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Regularity analyses and approximation of nonlocal variational equality and inequality problems ☆

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ABSTRACT

We consider linear and obstacle problems driven by a nonlocal integral operator, for which nonlocal interactions are restricted to a ball of finite radius. These types of operators are used to model anomalous diffusion and, for a special choice of the integral kernels, reduce to the fractional Laplace operator on a bounded domain. By means of a nonlocal vector calculus we recast the problems in a weak form, leading to corresponding nonlocal variational equality and inequality problems. We prove optimal regularity results for both problems, including a higher regularity of the solution and the Lagrange multiplier. Based on the regularity results, we analyze the convergence of finite element approximations for a linear problem and illustrate the theoretical findings by numerical results.

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1. Introduction

We consider the analysis and approximation of a linear diffusion problem and a related obstacle problem involving the nonlocal operator

$$-\mathcal{L}_\delta u(\mathbf{x}) := 2 \int_{B_\delta(\mathbf{x})} (u(\mathbf{x}) - u(\mathbf{y}))\gamma(\mathbf{x},\mathbf{y})\,\mathrm{d}\mathbf{y}, \quad \mathbf{x} \in \Omega, \tag{1.1}$$

where the kernel $\gamma(\mathbf{x},\mathbf{y}) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is a non-negative symmetric function, $\Omega \subset \mathbb{R}^n$ is a bounded domain, and $B_\delta(\mathbf{x})$ is a ball in $\mathbb{R}^n$ of a radius $\delta > 0$ centered at $\mathbf{x}$. The operator $\mathcal{L}_\delta$ is nonlocal because the value of the function u at a point $\mathbf{x}$ is defined by the contributions of u at other points $\mathbf{y}$ separated from $\mathbf{x}$ by a finite distance. The parameter δ determines the extent of the nonlocal interactions.

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If nonlocal interactions become infinite ($\delta \to \infty$) and the kernel $\gamma = 1/|\mathbf{x}-\mathbf{y}|^{n+2s}$ up to some scaling factor, the nonlocal operator $-\mathcal{L}_\delta$ reduces to the fractional Laplace operator $(-\Delta)^s$, $s \in (0,1)$, on a bounded domain; see [13]. In the case $s \in [1/2, 1)$, the integral (2.2) should be understood in the Cauchy principal value sense.

In addition to the fractional Laplace kernel, other kernels have been actively exploited in different applications, e.g., peridynamics [32], machine learning [29], and image analysis [7,17]. Furthermore, an extension to nonlocal convection diffusion models have also been investigated [12,34].

Nonlocal diffusion operators also arise in mathematical finance applications such as option pricing; see, e.g., [11]. In this context, a nonlocal diffusion operator is used to model the behaviour of the log-asset price that is not exclusively driven by a Brownian motion, but instead follows a jump-diffusion process, e.g., the Merton model [26].

The (non-)local obstacle problem is often associated with the optimal stopping time problem in stochastic control, see, e.g., [2], that, in the context of option pricing, is related to the pricing of American put options.

In this paper, we analyze the stationary linear and obstacle problem associated with the nonlocal operator $-\mathcal{L}_\delta$ and also analyze finite element discretizations of these problems. A main contribution of this work is to derive improved Sobolev regularity for the solution of both problems. Such results are necessary, e.g., for the error analysis of finite element approximation.

We first consider, for a given function f, the following linear nonlocal problem with homogeneous volume constraints:

$$-\mathcal{L}_\delta u(\mathbf{x}) = f(\mathbf{x}), \qquad \text{for } \mathbf{x} \in \Omega, \tag{1.2a}$$

$$u(\mathbf{x}) = 0, \qquad \text{for } \mathbf{x} \in \Omega_\delta, \tag{1.2b}$$

where $\Omega_\delta \subset \mathbb{R}^n \setminus \Omega$ is the δ-neighbourhood of Ω. Volume constraints such as (1.2b) (that is the nonlocal analog of the Dirichlet boundary condition imposed on $\partial\Omega$ in the local partial differential equation case) are essential for the well posedness of the nonlocal problem. The variational analysis and finite element approximation of the problem are possible due to the recently developed nonlocal vector calculus [15,16].

For the problem (1.2), we restrict attention to the kernel related to the fractional Laplacian. The Sobolev regularity of the fractional Laplace problem on bounded smooth domains in terms of the Sobolev regularity of the data is considered in [20] using Hörmander's theory for pseudo-differential operators. On less smooth domains and under stronger regularity assumption on the data, similar regularity estimates in (weighted) Sobolev spaces have been established in [1] by building upon Hölder regularity results for the fractional Laplace problem [28]. However, the corresponding results for nonlocal operators (1.1) related to the *truncated* fractional Laplacian are not available in the literature although they are necessary for the finite element analysis; see, e.g., [15]. Our aim is to fill this gap and derive the corresponding regularity estimates for truncated kernels. We do so by relying upon results from [20] and by estimating the error terms arising from truncation; see Theorem 3.4. We also develop the discretization of the problem by linear finite elements and derive the corresponding a priori error estimates.

In the second part of the paper, we consider the obstacle problem in which, in contrast to the linear model (1.2), the solution $u(\mathbf{x})$ is additionally constrained by the given obstacle functional $\psi(\mathbf{x})$ and solves the following set of inequalities:

$$-\mathcal{L}_\delta u(\mathbf{x}) \geq f(\mathbf{x}), \qquad \text{for } \mathbf{x} \in \Omega, \tag{1.3a}$$

$$u(\mathbf{x}) \geq \psi(\mathbf{x}), \qquad \text{for } \mathbf{x} \in \Omega, \tag{1.3b}$$

$$\left(-\mathcal{L}_\delta u(\mathbf{x}) - f(\mathbf{x})\right)\left(u(\mathbf{x}) - \psi(\mathbf{x})\right) = 0, \qquad \text{for } \mathbf{x} \in \Omega, \tag{1.3c}$$

$$u(\mathbf{x}) = 0, \qquad \text{for } \mathbf{x} \in \Omega_\delta. \tag{1.3d}$$

The well-posedness of this problem and the convergence of the finite element approximation have been established in [21] for the fractional Laplace and integrable kernels. We analyze this problem in a variational framework, considering instead of the primal variational inequality an equivalent variational saddle-point formulation. An additional Lagrange multiplier variable is introduced that physically can be interpreted as a contact force on the obstacle. The well posedness of the nonlocal (and local) problem require only low regularity assumptions on the Lagrange multiplier. In the local case, it is well known that for sufficiently regular right-hand sides and obstacle function, the regularity of the Lagrange multiplier can be improved together with the primal solution. However, for the kernels under consideration in this work, similar results are not available. In this paper, for a general class of kernels, we are able to derive improved regularity results for the Lagrange multiplier under certain regularity assumptions on the data; see Theorem 4.4. For the proof, we follow the penalty approach developed for the analysis of the local variational inequalities [24], and derive the analogue of the Lewy-Stampacchia dual estimates [25] for the nonlocal case. Combining these with the new regularity estimates we derive for the problem (1.2), we are also able to show an improved regularity of the primal solution for the truncated fractional Laplace kernel. In addition, for this type of kernel, we show the convergence of the nonlocal solution to the solution of the corresponding variational inequality for the fractional Laplace operator.

We comment on other works related to the regularity of the obstacle problem for the fractional Laplacian. Similar Lewy-Stampacchia type estimates are obtained in [31] in the context of an abstract framework of (non-)local operators. However, the case of the truncated kernel is not covered by their analysis and higher regular data is assumed together with an additional restriction on the power of the fractional Laplacian. The regularity of the obstacle problem measured in Hölder and Lipschitz spaces is extensively studied in, e.g., [33,8]. During the publication process of this manuscript, two other preprints on fractional obstacle problems have been made available: In [5] the finite element approximation of the obstacle problem for the fractional Laplacian is studied, and error estimates are proven. Regularity results for the solution are derived in weighted Sobolev spaces, however, under additional regularity assumptions on the right-hand side (Hölder continuity) and the obstacle. In [4] the regularity of the obstacle problem involving integro-differential operators with the fractional Laplacian as the nonlocal term is studied, and a finite element based discretization is proposed and analyzed. In the purely nonlocal case, the same regularity for the solution is proven (as in Corollary 4.5). However, the underlying result from Theorem 4.4 is valid for a general class of nonlocal operators.

The rest of the paper is structured as follows. In Section 2, we introduce the necessary function spaces and recall some preliminary results, which are used in the rest of the paper. Section 3 covers the analysis for the linear problem (1.2) and the corresponding regularity results are derived in Section 3.1. In Section 3.2, we present a finite element discretization for the linear problem and derive the associated a priori error estimates. The nonlocal variational inequality formulation of problem (1.3) is considered in Section 4. For the fractional Laplace kernel, we show, in Section 4.1, the convergence of the nonlocal solution to the fractional Laplacian solution. In Section 4.2, we derive the improved regularity for the Lagrange multiplier and the corresponding primal solution. We discuss the finite element discretization of the nonlocal variational inequality in Section 4.3. Numerical results that illustrate our theoretical findings are given in Section 5 for one-dimensional linear and obstacle problems.

2. Preliminaries

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a Lipschitz boundary. Following [15], we introduce a truncated kernel $\gamma_\delta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ that is a non-negative symmetric function and that, for all $\mathbf{x} \in \mathbb{R}^n$, satisfies the following conditions:

$$\begin{cases} \gamma_\delta(\mathbf{x},\mathbf{y}) \geq 0 & \forall \mathbf{y} \in B_\delta(\mathbf{x}) \\ \gamma_\delta(\mathbf{x},\mathbf{y}) \geq \gamma_0 > 0 & \forall \mathbf{y} \in B_{\delta/2}(\mathbf{x}) \\ \gamma_\delta(\mathbf{x},\mathbf{y}) = 0 & \forall \mathbf{y} \in \mathbb{R}^n \setminus B_\delta(\mathbf{x}), \end{cases} \tag{2.1}$$

where γ_0 is a positive constant and $B_\delta(\mathbf{x})$ is the ball of radius δ centered at $\mathbf{x}$:

$$B_\delta(\mathbf{x}) := \{\mathbf{y} \in \mathbb{R}^n \colon |\mathbf{x}-\mathbf{y}| \leq \delta\}.$$

These conditions imply that the nonlocal interactions are limited to a ball of a radius δ, referred as the interaction radius. Often, we will require that $\gamma_\delta(\mathbf{x},\mathbf{y})$ is radial, i.e., $\gamma_\delta(\mathbf{x},\mathbf{y}) = \gamma_\delta(|\mathbf{x}-\mathbf{y}|)$, where by a slight abuse of notation the radial function is also denoted by γ_δ. For $\delta > 0$, we define an interaction domain $\Omega_\delta \subset \mathbb{R}^d \setminus \Omega$ corresponding to Ω as follows:

$$\Omega_\delta := \{\mathbf{y} \in \mathbb{R}^d \setminus \Omega \colon \gamma_\delta(\mathbf{x},\mathbf{y}) \neq 0, \quad \mathbf{x} \in \Omega\}.$$

In terms of these notations, we can express the operator (1.1) as

$$-\mathcal{L}_\delta u(\mathbf{x}) := 2 \int_{\Omega \cup \Omega_\delta} (u(\mathbf{x}) - u(\mathbf{y}))\gamma_\delta(\mathbf{x},\mathbf{y})\, d\mathbf{y}, \quad \mathbf{x} \in \Omega. \tag{2.2}$$

We note, for the special case $\delta = \infty$ we also denote by $-\mathcal{L}_\infty$ the operator (2.2) corresponding to kernel γ_∞. Because the kernel $\gamma_\delta(\mathbf{x},\mathbf{y})$ determines the qualitative nature of the solution problem, the model formulations (1.2) and (1.3) can be used to cover a large class of problems. We analyze several examples for the kernels.

