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Convexity according to a pair of quasi-arithmetic means and inequalities [☆]

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ABSTRACT

We consider a class of generalized convex functions, which are defined according to a pair of quasi-arithmetic means and called $(\mathcal{M}_\phi, \mathcal{M}_\psi)$-convex functions, and establish various Fejér type inequalities for such a function class. These inequalities not merely provide a natural and intrinsic characterization of the $(\mathcal{M}_\phi, \mathcal{M}_\psi)$-convex functions, but actually offer a generalization and refinement of the most part of the concrete Hermite-Hadamard and Fejér type inequalities obtained in earlier studies for different kinds of convexity and fractional integrals. Applications to inequalities involving the gamma function and special means are also included.

1. Introduction

Recall that the Hermite-Hadamard inequality, which was first noticed by Hermite [18] in 1883 and rediscovered ten years later by Hadamard [16], gives us a lower and an upper estimations for the integral mean value of any convex function on a closed interval, involving the midpoint and the endpoints of the domain. More precisely, if $f : [a, b] \to \mathbb{R}$ is convex, then

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2}. \tag{1.1}$$

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Quite interesting, this double inequality is not merely a consequence of convexity, but also characterizes it; that is, every continuous function satisfies either its left- or right-side on any subinterval of the domain, then the function is convex (see [40, Theorem 1]).

The Hermite-Hadamard inequality has evoked the interest of many mathematicians and becomes an important cornerstone in mathematical analysis and optimization. Many classical results related to this inequality can be found in the monograph of Pečarić, Proschan and Tong [35]. Especially, in the last two decades it has received much attention. There is, in fact, a growing literature providing new proofs, extensions and considering its refinements, generalizations, numerous interpolations and applications in the theory of special means. The monograph of Dragomir and Pearce [13] gives a comprehensive review of this literature.

Observe that most of the works dealing with the Hermite-Hadamard inequality have been obtained by studying a specific kind of convexity (see, for example, [5,9–12,14,15,17,20,22,29,30,33,34,42–47]) or/and utilizing certain fractional integrals (see, for instance, [2,6,8,21,23,25,26,36,39]). This observation, when looked at from a more general point of view, leads to a new generalization that includes a wide class of known results as special cases. For this purpose, we need to consider a class of generalized convex functions which, in particular, recover the ones used in earlier studies, and at the same time, offer an effective method to deal with generalized fractional integrals.

Notice that the first and last terms in (1.1) can be written as

$$f\left(\frac{a+b}{2}\right) = f(A(a,b)) \quad \text{and} \quad \frac{f(a)+f(b)}{2} = A(f(a), f(b)),$$

where $A(a,b)$ stands for the arithmetic mean of a and b. Then (1.1) is an interpolating inequality for

$$f(A(a,b)) \leq A(f(a), f(b)),$$

which is used to define the midpoint, or Jensen's, convex functions. Therefore, to generalize the concept of convexity, it is natural to replace the arithmetic means, A, in the above inequality by a pair of more general means. The quasi-arithmetic means defined below seem to be the best adapted to our goal.

Let $I \subset \mathbb{R}$ be an open interval and $\phi : I \to \mathbb{R}$ be a continuous and strictly monotonic function. The quasi-arithmetic mean of $a \in I$ and $b \in I$ with weight $\alpha \in [0,1]$ is denoted by $\mathcal{M}_\phi(a,b;\alpha)$ and is defined by

$$\mathcal{M}_\phi(a,b;\alpha) = \phi^{-1}\left(\alpha\phi(a) + (1-\alpha)\phi(b)\right)$$

(see, for instance, [7, Chapter IV]). By monotonicity, it follows that

$$\min\{a,b\} \leq \mathcal{M}_\phi(a,b;\alpha) \leq \max\{a,b\}.$$

We make the convention to omit the weight α when $\alpha = 1/2$; that is,

$$\mathcal{M}_\phi(a,b) = \mathcal{M}_\phi(a,b;1/2).$$

We emphasize that if $I \subset (0,\infty)$, then the quasi-arithmetic means $\mathcal{M}_\phi(a,b)$ include the power means of order $p \in \mathbb{R}$, which correspond to the choice

$$\phi(x) = \begin{cases} x^p & \text{if} \quad p \neq 0, \\ \ln x & \text{if} \quad p = 0. \end{cases}$$

The most used in applications are the power means of order $1, 0$ and -1, usually known as the arithmetic, geometric and harmonic means, respectively.

Here and subsequently, I and J denote open intervals in the real line $\mathbb{R}$, $\phi : I \to \mathbb{R}$ and $\psi : J \to \mathbb{R}$ are continuous and strictly monotonic functions. With the quasi-arithmetic means $\mathcal{M}_\phi$ and $\mathcal{M}_\psi$ in hand, we are now in a position to generalize the notion of convexity. According to Aumann [4] (see also [32]), a function $f : I \to J$ is said to be $(\mathcal{M}_\phi, \mathcal{M}_\psi)$-convex if it verifies the following analogue of Jensen's inequality:

$$f(\mathcal{M}_\phi(a, b; \alpha)) \le \mathcal{M}_\psi(f(a), f(b); \alpha) \tag{1.2}$$

for all $a, b \in I$ and $\alpha \in [0, 1]$. Especially, we say that f is $\mathcal{M}_\psi$-convex if it satisfies (1.2) with $\phi(x) = x$. If inequality (1.2) works in the opposite way, f is called $(\mathcal{M}_\phi, \mathcal{M}_\psi)$-concave. It is worth pointing out that the $(\mathcal{M}_\phi, \mathcal{M}_\psi)$-convex functions cover a very large variety of functions playing an important role in mathematics such as

- the usual convex functions if we take $\phi(x) = x$ and $\psi(x) = x$,
- the log-convex functions if we choose $\phi(x) = x$ and $\psi(x) = \ln x$,
- the r-convex functions if we take $\phi(x) = x$ and

$$\psi(x) = \begin{cases} x^r & \text{if} \quad r \ne 0, \\ \ln x & \text{if} \quad r = 0, \end{cases}$$

- the harmonically convex functions if we choose $\phi(x) = 1/x$ and $\psi(x) = x$,
- the harmonically log-convex functions if we take $\phi(x) = 1/x$ and $\psi(x) = \ln x$,
- the harmonically r-convex functions if we take $\phi(x) = 1/x$ and

$$\psi(x) = \begin{cases} x^r & \text{if} \quad r \ne 0, \\ \ln x & \text{if} \quad r = 0, \end{cases}$$

- the p-convex functions if we take $\phi(x) = x^p$ and $\psi(x) = x$, and
- the multiplicatively convex functions if we take $\phi(x) = \ln x$ and $\psi(x) = \ln x$.

Next, we propose an effective procedure to establish Hermite-Hadamard's inequalities for generalized fractional integrals. Recall that a weighted version of (1.1) was developed by Fejér [14], who showed that if $f : [a, b] \to \mathbb{R}$ is convex and $g : [a, b] \to [0, \infty)$ is integrable, with $\int_a^b g(x)dx > 0$, and symmetric to $(a + b)/2$, i.e., $g(x) = g(a + b - x)$ for all $x \in [a, b]$, then

$$f\left(\frac{a+b}{2}\right) \le \frac{\int_a^b f(x)g(x)dx}{\int_a^b g(x)dx} \le \frac{f(a)+f(b)}{2}. \tag{1.3}$$

It should be suggested that if $g(x) = [(b - x)^{\nu-1} + (x - a)^{\nu-1}]/\Gamma(\nu)$, with $\nu > 0$, then (1.3) yields the following Hermite-Hadamard inequalities for the Riemann-Liouville fractional integrals due to Sarikaya et al. [39, Theorem 2]:

$$f\left(\frac{a+b}{2}\right) \le \frac{\Gamma(\nu+1)}{2(b-a)^\nu}\left[\mathcal{I}^\nu_{a+}[f](b) + \mathcal{I}^\nu_{b-}[f](a)\right] \le \frac{f(a)+f(b)}{2},$$

where

$$\mathcal{I}^\nu_{a+}[f](x) := \frac{1}{\Gamma(\nu)} \int_a^x (x-y)^{\nu-1} f(y)dy, \quad x > a,$$

and

$$\mathcal{I}_{b-}^{\nu}[f](x) := \frac{1}{\Gamma(\nu)} \int_{x}^{b} (y-x)^{\nu-1} f(y)dy, \quad x < b.$$

In this way one may develop further inequalities for various generalized fractional integrals.

Accordingly, the aim of the present paper is to deal with interpolating inequalities of Fejér type for (1.2), which not only provide a natural and intrinsic characterization of the $(\mathcal{M}_\phi, \mathcal{M}_\psi)$-convex functions, but also agree with a wide class of known inequalities of Hermite-Hadamard and Fejér type for different kinds of convexity and fractional integrals existing in the literature as special cases. As applications, we establish some inequalities involving the gamma function and special means and apply them to deduce two well-known asymptotic formulas for the gamma function and a refinement of Young's inequality. The concepts and techniques of this paper may stimulate further research in this fascinating area.

2. Fejér type inequalities for $(\mathcal{M}_\phi, \mathcal{M}_\psi)$-convex functions

In what follows, let $f : I \to J$ be $(\mathcal{M}_\phi, \mathcal{M}_\psi)$-convex, $a, b \in I$ with $a < b$, $\alpha \in (0, 1)$, and let $w_1, w_2 : [0, 1] \to [0, \infty)$ be integrable, with $\int_0^s w_1(t)dt > 0$ and $\int_s^1 w_2(t)dt > 0$ for all $s \in (0, 1)$. For simplicity of notation, we will write $\mathcal{L}(t) = \mathcal{M}_\phi(a, \mathcal{M}_\phi(a, b; \alpha); t)$ and $\mathcal{R}(t) = \mathcal{M}_\phi(b, \mathcal{M}_\phi(a, b; \alpha); t)$ for $t \in [0, 1]$.

Theorem 2.1. *Let $\mathcal{F}, \mathcal{G} : [0, 1] \to \mathbb{R}$ be defined by*

$$\mathcal{F}(t) = \mathcal{M}_\psi(f \circ \mathcal{L}(t), f \circ \mathcal{R}(t); \alpha) \quad \textit{and} \quad \mathcal{G}(t) = \mathcal{M}_\psi(\mathcal{F}(1), \mathcal{F}(0); t),$$

respectively.

