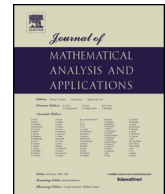




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# Solitary waves for one-dimensional nematicon equations

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## ABSTRACT

In this paper, we study the one-dimensional nematicon equations which model the propagation of coherent and polarized light in nonlocal nematic liquid crystals. The existences of local and global solutions are derived first upon applying the Strichartz's estimates. Then the existence of ground state solitary wave solutions is proved by using the concentration-compactness technique and the critical point theory.

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## 1. Introduction

Nematic liquid crystals (NLCs) consist of rod-like molecules exhibiting both orientational alignment and strong optical nonlinearities due to large refractive index anisotropy. When an external electric field is applied, the optical characteristics of NLCs will change and enable macroscopic reorientation of molecules. It has been observed also that the light incident modifies the electric permittivity tensor of NCLs and leads to reorientational nonlinearity. When studying laser propagation in NLCs, researchers have observed recently the so-called nematicons, i.e., the stable spatial optical solitons in NLCs [3,5,12,15]. Due to their potential applications in all-optical switches, light waveguide, integrated optics and optical computing technologies, the study of nematicons has attracted many attentions in recent years (see [15] and references therein).

Consider the propagation of a linearly polarized optical beam through a planar cell filled with undoped NLCs. Assume the optical beam propagates in the  $z$ -direction, which is perpendicular to the confining interfaces on the  $(x, y)$ -plane. The governing equations for the propagation of optical beam through the nematic liquid cell are given by [3,5,7,12]

$$\begin{cases} i\partial_z E + \frac{1}{2}\Delta_{x,y} E + E \sin 2\theta = 0, & (x, y) \in \mathbb{R}^2, z \in \mathbb{R}, \\ \nu \Delta_{x,y} \theta - q \sin 2\theta = -2|E|^2 \cos 2\theta, & (x, y) \in \mathbb{R}^2, z \in \mathbb{R}, \end{cases} \quad (1.1)$$

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where  $\Delta_{x,y} = \partial_{xx} + \partial_{yy}$ ,  $E(x, y, z)$  is the optical field,  $\theta$  corresponds to the perturbation of the director angle from the pre-tilt angle,  $\nu$  measures the elasticity of NLCs, and  $q$  is related to the energy of the static electric field which pre-tilts NLCs.

System (1.1) provides a very good agreement with experimental data [14,16]. The regularizing effect of nonlocal nonlinearity makes the system (1.1) an interesting laboratory for studying solitary waves [2,17]. Since it is almost impossible to solve (1.1) analytically, one has to solve this system by numerical methods or approximation methods. For the highly nonlocal case, i.e.,  $\nu$  is large, the response of the director to the electric field is nonlocal due to the slow decay of the crystal distortion produced by the optical beam. The system (1.1) can be approximated by the following system

$$\begin{cases} i\partial_z E + \frac{1}{2}\Delta_{x,y}E + 2\theta E = 0, & (x, y) \in \mathbb{R}^2, z \in \mathbb{R}, \\ \nu\Delta_{x,y}\theta - 2q\theta = -2|E|^2, & (x, y) \in \mathbb{R}^2, z \in \mathbb{R}. \end{cases} \quad (1.2)$$

Consider the nematicon equations (1.2) for only one transverse dimension  $x$ , i.e., independent of variable  $y$ . (1.2) is then reduced to the following one-dimensional nematicon equations

$$\begin{cases} i\partial_z E + \frac{1}{2}\partial_{xx}E + 2\theta E = 0, & x \in \mathbb{R}, z \in \mathbb{R}, \\ \nu\partial_{xx}\theta - 2q\theta = -2|E|^2, & x \in \mathbb{R}, z \in \mathbb{R}. \end{cases} \quad (1.3)$$

It is straightforward to solve the second equation of system (1.3) and obtain the following expression,

$$\theta(x) = \frac{2}{\nu}(G^\gamma * |E|^2)(x) = \frac{2}{\nu} \int_{\mathbb{R}} \frac{1}{2\gamma} e^{-\gamma|x-x'|} |E(x', z)|^2 dx', \quad (1.4)$$

where  $\gamma = (\frac{2q}{\nu})^{\frac{1}{2}}$ ,  $G^\gamma(x) = \frac{1}{2\gamma}e^{-\gamma|x|}$  is the fundamental solution [8] of

$$(-\partial_{xx} + \gamma^2)\theta = \delta(x).$$

By substituting the expression of  $\theta(x)$  into the first equation of system (1.3), we obtain the following nonlinear nonlocal Schrödinger equation,

$$i\partial_z E + \frac{1}{2}\partial_{xx}E + \frac{4}{\nu}(G^\gamma * |E|^2)E = 0, \quad x \in \mathbb{R}, z \in \mathbb{R}. \quad (1.5)$$

Note that equation (1.5), together with (1.4), is equivalent to system (1.3). It is interesting to point out that Marchant and Smyth [10] studied the following approximate equation

$$i\partial_z E + \frac{1}{2}\partial_{xx}E + \frac{\gamma}{2q}e^{-\gamma|x|} \left( \int_{\mathbb{R}} |E(x', z)|^2 dx' \right) E = 0, \quad x \in \mathbb{R}, z \in \mathbb{R}, \quad (1.6)$$

and obtained an asymptotic bulk solitary wave solution, which leads to both the ground state and symmetric excited states with multiple peaks.

