$

$$\begin{aligned} [(\lambda + b^2)^{-1}, a]_{-} &= -(\lambda + b^2)^{-1}b[a, b]_{+}(\lambda + b^2)^{-1} \\ &\quad + (\lambda + b^2)^{-1}[a, b]_{+}b(\lambda + b^2)^{-1}. \end{aligned} \quad (\text{A.6})$$

Proof. We have

$$(\lambda + b)a = ba + \tau ab - \tau ab + a\lambda = [b, a]_{\tau} + a(\lambda - \tau b).$$

Now multiply from the left by $(\lambda + b)^{-1}$ and from the right by $(\lambda - \tau b)^{-1}$ to obtain the first identity.

The second and third identity follow by applying the first identity to b^2, a and then the Leibniz rule to $[b^2, a]_{-} = b[b, a]_{-} + [b, a]_{-}b = b[b, a]_{+} - [b, a]_{+}b$. \square

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