1. *The functions $\mathcal{F}$ and $\mathcal{G}$ are $\mathcal{M}_\psi$-convex, increasing on $[0, 1]$, and*

$$\begin{aligned} &\mathcal{F}(0) = \mathcal{G}(0) = f(\mathcal{M}_\phi(a, b; \alpha)), \\ &\mathcal{F}(t) \le \mathcal{G}(t), \quad t \in (0, 1), \\ &\mathcal{F}(1) = \mathcal{G}(1) = \mathcal{M}_\psi(f(a), f(b); \alpha). \end{aligned} \tag{2.1}$$

2. *For $s \in (0, 1]$, define*

$$\mathcal{I}_1(s) = \psi^{-1}\left(\frac{\int_0^s \psi \circ \mathcal{F}(t) w_1(t)dt}{\int_0^s w_1(t)dt}\right) \quad \textit{and} \quad \beta_1(s) = \frac{\int_0^s t w_1(t)dt}{\int_0^s w_1(t)dt}.$$

Then $\mathcal{F} \circ \beta_1$, $\mathcal{I}_1$, and $\mathcal{G} \circ \beta_1$ increase on $(0, 1]$ and satisfy

$$\begin{gathered} \lim_{s\to 0^+} \mathcal{F} \circ \beta_1(s) = \lim_{s\to 0^+} \mathcal{I}_1(s) = \lim_{s\to 0^+} \mathcal{G} \circ \beta_1(s) = f(\mathcal{M}_\phi(a, b; \alpha)), \\ \mathcal{F} \circ \beta_1(s) \le \mathcal{I}_1(s) \le \mathcal{G} \circ \beta_1(s) \le \mathcal{F}(s), \quad s \in (0, 1]. \end{gathered} \tag{2.2}$$

3. *For $s \in [0, 1)$, define*

$$\mathcal{I}_2(s) = \psi^{-1}\left(\frac{\int_s^1 \psi \circ \mathcal{F}(t) w_2(t)dt}{\int_s^1 w_2(t)dt}\right) \quad \textit{and} \quad \beta_2(s) = \frac{\int_s^1 t w_2(t)dt}{\int_s^1 w_2(t)dt}.$$

Then $\mathcal{F} \circ \beta_2$, $\mathcal{I}_2$, and $\mathcal{G} \circ \beta_2$ increase on $[0, 1)$ and satisfy

$$\begin{aligned}\mathcal{G}(s) \leq \mathcal{F} \circ \beta_2(s) \leq \mathcal{I}_2(s) \leq \mathcal{G} \circ \beta_2(s), \quad s \in [0,1), \\ \lim_{s \to 1^-} \mathcal{F} \circ \beta_2(s) = \lim_{s \to 1^-} \mathcal{I}_2(s) = \lim_{s \to 1^-} \mathcal{G} \circ \beta_2(s) = \mathcal{M}_\psi(f(a), f(b); \alpha).\end{aligned} \tag{2.3}$$

If, in addition, $w_1 = w_2$, then $\mathcal{I}_1(1) = \mathcal{I}_2(0)$.

Before proving the theorem, let us mention three lemmas that will be imperative to the proof of our main result.

The first lemma, called Aczél correspondence principle [1] (see also [32, Lemma A.2.2]), reduces the $(\mathcal{M}_\phi, \mathcal{M}_\psi)$-convexity to the usual convexity of a function derived via a change of variable and a change of function.

Lemma 2.2 *(Aczél correspondence principle). If ψ is increasing on J, then f is $(\mathcal{M}_\phi, \mathcal{M}_\psi)$-convex on I if and only if $\psi \circ f \circ \phi^{-1}$ is convex on $\phi(I)$. If ψ is decreasing on J, then f is $(\mathcal{M}_\phi, \mathcal{M}_\psi)$-convex on I if and only if $\psi \circ f \circ \phi^{-1}$ is concave on $\phi(I)$.*

The following lemma provides a useful inequality related to convex functions, which generalizes the result of Hwang, Tseng and Yang given in [19, Lemma].

Lemma 2.3. *Let $H : [A, B] \subset \mathbb{R} \to \mathbb{R}$ be a convex function and let $\beta \in [0, 1]$. Then for any $C, D \in [A, B]$, with $\beta A + (1 - \beta)B = \beta C + (1 - \beta)D$, one has*

$$\beta H(C) + (1 - \beta)H(D) \leq \beta H(A) + (1 - \beta)H(B). \tag{2.4}$$

Proof. If $A = B$, then there is nothing to prove. Otherwise, write

$$C = \frac{B-C}{B-A}A + \frac{C-A}{B-A}B \quad \text{and} \quad D = \frac{B-D}{B-A}A + \frac{D-A}{B-A}B.$$

Using the assumption $\beta A + (1 - \beta)B = \beta C + (1 - \beta)D$, we get

$$\beta\frac{B-C}{B-A} + (1-\beta)\frac{B-D}{B-A} = \beta \quad \text{and} \quad \beta\frac{C-A}{B-A} + (1-\beta)\frac{D-A}{B-A} = 1-\beta.$$

Together with the convexity of H, we obtain

$$\begin{aligned}\beta H(C) + (1-\beta)H(D) &\leq \beta\left[\frac{B-C}{B-A}H(A) + \frac{C-A}{B-A}H(B)\right] + (1-\beta)\left[\frac{B-D}{B-A}H(A) + \frac{D-A}{B-A}H(B)\right] \\ &= \beta H(A) + (1-\beta)H(B),\end{aligned}$$

which is the desired conclusion. □

Lemma 2.4. *Let $P : [0, 1] \to \mathbb{R}$ be continuous and increasing.*

1. *For $s \in (0, 1]$, define*

$$P_1(s) = \frac{\int_0^s P(t)w_1(t)dt}{\int_0^s w_1(t)dt}.$$

Then P_1 is increasing on $(0, 1]$, with

$$\lim_{s \to 0^+} P_1(s) = P(0) \leq P_1(s) \leq P(s), \quad s \in (0, 1]. \tag{2.5}$$

2. *Similarly, for $s \in [0, 1)$, define*

$$P_2(s) = \frac{\int_s^1 P(t)w_2(t)dt}{\int_s^1 w_2(t)dt}.$$

Then P_2 is increasing on $[0, 1)$, with

$$P(s) \le P_2(s) \le P(1) = \lim_{s \to 1^-} P_2(s), \quad s \in [0, 1).$$

The principal significance of the lemma is that it allows one to establish various weighted interpolating inequalities for a continuous and monotonic function.

Proof. We give only the proof for the first assertion of the lemma, as the proof of the other one is similar. To show P_1 is increasing on $(0, 1]$, suppose that $0 < s_1 < s_2 \le 1$. Since P is increasing and w_1 is nonnegative on $[0, 1]$, it follows that

$$\int_0^{s_1} P(u)w_1(u)du \le P(s_1) \int_0^{s_1} w_1(u)du \tag{2.6}$$

and

$$P(s_1) \int_{s_1}^{s_2} w_1(v)dv \le \int_{s_1}^{s_2} P(v)w_1(v)dv. \tag{2.7}$$

Multiplying (2.6) by $\int_{s_1}^{s_2} w_1(v)dv \ge 0$ and using (2.7) leads to

$$\int_0^{s_1} P(u)w_1(u)du \int_{s_1}^{s_2} w_1(v)dv \le \int_{s_1}^{s_2} P(v)w_1(v)dv \int_0^{s_1} w_1(u)du. \tag{2.8}$$

Since

$$\int_{s_1}^{s_2} w_1(v)dv = \int_0^{s_2} w_1(t)dt - \int_0^{s_1} w_1(u)du$$

and

$$\int_{s_1}^{s_2} P(v)w_1(v)dv = \int_0^{s_2} P(t)w_1(t)dt - \int_0^{s_1} P(u)w_1(u)du,$$

it follows from (2.8) that

$$\int_0^{s_1} P(u)w_1(u)du \int_0^{s_2} w_1(t)dt \le \int_0^{s_2} P(t)w_1(t)dt \int_0^{s_1} w_1(u)du. \tag{2.9}$$

Dividing (2.9) by $(\int_0^{s_2} w_1(t)dt)(\int_0^{s_1} w_1(u)du) > 0$, we obtain

$$P_1(s_1) \le P_1(s_2),$$

which implies that P_1 is increasing on $(0, 1]$ as required.

We now show (2.5). Since P is increasing on $[0, 1]$, it follows that

$$P(0) \le P_1(s) \le P(s), \quad s \in (0, 1].$$

The continuity of P thus yields

$$\lim_{s \to 0^+} P_1(s) = P(0),$$

which completes the proof. □

We are now in a position to prove the theorem.

Proof of Theorem 2.1. Since ψ is strictly monotonic, we need to examine two possibilities of ψ. Assume first that ψ is strictly increasing on J. But then, because ψ is also continuous on J, ψ^{-1} is continuous and strictly increasing on $\psi(J)$. Furthermore, by Aczél correspondence principle, $\psi \circ f \circ \phi^{-1}$ is convex on $\phi(I)$.