In this paper, we investigate first the local and global well-posedness of the Cauchy problem of nematicon equation (1.5). We will prove the existence of local and global solutions of (1.5) by using the Strichartz's estimates [4], and establish the conservation laws. Then we study the solitary wave solutions of equation (1.5), i.e., solutions of the form

$$E(x, z) = e^{-i\omega z} u(x), \quad \omega \in \mathbb{R}. \quad (1.7)$$

Equation (1.5) is then reduced to

$$-\frac{1}{2}\partial_{xx}u + \omega u = \frac{4}{\nu}(G^\gamma * |u|^2)u, \quad x \in \mathbb{R}. \quad (1.8)$$

We will prove the existence of ground state solutions for equation (1.8) by using the concentration-compactness technique [9] and the critical point theory [11].

## 2. Preliminary setting and main results

In this section, we introduce some basic notations and lemmas which will be used in subsequent sections. Let  $L^p(\mathbb{R})$  denote the usual Lebesgue space for  $p \geq 1$ , and  $H^1(\mathbb{R})$  denote the usual Sobolev space. For  $I \subseteq \mathbb{R}$  and a Banach space  $X$ , let

$$L^p(I; X) = \{E : I \rightarrow X \mid \|E(\cdot)\|_X \in L^p(I)\}.$$

The Cauchy problem for the nematicon equation (1.5) with initial data  $E(0) = E_0 \in H^1(\mathbb{R})$  is equivalent to the following integral equation

$$E(z) = e^{izA}E_0 - i \int_0^z e^{i(z-\tau)A}F(E(\tau))d\tau, \quad (2.1)$$

where  $A = -\frac{1}{2}\partial_{xx}$  and

$$F(E) := \frac{4}{\nu}(G^\gamma * |E|^2)E.$$

Following the notion introduced in [4], an pair of indices  $(q, r)$  with  $2 \leq q, r \leq \infty$  is admissible for one-dimensional space  $\mathbb{R}$  if  $0 \leq \frac{2}{q} = \frac{1}{2} - \frac{1}{r} \leq 1$ .

**Lemma 2.1** (Strichartz estimates [4,6]). *For any  $z > 0$ , the following properties hold.*

(a) *Let  $\varphi \in L^2(\mathbb{R})$ . For any admissible pair  $(q, r)$ , there exists a constant  $c = c(z, q, r)$  such that*

$$\|e^{itA}\varphi\|_{L^q((-z, z); L^r)} \leq c\|\varphi\|_{L^2}; \quad (2.2)$$

(b) *Let  $I \subset (-z, z)$  be an interval and  $z_0 \in \bar{I}$ . For any admissible pairs  $(q, r)$  and  $(m, n)$ , there exists a constant  $c = c(z, q, r, m, n)$  such that*

$$\left\| \int_{z_0}^z e^{i(z-s)A}F(s)ds \right\|_{L^q(I; L^r)} \leq c\|F\|_{L^{m'}(I; L^{n'})}, \quad (2.3)$$

for every  $F \in L^{n'}(I; L^{m'})$ , where  $m', n'$  are the dual exponents by  $\frac{1}{m} + \frac{1}{m'} = 1$  and  $\frac{1}{n} + \frac{1}{n'} = 1$ .  $\square$

**Lemma 2.2** (Gagliardo-Nirenberg's inequality [4]). *For any  $E \in H^1(\mathbb{R})$ , we have*

$$\|E\|_{L^p} \leq c\|E\|_{L^2}^{(1-a)}\|\partial_x E\|_{L^2}^a,$$

where  $2 \leq p \leq \infty$ ,  $a = \frac{1}{2} - \frac{1}{p}$ , and constant  $c$  depends only on  $p$  and  $a$ .  $\square$

Now, we state our main results of this paper.

**Theorem 2.1** (Global existence of  $H^1$ -solutions). *For any  $E(\cdot, 0) = E_0 \in H^1(\mathbb{R})$ , there exists a unique solution  $E \in C(\mathbb{R}, H^1(\mathbb{R})) \cap C^1(\mathbb{R}, H^{-1}(\mathbb{R}))$  for the nematicon equation (1.5).  $\square$*

For equation (1.8), define functional  $\Phi : H^1(\mathbb{R}) \rightarrow \mathbb{R}$  by

$$\Phi(u) = \frac{1}{2} \int_{\mathbb{R}} |\partial_x u|^2 dx + \omega \int_{\mathbb{R}} |u|^2 dx - \frac{2}{\nu} \int_{\mathbb{R}} (G^\gamma * |u|^2) |u|^2 dx. \quad (2.4)$$

It is straightforward to verify that  $\Phi \in C^1(H^1(\mathbb{R}), \mathbb{R})$  and

$$(\Phi'(u), v) = \int_{\mathbb{R}} \partial_x u \partial_x \bar{v} dx + 2\omega \int_{\mathbb{R}} u \bar{v} dx - \frac{8}{\nu} \int_{\mathbb{R}} (G^\gamma * |u|^2) u \bar{v} dx, \quad \forall v \in H^1(\mathbb{R}). \quad (2.5)$$

Thus the critical points of  $\Phi(u)$  in  $H^1(\mathbb{R})$  correspond to the weak solutions of the equation (1.8).

For any  $\lambda > 0$  and  $u \in H^1(\mathbb{R})$ , let  $u_\lambda(x) = u(x + \lambda)$ . Then

$$(G^\gamma * |u_\lambda|^2)(x) = \int_{\mathbb{R}} \frac{1}{2\gamma} e^{-\gamma|x-x'|} |u_\lambda(x')|^2 dx' = \int_{\mathbb{R}} \frac{1}{2\gamma} e^{-\gamma|x+\lambda-x'|} |u(x')|^2 dx' = (G^\gamma * |u|^2)(x + \lambda).$$

Consequently, we have  $\Phi(u_\lambda) = \Phi(u)$  for any  $\lambda \in \mathbb{R}$ . Thus  $\Phi$  is translation invariant.