2.1. Kernels

Although we often focus on the kernels γ_δ defined in Case 1 below that are related to the fractional Laplace operator, we also consider other possible choices for γ_δ.

Case 1. (*Fractional Laplacian-type kernels.*) For $s \in (0,1)$, $\delta > 0$, and some positive constants $\gamma_1, \gamma_2 > 0$, consider kernels which are proportional to $1/|\mathbf{x}-\mathbf{y}|^{n+2s}$, namely, for all $\mathbf{x} \in \Omega \cup \Omega_\delta$

$$\frac{\gamma_1}{|\mathbf{x}-\mathbf{y}|^{n+2s}} \leq \gamma_\delta(\mathbf{x},\mathbf{y}) \leq \frac{\gamma_2}{|\mathbf{x}-\mathbf{y}|^{n+2s}} \quad \text{for } \mathbf{y} \in B_\delta(\mathbf{x}).$$

As an example, consider the kernel

$$\gamma_\delta(\mathbf{x},\mathbf{y}) = \begin{cases} \dfrac{\sigma(\mathbf{x},\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{n+2s}}, & \mathbf{y} \in B_\delta(\mathbf{x}), \\ 0, & \text{otherwise} \end{cases} \tag{2.3}$$

with $\sigma(\mathbf{x},\mathbf{y})$ a symmetric function that is bounded from below and above by positive constants. For $\delta = +\infty$, as a particular instance of (2.3), we consider the kernel corresponding to the fractional Laplace operator (2.5):

$$\gamma_\infty(\mathbf{x},\mathbf{y}) = \frac{c_{n,s}}{2|\mathbf{x}-\mathbf{y}|^{n+2s}}, \quad c_{n,s} = \frac{2^{2s} s \Gamma(s+\frac{n}{2})}{\pi^{n/2}\Gamma(1-s)}, \tag{2.4}$$

where $\Gamma(\cdot)$ denotes the Gamma function. Then, the operator $-\mathcal{L}_\infty$ corresponds to the integral definition of the fractional Laplace operator

$$(-\Delta)^s u(\mathbf{x}) := c_{n,s} \int_{\mathbb{R}^n} \frac{u(\mathbf{x}) - u(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{n+2s}} \, \mathrm{d}\mathbf{y}, \quad 0 < s < 1. \tag{2.5}$$

The fractional Laplace Poisson problem on bounded domains with homogeneous volume constraints takes the form

$$(-\Delta)^s u(\mathbf{x}) = f(\mathbf{x}) \qquad \text{for } \mathbf{x} \in \Omega, \tag{2.6a}$$

$$u(\mathbf{x}) = 0 \qquad \text{for } \mathbf{x} \in \mathbb{R}^n \setminus \Omega. \tag{2.6b}$$

In [13] the convergence of the nonlocal solution of (1.2) to the solution of the fractional Laplacian problem (2.6) is shown.

Case 2. (*Square-integrable kernels.*) There exists a constant γ_3, such that

$$\int_{\Omega \cup \Omega_\delta} \gamma_\delta^2(\mathbf{x}, \mathbf{y}) \, \mathrm{d}\mathbf{y} \le \gamma_3^2, \quad \forall \mathbf{x} \in \Omega.$$

Case 3. (*Peridynamic-type kernel.*) There exist constants γ_4, $\gamma_5 > 0$ such that

$$\frac{\gamma_4}{|\mathbf{x}-\mathbf{y}|} \le \gamma_\delta(\mathbf{x}, \mathbf{y}) \le \frac{\gamma_5}{|\mathbf{x}-\mathbf{y}|} \quad \text{for } \mathbf{y} \in B_\delta(\mathbf{x}). \tag{2.7}$$

As an example, consider the kernel

$$\gamma_\delta(\mathbf{x}, \mathbf{y}) = \begin{cases} \dfrac{\xi(\mathbf{x}, \mathbf{y})}{|\mathbf{x}-\mathbf{y}|}, & \mathbf{y} \in B_\delta(\mathbf{x}), \\ 0, & \text{otherwise} \end{cases} \tag{2.8}$$

with $\xi(\mathbf{x}, \mathbf{y})$ a symmetric function bounded from below and above by positive constants. The kernel (2.8) is integrable for $n > 1$ and square-integrable for $n > 2$ so that in the latter case, the kernel (2.7) is an example of the Case 2 type kernels.

Note that peridynamics is a nonlocal continuum model for solid mechanics featuring kernels with this type singularity, but, of course, with vector-valued displacement functions to be solved for. However, the scalar case we consider here and the vector case are entirely similar with respect to the regularity and other features of their solutions.

2.2. Nonlocal function spaces

We introduce the function spaces used in this work and review some of the important properties of these spaces.

For a general open set $\widetilde{\Omega} \subset \mathbb{R}^n$, we denote by $L^2(\widetilde{\Omega})$ the standard space of square integrable functions on $\widetilde{\Omega}$. The fractional Sobolev space for $s \in (0, 1)$ is defined as

$$H^s(\widetilde{\Omega}) := \left\{ v \in L^2(\widetilde{\Omega}) \colon |v|_{H^s(\widetilde{\Omega})} < \infty \right\}$$

with Gagliardo seminorm

$$|v|^2_{H^s(\widetilde{\Omega})} := \int_{\widetilde{\Omega}} \int_{\widetilde{\Omega}} \frac{|v(\mathbf{x}) - v(\mathbf{y})|^2}{|\mathbf{x}-\mathbf{y}|^{n+2s}} \, \mathrm{d}\mathbf{y} \, \mathrm{d}\mathbf{x}.$$

The space $H^s(\widetilde{\Omega})$ is a Hilbert space that is endowed with the norm

$$\|v\|_{H^s(\widetilde{\Omega})} = \|v\|_{L^2(\widetilde{\Omega})} + |v|_{H^s(\widetilde{\Omega})}.$$

For $\sigma > 1$ not an integer, we define $H^\sigma(\widetilde{\Omega})$, $\sigma = m + s$, $m \in \mathbb{N}$, $s \in (0,1)$, as

$$H^\sigma(\widetilde{\Omega}) := \{v \in H^m(\widetilde{\Omega}) \colon |D^\alpha v| \in H^s(\widetilde{\Omega}), \text{ for } |\alpha| = m\},$$

which is equipped with the norm

$$\|v\|_{H^\sigma(\widetilde{\Omega})} = \|v\|_{H^m(\widetilde{\Omega})} + \sum_{|\alpha|=m} |D^\alpha v|_{H^s(\widetilde{\Omega})}.$$

Additionally, for $\widetilde{\Omega}$ such that $\overline{\Omega} \subset \widetilde{\Omega}$ and $s > 0$, we define the space incorporating the volume constraints given by

$$H^s_\Omega(\widetilde{\Omega}) := \{v \in H^s(\widetilde{\Omega}) \colon v = 0 \text{ on } \widetilde{\Omega} \setminus \Omega\},$$

that is endowed with the norm of $H^s(\overline{\Omega})$, i.e., $\|v\|_{H^s_\Omega(\widetilde{\Omega})} = \|v\|_{H^s(\widetilde{\Omega})}$.

We define the restriction operator $r_{\widetilde{\Omega}} : H^s_\Omega(\mathbb{R}^n) \to H^s_\Omega(\widetilde{\Omega})$ by $r_{\widetilde{\Omega}} u = u\big|_{\widetilde{\Omega}}$ and the extension operator $e_{\widetilde{\Omega}} : H^s_\Omega(\widetilde{\Omega}) \to H^s_\Omega(\mathbb{R}^n)$ by

$$e_{\widetilde{\Omega}} u = \begin{cases} u & \text{on } \widetilde{\Omega}, \\ 0 & \text{on } \mathbb{R}^n \setminus \widetilde{\Omega}, \end{cases}$$

i.e., continuation of u by zero outside $\widetilde{\Omega}$. The restriction and extension operators are linear continuous mappings, and $H^s_\Omega(\widetilde{\Omega})$ and $H^s_\Omega(\mathbb{R}^n)$ are isomorphic.

For variational analyses of the problems (1.2) and (1.3), we define the bilinear form for γ_δ satisfying (2.1) and defined in one of Case 1, Case 2, or Case 3 as

$$a(u,v) := \int\limits_{\Omega\cup\Omega_\delta} \int\limits_{\Omega\cup\Omega_\delta} (u(\mathbf{x}) - u(\mathbf{y}))\,(v(\mathbf{x}) - v(\mathbf{y}))\,\gamma_\delta(\mathbf{x},\mathbf{y})\,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x}.$$

The associated energy and constrained energy spaces are defined as

$$X := \{v \in L^2(\Omega\cup\Omega_\delta) \colon\ a(u,v) < \infty\} \quad \text{and} \quad V := \{v \in X \colon\ v = 0 \quad \text{a.e. on } \Omega_\delta\}.$$

Additionally, we have the constrained L^2-space

$$L^2_\Omega(\Omega\cup\Omega_\delta) := \{v \in L^2(\Omega\cup\Omega_\delta) \colon\ v = 0 \quad \text{a.e. on } \Omega_\delta\},$$

which is isometrically isomorphic to $L^2(\Omega)$. For Case 1 and $s \in (0,1)$, the nonlocal space X is equivalent to the fractional Sobolev space $H^s(\Omega\cup\Omega_\delta)$ and V is equivalent to $H^s_\Omega(\Omega\cup\Omega_\delta)$. For Case 2 and Case 3 ($n > 2$) the space X is equivalent to $L^2(\Omega\cup\Omega_\delta)$ and V to $L^2_\Omega(\Omega\cup\Omega_\delta)$; see [15]. For Case 3 ($n = 1$), it is not known that the energy space V is equivalent to any Sobolev space. However, it is a separable Hilbert space and a dense subspace of $L^2(\Omega)$, see, e.g., [14, Theorem 3.14].