1. To show $\mathcal{F}$ is $\mathcal{M}_\psi$-convex on $[0, 1]$, it suffices to show that $\psi \circ \mathcal{F}$ is convex on $[0, 1]$. We have

$$\psi \circ \mathcal{F}(t) = \alpha\psi \circ f \circ \phi^{-1}(A(t)) + (1 - \alpha)\psi \circ f \circ \phi^{-1}(B(t)),$$

where

$$A(t) := t\phi(a) + (1 - t)(\alpha\phi(a) + (1 - \alpha)\phi(b)) \tag{2.10}$$

and

$$B(t) := t\phi(b) + (1 - t)(\alpha\phi(a) + (1 - \alpha)\phi(b)). \tag{2.11}$$

Since $\psi \circ f \circ \phi^{-1}$ is convex on $\phi([a, b])$, $A(t)$ and $B(t)$ are linear on $[0, 1]$, it follows that $\psi \circ \mathcal{F}$ is convex on $[0, 1]$ as claimed. The $\mathcal{M}_\psi$-convexity of $\mathcal{G}$ on $[0, 1]$ immediately follows from the definition of $\mathcal{G}$. Next, it is easily seen that

$$\mathcal{F}(0) = \mathcal{G}(0) = f(\mathcal{M}_\phi(a, b; \alpha)),$$
$$\mathcal{F}(1) = \mathcal{G}(1) = \mathcal{M}_\psi(f(a), f(b); \alpha).$$

Now, by the convexity of $\psi \circ f \circ \phi^{-1}$,

$$\psi \circ f \circ \phi^{-1}(A(t)) \le t\psi \circ f(a) + (1 - t)\psi \circ f(\mathcal{M}_\phi(a, b; \alpha))$$

and

$$\psi \circ f \circ \phi^{-1}(B(t)) \le t\psi \circ f(b) + (1 - t)\psi \circ f(\mathcal{M}_\phi(a, b; \alpha)).$$

We thus get

$$\psi \circ \mathcal{F}(t) \le t\psi(\mathcal{M}_\psi(f(a), f(b); \alpha)) + (1 - t)\psi \circ f(\mathcal{M}_\phi(a, b; \alpha)) = \psi \circ \mathcal{G}(t)$$

and, because ψ^{-1} is increasing on $\psi(J)$,

$$\mathcal{F}(t) \le \mathcal{G}(t), \quad t \in [0,1],$$

whence (2.1) is verified.
We proceed to show that $\mathcal{F}$ is increasing. To this end, suppose that $0 < t < r \le 1$. By the $(\mathcal{M}_\phi, \mathcal{M}_\psi)$-convexity of f,

$$\mathcal{F}(t) = \mathcal{M}_\psi(f(\mathcal{L}(t)), f(\mathcal{R}(t)); \alpha) \ge f(\mathcal{M}_\phi(\mathcal{L}(t), \mathcal{R}(t); \alpha)) = \mathcal{F}(0),$$

which yields

$$\psi \circ \mathcal{F}(t) \ge \psi \circ \mathcal{F}(0).$$

Together with the convexity of $\psi \circ \mathcal{F}$, this gives

$$\frac{\psi \circ \mathcal{F}(r) - \psi \circ \mathcal{F}(t)}{r - t} \ge \frac{\psi \circ \mathcal{F}(t) - \psi \circ \mathcal{F}(0)}{t - 0} \ge 0,$$

which implies that $\psi \circ \mathcal{F}$ is increasing on $[0,1]$. Since ψ^{-1} is increasing on $\psi(J)$, we conclude that $\mathcal{F}$ is increasing on $[0,1]$ as desired. Since

$$\psi \circ \mathcal{G}(t) = t[\psi(\mathcal{M}_\psi(f(a), f(b); \alpha)) - \psi \circ f(\mathcal{M}_\phi(a, b; \alpha))] + \psi \circ f(\mathcal{M}_\phi(a, b; \alpha))$$

and

$$\psi(\mathcal{M}_\psi(f(a), f(b); \alpha)) - \psi \circ f(\mathcal{M}_\phi(a, b; \alpha)) \ge 0,$$

it follows that $\psi \circ \mathcal{G}$, and so does $\mathcal{G}$, increases on $[0,1]$.

2. Applying Lemma 2.4 for $P = \psi \circ \mathcal{F}$, we conclude that $\psi \circ \mathcal{I}_1$ is increasing on $(0,1]$, with

$$\lim_{s \to 0^+} \psi \circ \mathcal{I}_1(s) = \psi \circ \mathcal{F}(0) = \psi \circ f(\mathcal{M}_\phi(a, b; \alpha)).$$

Since ψ^{-1} is continuous and strictly increasing on $\psi(J)$, it follows that $\mathcal{I}_1$ is increasing on $(0,1]$ and

$$\lim_{s \to 0^+} \mathcal{I}_1(s) = f(\mathcal{M}_\phi(a, b; \alpha)).$$

Again, by Lemma 2.4, β_1 is increasing on $(0,1]$, with

$$\lim_{s \to 0^+} \beta_1(s) = 0 \le \beta_1(s) \le s, \quad s \in (0,1].$$

Thus, the first part of the theorem asserts that $\mathcal{F} \circ \beta_1$ and $\mathcal{G} \circ \beta_1$ are well-defined, increasing on $(0,1]$ and

$$\lim_{s \to 0^+} \mathcal{F} \circ \beta_1(s) = \lim_{s \to 0^+} \mathcal{G} \circ \beta_1(s) = f(\mathcal{M}_\phi(a, b; \alpha)).$$

Our next goal is to show the inequalities in (2.2). Fix $s \in (0,1]$. Applying Jensen's inequality (see, for example, [35, Chapter 2]) to the convex function $\psi \circ \mathcal{F}$ on the interval $[0,s]$ with respect to the measure $w_1(t)dt$, we obtain

$$\psi \circ \mathcal{F}\left(\frac{\int_0^s tw_1(t)dt}{\int_0^s w_1(t)dt}\right) \leq \frac{\int_0^s \psi \circ \mathcal{F}(t)w_1(t)dt}{\int_0^s w_1(t)dt},$$

which yields

$$\mathcal{F} \circ \beta_1(s) \leq \mathcal{I}_1(s).$$

From what has already been proved, it follows that

$$\frac{\int_0^s \psi \circ \mathcal{F}(t)w_1(t)dt}{\int_0^s w_1(t)dt} \leq \frac{\int_0^s \psi \circ \mathcal{G}(t)w_1(t)dt}{\int_0^s w_1(t)dt} = \psi \circ \mathcal{G} \circ \beta_1(s),$$

which, as the function ψ^{-1} is increasing, implies

$$\mathcal{I}_1(s) \leq \mathcal{G} \circ \beta_1(s).$$

It remains to show

$$\mathcal{G} \circ \beta_1(s) \leq \mathcal{F}(s).$$

We utilize Lemma 2.3, with $H = \psi \circ f \circ \phi^{-1}$, $A = \min\{A(s), B(s)\}$, $B = \max\{A(s), B(s)\}$, $C = \min\{A(\beta_1(s)), B(\beta_1(s))\}$, $D = \max\{A(\beta_1(s)), B(\beta_1(s))\}$, and

$$\beta = \begin{cases} \alpha & \text{if} \quad A(s) \leq B(s), \\ 1-\alpha & \text{if} \quad A(s) > B(s), \end{cases}$$

where $A(\cdot)$ and $B(\cdot)$ are as in (2.10) and (2.11), respectively. To do this, we need to ensure that $C, D \in [A, B]$, with $\beta A + (1-\beta)B = \beta C + (1-\beta)D$. But this immediately follows from the fact that

$$\beta A + (1-\beta)B = \beta C + (1-\beta)D = \alpha\phi(a) + (1-\alpha)\phi(b)$$

and

$$B - A = s|\phi(b) - \phi(a)| \geq \beta_1(s)|\phi(b) - \phi(a)| = D - C.$$

A computation shows that

$$\psi \circ \mathcal{G} \circ \beta_1(s) = \beta H(C) + (1-\beta)H(D)$$

and

$$\psi \circ \mathcal{F}(s) = \beta H(A) + (1-\beta)H(B).$$

On account of (2.4), we have

$$\psi \circ \mathcal{G} \circ \beta_1(s) \leq \psi \circ \mathcal{F}(s),$$

which establishes the desired inequality.

3. We proceed similarly as in the proof of part 2, with β_1 and $(0, s]$, respectively, replaced by β_2 and $[s, 1)$, we can assert that $\mathcal{F} \circ \beta_2$, $\mathcal{I}_2$, and $\mathcal{G} \circ \beta_2$ increase on $[0, 1)$ and (2.3) follows. If $w_1 = w_2$, then $\mathcal{I}_1(1) = \mathcal{I}_2(0)$, which is clear from the definitions of $\mathcal{I}_1$ and $\mathcal{I}_2$.

Finally, the same proof remains valid for the case when ψ is decreasing. $\square$

It is interesting to note that Theorem 2.1 is not merely a consequence of $(\mathcal{M}_\phi, \mathcal{M}_\psi)$-convexity, but actually characterizes it. More precisely, we propose the following corollary whose proof is adapted from Theorem 2.1, Aczél correspondence principle, Jensen's criterion of convexity [32, Theorem 1.1.8], and characterizations of convexity via Hermite-Hadamard's inequality [40, Theorem 1].

Corollary 2.5. *Given a continuous function $f : I \to J$, the following assertions are equivalent:*

(1) *The function f is $(\mathcal{M}_\phi, \mathcal{M}_\psi)$-convex;*
(2) *For all elements $a < b$ of I, the function $\mathcal{F}$, with $\alpha = 1/2$, is increasing on $[0, 1]$;*
(3) *For all elements $a < b$ of I, the function $\mathcal{I}_1$, with $\alpha = 1/2$ and $w_1 = 1$, is increasing on $(0, 1]$;*
(4) *For all elements $a < b$ of I, it holds that*

$$f(\mathcal{M}_\phi(a,b)) \le \psi^{-1}\left(\frac{1}{\phi(b)-\phi(a)}\int_a^b \psi\circ f(x)d\phi(x)\right);$$

(5) *For all elements $a < b$ of I, the function $\mathcal{I}_2$, with $\alpha = 1/2$ and $w_2 = 1$, is increasing on $[0, 1)$;*
(6) *For all elements $a < b$ of I, it holds that*

$$\psi^{-1}\left(\frac{1}{\phi(b)-\phi(a)}\int_a^b \psi\circ f(x)d\phi(x)\right) \le \mathcal{M}_\psi(f(a), f(b));$$

(7) *For all elements $a < b$ of I, the function $\mathcal{G}$, with $\alpha = 1/2$, is increasing on $[0, 1]$.*

Proof. By Theorem 2.1, the implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4), (1) $\Rightarrow$ (5) $\Rightarrow$ (6), and (1) $\Rightarrow$ (7) hold. It remains to show (4) $\Rightarrow$ (1), (6) $\Rightarrow$ (1), and (7) $\Rightarrow$ (1). Without loss of generality, we may assume that ψ is increasing on J. By Aczél correspondence principle, it suffices to show $\psi\circ f\circ\phi^{-1}$ is convex on $\phi(I)$ provided that one of the conditions (4), (6), and (7) occurs. Since ϕ is continuous and strictly monotonic on I, ϕ^{-1} is continuous and strictly monotonic on $\phi(I)$. Now, the continuity of ψ, f, and ϕ^{-1} implies that of $\psi\circ f\circ\phi^{-1}$. Clearly, (7) asserts that $\psi\circ f\circ\phi^{-1}$ is midpoint convex on $\phi(I)$. By Jensen's criterion of convexity, $\psi\circ f\circ\phi^{-1}$ is convex on $\phi(I)$. Finally, the conditions (4) and (6) show that the Hermite-Hadamard inequality verifies for the continuous function $\psi\circ f\circ\phi^{-1}$ on $\phi(I)$. Therefore, it follows from [40, Theorem 1] that $\psi\circ f\circ\phi^{-1}$ is convex on $\phi(I)$, which completes the proof. $\square$

Let us now mention another important consequence of Theorem 2.1. It should be pointed out that a variety of Fejér type inequalities for $(\mathcal{M}_\phi, \mathcal{M}_\psi)$-convex functions can be produced by choosing various weights, w_1 and w_2. For instance, let us choose

$$w_j(t) = (1-\alpha)g_j\circ\mathcal{L}(t) + \alpha g_j\circ\mathcal{R}(t), \quad t \in [0,1],$$

where $g_j : [a, b] \to [0, \infty)$, for $j = 1, 2$, are given to satisfy

$$\frac{1-\alpha}{\alpha}g_1\circ\mathcal{L}(t) = \frac{\alpha}{1-\alpha}g_1\circ\mathcal{R}(t), \quad t \in [0,s] \tag{2.12}$$

and

$$\frac{1-\alpha}{\alpha}g_2\circ\mathcal{L}(t) = \frac{\alpha}{1-\alpha}g_2\circ\mathcal{R}(t), \quad t \in [s,1]. \tag{2.13}$$

Notice that if $\alpha = 1/2$ and $\phi(x) = x$, then the assumptions (2.12) and (2.13) reduce to the ones that g_1 and g_2 are symmetric to $(a+b)/2$.