**Definition 2.1.**  $u_0 \in H^1(\mathbb{R})$  is called a ground state solution of equation (1.8) if

$$\Phi(u_0) = \inf\{\Phi(u), u \in \mathcal{M}\}, \quad (2.6)$$

where  $\mathcal{M} = \{u \in H^1(\mathbb{R}) \mid u \text{ is a nontrivial critical point of } \Phi(u)\}$ .  $\square$

By using the concentration-compactness technique introduced by Lions [9], we have the following result on ground state solutions of equation (1.8).

**Theorem 2.2** (Existence of ground state solutions). *Let  $\omega > 0$ . There exists a ground state solution for equation (1.8). Consequently, there exists a ground state solitary wave solution for the nematicon equation (1.5).  $\square$*

### 3. Well-posedness of the Cauchy problem

In this section, we study the well-posedness of the Cauchy problem for the nematicon equation (1.5) with initial data  $E(0) = E_0 \in H^1(\mathbb{R})$ . Since the Cauchy problem is equivalent to the integral equation (2.1), we will investigate the local and global existences of solutions of (2.1). We prove first the local existence of  $H^1$ -solutions by the contraction mapping principle and Lemma 2.1. Define the following map

$$P(E)(z, x) = (e^{izA} E_0)(x) - i \int_0^z e^{i(z-\tau)A} F(E(\tau)) d\tau. \quad (3.1)$$

**Lemma 3.1** (Local existence of  $H^1$ -solutions). *For any  $\rho > 0$ , there exists a  $z(\rho) > 0$  such that, for any  $E_0 \in H^1(\mathbb{R})$  with  $\|E_0\|_{H^1} \leq \rho$ , the nematicon equations (1.5) has a unique solution  $E \in C(I, H^1(\mathbb{R}))$  where  $I = [-z(\rho), z(\rho)]$ .*

**Proof.** Note that  $G^\gamma(x) = \frac{1}{2^\gamma} e^{-\gamma|x|}$ . By Theorem 6.23 in [8],  $G^\gamma(x)$  is symmetric decreasing, and satisfies the following properties,

$$G^\gamma(x) > 0 \text{ for all } x \in \mathbb{R}; \quad G^\gamma(x) \rightarrow \frac{1}{2^\gamma}, \text{ as } x \rightarrow 0; \text{ and } \frac{-\log G^\gamma(x)}{\gamma|x|} \rightarrow 1, \text{ as } |x| \rightarrow \infty. \quad (3.2)$$

Hence, by (3.2),  $G^\gamma \in L^{p_0}(\mathbb{R})$  for any  $1 \leq p_0 \leq \infty$ . By the Sobolev embedding theorem [1], we have  $H^1(\mathbb{R}) \hookrightarrow L^r(\mathbb{R})$  and  $L^{r'}(\mathbb{R}) \hookrightarrow H^{-1}(\mathbb{R})$ , where  $r > 1$ ,  $r' > 1$  and  $\frac{1}{r'} + \frac{1}{r} = 1$ . By applying the Young and Hölder's inequalities, we have, for any  $u, v, w \in H^1(\mathbb{R})$ ,

$$\|(G^\gamma * (u\bar{v}))w\|_{L^{r'}} \leq \|G^\gamma\|_{L^{p_0}} \|u\|_{L^r} \|v\|_{L^r} \|w\|_{L^r}, \quad (3.3)$$

where  $r = \frac{4p_0}{2p_0-1}$ ,  $\frac{1}{r'} + \frac{1}{r} = 1$ . For any  $E_1, E_2 \in H^1(\mathbb{R})$ , we have

$$\begin{aligned} & \|F(E_1) - F(E_2)\|_{L^{r'}} \\ &= \frac{4}{\nu} \|(G^\gamma * |E_1|^2)E_1 - (G^\gamma * |E_2|^2)E_2\|_{L^{r'}} \\ &\leq \frac{4}{\nu} [\|(G^\gamma * ((E_1 - E_2)\bar{E}_1))E_1\|_{L^{r'}} + \|(G^\gamma * (E_1(\bar{E}_1 - \bar{E}_2)))E_1\|_{L^{r'}} + \|(G^\gamma * |E_2|^2)(E_1 - E_2)\|_{L^{r'}}] \end{aligned}$$

By applying (3.3), for any  $M > 0$ , there exists a  $C(M) > 0$  such that

$$\|F(E_1) - F(E_2)\|_{L^{r'}} \leq C(M)\|E_1 - E_2\|_{L^r}, \text{ for all } E_1, E_2 \in L^r(\mathbb{R}) \text{ with } \|E_1\|_{H^1}, \|E_2\|_{H^1} \leq M. \quad (3.4)$$

Hence  $F \in C(L^r(\mathbb{R}), L^{r'}(\mathbb{R}))$ . Let  $\rho > 0$  be given and  $\|E_0\|_{H^1} \leq \rho$ . Let  $M$  be chosen such that  $M > \rho$ . By Lemma 2.1, one can choose a small  $z(\rho) > 0$  such that

- (i) if  $E \in C(I, H^1(\mathbb{R}))$  with  $\|E(z)\|_{H^1} \leq M$  for  $z \in I$ , then  $P(E) \in C(I, H^1(\mathbb{R}))$  with  $\|P(E)(z)\|_{H^1} \leq M$  for  $z \in I$ ;
- (ii) mapping  $P$  is a contraction mapping in  $C(I, H^1(\mathbb{R}))$ ,

where  $I = [-z(\rho), z(\rho)]$ . Thus, by the Banach contraction mapping principle, there is a unique  $E \in C(I, H^1(\mathbb{R}))$  such that  $P(E) = E$ . Consequently, the Cauchy problem of the nematicon equation (1.5) admits a unique solution in  $C(I, H^1(\mathbb{R}))$ .