Hence, X and V are Hilbert spaces equipped with the inner product and norms

$$(u,v)_V := a(u,v), \qquad \|v\|^2_V := a(v,v), \quad \|v\|^2_X = \|v\|^2_{L^2(\Omega\cup\Omega_\delta)} + \|v\|^2_V.$$

For Case 1 and some positive constants C_1 and C_2 and $s \in (0,1)$, we have the norm equivalence [15]:

$$C_1\|v\|_{H^s(\Omega\cup\Omega_\delta)} \le \|v\|_V \le C_2\|v\|_{H^s(\Omega\cup\Omega_\delta)} \qquad \forall v \in V. \tag{2.9}$$

Moreover, for Case 1 the space V is equivalent to $H^s_\Omega(\mathbb{R}^n)$, $s \in (0,1)$, and the following norm equivalence

$$C_3\|v\|_{H^s_\Omega(\mathbb{R}^n)} \le \|v\|_V \le C_4\|v\|_{H^s_\Omega(\mathbb{R}^n)} \quad \forall v \in V \tag{2.10}$$

holds. This also implies that by means of the extension and restriction operators, we can always extend $u \in V$ to $\widetilde{u} := e_{\Omega\cup\Omega_\delta}u \in H^s_\Omega(\mathbb{R}^n)$, and vice-versa, for any $u \in H^s_\Omega(\mathbb{R}^n)$ we can restrict it to $\widetilde{u} := r_{\Omega\cup\Omega_\delta}u \in V$. Therefore, by an abuse of notation, we often omit the notation of these operators, and simply write u if it is clear from the context.

We state another important result from [15] for the nonlocal space V.

Lemma 2.1 *(Nonlocal Poincaré inequality). Let a kernel γ_δ satisfy (2.1) and defined in either Case 1, Case 2 or Case 3. Then there exists a constant $C_P > 0$ such that the following holds*

$$\|v\|_{L^2(\Omega\cup\Omega_\delta)} \le C_P\|v\|_V \quad \forall v \in V. \tag{2.11}$$

We denote by V' the dual space of V, and by $\langle\cdot,\cdot\rangle$ the extended $L^2_\Omega(\Omega\cup\Omega_\delta)$ duality pairing between these spaces. For any $f \in V'$ we define the dual norm as

$$\|f\|_{V'} := \sup_{\substack{v\in V\\ v\neq 0}} \frac{\int_\Omega f\ v\,\mathrm{d}\mathbf{x}}{\|v\|_V}.$$

In a similar way, we define $H^{-s}(\Omega) = (H^s_\Omega(\mathbb{R}^n))'$.

3. Linear nonlocal problem

Using the nonlocal Green's first identity [15], we obtain the following weak formulation of the linear nonlocal problem (1.2): For a given $f \in V'$, find $u \in V$ such that

$$a(u,v) = \langle f, v\rangle \quad \forall v \in V. \tag{3.1}$$

By the Lax-Milgram theorem, the problem (3.1) admits a unique solution. Moreover, there exists a constant $C > 0$ such that solution satisfies

$$\|u\|_V \le C\|f\|_{V'}. \tag{3.2}$$

Taking into account the equivalence of the nonlocal energy and fractional Sobolev spaces (2.9), the estimate (3.2) for Case 1 implies that for any $f \in H^{-s}(\Omega)$,

$$\|u\|_{H^s_\Omega(\Omega\cup\Omega_\delta)} \le C\|f\|_{H^{-s}(\Omega)} \quad \text{for } 0 < s < 1.$$

For some cases, we can consider the special case $\delta = +\infty$ and we recall the corresponding integral kernel and nonlocal operator are denoted by γ_∞ and $-\mathcal{L}_\infty$, respectively. We introduce a kernel

$$\gamma_{\delta^c} := \gamma_\infty - \gamma_\delta.$$

For all $u, v \in V$ and $\delta > 0$, we define the linear bounded operator $A : V \to V'$ associated with the bilinear form $a(\cdot,\cdot)$ given by

$$\langle Au, v\rangle = a(u,v) \quad \forall u, v \in V. \tag{3.3}$$

Analogously, for $\delta = +\infty$, we introduce the bilinear form $a_\infty : V \times V \to \mathbb{R}$, corresponding to the kernel γ_∞ as follows

$$a_\infty(u,v) := \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} (u(\mathbf{x}) - u(\mathbf{y}))\,(v(\mathbf{x}) - v(\mathbf{y}))\,\gamma_\infty(\mathbf{x},\mathbf{y})\,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x}, \tag{3.4}$$

and the associated operator $A_\infty : V \to V'$ defined as $\langle A_\infty u, v\rangle = a_\infty(u,v)$. In order to ensure that $a_\infty(\cdot,\cdot)$ is well-defined, we restrict attention to Case 1, and Case 2, under an additional inegrability assumption on the kernel γ_∞, specifically radial kernels with $\gamma_\infty(|\cdot|) \in L^1(\mathbb{R}^n \setminus B_\delta(0))$ for some $\delta > 0$. The following establishes a useful relation between the nonlocal operators corresponding to kernels γ_δ and γ_∞.

Proposition 3.1. *For $\delta > 0$, let the kernel γ_δ is radial and satisfies (2.1) and, furthermore, let $\gamma_{\delta^c} \in L^1(\mathbb{R}^n)$ for some $\delta > 0$. Then, we have that*

$$-\mathcal{L}_\delta u = -\mathcal{L}_\infty u - C(\delta,n)u + 2u * \gamma_{\delta^c},$$

where $C(\delta,n) = 2\|\gamma_{\delta^c}\|_{L^1(\mathbb{R}^n)}$ and $*$ denotes the convolution operator on $\mathbb{R}^n$. In particular, for the fractional Laplacian, i.e. Case 1 with $\sigma(\mathbf{x},\mathbf{y}) = c_{n,s}/2$, we have that $C(\delta,n) = C(\delta,n,s)$

$$C(\delta,n,s) = \frac{c_{n,s}\pi^{n/2}}{\Gamma(n/2)\delta^{2s}s}. \tag{3.5}$$

Moreover, if $\delta > \mathrm{diam}|\Omega|$, we have that $u * \gamma_{\delta^c} = 0$.

Proof. The result follows directly from the definitions of the operators $\mathcal{L}_\delta$ and $\mathcal{L}_\infty$. □

Remark 1. From the implementation point of view, Proposition 3.1 provides a useful relation between, e.g., the nonlocal and fractional Laplace operators, as it allows to assemble only one of those matrices and then subtract or add a mass matrix term correspondingly.

3.1. Regularity study for the linear nonlocal problem

We derive a regularity result for the linear nonlocal problem (1.2) corresponding to Case 1 kernels.

Consider the following weak formulation of the problem (2.6): for given $f \in H^{-s}(\Omega)$, find $u \in H^s_\Omega(\mathbb{R}^n)$ such that

$$\frac{c_{n,s}}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{u(\mathbf{y}) - u(\mathbf{x})}{|\mathbf{x}-\mathbf{y}|^{n+2s}}(v(\mathbf{y}) - v(\mathbf{x}))\,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x} = \int_\Omega fv\,\mathrm{d}\mathbf{x} \quad \forall v \in H^s_\Omega(\mathbb{R}^n). \tag{3.6}$$

We recall the regularity result for the fractional Laplace problem on bounded smooth domains stated in [20] in terms of Hörmander μ-spaces; see also, e.g., [3], for the reinterpretation of this result. For an earlier result in a less general framework; see [35].

Theorem 3.2 *(Fractional Laplace problem on bounded domains.). Let Ω be a domain with C^∞ boundary $\partial\Omega$, and for $s \in (0,1)$, let $f \in H^r(\Omega)$, $r \geq -s$ and let $u \in H^s_\Omega(\mathbb{R}^n)$ be the solution of the fractional Laplace problem (3.6). Then, for any $\varepsilon > 0$, we have the regularity estimate*

$$|u|_{H^{s+\alpha}_\Omega(\mathbb{R}^n)} \leq C\|f\|_{H^r(\Omega)},$$

where $\alpha = \min\{s + r, 1/2 - \varepsilon\}$.

We note that independently of the smoothness of the right-hand side f, we cannot expect the solution u to be any smoother than $H_\Omega^{s+1/2-\varepsilon}(\mathbb{R}^n)$; see, e.g., [3] for a counterexample.

Before extending this result to the linear nonlocal problem (1.2), we first provide an auxiliary result that will be useful for the main proof of Theorem 3.4.

Proposition 3.3. *Let* $\omega \in H_\Omega^r(\mathbb{R}^n)$ *for* $r \in [0,1]$, *where for* $r = 0$, $H_\Omega^0(\mathbb{R}^n) \equiv L^2(\mathbb{R}^n)$, *and let* $g := \omega * \gamma_{\delta^c}$. *Then,* $g \in H_\Omega^r(\mathbb{R}^n)$, $r \geq 0$, *and for* $C = \|\gamma_{\delta^c}\|_{L^1(\mathbb{R}^n)} = \frac{\pi^{n/2}}{\Gamma(n/2)\delta^{2s}s}$ *it holds*

$$\|g\|_{H^r(\mathbb{R}^n)} \leq C\|\omega\|_{H^r(\mathbb{R}^n)}. \tag{3.7}$$

Proof. We note that for $r = 0$ the above result is a direct consequence of Young's inequality. For $r = 1$ we can apply Young's inequality to $\nabla(\omega * \gamma_{\delta^c}) = \nabla\omega * \gamma_{\delta^c}$. For $r \in (0,1)$ the inequality (3.7) can be obtained utilizing the fact that $H^r(\mathbb{R}^n)$ can be identified with an interpolation space of $L^2(\mathbb{R}^n)$ and $H^1(\mathbb{R}^n)$. However, for the convenience of the reader, we provide a short proof that follows directly using the definition of the Gagliardo seminorm. For all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ we have the estimate

$$\begin{aligned}
|g(\mathbf{y}) - g(\mathbf{x})| &= \left| \int\limits_{\mathbb{R}^n \setminus B_\delta(\mathbf{y})} \frac{\omega(\mathbf{z})}{|\mathbf{y}-\mathbf{z}|^{n+2s}}\,\mathrm{d}\mathbf{z} - \int\limits_{\mathbb{R}^n \setminus B_\delta(\mathbf{x})} \frac{\omega(\mathbf{z})}{|\mathbf{x}-\mathbf{z}|^{n+2s}}\,\mathrm{d}\mathbf{z} \right| \\
&\leq \sqrt{\int\limits_{\mathbb{R}^n \setminus B_\delta(0)} \frac{(\omega(\mathbf{y}-\mathbf{z}) - \omega(\mathbf{x}-\mathbf{z}))^2}{|\mathbf{z}|^{n+2s}}\,\mathrm{d}\mathbf{z}} \sqrt{\int\limits_{\mathbb{R}^n \setminus B_\delta(0)} \frac{\mathrm{d}\mathbf{z}}{|\mathbf{z}|^{n+2s}}}.
\end{aligned}$$

Taking into account that the last term can be computed exactly, for $r > 0$, we obtain

$$\begin{aligned}
|g|^2_{H^r(\mathbb{R}^n)} &= \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} \frac{(g(\mathbf{y}) - g(\mathbf{x}))^2}{|\mathbf{y}-\mathbf{x}|^{n+2r}}\,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x} \\
&\leq \frac{\pi^{n/2}}{\Gamma(n/2)\delta^{2s}s} \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n \setminus B_\delta(0)} \frac{(\omega(\mathbf{y}-\mathbf{z}) - \omega(\mathbf{x}-\mathbf{z}))^2}{|\mathbf{y}-\mathbf{x}|^{n+2r}|\mathbf{z}|^{n+2s}}\,\mathrm{d}\mathbf{z}\,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x} \\
&= \frac{\pi^{n/2}}{\Gamma(n/2)\delta^{2s}s} \int\limits_{\mathbb{R}^n \setminus B_\delta(0)} \frac{1}{|\mathbf{z}|^{n+2s}}\,\mathrm{d}\mathbf{z} \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} \frac{(\omega(\mathbf{y}) - \omega(\mathbf{x}))^2}{|\mathbf{y}-\mathbf{x}|^{n+2r}}\,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x} \\
&= \left(\frac{\pi^{n/2}}{\Gamma(n/2)\delta^{2s}s}\right)^2 |\omega|^2_{H^r(\mathbb{R}^n)}. \quad \square
\end{aligned}$$

We now prove the main result of this section.