A computation, using (2.12) and $\mathcal{L}(0) = \mathcal{R}(0)$, forces

$$\int_0^s w_1(t)dt = (1-\alpha)\int_0^s g_1 \circ \mathcal{L}(t)dt + \alpha \int_0^s g_1 \circ \mathcal{R}(t)dt$$
$$= \frac{1}{\phi(b)-\phi(a)} \int_{\mathcal{L}(s)}^{\mathcal{R}(s)} g_1(x)d\phi(x),$$
$$\int_0^s \psi \circ \mathcal{F}(t) w_1(t)dt = (1-\alpha)\int_0^s \psi \circ f \circ \mathcal{L}(t) g_1 \circ \mathcal{L}(t)dt + \alpha \int_0^s \psi \circ f \circ \mathcal{R}(t) g_1 \circ \mathcal{R}(t)dt$$
$$= \frac{1}{\phi(b)-\phi(a)} \int_{\mathcal{L}(s)}^{\mathcal{R}(s)} \psi \circ f(x) g_1(x)d\phi(x),$$

and hence

$$\mathcal{I}_1(s) = \psi^{-1}\left(\frac{\int_{\mathcal{L}(s)}^{\mathcal{R}(s)} \psi \circ f(x) g_1(x)d\phi(x)}{\int_{\mathcal{L}(s)}^{\mathcal{R}(s)} g_1(x)d\phi(x)}\right).$$

Similarly, by (2.13), $\mathcal{L}(1) = a$, and $\mathcal{R}(1) = b$,

$$\mathcal{I}_2(s) = \psi^{-1}\left(\frac{\int_a^{\mathcal{L}(s)} \psi \circ f(x) g_2(x)d\phi(x) + \int_{\mathcal{R}(s)}^b \psi \circ f(x) g_2(x)d\phi(x)}{\int_a^{\mathcal{L}(s)} g_2(x)d\phi(x) + \int_{\mathcal{R}(s)}^b g_2(x)d\phi(x)}\right).$$

Together with the aid of Theorem 2.1, we establish the following corollary.

Corollary 2.6. *Suppose that $g_1, g_2 : [a,b] \to [0,\infty)$ are integrable, with $\int_0^s g_1 \circ \mathcal{L}(t)dt > 0$ and $\int_s^1 g_2 \circ \mathcal{R}(t)dt > 0$ for all $s \in (0,1)$, and satisfy (2.12) and (2.13). Then, for $s \in (0,1)$,*

$$\begin{aligned} f(\mathcal{M}_\phi(a,b;\alpha)) &\le \mathcal{F}\left(\frac{\int_0^s t g_1 \circ \mathcal{L}(t)dt}{\int_0^s g_1 \circ \mathcal{L}(t)dt}\right) \le \psi^{-1}\left(\frac{\int_{\mathcal{L}(s)}^{\mathcal{R}(s)} \psi \circ f(x) g_1(x)d\phi(x)}{\int_{\mathcal{L}(s)}^{\mathcal{R}(s)} g_1(x)d\phi(x)}\right) \\ &\le \mathcal{G}\left(\frac{\int_0^s t g_1 \circ \mathcal{L}(t)dt}{\int_0^s g_1 \circ \mathcal{L}(t)dt}\right) \le \mathcal{F}(s) \le \mathcal{G}(s) \le \mathcal{F}\left(\frac{\int_s^1 t g_2 \circ \mathcal{L}(t)dt}{\int_s^1 g_2 \circ \mathcal{L}(t)dt}\right) \\ &\le \psi^{-1}\left(\frac{\int_a^{\mathcal{L}(s)} \psi \circ f(x) g_2(x)d\phi(x) + \int_{\mathcal{R}(s)}^b \psi \circ f(x) g_2(x)d\phi(x)}{\int_a^{\mathcal{L}(s)} g_2(x)d\phi(x) + \int_{\mathcal{R}(s)}^b g_2(x)d\phi(x)}\right) \\ &\le \mathcal{G}\left(\frac{\int_s^1 t g_2 \circ \mathcal{L}(t)dt}{\int_s^1 g_2 \circ \mathcal{L}(t)dt}\right) \le \mathcal{M}_\psi(f(a), f(b);\alpha). \end{aligned} \tag{2.14}$$

Remark 2.7. It turns out that a great deal of existing inequalities of Hermite-Hadamard and Fejér type for different kinds of convexity can be deduced from Corollary 2.6.

1. Let us first consider $\phi(x) = x$ and $\psi(x) = x$. If $\alpha = 1/2$ and $g_1 = g_2 = 1$, then (2.14) implies

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{2}\left[f\left(\frac{5a+3b}{8}\right) + f\left(\frac{3a+5b}{8}\right)\right] \le \frac{2}{b-a}\int\limits_{(3a+b)/4}^{(a+3b)/4} f(x)dx$$
$$\le \frac{1}{4}\left[\frac{f(a)+f(b)}{2}\right] + \frac{3}{4}f\left(\frac{a+b}{2}\right) \le \frac{1}{2}\left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)\right]$$
$$\le \frac{1}{b-a}\int\limits_a^b f(x)dx \le \frac{1}{2}\left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right] \le \frac{f(a)+f(b)}{2},$$

   which offers a refinement of (1.1) and the ones due to Hammer [17], Dragomir, Milošević and Sándor [11]. In the meanwhile, the general form (2.14) extends, generalizes and refines (1.3) and many known inequalities obtained by Brenner and Alzer [5], Dragomir [10], Lupaş [30], Tseng, Hwang and Dragomir [42,43], Vasić and Lacković [44,45], Yang and Hong [46], Yang and Tseng [47].
2. Moreover, if we choose $\phi(x) = x$ and

$$\psi(x) = \begin{cases} x^r & \text{if} \quad r \neq 0, \\ \ln x & \text{if} \quad r = 0, \end{cases}$$

   then inequalities in (2.14) recover Hermite-Hadamard type inequalities for log-convex functions established by Dragomir and Mond [12] as well as for r-convex functions given by Gill, Pearce and Pečarić [15].
3. Next, if $\phi(x) = x^p$ with $p \neq 0$ and $\psi(x) = x$, then (2.14) derives various inequalities of Hermite-Hadamard and Fejér type for not only harmonically convex functions obtained by Chen and Wu [9], İşcan [20], but also p-convex functions proved by İşcan [22], Kunt and İşcan [29].
4. Finally, if we choose $\phi(x) = 1/x$ and

$$\psi(x) = \begin{cases} x^r & \text{if} \quad r \neq 0, \\ \ln x & \text{if} \quad r = 0, \end{cases}$$

   then (2.14) offers a refinement of the inequalities due to Noor, Noor and Awan applied to harmonically log-convex functions [33] and harmonically r-convex functions [34].

The important point to note here is that for a suitable choice of the weights, g_1 and g_2, Corollary 2.6 provides a wide class of Fejér type inequalities for $(\mathcal{M}_\phi, \mathcal{M}_\psi)$-convex functions utilizing various integral operators and fractional integrals. For this purpose, let us consider a kernel, say $K : \phi(I) \times \phi(I) \to [0, \infty)$, and define

$$\mathcal{K}^{\phi}_{a+}[f](x) := \int\limits_a^x K(\phi(x), \phi(y))f(y)d\phi(y), \quad x > a, \tag{2.15}$$

and

$$\mathcal{K}^{\phi}_{b-}[f](x) := \int\limits_x^b K(\phi(x), \phi(y))f(y)d\phi(y), \quad x < b \tag{2.16}$$

as long as the integrals exist and are finite.

Remark 2.8. We emphasize that our definition agrees with many known fractional integrals existing in the literature as special cases.

1. Let us first consider

$$K(u,v) = \frac{1}{\Gamma(\nu)}|u-v|^{\nu-1}, \quad u,v \in \phi(I),$$

where $\nu > 0$. Then integral operators (2.15) and (2.16) become the fractional integrals of a function with respect to another function known from [28]:

$$\mathcal{I}^{\nu,\phi}_{a+}[f](x) := \frac{1}{\Gamma(\nu)} \int\limits_a^x |\phi(x)-\phi(y)|^{\nu-1} f(y)d\phi(y), \quad x > a,$$

and

$$\mathcal{I}^{\nu,\phi}_{b-}[f](x) := \frac{1}{\Gamma(\nu)} \int\limits_x^b |\phi(x)-\phi(y)|^{\nu-1} f(y)d\phi(y), \quad x < b.$$

These operators include the Riemann-Liouville [38], Hadamard [38], and Katugampola [27] fractional integrals, which correspond to the choice $\phi(x) = x$, $\phi(x) = \ln x$, and $\phi(x) = x^\rho/\rho$ with $\rho > 0$, respectively.