It follows also from the Young and Hölder's inequalities that, for any  $u, v, w, \phi \in H^1(\mathbb{R})$ ,

$$\left| \int_{\mathbb{R}} (G^\gamma * (u\bar{v}))w\bar{\phi} dx \right| \leq \|G^\gamma\|_{L^{p_0}} \|u\|_{L^r} \|v\|_{L^r} \|w\|_{L^r} \|\phi\|_{L^r},$$

where  $r = \frac{4p_0}{2p_0-1}$ . By the Sobolev embedding theorem, we also have  $F \in C(H^1(\mathbb{R}), H^{-1}(\mathbb{R}))$ . It follows from equation (1.5) that solution  $E(x, z)$  satisfies  $E \in C(I, H^1(\mathbb{R}))$  and  $\partial_z E \in C(I, H^{-1}(\mathbb{R}))$  where  $I = [-z(\rho), z(\rho)]$ .  $\square$

Define the mass  $m(E(\cdot, z))$  by

$$m(E(\cdot, z)) = \left( \int_{\mathbb{R}} |E(x, z)|^2 dx \right)^{1/2},$$

and the energy associated with the nematicon equation (1.5) by

$$H(E(\cdot, z)) = \frac{1}{2} \int_{\mathbb{R}} |\partial_x E|^2 dx - \frac{2}{\nu} \int_{\mathbb{R}} (G^\gamma * |E|^2) |E|^2 dx. \quad (3.5)$$

Then the solution  $E(x, z)$  obtained in Lemma (3.1) conserves both the mass  $m(E(\cdot, z))$  and the energy  $H(E(\cdot, z))$  [13]. Indeed, by multiplying (1.5) by  $\bar{E}(x, z)$ , integrating with respect to  $x$  on  $\mathbb{R}$  and taking the imaginary part, we have

$$\mathbf{Im} \left[ \int_{\mathbb{R}} i(\partial_z E) \bar{E} dx + \frac{1}{2} \int_{\mathbb{R}} (\partial_{xx} E) \bar{E} dx + \frac{4}{\nu} \int_{\mathbb{R}} (G^\gamma * |E|^2) E \bar{E} dx \right] = 0.$$

Then we obtain

$$\frac{\partial}{\partial z} \left( \int_{\mathbb{R}} |E(x, z)|^2 dx \right) = 0.$$

Hence

$$m(E(\cdot, z)) = m(E_0). \quad (3.6)$$

On the other hand, by multiplying (1.5) by  $2\partial_z \bar{E}$ , integrating with respect to  $x$  on  $\mathbb{R}$  and taking the real part, we have

$$\mathbf{Re} \left[ 2 \int_{\mathbb{R}} i\partial_z E \partial_z \bar{E} dx + \int_{\mathbb{R}} \partial_{xx} E \partial_z \bar{E} dx + \frac{4}{\nu} \int_{\mathbb{R}} (G^\gamma * |E|^2) E \partial_z \bar{E} dx \right] = 0.$$

A direct calculation yields

$$\frac{\partial}{\partial z} H(E(\cdot, z)) = 0.$$

Hence

$$H(E(\cdot, z)) = H(E_0). \quad (3.7)$$

**Proof of Theorem 2.1.** Let  $\lambda = m(E(\cdot, z))$ . By (3.6),  $\lambda$  remains constant along trajectory  $E(x, z)$ . By Lemma 2.2, we have

$$\begin{aligned} \int_{\mathbb{R}} (G^\gamma * |E|^2) |E(x, z)|^2 dx &\leq \|G^\gamma * |E|^2\|_{L^\infty} \|E\|_{L^2}^2 \\ &\leq \lambda \|G^\gamma\|_{L^{p_0}} \|E\|_{L^{2p'_0}}^2 \\ &\leq \lambda C \|G^\gamma\|_{L^{p_0}} \|\partial_x E\|_{L^2}^{2a} \|E\|_{L^2}^{2(1-a)}, \end{aligned}$$

with  $p_0 \geq 1$ ,  $\frac{1}{p_0} + \frac{1}{p'_0} = 1$  and  $a = \frac{1}{2p_0} < 1$ . Hence, we have

$$H(E(\cdot, z)) \geq \frac{1}{2} \|\partial_x E\|_{L^2}^2 - \lambda^{2-a} C \|G^\gamma\|_{L^{p_0}} \|\partial_x E\|_{L^2}^{2a}.$$

It follows from (3.6) and (3.7) that  $\|E(\cdot, z)\|_{H^1}$  must be remain bounded by a constant  $M_0$  for all  $z \in [-z(\rho), z(\rho)]$ , where  $M_0$  depends only on  $\|E_0\|_{H^1}$ . Thus the existence interval can be extended to  $\mathbb{R}$ , and  $\|E(\cdot, z)\|_{H^1} \leq M_0, \forall z \in \mathbb{R}$ .  $\square$

#### 4. Ground state solutions

In this section, we prove the existence of ground state solutions of equation (1.8) by using the concentration-compactness technique [9] and the Mountain Pass Lemma [11].