Theorem 3.4 *(Nonlocal problem with truncated interactions). Let* Ω *be a domain with* C^∞ *boundary* $\partial\Omega$, *and let* $f \in H^r(\Omega)$, $r \geq -s$, *and let* $u \in V$ *be the solution of* (3.1) *with the Case 1 kernel with* $\sigma(\mathbf{x}, \mathbf{y}) = \sigma$ *and* $\delta > 0$. *Then, for any* $\varepsilon > 0$, *there exists a constant* $C > 0$, *such that*

$$|u|_{H_\Omega^{s+\alpha}(\mathbb{R}^n)} \leq C\|f\|_{H^r(\Omega)}, \tag{3.8}$$

where $\alpha = \min\{s + r, 1/2 - \varepsilon\}$. *We note that by writing* u *in* (3.8), *we mean its extension by zero to* $\mathbb{R}^n$, $e_{\Omega\cup\Omega_\delta}u$.

Proof. Without loss of generality, we conduct the proof for $\sigma = c_{n,s}/2$. Then, using Proposition 3.1 we can write for all $v \in H^s_\Omega(\mathbb{R}^n)$

$$a_\infty(u,v) = a(u,v) + C(\delta,n,s)(u,v)_{L^2(\Omega)} - c_{n,s}(g,v)_{L^2(\mathbb{R}^n)}, \tag{3.9}$$

where $C(\delta,n,s)$ is defined in (3.5) and $g = u * \gamma_{\delta^c}$. Taking into account that for $f \in H^r(\Omega)$, $u \in V$ is a solution of (3.1), we can then rewrite (3.9) as

$$a_\infty(u,v) = (F,v)_{L^2(\Omega)}, \tag{3.10}$$

where $F = f + C(\delta,n,s)u - c_{n,s}g$. Let $-s \le r \le 0$ and recall that $H^0(\Omega) \equiv L^2(\Omega)$, then, by using Proposition 3.3, (2.11), and (3.2) we obtain that $F \in H^r(\Omega)$, and for $C > 0$ the following holds

$$\|F\|_{H^r(\Omega)} \le \|f\|_{H^r(\Omega)} + C\|u\|_{L^2(\Omega)} \le C\|f\|_{H^r(\Omega)}.$$

Now, applying Theorem 3.2 to (3.10) with the right-hand side F, we obtain that $u \in H^{s+\alpha_0}_\Omega(\mathbb{R}^n)$, where $\alpha_0 = \min\{s+r, 1/2-\varepsilon\}$ with $\varepsilon > 0$ and moreover, we have that

$$|u|_{H^{s+\alpha_0}_\Omega(\mathbb{R}^n)} \le C\|f\|_{H^r(\Omega)}, \tag{3.11}$$

for $C > 0$. For the case $r > 0$ we follow a boot-strapping procedure. First, we apply the embedding $H^r(\Omega) \hookrightarrow L^2(\Omega)$ together with the result for u above with $r = 0$. Using this, we further obtain that $F \in H^{\beta_1}(\Omega)$, $\beta_1 = \min\{r, s+\alpha_0\}$, and by Theorem 3.2, $u \in H^{s+\alpha_1}_\Omega(\mathbb{R}^n)$, where $\alpha_1 = \min\{s+r, 3s, 1/2-\varepsilon\}$, $\varepsilon > 0$. Using repeatedly the regularity result (3.11) from the previous step, Proposition 3.3, and the Sobolev imbedding theorem (see, e.g., [19]), we obtain the regularity estimate

$$|u|_{H^{s+\alpha_1}_\Omega(\mathbb{R}^n)} \le C\|F\|_{H^{\beta_1}(\Omega)} \le C\|f\|_{H^r(\Omega)},$$

for some constant $C > 0$. By iterating the previous arguments, we obtain that, for $m \in \mathbb{N}$, $F \in H^{\beta_m}(\Omega)$, $\beta_m = \min\{r, 2ms, s+1/2-\varepsilon\}$, $u \in H^{s+\alpha_m}_\Omega(\mathbb{R}^n)$, $\alpha_m = \min\{s+r, (1+2m)s, 1/2-\varepsilon\}$, and,

$$|u|_{H^{s+\alpha_m}_\Omega(\mathbb{R}^n)} \le C\|F\|_{H^{\beta_m}(\Omega)} \le C\|f\|_{H^r(\Omega)},$$

for $C > 0$. Clearly, for m large enough, we obtained that $\beta_m = \min\{r, s+1/2-\varepsilon\}$, and $\alpha_m = \min\{s+r, 1/2-\varepsilon\}$. Denoting by $\alpha = \alpha_m$, we have obtain (3.8). □

To the best of our knowledge, the regularity of the solution of the fractional Laplace problem (2.6) on less regular, e.g., Lipschitz, domains remains an active field of research. In [1], similar results as in Theorem 3.2 in terms of (weighted) Sobolev spaces are obtained for Lipschitz domains under the assumption of Hölder regularity on the data. However, these higher regularity assumptions on the data are not applicable to the regularity of the solution of the variational inequality considered in Section 4.

3.2. Discretization of the linear problem and a priori error estimates

Let Ω be a convex domain with C^∞-boundary (which is needed for the regularity results (3.8)). We partition $\Omega \cup \Omega_\delta$ into a shape regular quasi-uniform triangulation $\{\mathcal{T}_h\}_h$; see, e.g., [6,10]. We denote by h the maximum diameter of the elements $K \in \mathcal{T}_h$ and set $\overline{\Omega^h \cup \Omega^h_\delta} = \cup_{K\in\mathcal{T}_h}\overline{K}$ under the assumption that $\Omega^h \subset \Omega$ and $|\Omega \setminus \Omega^h| \le ch^2$, $|\Omega_\delta \setminus \Omega^h_\delta| \le ch^2$, which can be realized by requiring boundary nodes to lie on the boundary of Ω and Ω_δ, respectively.

We use the subspace $V_h \subset V$ of piecewise linear polynomials associated with $\mathcal{T}_h$ that satisfy a homogeneous Dirichlet volume constraint in Ω_δ^h, i.e.,

$$V_h = \{v_h \in C^0(\overline{\Omega \cup \Omega_\delta})\colon v_h|_K \in \mathcal{P}_1(K)\ \forall K \in \mathcal{T}_h,\ v_h = 0 \text{ on } (\Omega \cup \Omega_\delta) \setminus \Omega^h\}. \tag{3.12}$$

Let $\mathcal{J}_h^m$ denote the set of all interior nodes of Ω and $\mathcal{J}_h^l$ the set of all nodes. We can represent $X_h = \text{span}\{\phi_p,\ p \in \mathcal{J}_h^l\}$ and $V_h = \text{span}\{\phi_p,\ p \in \mathcal{J}_h^m\}$, where ϕ_p are the nodal Lagrange basis functions.

The discrete formulation of the linear problem (3.1) is: find $u_h \in V_h$ such that

$$a(u_h, v_h) = \langle f, v_h\rangle \quad \forall v \in V_h. \tag{3.13}$$

The existence and uniqueness of the solution of (3.13) directly follows from the conformity of the discrete spaces V_h. In addition, the best approximation property holds [15]:

$$\|u - u_h\|_V \le \min_{v_h \in V_h} \|u - v_h\|_V \to 0 \ \text{ as } h \to 0. \tag{3.14}$$

To obtain convergence estimates, we require that there exists a quasi-interpolation operator $\Pi_h\colon H_\Omega^s(\mathbb{R}^n) \to V_h$ such that

$$\|u - \Pi_h u\|_{H_\Omega^s(\mathbb{R}^n)} \le Ch^\alpha \|u\|_{H_\Omega^{s+\alpha}(\mathbb{R}^n)} \tag{3.15}$$

for $0 < \alpha < 1/2$, and some positive constant C independent of h. For instance, the Scott–Zhang interpolation operator [30] fulfils the desired properties; see [9], cf. also [1]. Then, exploiting the regularity of the solution and using the best approximation property (3.14), we obtain the following a priori error estimates for the finite element discretization for the linear problem (3.1).

Proposition 3.5. *Let Ω be a domain with C^∞ boundary $\partial\Omega$, and let for $f \in H^r(\Omega)$, $r \ge 0$, $u \in V$ be a solution of (3.1) with kernel γ_δ satisfying (2.1) and defined as in Case 1 with $\sigma(\mathbf{x}, \mathbf{y}) = \sigma$. Then, for any $\varepsilon > 0$ there exists a constant $C > 0$, independent of h, such that*

$$\|u - u_h\|_{H_\Omega^s(\mathbb{R}^n)} \le Ch^\alpha \|f\|_{H^r(\Omega)}, \tag{3.16a}$$

$$\|u - u_h\|_{L^2(\Omega)} \le Ch^{\alpha+\beta} \|f\|_{H^r(\Omega)}, \tag{3.16b}$$

where $\alpha = \min\{s + r, 1/2 - \varepsilon\}$, and $\beta = \min\{s, 1/2 - \varepsilon\}$.

Proof. The proof for the estimates in the energy norm (3.16a) is directly obtained by combining (3.14), (3.15), and the regularity estimates (3.8). The convergence estimate in the L^2-norm (3.16b) is obtained by applying a standard Aubin-Nitsche duality argument together with the regularity results (3.8). □

4. Nonlocal variational inequality

To derive the variational formulation of the obstacle problem (1.3) we introduce a set $\mathcal{K}$ of admissible solutions

$$\mathcal{K} := \{u \in V\colon\ u \ge \psi \text{ on } \Omega\},$$

for all $\psi \in X$ and assume that $\psi(\mathbf{x}) \le 0$ on Ω_δ. It is easy to see that $\mathcal{K}$ is closed, convex, and non-empty. Then, a weak formulation of (1.3) leads to the following variational inequality problem: find $u \in \mathcal{K}$ such that for a given $f \in V'$ it holds

$$a(u, v-u) \geq \langle f, v-u\rangle, \quad \forall v \in \mathcal{K}. \tag{4.1}$$

The problem admits a unique solution; see [18,21].