2. Next, if $\phi(x) = x$ and

$$K(x,y) = \frac{1}{\nu}\exp\left(-\frac{1-\nu}{\nu}|x-y|\right), \quad x,y \in [a,b],$$

with $0 < \nu < 1$, we obtain the following fractional integrals due to Ahmad et al. [2]:

$$\mathcal{J}^{\nu}_{a+}[f](x) := \frac{1}{\nu} \int\limits_a^x \exp\left(-\frac{1-\nu}{\nu}(x-y)\right) f(y)dy, \quad x > a,$$

and

$$\mathcal{J}^{\nu}_{b-}[f](x) := \frac{1}{\nu} \int\limits_x^b \exp\left(-\frac{1-\nu}{\nu}(y-x)\right) f(y)dy, \quad x < b.$$

3. We now consider $\phi(x) = x$ and

$$K(x,y) = \frac{1}{\Gamma(\nu)}|x-y|^{\nu-1}\ln^\mu \frac{\delta}{|x-y|}, \quad x,y \in [a,b],$$

where $\nu > 0$, $\mu \geq 0$, and $\delta > b-a$. Then (2.15) and (2.16) reduce to the operators with power-logarithmic kernels given in [38]:

$$\mathcal{I}^{\nu,\mu}_{a+}[f](x) := \frac{1}{\Gamma(\nu)} \int\limits_a^x (x-y)^{\nu-1} \ln^\mu\left(\frac{\delta}{x-y}\right) f(y)dy, \quad x > a,$$

and

$$\mathcal{I}_{b-}^{\nu,\mu}[f](x) := \frac{1}{\Gamma(\nu)} \int_x^b (y-x)^{\nu-1} \ln^{\mu}\left(\frac{\delta}{y-x}\right) f(y)dy, \quad x < b.$$

4. Finally, let us take $\phi(x) = x$ for $x \in [0,1]$, and let $\nu > 0$. If we choose

$$K(x,y) = F\left(\frac{|x-y|}{\nu}\right), \quad x, y \in [0,1],$$

where

$$F(z) := \left(\int_0^{\infty} \frac{z^{t-1}}{\Gamma(t)} dt\right) e^{-z}, \quad z > 0,$$

we recapture the fractional integrals of type (I) given in [24]:

$$\mathcal{H}_0^{\nu}[f](x) := \int_0^x F\left(\frac{x-y}{\nu}\right) f(y)dy, \quad x \in [0,1],$$

and

$$\mathcal{H}_1^{\nu}[f](x) := \int_x^1 F\left(\frac{y-x}{\nu}\right) f(y)dy, \quad x \in [0,1].$$

Instead, if we choose

$$K(x,y) = \frac{1}{\nu} E_1\left(\frac{|x-y|}{\nu}\right), \quad x, y \in [0,1],$$

where

$$E_1(z) := \int_z^{\infty} \frac{e^{-t}}{t} dt, \quad z > 0,$$

we deduce the fractional integrals of type (II) given in [24]:

$$\mathcal{S}_0^{\nu}[f](x) := \frac{1}{\nu} \int_0^x E_1\left(\frac{x-y}{\nu}\right) f(y)dy, \quad x \in [0,1],$$

and

$$\mathcal{S}_1^{\nu}[f](x) := \frac{1}{\nu} \int_x^1 E_1\left(\frac{y-x}{\nu}\right) f(y)dy, \quad x \in [0,1].$$

For a deeper discussion of these and more general operators we refer the reader to [2], [24], [27], [28], and [38].

In Corollary 2.6, for each $s \in [0,1]$, let us choose

$$g_1(x) = [K(\phi \circ \mathcal{R}(s), \phi(x)) + K(\phi \circ \mathcal{L}(s), \phi(x))]\, h_1(x), \quad x \in [\mathcal{L}(s), \mathcal{R}(s)]$$

and

$$g_2(x) = \begin{cases} K(\phi \circ \mathcal{L}(s), \phi(x)) h_2(x) & \text{if} \quad x \in [a, \mathcal{L}(s)], \\ K(\phi \circ \mathcal{R}(s), \phi(x)) h_2(x) & \text{if} \quad x \in [\mathcal{R}(s), b], \end{cases}$$

where $h_j : [a,b] \to [0,\infty)$, for $j = 1,2$, are given in such a way that the assumptions (2.12) and (2.13) are guaranteed, i.e.,

$$\begin{aligned} &\frac{1-\alpha}{\alpha}[K(\phi \circ \mathcal{R}(s), \phi \circ \mathcal{L}(t)) + K(\phi \circ \mathcal{L}(s), \phi \circ \mathcal{L}(t))] h_1 \circ \mathcal{L}(t) \\ =&\frac{\alpha}{1-\alpha}[K(\phi \circ \mathcal{R}(s), \phi \circ \mathcal{R}(t)) + K(\phi \circ \mathcal{L}(s), \phi \circ \mathcal{R}(t))] h_1 \circ \mathcal{R}(t), \quad t \in [0,s], \end{aligned} \tag{2.17}$$

and

$$\begin{aligned} &\frac{1-\alpha}{\alpha} K(\phi \circ \mathcal{L}(s), \phi \circ \mathcal{L}(t)) h_2 \circ \mathcal{L}(t) \\ =&\frac{\alpha}{1-\alpha} K(\phi \circ \mathcal{R}(s), \phi \circ \mathcal{R}(t)) h_2 \circ \mathcal{R}(t), \quad t \in [s,1]. \end{aligned} \tag{2.18}$$

In order to simplify these assumptions, it is necessary to put some restrictions on α and K. Let us first take $\alpha = 1/2$ and investigate a class of kernels, K, of the form

$$K(u,v) = k(|u-v|), \quad u, v \in \phi(I), \tag{2.19}$$

where $k : [0,\infty) \to [0,\infty)$ is given so that the integral operators (2.15) and (2.16) are well-defined. As one can see that the kernels used to define the fractional integrals indicated in Remark 2.8 are all of the form (2.19).

We check at once that

$$|\phi \circ \mathcal{L}(s) - \phi \circ \mathcal{L}(t)| = |\phi \circ \mathcal{R}(s) - \phi \circ \mathcal{R}(t)| = \frac{1}{2}|s-t||\phi(b) - \phi(a)|$$

and

$$|\phi \circ \mathcal{L}(s) - \phi \circ \mathcal{R}(t)| = |\phi \circ \mathcal{R}(s) - \phi \circ \mathcal{L}(t)| = \frac{1}{2}(s+t)|\phi(b) - \phi(a)|.$$

Consequently, (2.17) and (2.18) reduce to

$$h_1 \circ \mathcal{L}(t) = h_1 \circ \mathcal{R}(t), \quad t \in [0,s]$$

and

$$h_2 \circ \mathcal{L}(t) = h_2 \circ \mathcal{R}(t), \quad t \in [s,1],$$

respectively. This enables one to take

$$h(x) = \begin{cases} h_1(x) & \text{if} \quad x \in [\mathcal{L}(s), \mathcal{R}(s)], \\ h_2(x) & \text{otherwise.} \end{cases}$$

Put this way, we have

$$\beta_1(s) = \frac{\mathcal{K}^{\phi}_{\mathcal{L}(s)+}[\varphi h](\mathcal{R}(s)) + \mathcal{K}^{\phi}_{\mathcal{R}(s)-}[\varphi h](\mathcal{L}(s))}{\mathcal{K}^{\phi}_{\mathcal{L}(s)+}[h](\mathcal{R}(s)) + \mathcal{K}^{\phi}_{\mathcal{R}(s)-}[h](\mathcal{L}(s))}$$

and

$$\beta_2(s) = \frac{\mathcal{K}^{\phi}_{a+}[\varphi h](\mathcal{L}(s)) + \mathcal{K}^{\phi}_{b-}[\varphi h](\mathcal{R}(s))}{\mathcal{K}^{\phi}_{a+}[h](\mathcal{L}(s)) + \mathcal{K}^{\phi}_{b-}[h](\mathcal{R}(s))},$$

where

$$\varphi(x) := \left| \frac{\phi(a) + \phi(b) - 2\phi(x)}{\phi(b) - \phi(a)} \right|, \quad x \in I. \tag{2.20}$$

In summary, we get the following corollary.

Corollary 2.9. *Let φ be given by (2.20) and $\alpha = 1/2$. Suppose that $K : \phi(I) \times \phi(I) \to [0, \infty)$ is of the form (2.19) and $h : [a, b] \to [0, \infty)$ is integrable such that*

$$\frac{\mathcal{K}^{\phi}_{\mathcal{L}(s)+}[h](\mathcal{R}(s)) + \mathcal{K}^{\phi}_{\mathcal{R}(s)-}[h](\mathcal{L}(s))}{\phi(b) - \phi(a)} > 0 \quad \text{and} \quad \frac{\mathcal{K}^{\phi}_{a+}[h](\mathcal{L}(s)) + \mathcal{K}^{\phi}_{b-}[h](\mathcal{R}(s))}{\phi(b) - \phi(a)} > 0$$

for all $s \in (0, 1)$. If

$$h \circ \mathcal{L}(t) = h \circ \mathcal{R}(t), \quad t \in [0, 1], \tag{2.21}$$

then

$$\begin{aligned}
f(\mathcal{M}_\phi(a, b)) &\le \mathcal{F}\left(\frac{\mathcal{K}^{\phi}_{\mathcal{L}(s)+}[\varphi h](\mathcal{R}(s)) + \mathcal{K}^{\phi}_{\mathcal{R}(s)-}[\varphi h](\mathcal{L}(s))}{\mathcal{K}^{\phi}_{\mathcal{L}(s)+}[h](\mathcal{R}(s)) + \mathcal{K}^{\phi}_{\mathcal{R}(s)-}[h](\mathcal{L}(s))} \right) \\
&\le \psi^{-1}\left(\frac{\mathcal{K}^{\phi}_{\mathcal{L}(s)+}[(\psi \circ f)h](\mathcal{R}(s)) + \mathcal{K}^{\phi}_{\mathcal{R}(s)-}[(\psi \circ f)h](\mathcal{L}(s))}{\mathcal{K}^{\phi}_{\mathcal{L}(s)+}[h](\mathcal{R}(s)) + \mathcal{K}^{\phi}_{\mathcal{R}(s)-}[h](\mathcal{L}(s))} \right) \\
&\le \mathcal{G}\left(\frac{\mathcal{K}^{\phi}_{\mathcal{L}(s)+}[\varphi h](\mathcal{R}(s)) + \mathcal{K}^{\phi}_{\mathcal{R}(s)-}[\varphi h](\mathcal{L}(s))}{\mathcal{K}^{\phi}_{\mathcal{L}(s)+}[h](\mathcal{R}(s)) + \mathcal{K}^{\phi}_{\mathcal{R}(s)-}[h](\mathcal{L}(s))} \right) \le \mathcal{F}(s) \\
&\le \mathcal{G}(s) \le \mathcal{F}\left(\frac{\mathcal{K}^{\phi}_{a+}[\varphi h](\mathcal{L}(s)) + \mathcal{K}^{\phi}_{b-}[\varphi h](\mathcal{R}(s))}{\mathcal{K}^{\phi}_{a+}[h](\mathcal{L}(s)) + \mathcal{K}^{\phi}_{b-}[h](\mathcal{R}(s))} \right) \\
&\le \psi^{-1}\left(\frac{\mathcal{K}^{\phi}_{a+}[(\psi \circ f)h](\mathcal{L}(s)) + \mathcal{K}^{\phi}_{b-}[(\psi \circ f)h](\mathcal{R}(s))}{\mathcal{K}^{\phi}_{a+}[h](\mathcal{L}(s)) + \mathcal{K}^{\phi}_{b-}[h](\mathcal{R}(s))} \right) \\
&\le \mathcal{M}_\psi(f(a), f(b)).
\end{aligned} \tag{2.22}$$