Let  $\omega > 0$ , we may endow  $H^1(\mathbb{R})$  with the following norm

$$\|u\|_* = \left( \int_{\mathbb{R}} (|\partial_x u|^2 + 2\omega|u|^2) dx \right)^{1/2}, \quad u \in H^1(\mathbb{R}),$$

which is equivalent to the standard  $H^1(\mathbb{R})$ -norm. Then the functional defined in (2.4) can be expressed by

$$\Phi(u) = \frac{1}{2} \|u\|_*^2 - Q(u), \quad (4.1)$$

where

$$Q(u) = \frac{2}{\nu} \int_{\mathbb{R}} (G^\gamma * |u|^2) |u|^2 dx.$$

**Mountain Pass Lemma.** [11] Let  $X$  be a real Banach space,  $B_\rho(0) = \{v \in X \mid \|v\|_X < \rho\}$  and  $\partial B_\rho(0) = \{v \in X \mid \|v\|_X = \rho\}$ . Assume that  $J \in C^1(X, \mathbb{R})$  satisfies the following conditions:

- (a) for some  $v_0 \in X$ , there are constants  $\rho, \alpha > 0$  such that  $J|_{v_0 + \partial B_\rho(0)} \geq J(v_0) + \alpha$ ;
- (b) there is an  $e \in X \setminus (v_0 + \overline{B}_\rho(0))$  such that  $J(e) \leq J(v_0)$ .

Then there exists a sequence  $\{v_n\}$  in  $X$  such that  $J(v_n) \rightarrow \beta$  and  $J'(v_n) \rightarrow 0$  in  $X'$  as  $n \rightarrow \infty$ , where  $\beta$  can be characterized by

$$\beta = \inf_{\gamma \in \Gamma} \max_{v \in \gamma([0,1])} J(v),$$

where  $\Gamma = \{\gamma \in C([0,1], X) \mid \gamma(0) = v_0, \gamma(1) = e\}$ .  $\square$

We prove first that functional  $\Phi(u)$  satisfies the conditions of the Mountain Pass Lemma. Note that  $\Phi(0) = 0$  and  $\Phi \in C^1(H^1(\mathbb{R}), \mathbb{R})$ .

**Lemma 4.1.** There exist constants  $\rho > 0$  and  $\alpha > 0$  such that

$$\Phi(u) \geq \alpha, \quad \forall u \in \partial B_\rho(0),$$

where  $B_\rho(0) = \{u \in H^1(\mathbb{R}) \mid \|u\|_* < \rho\}$ .

**Proof.** By Lemma 2.2, we have

$$\begin{aligned} Q(u) &\leq C \|G^\gamma\|_{L^{p_0}} \|u\|_{L^2}^{2(2-a)} \|\partial_x u\|_{L^2}^{2a} \\ &\leq C (2\omega)^{-(2-a)} \|G^\gamma\|_{L^{p_0}} \|u\|_*^{2(2-a)} \|u\|_*^{2a} \\ &= C (2\omega)^{-(2-a)} \|G^\gamma\|_{L^{p_0}} \|u\|_*^4, \end{aligned} \quad (4.2)$$

where  $1 \leq p_0 \leq \infty$ ,  $0 < a = \frac{1}{2p_0} < 1$ . Then

$$\Phi(u) = \frac{1}{2}\|u\|_*^2 - Q(u) \geq \frac{1}{2}\|u\|_*^2 - C(2\omega)^{-(2-a)}\|G^\gamma\|_{L^{p_0}}\|u\|_*^4. \quad (4.3)$$

Thus there exist constants  $\rho > 0$  and  $\alpha > 0$  such that  $\Phi|_{\partial B_\rho(0)} \geq \alpha$ .  $\square$

**Lemma 4.2.** *There exists an  $e \in H^1(\mathbb{R})$  such that  $e \in H^1(\mathbb{R}) \setminus \overline{B}_\rho(0)$  and  $I(e) \leq \Phi(0) = 0$ .*

**Proof.** Let  $u_0 \in H^1(\mathbb{R})$  and  $u_0 \neq 0$ . For  $t > 0$ , we have

$$\Phi(tu_0) = \frac{t^2}{2}\|u_0\|_*^2 - \frac{t^4}{\nu} \int_{\mathbb{R}} (G^\gamma * |u_0|^2)|u_0|^2 dx \rightarrow -\infty, \quad \text{as } t \rightarrow +\infty.$$

Then, for a sufficiently large  $t_0 > 0$ , we have  $\Phi(t_0 u_0) < 0$  and  $t_0 u_0 \in H^1(\mathbb{R}) \setminus \overline{B}_\rho(0)$ . Thus one may choose  $e = t_0 u_0$ .  $\square$

Thus, by Lemmas 4.1 and 4.2, functional  $\Phi$  satisfies both conditions of the Mountain Pass Lemma. It follows from the Mountain Pass Lemma that there exists a sequence  $\{u_j\} \subset H^1(\mathbb{R})$  such that

$$\Phi(u_j) \rightarrow b, \text{ and } \Phi'(u_j) \rightarrow 0 \text{ in } H^{-1}(\mathbb{R}), \text{ as } j \rightarrow \infty, \quad (4.4)$$

where

$$b = \inf_{g \in \Gamma} \max_{t \in [0,1]} \Phi(g(t)) \quad (4.5)$$

and

$$\Gamma = \{g \in C([0,1]; H^1(\mathbb{R})) \mid g(0) = 0 \text{ and } g(1) = e\}. \quad (4.6)$$

**Lemma 4.3.** *Let  $\{u_j\} \subset H^1(\mathbb{R})$  be the sequence of satisfying (4.4). Then*

$$\lim_{j \rightarrow \infty} \|u_j\|_*^2 = 4b, \quad \lim_{j \rightarrow \infty} Q(u_j) = b. \quad (4.7)$$