By means of Lagrange multipliers, we restate the problem (4.1) in an equivalent mixed formulation or saddle point form. Let the space $W := V'$ and define a bilinear form $b : W \times V \to \mathbb{R}$ as a duality pairing $b(\eta, v) := \langle \eta, v\rangle$. In W, we define the set M (referred to as a dual cone) as

$$M := \{\eta \in W\colon\ b(\eta, v) \geq 0,\ v \in V,\ v \geq 0\}.$$

Obviously, the bilinear form $b(\eta, v)$ is bounded and inf-sup stable on $W \times V$, i.e.,

$$\beta = \inf_{\eta\in W, \eta\neq 0} \sup_{v\in V, v\neq 0} \frac{b(\eta, v)}{\|\eta\|_W \|v\|_V} \geq \beta_0 > 0.$$

In the present setting, we have $\beta = 1$. Then, the set $\mathcal{K}$ can be expressed equivalently as $\mathcal{K} = \{v \in V\colon\ b(\eta, v) \geq b(\eta, \psi),\ \forall \eta \in M\}$, and we arrive in an equivalent saddle point formulation (see, e.g., [23]): find $u \in V$ and $\lambda \in M$ such that

$$a(u, v) - b(\lambda, v) = \langle f, v\rangle, \qquad \forall v \in V, \tag{4.2a}$$

$$b(\eta - \lambda, u) \geq b(\eta - \lambda, \psi), \qquad \forall \eta \in M. \tag{4.2b}$$

4.1. Variational inequalities with the fractional Laplacian

It is easy to see, that, similarly as in the linear case, for γ_δ defined as in Case 1 with $\sigma(\mathbf{x}, \mathbf{y}) = c_{n,s}/2$ in (2.3), the solution of the nonlocal variational inequality (4.2) converges to the solution of the variational inequality with the fractional Laplacian as $\delta \to \infty$. Let

$$M_\infty := \{\eta \in H^{-s}(\Omega)\colon\ b(\eta, v) \geq 0,\ v \in H^s_\Omega(\mathbb{R}^n),\ v \geq 0\},$$

where $b(\cdot, \cdot)$ is defined as the extended $L^2_\Omega(\mathbb{R}^n)$ duality pairing on $H^{-s}(\Omega) \times H^s_\Omega(\mathbb{R}^n)$. Then, we consider the following variational inequality problem corresponding to the fractional Laplacian (2.5): for $0 < s < 1$ find $u \in H^s_\Omega(\mathbb{R}^n)$ and $\lambda \in M_\infty$ such that

$$a_\infty(u, v) - b(\lambda, v) = \langle f, v\rangle, \qquad \forall v \in H^s_\Omega(\mathbb{R}^n), \tag{4.3a}$$

$$b(\eta - \lambda, u) \geq b(\eta - \lambda, \psi), \qquad \forall \eta \in M_\infty, \tag{4.3b}$$

where $a_\infty(\cdot, \cdot)$ is defined in (3.4) with γ_∞ defined in (2.4), and $\psi \in H^s(\mathbb{R}^n)$, $\psi \leq 0$ on $\mathbb{R}^n \setminus \Omega$. Because the bilinear form $a(\cdot, \cdot)$ (or $a_\infty(\cdot, \cdot)$) defines an inner product on V (or $H^s_\Omega(\mathbb{R}^n)$), and $b(\cdot, \cdot)$ is inf-sup stable, the problem (4.2) (or (4.3)) admits a unique solution.

Proposition 4.1. *Let $(u_\infty, \lambda_\infty) \in H^s_\Omega(\mathbb{R}^n) \times M_\infty$ be a solution pair of the variational inequality with the fractional Laplace operator (4.3), and let $(u_\delta, \lambda_\delta) \in V \times M$ denote the solution of the nonlocal variational inequality (4.2) with γ_δ satisfying (2.1) and defined in (2.3) with $\sigma(\mathbf{x}, \mathbf{y}) = c_{n,s}/2$ and $\delta > \operatorname{diam}|\Omega|$. Then,*

$$\|u_\infty - u_\delta\|_{H^s_\Omega(\mathbb{R}^n)} \leq \frac{C_P}{C_3} C(\delta, n, s) \|u_\infty\|_{L^2(\Omega)},$$

where the constants C_P, C_3, and $C(\delta, n, s)$ are defined in (2.11), (2.10), and (3.5), respectively. Here, $C(\delta, n, s) \sim \delta^{-2s}$ and $C(\delta, n, s) \to 0$ as $\delta \to \infty$.

Proof. From Proposition 3.1 we obtain that for all $v \in H^s_\Omega(\mathbb{R}^n)$

$$a(u_\infty - u_\delta, v) + C(\delta, n, s)(u_\infty, v)_{L^2(\Omega)} - b(\lambda_\infty - \lambda_\delta, v) = 0.$$

Setting $v := u_\infty - u_\delta$ and using Cauchy-Schwarz inequality we obtain

$$\|u_\infty - u_\delta\|_V^2 \leq C(\delta, n, s)\|u_\infty\|_{L^2(\Omega)}\|u_\infty - u_\delta\|_{L^2(\Omega)} + b(\lambda_\infty - \lambda_\delta, u_\infty - u_\delta). \tag{4.4}$$

We note that the inequality constraint (4.3b) can be expressed as

$$b(\lambda_\infty, u_\infty) = b(\lambda_\infty, \psi), \quad \text{and} \quad b(\eta, u_\infty) \geq b(\eta, \psi), \quad \forall \eta \in M_\infty.$$

This property is obtained by taking $\eta = 0$ and $\eta = 2\lambda_\infty$ in (4.3b). Analogous properties hold for the solution pair $(u_\delta, \lambda_\delta) \in V \times M$ of (4.2). Then, taking into account the last properties and the equivalence of the dual cones M_∞ and M (in the sense, that for any $\eta \in M_\infty$ we have $\eta \in M$ and vice versa), we can estimate

$$b(\lambda_\infty - \lambda_\delta, u_\infty - u_\delta) \leq 0.$$

Combining the last estimate and using (2.11), the estimate (4.4) becomes

$$\|u_\infty - u_\delta\|_V \leq C_P C(\delta, n, s)\|u_\infty\|_{L^2(\Omega)}.$$

Finally, using the norm equivalence (2.10) we obtain the necessary result. □

Remark 2. We note that when $\psi = -\infty$, $\lambda_\delta = 0$ and $\lambda_\infty = 0$, and the variational inequality problem (4.2) or (4.3) reduces to the linear problem (3.1) or (3.6), respectively. Then, the error estimates in Proposition 4.1 also recover the error estimates, derived in [13] for the solutions of the corresponding linear problems. In fact, for both linear and variational inequality problems we obtain the same estimate and $C(\delta, n, s) \to 0$ as $\delta \to \infty$ at a rate $2s$.

4.2. Regularity study for nonlocal variational inequalities

In the previous section, we posed a variational inequality problem (4.2) under low regularity assumptions on the Lagrange multiplier $\lambda \in W = V'$. Now, we demonstrate that for a general class of kernels γ_δ satisfying (2.1), and with sufficiently regular ψ and f, we have that $\lambda \in L^2(\Omega)$.

Moreover, for a class of kernels corresponding to the fractional Laplacian, we also prove an improved regularity result for the primal solution u of (4.2) by extending the improved regularity result for the solution of the linear fractional Laplace problem. We derive the regularity estimates for the nonlocal variational inequality (4.2) by generalizing the Lewy-Stampacchia estimates for local variational inequalities; see [24]. We note that similar results have been established for the fractional Laplacian in [31], cf. also [27]. However, the case of truncated kernel is not covered by these analyses and the additional assumption $n > 2s$ is imposed. Here, we give an independent derivation of the results for a general class of truncated kernels γ_δ.

First, we state some results from the theory of monotone operators that are useful for our analyses.

Lemma 4.2 *(Minty Lemma, [24, Chapter III, Lemma 1.5]). Let $\mathcal{X}$ be a reflexive Banach space and $\mathbb{K} \subset \mathcal{X}$ be a closed convex set, and $A : \mathbb{K} \to \mathcal{X}'$ be monotone and continuous on finite-dimensional subspaces. Then, $u \in \mathbb{K}$ satisfies $\langle Au, v - u\rangle_{\mathcal{X}' \times \mathcal{X}} \geq 0$, $\forall v \in \mathbb{K}$, if and only if $u \in \mathbb{K}$ satisfies $\langle Av, v - u\rangle_{\mathcal{X}' \times \mathcal{X}} \geq 0$, $\forall v \in \mathbb{K}$.*

In what follows, we assume that

$$f \in L^2(\Omega) \quad \text{and} \quad (-\mathcal{L}_\delta \psi - f)^+\big|_\Omega \in L^2(\Omega), \tag{4.5}$$

where $(\cdot)^+ := \max(\cdot, 0)$, and, as before, $-\mathcal{L}_\delta$ is defined in (2.2) with kernels γ_δ satisfying (2.1), and defined in either Case 1, Case 2, or Case 3. To derive a regularity result, we study an approximation of the variational inequality by a penalized problem. For any $\varepsilon > 0$, introduce a sequence of penalty functions $\mathcal{H}_\varepsilon : \mathbb{R} \to [0, 1]$, which are assumed to be uniformly Lipschitz continuous, non-increasing, and bounded, $0 \le \mathcal{H}_\varepsilon(t) \le 1$. In particular, we can specify

$$\mathcal{H}_\varepsilon(t) = \begin{cases} 1 & \text{for } \ t \le 0, \\ 1 - t/\varepsilon & \text{for } \ 0 \le t \le \varepsilon, \\ 0 & \text{for } \ t \ge \varepsilon, \end{cases}$$

and consider the following penalized problem

$$\begin{aligned} -\mathcal{L}_\delta u_\varepsilon &= (-\mathcal{L}_\delta \psi - f)^+ \mathcal{H}_\varepsilon(u_\varepsilon - \psi) + f && \text{in } \Omega, \\ u_\varepsilon &= 0 && \text{in } \Omega_\delta. \end{aligned}$$

In a variational form, the above problem reads as follows: for a fixed $\varepsilon > 0$, find $u_\varepsilon \in V$, such that for all $v \in V$

$$a(u_\varepsilon, v) - \int\limits_{\Omega \cup \Omega_\delta} \left((-\mathcal{L}_\delta \psi - f)^+ \mathcal{H}_\varepsilon(u_\varepsilon - \psi) \right) v \, \mathrm{d}\mathbf{x} = \int\limits_\Omega f v \, \mathrm{d}\mathbf{x}. \tag{4.6}$$

Using the theory of nonlinear monotone operators, one obtains the existence and uniqueness of solutions of (4.6).