In particular, one has

$$
\begin{aligned}
f(\mathcal{M}_\phi(a,b)) &\le \mathcal{F}\left(\frac{\mathcal{K}^\phi_{a+}[\varphi h](b)+\mathcal{K}^\phi_{b-}[\varphi h](a)}{\mathcal{K}^\phi_{a+}[h](b)+\mathcal{K}^\phi_{b-}[h](a)}\right)\\
&\le \psi^{-1}\left(\frac{\mathcal{K}^\phi_{a+}[(\psi\circ f)h](b)+\mathcal{K}^\phi_{b-}[(\psi\circ f)h](a)}{\mathcal{K}^\phi_{a+}[h](b)+\mathcal{K}^\phi_{b-}[h](a)}\right)\\
&\le \mathcal{G}\left(\frac{\mathcal{K}^\phi_{a+}[\varphi h](b)+\mathcal{K}^\phi_{b-}[\varphi h](a)}{\mathcal{K}^\phi_{a+}[h](b)+\mathcal{K}^\phi_{b-}[h](a)}\right)\le \mathcal{M}_\psi(f(a),f(b))
\end{aligned}
\tag{2.23}
$$

and

$$
\begin{aligned}
f(\mathcal{M}_\phi(a,b)) &\le \mathcal{F}\left(\frac{\mathcal{K}^\phi_{a+}[\varphi h](\mathcal{M}_\phi(a,b))+\mathcal{K}^\phi_{b-}[\varphi h](\mathcal{M}_\phi(a,b))}{\mathcal{K}^\phi_{a+}[h](\mathcal{M}_\phi(a,b))+\mathcal{K}^\phi_{b-}[h](\mathcal{M}_\phi(a,b))}\right)\\
&\le \psi^{-1}\left(\frac{\mathcal{K}^\phi_{a+}[(\psi\circ f)h](\mathcal{M}_\phi(a,b))+\mathcal{K}^\phi_{b-}[(\psi\circ f)h](\mathcal{M}_\phi(a,b))}{\mathcal{K}^\phi_{a+}[h](\mathcal{M}_\phi(a,b))+\mathcal{K}^\phi_{b-}[h](\mathcal{M}_\phi(a,b))}\right)\\
&\le \mathcal{G}\left(\frac{\mathcal{K}^\phi_{a+}[\varphi h](\mathcal{M}_\phi(a,b))+\mathcal{K}^\phi_{b-}[\varphi h](\mathcal{M}_\phi(a,b))}{\mathcal{K}^\phi_{a+}[h](\mathcal{M}_\phi(a,b))+\mathcal{K}^\phi_{b-}[h](\mathcal{M}_\phi(a,b))}\right)\le \mathcal{M}_\psi(f(a),f(b)).
\end{aligned}
\tag{2.24}
$$

Remark 2.10. Through a proper choice of the functions ψ, ϕ, and K such as being indicated in Remark 2.8, (2.23) can be regarded as a generalization and refinement of several results obtained recently by Ahmad et al. [2], Budak [6], Chen and Katugampola [8], İşcan [21], İşcan and Wu [23], Jleli, O'Regan and Samet [25], Jleli and Samet [26], Peng, Wei and Wang [36], and Sarikaya et al. [39], while (2.22) and (2.24) are essentially new. As an illustration of our new results, let us briefly mention a special case of (2.24) when $\psi(x)=x$, $\phi(x)=x$, and

$$
K(x,y)=\frac{1}{\Gamma(\nu)}|x-y|^{\nu-1}\ln^\mu\frac{\delta}{|x-y|},\quad x,y\in[a,b],
$$

where $\nu>0$, $\mu\ge 0$, and $\delta>b-a$. Then the assumption (2.21) is nothing but the statement that h is symmetric to $(a+b)/2$. The function φ now becomes

$$
\varphi(x)=\left|\frac{a+b-2x}{b-a}\right|,\quad x\in[a,b],
$$

and so

$$
\frac{\mathcal{K}^\phi_{a+}[\varphi h](\mathcal{M}_\phi(a,b))+\mathcal{K}^\phi_{b-}[\varphi h](\mathcal{M}_\phi(a,b))}{\mathcal{K}^\phi_{a+}[h](\mathcal{M}_\phi(a,b))+\mathcal{K}^\phi_{b-}[h](\mathcal{M}_\phi(a,b))}=\frac{2\nu}{b-a}\frac{\mathcal{I}^{\nu+1,\mu}_{a+}[h](\frac{a+b}{2})}{\mathcal{I}^{\nu,\mu}_{a+}[h](\frac{a+b}{2})}=:\gamma.
$$

Taking into account (2.24), we obtain the following Fejér type inequalities for convex functions utilizing the operators with power-logarithmic kernels:

$$
\begin{aligned}
f\left(\frac{a+b}{2}\right) &\le \frac{1}{2}\left[f\left(\frac{1+\gamma}{2}a+\frac{1-\gamma}{2}b\right)+f\left(\frac{1-\gamma}{2}a+\frac{1+\gamma}{2}b\right)\right]\\
&\le \frac{\mathcal{I}^{\nu,\mu}_{a+}[fh](\frac{a+b}{2})+\mathcal{I}^{\nu,\mu}_{b-}[fh](\frac{a+b}{2})}{\mathcal{I}^{\nu,\mu}_{a+}[h](\frac{a+b}{2})+\mathcal{I}^{\nu,\mu}_{b-}[h](\frac{a+b}{2})}\\
&\le \gamma\left[\frac{f(a)+f(b)}{2}\right]+(1-\gamma)f\left(\frac{a+b}{2}\right)\le\frac{f(a)+f(b)}{2}.
\end{aligned}
\tag{2.25}
$$

In particular, on taking $h=1$, one has

$$\gamma = \frac{2\delta}{b-a}\left(\frac{\nu}{\nu+1}\right)^{\mu+1}\frac{\Gamma\left(\mu+1,(\nu+1)\ln\frac{2\delta}{b-a}\right)}{\Gamma\left(\mu+1,\nu\ln\frac{2\delta}{b-a}\right)}$$

and

$$\mathcal{I}_{a+}^{\nu;\mu}[1]\left(\frac{a+b}{2}\right)+\mathcal{I}_{b-}^{\nu;\mu}[1]\left(\frac{a+b}{2}\right)=2\frac{\delta^{\nu}\Gamma\left(\mu+1,\nu\ln\frac{2\delta}{b-a}\right)}{\nu^{\mu}\Gamma(\nu+1)},$$

where $\Gamma(\mu+1,x)$ denotes the upper incomplete gamma function defined as

$$\Gamma(\mu+1,x)=\int_x^{\infty}y^{\mu}e^{-y}dy.$$

By letting $\mu=0$, then (2.25) reduces to a new Fejér type inequality for convex functions via the Riemann-Liouville fractional integrals. Accordingly, our viewpoint sheds some new light on this field.

Remark 2.11. We have been working under the assumption that K is of the form (2.19) and $\alpha=1/2$. We will now show how to dispense with this assumption.

Notice first that (2.19) is assumed in order to simplify (2.17) and (2.18), so the kernel K does not necessarily have this property. Instead, we may assume that

$$K(u,\phi\circ\mathcal{L}(t))=K(u,\phi\circ\mathcal{R}(t)),\quad u\in\phi(I),t\in[0,1]. \tag{2.26}$$

If, in addition, $\alpha=1/2$ and h is subject to the condition (2.21), then (2.23) and (2.24) are still true.

Next, for $\alpha\in(0,1)$, it is required that

$$\frac{1-\alpha}{\alpha}h\circ\mathcal{L}(t)=\frac{\alpha}{1-\alpha}h\circ\mathcal{R}(t),\quad t\in[0,1]. \tag{2.27}$$

Set

$$\varphi_\alpha(x):=\begin{cases}1-\frac{1}{1-\alpha}\frac{\phi(x)-\phi(a)}{\phi(b)-\phi(a)} & \text{if}\quad x\in[a,\mathcal{M}_\phi(a,b;\alpha)],\\ 1-\frac{1}{\alpha}\frac{\phi(b)-\phi(x)}{\phi(b)-\phi(a)} & \text{if}\quad x\in[\mathcal{M}_\phi(a,b;\alpha),b].\end{cases}$$