**Proof.** By Lemma 4.1, we have  $b > 0$ . We show first that  $\{u_j\}$  is bounded in  $H^1(\mathbb{R})$ . Suppose this is not true. Then  $\{u_j\}$  has a subsequence, denoted by  $\{u_j\}$  also, such that  $\|u_j\|_* \rightarrow \infty$  as  $j \rightarrow \infty$ . It follows from (4.4) that

$$\Phi(u_j) = \frac{1}{2}\|u_j\|_*^2 - Q(u_j) \rightarrow b, \text{ and } \frac{\Phi'(u_j)u_j}{\|u_j\|_*} = \frac{\|u_j\|_*^2 - 4Q(u_j)}{\|u_j\|_*} \rightarrow 0, \text{ as } j \rightarrow \infty. \quad (4.8)$$

Then

$$\begin{aligned} 0 &= \lim_{j \rightarrow \infty} \frac{\|u_j\|_*^2 - 4Q(u_j)}{\|u_j\|_*^2} \\ &= \lim_{j \rightarrow \infty} \frac{4(\frac{1}{2}\|u_j\|_*^2 - Q(u_j)) - \|u_j\|_*^2}{\|u_j\|_*^2} \\ &= -1. \end{aligned}$$

Thus  $\{u_j\}$  is bounded in  $H^1(\mathbb{R})$ . Hence, by (4.4), we have

$$\lim_{j \rightarrow \infty} \Phi'(u_j)u_j = \lim_{j \rightarrow \infty} (\|u_j\|_*^2 - 4Q(u_j)) = 0 \quad (4.9)$$

Therefore, it follows from (4.8) and (4.9) that

$$\lim_{j \rightarrow \infty} \|u_j\|_*^2 = \lim_{j \rightarrow \infty} \left[ 4 \left( \frac{1}{2} \|u_j\|_*^2 - Q(u_j) \right) - (\|u_j\|_*^2 - 4Q(u_j)) \right] = 4b.$$

Consequently,  $\lim_{j \rightarrow \infty} Q(u_j) = b$ .  $\square$

Let  $\mathcal{P} = \{u \in H^1(\mathbb{R}) \setminus \{0\}, Q(u) > 0\}$  and  $\beta = \inf \left\{ \frac{\|u\|_*^4}{Q(u)}, u \in \mathcal{P} \right\}$ . Note that  $u \in \mathcal{M}$  implies that  $u$  is a nontrivial solution of (1.8). Then  $\|u\|_*^2 - 4Q(u) = 0$ . Hence  $Q(u) = \frac{1}{4}\|u\|_*^2 > 0$ , which implies  $u \in \mathcal{P}$ . Thus  $\mathcal{M} \subseteq \mathcal{P}$ .

**Lemma 4.4.**  $\inf\{\Phi(u), u \in \mathcal{M}\} \geq b$ .

**Proof.** For any  $u \in \mathcal{P}$ , by Lemma 2.2 and (4.2), we have

$$Q(u) \leq C(2\omega)^{-(2-a)} \|G^\gamma\|_{L^{p_0}} \|u\|_*^4.$$

Hence

$$\beta = \inf \left\{ \frac{\|u\|_*^4}{Q(u)}, u \in \mathcal{P} \right\} \geq C^{-1}(2\omega)^{2-a} (\|G^\gamma\|_{L^{p_0}})^{-1} > 0.$$

Let  $u \in \mathcal{P}$ . Then  $\Phi(tu) = \frac{t^2}{2}\|u\|_*^2 - t^4Q(u)$  on  $(0, \infty)$  has a unique critical point

$$t_0 = t_0(u) = \frac{\|u\|_*}{2Q^{\frac{1}{2}}(u)} > 0,$$

with the critical value

$$\Phi(t_0u) = \frac{\|u\|_*^4}{16Q(u)}.$$

Note that  $\lim_{t \rightarrow \infty} \Phi(tu) = -\infty$ . Let  $e = t_1u$  such that  $\Phi(e) < 0$ , where  $t_1 > t_0$ . By the definition of  $b$  given by (4.5), we have

$$b \leq \Phi(t_0u) = \frac{\|u\|_*^4}{16Q(u)}.$$

Thus

$$b \leq \inf \left\{ \frac{\|u\|_*^4}{16Q(u)}, u \in \mathcal{P} \right\} \leq \inf \left\{ \frac{\|u\|_*^4}{16Q(u)}, u \in \mathcal{M} \right\},$$

where we used the fact  $\mathcal{M} \subseteq \mathcal{P}$ . Since  $u \in \mathcal{M}$  a nontrivial solution of (1.8), we have  $\Phi'(u)u = 0$ , i.e.,  $\|u\|_*^2 - 4Q(u) = 0$ . Then

$$\Phi(u) = \frac{1}{2}\|u\|_*^2 - Q(u) = \frac{\|u\|_*^4}{16Q(u)}.$$

Thus we have  $\inf\{\Phi(u), u \in \mathcal{M}\} \geq b$ .  $\square$

**Proof of Theorem 2.2.** By the Sobolev embedding theorem [1], we have  $H^1(\mathbb{R}) \subset C_0(\mathbb{R})$ , where  $C_0(\mathbb{R}) = \{u \in C(\mathbb{R}) \mid \lim_{|x| \rightarrow \infty} u(x) = 0\}$ . Let  $\{u_j\}$  be the sequence satisfying (4.4)-(4.6). By Lemma 4.3,  $\{u_j\}$  is bounded in  $H^1(\mathbb{R})$ , and there exists a  $M > 0$  such that  $\|u_j\|_* \leq M$  for any  $j \geq 1$ .