Lemma 4.3. *Let (4.5) be satisfied. Then, for a fixed $\varepsilon > 0$, there exists a unique solution $u_\varepsilon \in V$ of the penalized problem (4.6). Moreover, there exists a positive constant $C > 0$ such that*

$$\|u_\varepsilon\|_V \le C \left(\|f\|_{L^2(\Omega)} + \|(-\mathcal{L}_\delta \psi + f)^+\|_{L^2(\Omega)} \right). \tag{4.7}$$

Proof. The proof follows the lines, e.g., of [24, Lemma 2.2]; we present it here for completeness purposes. First, noting that $\mathcal{H}_\varepsilon(u_\varepsilon - \psi) \in L^\infty(\Omega \cup \Omega_\delta)$, $\varepsilon > 0$, we define a nonlinear operator $A_\varepsilon : V \to V'$ by

$$\langle A_\varepsilon u_\varepsilon, v \rangle := a(u_\varepsilon, v) - \int\limits_{\Omega \cup \Omega_\delta} ((-\mathcal{L}_\delta \psi - f)^+ \mathcal{H}_\varepsilon(u_\varepsilon - \psi)) v \, \mathrm{d}\mathbf{x}. \tag{4.8}$$

It is easy to see that A_ε is strictly monotone and coercive on V. Indeed, taking into account that $\mathcal{H}_\varepsilon$ is non-increasing, we obtain that

$$\begin{aligned} \langle A_\varepsilon u_\varepsilon - A_\varepsilon v, u_\varepsilon - v \rangle &= a(u_\varepsilon - v, u_\varepsilon - v) \\ &\quad - \int\limits_{\Omega \cup \Omega_\delta} (-\mathcal{L}_\delta \psi - f)^+ (\mathcal{H}_\varepsilon(u_\varepsilon - \psi) - \mathcal{H}_\varepsilon(v - \psi))(u_\varepsilon - v) \, \mathrm{d}\mathbf{x} \ge \|u_\varepsilon - v\|_V^2, \end{aligned}$$

which shows that A_ε is strictly monotone and coercive. For a fixed $\varepsilon > 0$, let u_ε^n be a sequence, such that $u_\varepsilon^n \to u_\varepsilon$ strongly in V. Then, $A_\varepsilon u_\varepsilon^n \rightharpoonup A_\varepsilon u_\varepsilon$ weakly in V', which also implies that A_ε is continuous

on finite-dimensional subspaces of V; see [24, Definition 1.2]. Then, by existence results for monotone operators [24, Chapter III, Corollary 1.8], and the strict monotonicity of A_ε, we obtain the existence and uniqueness of u_ε of (4.6) for a fixed $\varepsilon > 0$. Then, the bound (4.7) directly follows from (3.2), (2.11), and (4.5). □

Next, under mild assumptions on the data, we state the main regularity result for a nonlocal variational inequality.

Theorem 4.4 *(Regularity of the Lagrange multiplier). Let f and ψ be such that the condition (4.5) holds, and let $(u, \lambda) \in V \times M$ be the unique solution pair of (4.2) for the kernels γ_δ, satisfying (2.1) and defined in either Case 1, Case 2, or Case 3. Then, $\mathcal{L}_\delta u \in L^2(\Omega)$, $\lambda \in L^2(\Omega)$, and $\lambda \leq (-\mathcal{L}_\delta \psi - f)^+$.*

Proof. First, we verify that $u_\epsilon \in \mathcal{K}$. Let $z(\mathbf{x}) = u_\varepsilon(\mathbf{x}) - \psi(\mathbf{x})$ and $z^-(\mathbf{x}) = (\psi(\mathbf{x}) - u_\varepsilon(\mathbf{x}))^+$, where $(z)^- = (-z)^+ = -\min(z, 0)$. Because $\psi(\mathbf{x}) \leq 0$ on Ω_δ, we obtain that $(\psi - u_\varepsilon)^+ \in V$ and $z^- \in V$. Indeed, for any $v \in V$, due to the Lipschitz continuity of the map $v \mapsto v^+$, we obtain

$$\int\limits_{(\Omega\cup\Omega_\delta)^2} |v^+(\mathbf{x}) - v^+(\mathbf{y})|^2 \gamma_\delta(\mathbf{x}, \mathbf{y})\, \mathrm{d}(\mathbf{x}, \mathbf{y}) \leq \int\limits_{(\Omega\cup\Omega_\delta)^2} |v(\mathbf{x}) - v(\mathbf{y})|^2 \gamma_\delta(\mathbf{x}, \mathbf{y})\, \mathrm{d}(\mathbf{x}, \mathbf{y}),$$

i.e., $v^+ \in X$ and also $v^+ \in V$. Next, we show that $z^- = 0$. Using (4.6) and the nonlocal Green's identity [15], we obtain that

$$a(u_\varepsilon - \psi, z^-) = \int\limits_{\Omega\cup\Omega_\delta} \left((-\mathcal{L}_\delta \psi - f)^+ \mathcal{H}_\varepsilon(u_\varepsilon - \psi) + f + \mathcal{L}_\delta \psi\right) z^- \,\mathrm{d}\mathbf{x}. \tag{4.9}$$

Let $\mathcal{O}_1 := \{\mathbf{x} \in \Omega \cup \Omega_\delta \colon z(\mathbf{x}) \geq 0\}$, $\mathcal{O}_2 := \{\mathbf{x} \in \Omega \cup \Omega_\delta \colon z(\mathbf{x}) < 0\}$. Then using the fact that $z = z^+ - z^-$, $z^- = 0$ on $\mathcal{O}_1$, $z^- > 0$ on $\mathcal{O}_2$ and $z^+ = 0$ on $\mathcal{O}_2$, $z^+ > 0$ on $\mathcal{O}_1$, we can estimate the left-hand side of (4.9) as follows

$$\begin{aligned} a(z, z^-) &= -a(z^-, z^-) + a(z^+, z^-) \\ &= -\|z^-\|_V^2 + \int\limits_{\Omega\cup\Omega_\delta} \int\limits_{\Omega\cup\Omega_\delta} (z^+(\mathbf{x}) - z^+(\mathbf{y}))(z^-(\mathbf{x}) - z^-(\mathbf{y}))\gamma_\delta(\mathbf{x}, \mathbf{y})\, \mathrm{d}\mathbf{y}\, \mathrm{d}\mathbf{x} \\ &= -\|z^-\|_V^2 - \int\limits_{\mathcal{O}_1} \int\limits_{\mathcal{O}_2} z^+(\mathbf{x}) z^-(\mathbf{y}) \gamma_\delta(\mathbf{x}, \mathbf{y})\, \mathrm{d}\mathbf{y}\, \mathrm{d}\mathbf{x} - \int\limits_{\mathcal{O}_2} \int\limits_{\mathcal{O}_1} z^+(\mathbf{y}) z^-(\mathbf{x}) \gamma_\delta(\mathbf{x}, \mathbf{y})\, \mathrm{d}\mathbf{y}\, \mathrm{d}\mathbf{x} \\ &= -\|z^-\|_V^2 - 2 \int\limits_{\mathcal{O}_1} \int\limits_{\mathcal{O}_2} z^+(\mathbf{x}) z^-(\mathbf{y}) \gamma_\delta(\mathbf{x}, \mathbf{y})\, \mathrm{d}\mathbf{y}\, \mathrm{d}\mathbf{x} \leq 0. \end{aligned} \tag{4.10}$$

The last estimate follows by splitting the inner and outer integrals onto $\mathcal{O}_1$ and $\mathcal{O}_2$ and using the values of z^+ and z^- on $\mathcal{O}_1$ and $\mathcal{O}_2$. On the other hand, taking into account the fact that $z^-(\mathbf{x}) = 0$ on $\mathcal{O}_1$ and $\mathcal{H}_\varepsilon(z) = 1$ on $\mathcal{O}_2$, the right-hand side of (4.9) can be estimated as

$$\begin{aligned} &\int\limits_{\Omega\cup\Omega_\delta} \left((-\mathcal{L}_\delta \psi - f)^+ \mathcal{H}_\varepsilon(z) + f + \mathcal{L}_\delta \psi\right) z^- \,\mathrm{d}\mathbf{x} \\ &\qquad = \int\limits_{\mathcal{O}_2} \left((-\mathcal{L}_\delta \psi - f)^+ + f + \mathcal{L}_\delta \psi\right) z^- \,\mathrm{d}\mathbf{x} \geq 0. \end{aligned}$$

Hence, $a(z, z^-) = 0$, and since both terms in (4.10) have the same sign, it follows that $\|z^-\|_V = 0$, and, hence, $u_\varepsilon(\mathbf{x}) \geq \psi(\mathbf{x})$ a.e. on $\Omega \cup \Omega_\delta$.

By Lemma 4.3, u_ε is bounded in V; then, by the Banach-Alaoglu theorem, there exists a subsequence $u_{\varepsilon'}$ that converges weakly in V, i.e., $u_{\varepsilon'} \rightharpoonup \widetilde{u}$, $\widetilde{u} \in V$. Because $u_\varepsilon \in \mathcal{K}$ and $\mathcal{K}$ is closed and convex, we obtain that $\widetilde{u} \in \mathcal{K}$. Next, we show that $\widetilde{u}$ solves (4.1). From Lemma 4.3, we know that

$$\langle A_{\varepsilon'} u_{\varepsilon'}, \nu \rangle = \int_\Omega f\nu \, \mathrm{d}\mathbf{x}, \quad \forall \nu \in V,$$

where $\varepsilon' > 0$ and the operator $A_{\varepsilon'}$ is defined in (4.8). Let $v \in \mathcal{K}$, and suppose $v \geq \psi + \varepsilon_0 d$, where $d(\mathbf{x}) = \mathrm{dist}(\mathbf{x}, \Omega_\delta)$, and $\varepsilon_0 > 0$. Then, by taking $\nu = u_{\varepsilon'} - v$, and applying Lemma 4.2, we obtain

$$\langle A_{\varepsilon'} v, v - u_{\varepsilon'} \rangle - \int_{\Omega \cup \Omega_\delta} f(v - u_{\varepsilon'}) \, \mathrm{d}\mathbf{x} \geq 0, \quad \text{for all } v \in \mathcal{K},\ v \geq \psi + \varepsilon_0 d. \tag{4.11}$$

For $\varepsilon < \varepsilon_0 d(\mathbf{x})$, $\mathcal{H}_\varepsilon(v(\mathbf{x}) - \psi(\mathbf{x})) = 0$; then, by letting $\varepsilon \to 0$, we obtain that $\mathcal{H}_\varepsilon(v - \psi) \to 0$ pointwise. Thus,

$$\int_\Omega (-\mathcal{L}_\delta \psi - f)^+ \mathcal{H}_\varepsilon(v - \psi)(v - u_\varepsilon) \, \mathrm{d}\mathbf{x} \to 0, \quad \text{as } \varepsilon \to 0,$$

is obtained by additionally using $\mathcal{H}_\varepsilon \leq 1$, the uniform boundedness of u_ε in $L^2(\Omega)$, and (4.5). Then, (4.11) becomes

$$a(v, v - \widetilde{u}) - \int_{\Omega \cup \Omega_\delta} f(v - \widetilde{u}) \, \mathrm{d}\mathbf{x} \geq 0, \quad \text{for all } v \in \mathcal{K},\ v \geq \psi + \varepsilon_0 d. \tag{4.12}$$

In order to show (4.12) for arbitrary $v \in \mathcal{K}$, we define $v_{\varepsilon_0} := \max(\psi + \varepsilon_0 d, v) \geq \psi + \varepsilon_0 d$. We note that, since $d(\mathbf{x})$ is Lipschitz continuous, and $d(\mathbf{x}) = 0$ on Ω_δ, we have that $v_{\varepsilon_0} \in V$. In addition, we can verify that $v_{\varepsilon_0} \to v$ strongly in V as $\varepsilon_0 \to 0$. Then, letting $\varepsilon_0 \to 0$ and applying Lemma 4.2 again we arrive at

$$a(\widetilde{u}, v - \widetilde{u}) - \int_{\Omega \cup \Omega_\delta} f(v - \widetilde{u}) \, \mathrm{d}\mathbf{x} \geq 0, \quad \text{for all } v \in \mathcal{K}.$$

Thus, $\widetilde{u} = u$ is the unique solution of (4.1), hence, $u_\varepsilon \rightharpoonup u$ in V.