Then, under the assumptions (2.26) and (2.27), we have

$$\begin{aligned}f(\mathcal{M}_\phi(a,b;\alpha)) &\le \mathcal{F}\left(\frac{\mathcal{K}_{a+}^{\phi}[\varphi_\alpha h](b)+\mathcal{K}_{b-}^{\phi}[\varphi_\alpha h](a)}{\mathcal{K}_{a+}^{\phi}[h](b)+\mathcal{K}_{b-}^{\phi}[h](a)}\right)\\ &\le \psi^{-1}\left(\frac{\mathcal{K}_{a+}^{\phi}[(\psi\circ f)h](b)+\mathcal{K}_{b-}^{\phi}[(\psi\circ f)h](a)}{\mathcal{K}_{a+}^{\phi}[h](b)+\mathcal{K}_{b-}^{\phi}[h](a)}\right)\\ &\le \mathcal{G}\left(\frac{\mathcal{K}_{a+}^{\phi}[\varphi_\alpha h](b)+\mathcal{K}_{b-}^{\phi}[\varphi_\alpha h](a)}{\mathcal{K}_{a+}^{\phi}[h](b)+\mathcal{K}_{b-}^{\phi}[h](a)}\right)\le\mathcal{M}_\psi(f(a),f(b);\alpha)\end{aligned} \tag{2.28}$$

and

$$\begin{aligned} f(\mathcal{M}_\phi(a,b;\alpha)) &\le \mathcal{F}\left(\frac{\mathcal{K}^\phi_{a+}[\varphi_\alpha h](\mathcal{M}_\phi(a,b;\alpha)) + \mathcal{K}^\phi_{b-}[\varphi_\alpha h](\mathcal{M}_\phi(a,b;\alpha))}{\mathcal{K}^\phi_{a+}[h](\mathcal{M}_\phi(a,b;\alpha)) + \mathcal{K}^\phi_{b-}[h](\mathcal{M}_\phi(a,b;\alpha))}\right) \\ &\le \psi^{-1}\left(\frac{\mathcal{K}^\phi_{a+}[(\psi\circ f)h](\mathcal{M}_\phi(a,b;\alpha)) + \mathcal{K}^\phi_{b-}[(\psi\circ f)h](\mathcal{M}_\phi(a,b;\alpha))}{\mathcal{K}^\phi_{a+}[h](\mathcal{M}_\phi(a,b;\alpha)) + \mathcal{K}^\phi_{b-}[h](\mathcal{M}_\phi(a,b;\alpha))}\right) \\ &\le \mathcal{G}\left(\frac{\mathcal{K}^\phi_{a+}[\varphi_\alpha h](\mathcal{M}_\phi(a,b;\alpha)) + \mathcal{K}^\phi_{b-}[\varphi_\alpha h](\mathcal{M}_\phi(a,b;\alpha))}{\mathcal{K}^\phi_{a+}[h](\mathcal{M}_\phi(a,b;\alpha)) + \mathcal{K}^\phi_{b-}[h](\mathcal{M}_\phi(a,b;\alpha))}\right) \\ &\le \mathcal{M}_\psi(f(a),f(b);\alpha). \end{aligned} \tag{2.29}$$

Instead of using the conditions (2.26) and (2.27), we also obtain (2.28) under the more general assumption that

$$\begin{aligned} &\frac{1-\alpha}{\alpha}[K(\phi(b),\phi\circ\mathcal{L}(t)) + K(\phi(a),\phi\circ\mathcal{L}(t))]h\circ\mathcal{L}(t) \\ =&\frac{\alpha}{1-\alpha}[K(\phi(b),\phi\circ\mathcal{R}(t)) + K(\phi(a),\phi\circ\mathcal{R}(t))]h\circ\mathcal{R}(t), \quad t\in[0,1]. \end{aligned}$$

Similarly, (2.29) is still true if it is just assumed that

$$\begin{aligned} &\frac{1-\alpha}{\alpha}K(\phi(\mathcal{M}_\phi(a,b;\alpha)),\phi\circ\mathcal{L}(t))h\circ\mathcal{L}(t) \\ =&\frac{\alpha}{1-\alpha}K(\phi(\mathcal{M}_\phi(a,b;\alpha)),\phi\circ\mathcal{R}(t))h\circ\mathcal{R}(t), \quad t\in[0,1]. \end{aligned}$$

Remark 2.12. All inequalities indicated hold in the reversed direction if f is $(\mathcal{M}_\phi, \mathcal{M}_\psi)$-concave. Moreover, it is possible to establish other versions of Fejér type inequalities assuming f to be Lipschitz, differentiable with f' either bounded or $(\mathcal{M}_\phi, \mathcal{M}_\psi)$-convex, or twice differentiable with f'' bounded, but we will not develop this point here. However, the techniques of the present paper may motivate further research in this fascinating area.

3. Applications to inequalities involving the gamma function and special means

In this section, we explore two more applications of the main theorem to inequalities involving the gamma function and special means.

We first state some inequalities involving the gamma function, Γ, defined by the integral representation

$$\Gamma(z) = \int_0^\infty y^{z-1}e^{-y}dy, \quad \Re(z) > 0.$$

It is well-known that the second derivative of the function $z \mapsto \ln\Gamma(z)$ can be expressed in terms of the series (see, for instance, [31])

$$\frac{d^2}{dz^2}\ln\Gamma(z) = \frac{1}{z^2} + \frac{1}{(z+1)^2} + \frac{1}{(z+2)^2} + \cdots, \quad z \ne 0,-1,-2,...,$$

so the gamma function is log-convex on $(0,\infty)$. Fix $a > 0$. Applying Theorem 2.1 for $f(x) = \Gamma(x)$, $\phi(x) = x$, $\psi(x) = \ln x$, $b = a+1$, and $\alpha = 1/2$, using Raabe's formula [37],

$$\int_a^{a+1} \ln\Gamma(x)dx = \ln\sqrt{2\pi} + a\ln a - a,$$

we obtain the following result.

Corollary 3.1.

1. *The functions*

$$\mathcal{F}_1(t) := \sqrt{\Gamma\left(a+\frac{1-t}{2}\right)\Gamma\left(a+\frac{1+t}{2}\right)}$$

and

$$\mathcal{G}_1(t) := \left[\sqrt{\Gamma(a)\Gamma(a+1)}\right]^t \left[\Gamma\left(a+\frac{1}{2}\right)\right]^{1-t}$$

are log-convex and increasing on $[0,1]$, *with*

$$\Gamma\left(a+\frac{1}{2}\right) \le \mathcal{F}_1(t) \le \mathcal{G}_1(t) \le \sqrt{\Gamma(a)\Gamma(a+1)}$$

for all $t \in [0,1]$.

2. *The function*

$$\mathcal{P}_1(s) := \exp\left(\frac{1}{s}\int\limits_{a+(1-s)/2}^{a+(1+s)/2} \ln\Gamma(x)dx\right)$$

is increasing on $(0,1]$, *with*

$$\lim_{s\to 0^+} \mathcal{P}_1(s) = \Gamma\left(a+\frac{1}{2}\right), \quad \mathcal{P}_1(1) = \sqrt{2\pi}\left(\frac{a}{e}\right)^a,$$

and

$$\mathcal{F}_1(s/2) \le \mathcal{P}_1(s) \le \mathcal{G}_1(s/2) \le \mathcal{F}_1(s), \quad s \in (0,1].$$

3. *The function*

$$\mathcal{Q}_1(s) := \exp\left(\frac{1}{1-s}\left[\int\limits_a^{a+(1-s)/2} \ln\Gamma(x)dx + \int\limits_{a+(1+s)/2}^{a+1} \ln\Gamma(x)dx\right]\right)$$

is increasing on $[0,1)$, *with*

$$\mathcal{Q}_1(0) = \sqrt{2\pi}\left(\frac{a}{e}\right)^a, \quad \lim_{s\to 1^-} \mathcal{Q}_1(s) = \sqrt{\Gamma(a)\Gamma(a+1)},$$

and

$$\mathcal{G}_1(s) \le \mathcal{F}_1\left(\frac{1+s}{2}\right) \le \mathcal{Q}_1(s) \le \mathcal{G}_1\left(\frac{1+s}{2}\right), \quad s \in [0,1).$$

4. *In particular,*

$$\Gamma\left(a+\frac{1}{2}\right) \le \mathcal{F}_1(1/2) \le \sqrt{2\pi}\left(\frac{a}{e}\right)^a \le \mathcal{G}_1(1/2) \le \sqrt{\Gamma(a)\Gamma(a+1)}.$$

Consequently,

$$\frac{\Gamma(a+1/2)}{\Gamma(a)a^{1/2}} \le \frac{\sqrt{2\pi a}\,(a/e)^a}{\Gamma(a+1)} \le \sqrt{\frac{\Gamma(a+1/2)}{\Gamma(a)a^{1/2}}} \le 1 \le \sqrt{\frac{\Gamma(a+1/2)}{\Gamma(a)a^{1/2}}}\sqrt[4]{1+\frac{1}{2a}},$$

which deduces

$$\lim_{a\to\infty} \frac{\Gamma(a+1/2)}{\Gamma(a)a^{1/2}} = 1$$

and the famous Stirling's asymptotic formula for the gamma function

$$\Gamma(a+1) \approx \sqrt{2\pi a}\left(\frac{a}{e}\right)^a \quad \textit{as} \quad a\to\infty.$$

Remark 3.2. As noted by Trif [41], the gamma function is also multiplicatively convex on $[x_0,\infty)$, where x_0 is the unique positive solution of the equation

$$x\frac{d^2}{dx^2}\ln\Gamma(x) + \frac{d}{dx}\ln\Gamma(x) = 0.$$

Accordingly, one may develop further inequalities related to the gamma function by applying Theorem 2.1 for $f(x)=\Gamma(x)$, $\phi(x)=\ln x$, and $\psi(x)=\ln x$ on $[a,b]\subset[x_0,\infty)$.

Similar considerations may apply to other special functions provided that these functions are $(\mathcal{M}_\phi,\mathcal{M}_\psi)$-convex. We refer the reader to [32, Appendix A] for various examples of such a function.

The remainder of the paper is devoted to an application to special means. Let $0<a<b$ and $\alpha\in(0,1)$. Recall that the weighted arithmetic, weighted geometric and logarithmic means of a and b are defined by

$$A_\alpha(a,b) := \alpha a + (1-\alpha)b, \quad G_\alpha(a,b) := a^\alpha b^{1-\alpha}, \quad \text{and} \quad L(a,b) := \frac{b-a}{\ln b - \ln a},$$

respectively. If we choose in Theorem 2.1, $\phi(x)=\ln x$, $\psi(x)=x$, $f(x)=x$ for $x\in[a,b]$, and $w_1(t)=w_2(t)=1$ for $t\in[0,1]$, we obtain the following corollary.

Corollary 3.3.