It follows from Lemma 4.3 also that there exists a  $j_0 > 0$  such that  $Q(u_j) \geq \frac{b}{2} > 0$  for  $j \geq j_0$ . By the Holder's inequality, we have

$$0 < (G^\gamma * |u_j|^2)(x) \leq \|u_j\|_{L^\infty} \|G^\gamma\|_{L^2} \|u_j\|_{L^2} \leq M(2\omega)^{-1} \|u_j\|_{L^\infty} \|G^\gamma\|_{L^2}.$$

Thus, for  $j \geq j_0$ ,

$$\frac{b}{2} \leq Q(u_j) = \frac{2}{\nu} \int_{\mathbb{R}} (G^\gamma * |u_j|^2) |u_j|^2 dx \leq M(2\omega)^{-1} \|u_j\|_{L^\infty} \|G^\gamma\|_{L^2} \|u_j\|_{L^2}^2 \leq M^3(2\omega)^{-3} \|u_j\|_{L^\infty} \|G^\gamma\|_{L^2}.$$

Then

$$\|u_j\|_{L^\infty} \geq \frac{(2\omega)^3 b}{2M^3 \|G^\gamma\|_{L^2}} > 0, \quad \forall j \geq j_0. \quad (4.10)$$

For each  $j \geq j_0$ , let  $s_j \in \mathbb{R}$  such that  $|u_j(s_j)| = \|u_j\|_{L^\infty}$ . Define  $\tilde{u}_j(x) = u_j(x + s_j)$  for  $j \geq j_0$ . Then  $\{\tilde{u}_j\}$  is bounded in  $H^1(\mathbb{R})$  and has a weakly convergent subsequence in  $H^1(\mathbb{R})$  [9]. Denote this subsequence again by  $\{\tilde{u}_j\}$ . Suppose  $\{\tilde{u}_j\}$  converges to  $u_0 \in H^1(\mathbb{R})$  weakly. By (4.10) and the weakly lower semicontinuity of  $\|\cdot\|_{L^\infty}$ , we have

$$\|u_0\|_{L^\infty} \geq \frac{(2\omega)^3 b}{2M^3 \|G^\gamma\|_{L^2}} > 0.$$

Thus  $u_0 \neq 0$ . We will prove next  $u_0$  is a critical point of functional  $\Phi$ , i.e.,  $\Phi'(u_0) = 0$ .

Note that  $\Phi$  is translation invariant, we have  $\{\tilde{u}_j\}$  satisfies (4.4), i.e.,

$$\Phi(\tilde{u}_j) \rightarrow b, \text{ and } \Phi'(\tilde{u}_j) \rightarrow 0, \text{ as } j \rightarrow \infty. \quad (4.11)$$

Since  $\lim_{j \rightarrow \infty} \tilde{u}_j = u_0$  weakly in  $H^1(\mathbb{R})$ , we have

$$\lim_{j \rightarrow \infty} \left[ \int_{\mathbb{R}} \nabla \tilde{u}_j \nabla \bar{\phi} dx + 2\omega \int_{\mathbb{R}} \tilde{u}_j \bar{\phi} dx \right] = \int_{\mathbb{R}} \nabla u_0 \nabla \bar{\phi} dx + 2\omega \int_{\mathbb{R}} u_0 \bar{\phi} dx, \quad \phi \in H^1(\mathbb{R}). \quad (4.12)$$

Since  $\{\tilde{u}_j\}$  is bounded in  $H^1(\mathbb{R})$ , there exists a constant  $N_0 > 0$  such that  $\|\tilde{u}_j\|_{L^2} \leq N_0$  for  $j \geq j_0$  and  $\|u_0\|_{L^2} \leq N_0$ . For any  $\varepsilon > 0$ , we can choose a compact set  $K \subset \mathbb{R}$  such that

$$\|\phi\|_{L^2(\mathbb{R} \setminus K)} \leq \frac{\varepsilon}{4N_0^3 \|G^\gamma\|_{L^2}}.$$

Then, by using the Holder's inequality,

$$\begin{aligned} & \left| \int_{\mathbb{R} \setminus K} (G^\gamma * |\tilde{u}_j|^2) \tilde{u}_j \bar{\phi} dx - \int_{\mathbb{R} \setminus K} (G^\gamma * |u_0|^2) u_0 \bar{\phi} dx \right| \\ & \leq \|G^\gamma\|_{L^2} (\|\tilde{u}_j\|_{L^2}^3 + \|u_0\|_{L^2}^3) \|\phi\|_{L^2(\mathbb{R} \setminus K)} \\ & \leq 2N_0^3 \|G^\gamma\|_{L^2} \|\phi\|_{L^2(\mathbb{R} \setminus K)} \leq \frac{\varepsilon}{2}. \end{aligned}$$

By the Sobolev embedding theorem [1],  $\lim_{j \rightarrow \infty} \tilde{u}_j = \tilde{u}$  weakly in  $H^1(\mathbb{R})$  implies  $\lim_{j \rightarrow \infty} \tilde{u}_j(x) = u_0(x)$  uniformly on  $K$ . Then there exists a  $j_1 \geq j_0$  such that

$$|\tilde{u}_j(x) - u_0(x)| \leq \frac{\varepsilon}{6\|G^\gamma\|_{L^2} N_0^2 \|\phi\|_{L^2}}, \quad \forall x \in K, \quad j \geq j_1.$$