For $\varepsilon > 0$ define $\lambda_\varepsilon \in V'$ as $\lambda_\varepsilon := Au_\varepsilon - A_\varepsilon u_\varepsilon = Au_\varepsilon - f$, where $A : V \to V'$ and is defined in (3.3). Now letting $\varepsilon \to 0$, we know that $u_\varepsilon \rightharpoonup u$ in V, then $\lambda_\varepsilon \rightharpoonup Au - f$ weakly in V'. Now, denoting $\lambda = Au - f$, where λ is a solution of (4.2), we obtain that $\lambda_\varepsilon \rightharpoonup \lambda$ in V'. On the other hand, we know that

$$\langle \lambda_\varepsilon, v \rangle = \int_\Omega \left((-\mathcal{L}_\delta \psi - f)^+ \mathcal{H}_\varepsilon(u_\varepsilon - \psi) \right) v \, \mathrm{d}\mathbf{x} \quad \forall v \in V,$$

and hence λ_ε can be identified with an element in $L^2(\Omega)$. Furthermore, for some constant $C > 0$, independent of ε, $\|\lambda_\varepsilon\|_{L^2(\Omega)} \leq C$. In fact, the following holds pointwise almost everywhere in $L^2(\Omega)$:

$$\lambda_\varepsilon = (-\mathcal{L}_\delta \psi - f)^+ \mathcal{H}_\varepsilon(u_\varepsilon - \psi) \leq (-\mathcal{L}_\delta \psi - f)^+. \tag{4.13}$$

Hence, there exists a subsequence $\lambda_{\varepsilon'} \rightharpoonup \lambda$ in $L^2(\Omega)$. Because $(u, \lambda) \in V \times V'$ is the unique solution pair of (4.2), it follows that $\lambda_\varepsilon \rightharpoonup \lambda$ in $L^2(\Omega)$ and the weak limits in V' and $L^2(\Omega)$ coincide, thus $\lambda \in L^2(\Omega)$, and, additionally, the pointwise relation (4.13) is preserved in the limit. This concludes the proof. □

Corollary 4.5 *(Improved regularity of the solution). Let Ω be a domain with C^∞ boundary $\partial\Omega$, and let f and ψ be such that the condition (4.5) holds, and with the kernel γ_δ defined in (2.3) for $\delta > 0$ and $\sigma(\mathbf{x}, \mathbf{y}) = \sigma$. Then, for the solution pair $(u, \lambda) \in V \times M$ of the variational inequality (4.2) we obtain that $\mathcal{L}_\delta u \in L^2(\Omega)$, $\lambda \in L^2(\Omega)$ and $u \in H^{s+\alpha}_\Omega(\mathbb{R}^n)$, where $\alpha = \min\{s, 1/2 - \varepsilon\}$ for any $\varepsilon > 0$.*

Proof. From the Theorem 4.4 we obtain that $\lambda \in L^2(\Omega)$, and then we apply the regularity results (3.8) for u as a solution of a linear problem with the right hand-side $\lambda + f \in L^2(\Omega)$. □

4.3. Discretization of the nonlocal variational inequality

We subdivide $\Omega \cup \Omega_\delta$ by a triangulation $\{\mathcal{T}_h\}_h$ in the same manner as described in Section 3.2. Similarly, the primal space V is approximated by V_h defined as in (3.12).

For the discrete Lagrange multiplier space W_h, we use discontinuous piecewise linear biorthogonal basis functions, defined with respect to the same mesh as the basis functions of V_h; see [37]. Following the same notation as in Section 3.2, we define $W_h = \operatorname{span}\{\xi_q,\ q \in \mathcal{J}_h^m\}$, where ξ_p, $p \in \mathcal{J}_h^l$, satisfy a local biorthogonality relation

$$\int_K \xi_q \phi_p = \delta_{p,q} \int_K \phi_p \geq 0.$$

Then, the discrete Lagrange multiplier cone $M_h \subset W_h$ is defined as

$$M_h = \operatorname{span}_+\{\xi_q\}_{q\in\mathcal{J}_h^m} := \Big\{\eta \in W_h \colon \eta = \sum_{q\in\mathcal{J}_h^m} \alpha_q \xi_q,\ \alpha_q \geq 0\Big\}.$$

For these settings, we can guarantee the inf-sup stability of $b(\cdot,\cdot)$ on the pair of discrete spaces $W_h \times V_h$. We remark that $\xi_q \notin M$, i.e., $M_h \not\subset M$, so that we obtain a non-conforming approximation with respect to the Lagrange multiplier.

The approximation of (4.2) by the discrete saddle-point problem reads as follows: find $u_h \in V_h$ and $\lambda_h \in M_h$ such that

$$a(u_h, v_h) - b(\lambda_h, v_h) = \langle f, v_h\rangle \qquad \forall v_h \in V_h, \tag{4.14a}$$

$$b(\eta_h - \lambda_h, u_h) \geq b(\eta_h - \lambda_h, \psi) \qquad \forall \eta_h \in M_h. \tag{4.14b}$$

Because the discrete inf-sup stability of $b(\cdot,\cdot)$ holds on $W_h \times V_h$, we also have the existence and uniqueness of the solution of the discrete problem (4.14). The problem (4.14) can be solved by a semi-smooth Newton method, and, in particular, we can employ the primal-dual active set strategy [22] that is known to be locally superlinearly convergent.

5. Numerical results

In this section, we present numerical results for the nonlocal variational equality and inequality problems that illustrate our theoretical findings and provide an outlook on open questions and possible future directions.

5.1. Model settings

We consider one-dimensional problems, where the computational domain Ω is set to $\Omega = (0, 1)$ and is discretized with the uniform mesh of a mesh size $h = 1/N$, $N \in \mathbb{N}$. Unless otherwise stated, we set $h = 2^{-9}$. The interaction domain is then given by $\Omega_\delta = (-\delta, 0) \cup (1, 1 + \delta)$.

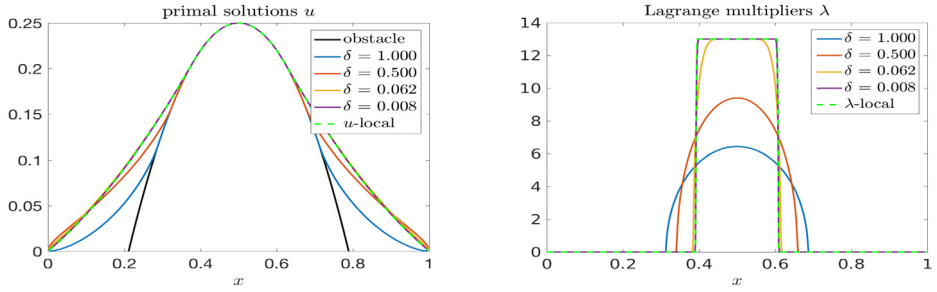


Fig. 1. Solutions (u_h, λ_h) for different interaction radii δ for $s = 0.5$, $\sigma = (1-s)/\delta^{2-2s}$, $f = -1$, and ψ defined in (5.1) and also for the corresponding local problem. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

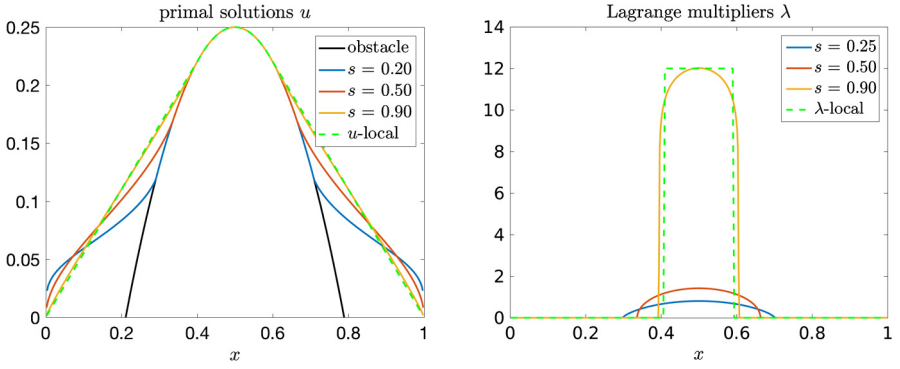


Fig. 2. Solutions (u_h, λ_h) for different values of $s \in (0,1)$ and $f = 0$, $\sigma = 1/2\delta^2$, $\delta = 2$, and ψ defined in (5.1) and also for the corresponding local problem.

Due to the nonlocality of the problems, the corresponding matrices, in general, are not sparse (in contrast to the finite element matrices for a local problem). This also increases the computational cost of assembling the matrices, and eventually of solving the discrete problems. In certain cases, e.g., when Ω is a $n-$dimensional hyper-rectangle and the kernel is translation invariant, the matrix possesses a multilevel Toeplitz structure for regular grids. In this case, the assembly time can be significantly reduced by only computing the first row of the matrix; see [36].

Comparison to the local variational inequality

First, we consider solutions of the variational inequality (4.14) with γ_δ defined as in *Case 1* with $\sigma = (2-2s)/\delta^{2-2s}$ and an obstacle functional ψ defined as

$$\psi = -3(x-0.5)^2 + 0.25. \tag{5.1}$$

Snapshots of solutions for different values of $\delta > 0$ are presented in Fig. 1. We also plot the solution of the local obstacle problem, i.e., for $-\mathcal{L}_\delta = -\Delta$. As we can see, as $\delta \to 0$, the nonlocal solutions u_h and λ_h converge to the solution of the local variational inequality, as is expected. We also note that $\psi \in H^{2s}(\Omega)$, $s \in (0,1)$ and it can be easily verified that the condition (4.5) is fulfilled. Hence, $\lambda \in L^2(\Omega)$ according to Theorem 4.4, which is also confirmed by the numerical examples.

In addition, we can make another interesting observation. It is well known that for local obstacle problems, the Lagrange multiplier has jumps, as is also clearly seen from Fig. 1. In contrast, for the nonlocal problem, we observe some "smoothing" effect with respect to increasing the interaction radius δ.