1. *The functions*

$$\mathcal{F}_2(s) := G_\alpha^{1-s}(a,b)A_\alpha(a^s,b^s) \quad \textit{and} \quad \mathcal{G}_2(s) := A_s(A_\alpha(a,b),G_\alpha(a,b))$$

are convex and increasing on $[0,1]$*, with*

$$\mathcal{F}_2(0)=\mathcal{G}_2(0)=G_\alpha(a,b) \quad \textit{and} \quad \mathcal{F}_2(1)=\mathcal{G}_2(1)=A_\alpha(a,b).$$

2. *The functions*

$$\mathcal{P}_2(s) := A_\alpha(L(a^sG_\alpha^{1-s}(a,b),G_\alpha(a,b)),L(b^sG_\alpha^{1-s}(a,b),G_\alpha(a,b)))$$

and

$$\mathcal{Q}_2(s) := A_\alpha(L(a, a^s G_\alpha^{1-s}(a,b)), L(b, b^s G_\alpha^{1-s}(a,b)))$$

are increasing on $[0,1]$*, with*

$$\mathcal{P}_2(0) = G_\alpha(a,b), \quad \mathcal{P}_2(1) = \mathcal{Q}_2(0), \quad \text{and} \quad \mathcal{Q}_2(1) = A_\alpha(a,b).$$

3. *Furthermore, for each* $s \in [0,1]$*, we have*

$$\begin{aligned} G_\alpha(a,b) &\leq \mathcal{F}_2(s/2) \leq \mathcal{P}_2(s) \leq \mathcal{G}_2(s/2) \leq \mathcal{F}_2(s) \leq \mathcal{G}_2(s) \\ &\leq \mathcal{F}_2\left(\frac{1+s}{2}\right) \leq \mathcal{Q}_2(s) \leq \mathcal{G}_2\left(\frac{1+s}{2}\right) \leq A_\alpha(a,b). \end{aligned} \tag{3.1}$$

Remark 3.4. Notice that (3.1) provides a refinement of the well-known Young inequality, which is equivalent to $G_\alpha(a,b) \leq A_\alpha(a,b)$. Likewise, one can derive a refinement of the inequality $H_\alpha(a,b) \leq G_\alpha(a,b)$ by considering $\phi(x) = 1/x$, $\psi(x) = \ln x$, and $f(x) = x$, where $H_\alpha(a,b)$ is the weighted harmonic mean of a and b defined by

$$H_\alpha(a,b) := \frac{1}{\frac{\alpha}{a} + \frac{1-\alpha}{b}}.$$

Also, with an appropriate selection of the functions ϕ, ψ, and f in Theorem 2.1, it is possible to derive similar inequalities involving the power and generalized logarithmic means. For a deeper discussion of inequalities for means of two variables, we refer the reader to [3] and [13, Chapter 2].

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References

[1] J. Aczél, The notion of mean values, Norske Vid. Selsk. Forhdl. 19 (1947) 83–86.
[2] B. Ahmad, A. Alsaedi, M. Kirane, B.T. Torebek, Hermite-Hadamard, Hermite-Hadamard-Fejér, Dragomir-Agarwal and Pachpatte type inequalities for convex functions via new fractional integrals, J. Comput. Appl. Math. 353 (2019) 120–129.
[3] H. Alzer, S.-L. Qiu, Inequalities for means in two variables, Arch. Math. 80 (2003) 201–215.
[4] G. Aumann, Konvexe Funktionen und die Induktion bei Ungleichungen zwischen Mittelwerten, Sitzungsber., Bayer. Akad. Wiss., Math.-Naturwiss. Kl. 1933 (1933) 403–415.
[5] J.L. Brenner, H. Alzer, Integral inequalities for concave functions with applications to special functions, Proc. R. Soc. Edinb., Sect. A, Math. 118 (1991) 173–192.
[6] H. Budak, On Fejér type inequalities for convex mappings utilizing fractional integrals of a function with respect to another function, Results Math. 74 (2019) 29, https://doi.org/10.1007/s00025-019-0960-8.
[7] P.S. Bullen, Handbook of Means and Their Inequalities, Kluwer Academic Publishers Group, Dordrecht, 2003.
[8] H. Chen, U.N. Katugampola, Hermite-Hadamard and Hermite-Hadamard-Fejér type inequalities for generalized fractional integrals, J. Math. Anal. Appl. 446 (2017) 1274–1291.
[9] F. Chen, S. Wu, Fejér and Hermite-Hadamard type inequalities for harmonically convex functions, J. Appl. Math. 2014 (2014) 386806.
[10] S.S. Dragomir, Two mappings in connection to Hadamard's inequalities, J. Math. Anal. Appl. 167 (1992) 49–56.
[11] S.S. Dragomir, D.S. Milošević, J. Sándor, On some refinements of Hadamard's inequalities and applications, Univ. Belgrad Publ. Elek. Fak. Sci. Math. 4 (1993) 3–10.
[12] S.S. Dragomir, B. Mond, Integral inequalities of Hadamard's type for log-convex functions, Demonstr. Math. 31 (1998) 354–364.
[13] S.S. Dragomir, C.E.M. Pearce, Selected Topics on Hermite-Hadamard Inequalities and Applications, RGMIA Monographs, Victoria University, 2002.
[14] L. Fejér, Über die Fourierreihen, II, Math. Naturwiss. Anz. Ungar. Akad. Wiss. 24 (1906) 369–390.

[15] P.M. Gill, C.E.M. Pearce, J. Pečarić, Hadamard's inequality for r-convex functions, J. Math. Anal. Appl. 215 (1997) 461–470.

[16] J. Hadamard, Étude sur les propriétés des fonctions entiéres et en particulier d'une fonction considérée par Riemann, J. Math. Pures Appl. 58 (1893) 171–215.

[17] P.C. Hammer, The midpoint method of numerical integration, Math. Mag. 31 (1958) 193–195.

[18] Ch. Hermite, Sur deux limites d'une intégrale dé finie, Mathesis 3 (1883) 82.

[19] D.-Y. Hwang, K.-L. Tseng, G.-S. Yang, Some Hadamard's inequalities for co-ordinated convex functions in the rectangle from the plane, Taiwan. J. Math. 11 (2007) 63–73.

[20] İ. İşcan, Hermite-Hadamard type inequalities for harmonically convex functions, Hacet. J. Math. Stat. 43 (2014) 935–942.

[21] İ. İşcan, Hermite-Hadamard-Fejér type inequalities for convex functions via fractional integrals, Stud. Univ. Babeş–Bolyai, Math. 60 (2015) 355–366.

[22] İ. İşcan, Hermite-Hadamard type inequalities for p-convex functions, Int. J. Anal. Appl. 11 (2016) 137–145.

[23] İ. İşcan, S. Wu, Hermite-Hadamard type inequalities for harmonically convex functions via fractional integrals, Appl. Math. Comput. 238 (2014) 237–244.

[24] M. Jleli, M. Kirane, B. Samet, A derivative concept with respect to an arbitrary kernel and applications to fractional calculus, Math. Methods Appl. Sci. 42 (2019) 137–160.

[25] M. Jleli, D. O'Regan, B. Samet, On Hermite-Hadamard type inequalities via generalized fractional integrals, Turk. J. Math. 40 (2016) 1221–1230.

[26] M. Jleli, B. Samet, On Hermite-Hadamard type inequalities via fractional integrals of a function with respect to another function, J. Nonlinear Sci. Appl. 9 (2016) 1252–1260.

[27] U.N. Katugampola, New approach to generalized fractional derivatives, Bull. Math. Anal. Appl. 6 (2014) 1–15.

[28] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, Theory and Applications of Fractional Differential Equations, North-Holland Mathematics Studies, vol. 204, Elsevier Sci. B.V., Amsterdam, 2006.

[29] M. Kunt, İ. İşcan, Hermite-Hadamard-Fejér type inequalities for p-convex functions, Arab J. Math. Sci. 23 (2017) 215–230.

[30] A. Lupaş, A generalisation of Hadamard's inequalities for convex functions, Univ. Beogr. Publ. Elektroteh. Fak. Ser. Mat. Fiz. 544–576 (1976) 115–121.

[31] M. Merkle, Logarithmic convexity and inequalities for the gamma function, J. Math. Anal. Appl. 203 (1996) 369–380.

[32] C.P. Niculescu, L.-E. Persson, Convex Functions and Their Applications: A Contemporary Approach, second ed., Springer International Publishing AG, part of Springer Nature, Switzerland, 2018.

[33] M.A. Noor, K.I. Noor, M.U. Awan, Some characterizations of harmonically log-convex functions, Proc. Jangjeon Math. Soc. 17 (2014) 51–61.

[34] M.A. Noor, K.I. Noor, M.U. Awan, Some new estimates of Hermite-Hadamard inequalities via harmonically r-convex functions, Matematiche 71 (2016) 117–127.

[35] J.E. Pečarić, F. Proschan, Y.C. Tong, Convex Functions, Partial Orderings and Statistical Applications, Mathematics in Science and Engineering, vol. 187, Academic Press, Boston, 1992.

[36] S. Peng, W. Wei, J.-R. Wang, On the Hermite-Hadamard inequalities for convex functions via Hadamard fractional integrals, Facta Univ., Ser. Math. Inform. 29 (2014) 55–75.

[37] J.L. Raabe, Angenäherte Bestimmung der Factorenfolge $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 ... n = \Gamma(n+1) = \int x^n e^{-x} dx$, wenn n eine sehr grosse Zahl ist, J. Reine Angew. Math. 25 (1840) 146–159.

[38] S.G. Samko, A.A. Kilbas, O.I. Marichev, Fractional Integrals and Derivatives: Theory and Applications, Gordon and Breach Science Publishers, Amsterdam, 1993.

[39] M.Z. Sarikaya, E. Set, H. Yaldiz, N. Başak, Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities, Math. Comput. Model. 57 (2013) 2403–2407.

[40] T. Trif, Characterizations of convex functions of a vector variable via Hermite-Hadamard's inequality, J. Math. Inequal. 2 (2008) 37–44.

[41] T. Trif, Convexity of the gamma function with respect to Hölder means, in: Y.J. Cho, J.K. Kim, S.S. Dragomir (Eds.), Inequality Theory and Applications, vol. 3, Nova Science Publishers, New York, 2003, pp. 189–195.

[42] K.-L. Tseng, S.-R. Hwang, S.S. Dragomir, Fejér-type inequalities (I), J. Inequal. Appl. 2010 (2010) 531976.

[43] K.-L. Tseng, S.-R. Hwang, S.S. Dragomir, Fejér-type inequalities (II), Math. Slovaca 67 (2017) 109–120.

[44] P.M. Vasić, I.B. Lacković, On an inequality for convex functions, Univ. Beogr. Publ. Elektroteh. Fak. Ser. Mat. Fiz. 461–497 (1974) 63–66.

[45] P.M. Vasić, I.B. Lacković, Some complements to the paper "On an inequality for convex functions", Univ. Beogr. Publ. Elektroteh. Fak. Ser. Mat. Fiz. 544–576 (1976) 59–62.

[46] G.-S. Yang, M.X. Hong, A note on Hadamard's inequality, Tamkang J. Math. 28 (1997) 33–37.

[47] G.-S. Yang, K.-L. Tseng, On certain integral inequalities related to Hermite-Hadamard inequalities, J. Math. Anal. Appl. 239 (1999) 180–187.