Then, by using the Holder's inequality,

$$\begin{aligned} & \left| \int_K (G^\gamma * |\tilde{u}_j|^2) \tilde{u}_j \bar{\phi} dx - \int_K (G^\gamma * |u_0|^2) u_0 \bar{\phi} dx \right| \\ & \leq \left| \int_K [G^\gamma * ((\tilde{u}_j - u_0) \tilde{u}_j)] \tilde{u}_j \bar{\phi} dx \right| + \left| \int_K [G^\gamma * (u_0(\tilde{u}_j - u_0))] \tilde{u}_j \bar{\phi} dx \right| + \left| \int_K (G^\gamma * |u_0|^2) (\tilde{u}_j - u_0) \bar{\phi} dx \right| \\ & \leq \frac{\varepsilon}{6\|G^\gamma\|_{L^2} N_0^2 \|\phi\|_{L^2}} (\|G^\gamma\|_{L^2} \|\tilde{u}_j\|_{L^2}^2 \|\phi\|_{L^2} + \|G^\gamma\|_{L^2} \|\tilde{u}_j\|_{L^2} \|u_0\|_{L^2} \|\phi\|_{L^2} + \|G^\gamma\|_{L^2} \|u_0\|_{L^2}^2 \|\phi\|_{L^2}) \leq \frac{\varepsilon}{2}. \end{aligned}$$

Then, for  $j \geq j_1$  and  $\phi \in H^1(\mathbb{R})$ , we have

$$\left| \int_{\mathbb{R}} (G^\gamma * |\tilde{u}_j|^2) \tilde{u}_j \bar{\phi} dx - \int_{\mathbb{R}} (G^\gamma * |u_0|^2) u_0 \bar{\phi} dx \right| \leq \varepsilon. \quad (4.13)$$

Thus

$$\begin{aligned} 0 &= \lim_{j \rightarrow \infty} \Phi'(\tilde{u}_j) \phi \\ &= \lim_{j \rightarrow \infty} \left[ \int_{\mathbb{R}} \nabla \tilde{u}_j \nabla \bar{\phi} dx + 2\omega \int_{\mathbb{R}} \tilde{u}_j \bar{\phi} dx - \frac{8}{\nu} \int_{\mathbb{R}} (G^\gamma * |\tilde{u}_j|^2) \tilde{u}_j \bar{\phi} dx \right] \\ &= \int_{\mathbb{R}} \nabla u_0 \nabla \bar{\phi} dx + 2\omega \int_{\mathbb{R}} u_0 \bar{\phi} dx - \frac{8}{\nu} \int_{\mathbb{R}} (G^\gamma * |u_0|^2) u_0 \bar{\phi} dx \\ &= \Phi'(u_0) \phi, \quad \phi \in H^1(\mathbb{R}). \end{aligned}$$

Therefore  $\Phi'(u_0) = 0$ . We will prove next that  $\Phi(u_0) = b$ .

By replacing  $\phi = u_0$  in (4.13), we have

$$\lim_{j \rightarrow \infty} \int_{\mathbb{R}} (G^\gamma * |\tilde{u}_j|^2) \tilde{u}_j \bar{u}_0 dx = \int_{\mathbb{R}} (G^\gamma * |u_0|^2) |u_0|^2 dx.$$

By using the fact that, for any compact set  $K \subset \mathbb{R}$ ,  $\lim_{j \rightarrow \infty} \tilde{u}_j(x) = u_0(x)$  uniformly on  $K$ , and applying the same procedure for proving (4.13), we have

$$\lim_{j \rightarrow \infty} \int_{\mathbb{R}} (G^\gamma * |\tilde{u}_j|^2) \tilde{u}_j (\tilde{u}_j - \bar{u}_0) dx = 0.$$

Then

$$\begin{aligned} & \lim_{j \rightarrow \infty} \left[ \int_{\mathbb{R}} (G^\gamma * |\tilde{u}_j|^2) |\tilde{u}_j|^2 dx - \int_{\mathbb{R}} (G^\gamma * |u_0|^2) |u_0|^2 dx \right] \\ &= \lim_{j \rightarrow \infty} \int_{\mathbb{R}} (G^\gamma * |\tilde{u}_j|^2) \tilde{u}_j (\tilde{u}_j - \bar{u}_0) dx + \lim_{j \rightarrow \infty} \left[ \int_{\mathbb{R}} (G^\gamma * |\tilde{u}_j|^2) \tilde{u}_j \bar{u}_0 dx - \int_{\mathbb{R}} (G^\gamma * |u_0|^2) |u_0|^2 dx \right] = 0. \end{aligned}$$

By Lemma 4.3 and the translation invariance of  $Q(u)$ , we have

$$b = \lim_{j \rightarrow \infty} Q(u_j) = \lim_{j \rightarrow \infty} Q(\tilde{u}_j) = Q(u_0).$$

Note that  $\Phi'(u_0)u_0 = 0$  gives  $\|u_0\|_*^2 - 4Q(u_0) = 0$ . Thus  $\|u_0\|_*^2 = 4b$ . Therefore  $\Phi(u_0) = \frac{1}{2}\|u_0\|_*^2 - Q(u_0) = b$ . Note that  $u_0 \in \mathcal{M}$ . Thus  $\inf\{\Phi(u), u \in \mathcal{M}\} \leq b$ . By Lemma 4.4, we have  $\inf\{\Phi(u), u \in \mathcal{M}\} = b$ . Therefore  $u_0$  is a ground state solution of (1.8).  $\square$

**Remark.** One can show that the ground state solution  $u_0$  is positive on  $\mathbb{R}$  by the maximum principle.

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