In Fig. 2, solutions of (4.14) for different values of $s \in (0,1)$ with $\delta = 2$ and γ_δ defined as in *Case 1* with $\sigma = 1/2\delta^2$. As is expected, for $s \to 1$, the solution of the nonlocal variational inequality converges to the solution of the local problem. Similarly, as in the previous example, we also observe a "smoothing"

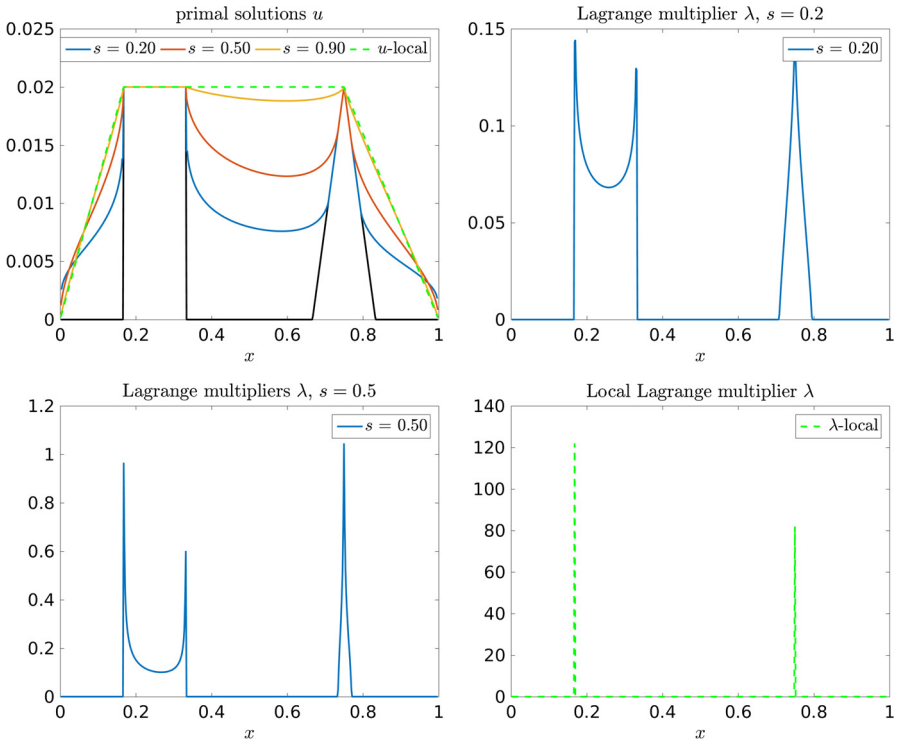


Fig. 3. The solution (u_h, λ_h) for different values of $s \in (0,1)$ and $f = 0$, $\sigma = 1/2\delta^2$, $\delta = 2$, and ψ defined in (5.2).

effect in the Lagrange multipliers for decreasing s. Our theory provides $L^2(\Omega)$ regularity of the Lagrange multipliers; however from both examples, the numerical results suggest a possible higher regularity of the nonlocal Lagrange multipliers in contrast to the local ones. This question is left for future investigation.

Next, we consider a less regular obstacle functional

$$\psi = \begin{cases} 0.02, & \text{for } x \in [1/6, 2/6], \\ 0.24(x - 2/3), & \text{for } x \in [2/3, 3/4], \\ 0.24(5/6 - x), & \text{for } x \in [3/4, 5/6], \\ 0, & \text{elsewhere.} \end{cases} \tag{5.2}$$

For ψ defined in (5.2), γ_δ defined as in *Case 1* with $\sigma = 1/2\delta^2$, and $\delta = 2$, we plot the solution for different values of $s \in (0,1)$ in Fig. 3. In the same figure, we also plot the solution of the local problem.

In contrast to the previous example, in this case ψ is not regular. In fact, while $\psi|_{[1/2,1]}$ belongs to $H^1(\Omega)$, the remaining part $\psi|_{[0,1/2)}$ is only in $H^{1/2-\varepsilon}(\Omega)$. Thus, only for $s < 1/4$ we have that $\psi \in H^{2s}(\Omega)$, which implies that the condition (4.5) holds, and hence the Lagrange multiplier of the nonlocal variational inequality is in $L^2(\Omega)$. This appears to correspond to the numerical results given on the top-right in Fig. 3. In contrast, for $s = 0.5$ the right part $\psi|_{[1/2,1]}$ still belongs to $H^{2s}(\Omega)$, however, the left part $\psi|_{[0,1/2)}$ does not. Correspondingly, on the bottom-left picture in Fig. 3, we can only expect the right part of the Lagrange multiplier to be in $L^2(\Omega)$. Finally, for the local problem the Lagrange multiplier is a sum of Dirac delta functions and is not in $L^2(\Omega)$.

Comparison to the variational inequality with the fractional Laplacian

We study the convergence of the solution of the nonlocal variational inequality (4.14) to the solution of the variational inequality with the fractional Laplace operator (4.3). Because the analytic solution of the variational inequality is in general not known, we compare the corresponding finite element surrogates. In

Table 1
Convergence of the energy error $\|u_h^\delta - u_h^{FL}\|_V$ and L^2-error $\|u_h^\delta - u_h^{FL}\|_{L^2(\Omega\cup\Omega_\delta)}$ with respect to the interaction radii δ and fixed $s = 0.5$, and the corresponding convergence rates.

δ	Energy error	Rate in V-norm	L^2-error	Rate in L^2-norm
2^3	9.7e–03	–	1.6e–03	–
2^4	4.8e–03	1.01	8.0e–04	1.02
2^5	2.4e–03	1.00	3.9e–04	1.00
2^6	1.2e–03	1.00	2.0e–04	1.00
2^7	6.0e–04	1.00	9.8e–04	1.00
2^8	2.9e–04	1.00	4.9e–05	1.00

Table 2
Linear problem: Convergence of the energy error $\|u_h - \bar{u}\|_V$ and L^2-error $\|u_h - \bar{u}\|_{L^2(\Omega\cup\Omega_\delta)}$ with respect to the grid size h, and the corresponding convergence rates, with $s = 0.5$, $\delta = 0.5$, $f = 1$.

h	Energy error	Rate in V-norm	L^2-error	Rate in L^2-norm
2^{-3}	9.27e–02	–	1.00e–02	–
2^{-4}	6.59e–02	0.492	5.21e–03	0.950
2^{-5}	4.66e–02	0.498	2.67e–03	0.963
2^{-6}	3.27e–02	0.513	1.34e–03	0.992
2^{-7}	2.24e–02	0.544	6.54e–04	1.041

Table 3
Linear problem: Rates of convergence of the error in the energy norm $\|u_h - \bar{u}\|_V$ and L^2-norm $\|u_h - \bar{u}\|_{L^2(\Omega\cup\Omega_\delta)}$ with respect to the grid size h for different s and fixed $\delta = 1$. The reference solution is computed on the fine mesh with $h = 2^{-12}$.

h \ s	Energy error				L^2-error			
	0.1	0.25	0.5	0.75	0.1	0.25	0.5	0.75
2^{-5}	0.48	0.49	0.50	0.55	0.68	0.79	0.97	1.15
2^{-6}	0.49	0.49	0.50	0.53	0.67	0.79	0.97	1.11
2^{-7}	0.50	0.50	0.51	0.52	0.66	0.79	0.98	1.20
2^{-8}	0.51	0.51	0.52	0.53	0.66	0.80	1.00	1.11
2^{-9}	0.54	0.54	0.54	0.55	0.68	0.82	1.04	1.15

Table 1, we report the approximation errors between the nonlocal solution u_h^δ of (4.14) with γ_δ defined in (2.3) with $\sigma(\mathbf{x},\mathbf{y}) = c_{n,s}/2$, and the finite element approximation u_h^{FL} of (4.3) for different $\delta > 0$, fixed $s = 0.5$ and $h = 2^{-9}$. We observe the rate one for both, the energy and L^2 errors. This illustrates the theoretical findings in Proposition 4.1, i.e., the error is proportional to $O(\delta^{-2s})$, $s \in (0,1)$.

Convergence of the finite element approximation for a linear problem

Now, we study the convergence of the finite element approximation for the linear problem (3.1) with respect to the mesh size h. We consider the error between the finite element solution u_h of (3.13) on different grid levels and the surrogate $\bar{u}$, which is also a finite element solution computed on a fine mesh of a grid size $h = 2^{-12}$. Here, we consider γ_δ defined in (2.3) with $\sigma = 1$.

In Table 2 we present convergence results for $f = 1$, $s = 0.5$, and $\delta = 0.5$. Because for these settings the solution admits $H^{1-\varepsilon}(\Omega\cup\Omega_\delta)$ regularity, $\varepsilon > 0$, we observe an $O(h^{0.5})$ rate of convergence in the energy norm and $O(h)$ in the L^2 norm, which illustrates the theoretical results in Proposition 3.5.

To study the convergence behaviour of the finite element approximation with respect to different $s \in (0,1)$, we report the corresponding rates of convergence in Table 3. Here, as previously, the results illustrates the theoretical ones (3.16).

Table 4
Variational inequality: Convergence of the energy error $\|u_h - \bar{u}\|_V$ and L^2-error $\|u_h - \bar{u}\|_{L^2(\Omega\cup\Omega_\delta)}$ with respect to the grid size h, and the corresponding convergence rates with $s = 0.5$, $\delta = 1$, $f = 0$, $\sigma = 1$, and ψ defined in (5.1). The reference solution is computed on the fine mesh with $h = 2^{-12}$.

h	Energy error	Rate in V-norm	L^2-error	Rate in L^2-norm
2^{-5}	3.61e–02	0.556	1.62e–03	1.200
2^{-6}	2.54e–02	0.508	8.22e–04	0.985
2^{-7}	1.78e–02	0.508	4.18e–04	0.976
2^{-8}	1.24e–02	0.520	2.10e–04	0.987
2^{-9}	8.50e–03	0.548	1.02e–04	1.041

Table 5
Variational inequality: Rates of convergence of the error in the energy norm $\|u_h - \bar{u}\|_V$ and L^2-norm $\|u_h - \bar{u}\|_{L^2(\Omega\cup\Omega_\delta)}$ with respect to the grid size h for different s and fixed $\delta = 1$. The reference solution is computed on the fine mesh with $h = 2^{-12}$.

h \ s	Energy error				L^2-error			
	0.1	0.25	0.5	0.75	0.1	0.25	0.5	0.75
2^{-5}	0.52	0.49	0.56	0.70	0.69	0.80	1.20	1.40
2^{-6}	0.50	0.49	0.51	0.58	0.67	0.77	0.98	1.23
2^{-7}	0.50	0.50	0.51	0.53	0.65	0.77	0.97	1.12
2^{-8}	0.51	0.51	0.52	0.53	0.65	0.78	0.98	1.15
2^{-9}	0.54	0.54	0.55	0.55	0.67	0.81	1.04	1.18

Convergence of the finite element approximation for the variational inequality

We also report on the convergence of the finite element method for the nonlocal variational inequality (4.2). Here, as before, the surrogate solution $\bar{u}$, which is a finite element solution, computed on a fine mesh of grid size $h = 2^{-12}$, and the kernel γ_δ defined in (2.3) with $\sigma = 1$. For $f = 0$, $s = 0.5$, $\delta = 1$, and an obstacle functional ψ defined in (5.1), the convergence of the energy and L^2 errors together with the corresponding rates are presented in Table 4. For other values of $s \in (0, 1)$ we also report the convergence rates in Table 5. As we can observe, the numerical results suggest the same convergence order for u_h as in the case of the linear problem.

6. Concluding remarks

In this paper, we provide regularity results for nonlocal linear and obstacle problems. For both cases, we have extended the regularity results [20] to the truncated fractional Laplace kernels. For a general class of kernels, we derive an improved regularity estimate for the Lagrange multiplier. We present the discretization of both problems by a finite element method and prove a priori error estimates for the linear problem. The development of the a priori error estimates for nonlocal obstacle problem is beyond the scope of this paper and is a subject for future investigation.

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