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Uniformly bounded weak and classical solutions to a singular parabolic system and applications ☆

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ABSTRACT

Equations of both elliptic and parabolic type featuring singular nonlinearities have appeared in numerous works throughout the years. In this work, we consider a time dependent problem featuring nonlinearities of the form $u^{-p}v^{-q}$, $u^{-r}v^{-s}$ subject to homogeneous Dirichlet boundary conditions and prove the existence of uniformly bounded weak and classical solutions under appropriate conditions on p, q, r, s. These results can, in some ways, be seen as a generalization of the results presented in [9]. These results are obtained using a functional method motivated by works found in [8], [22], [6] etc., and the boundary behaviour of a fundamental singular elliptic equation described in [16].

1. Introduction

1.1. Motivation

In this paper, we consider a system of coupled reaction-diffusion equations taking the following form:

$$\begin{cases} u_t = d\Delta u + \frac{1}{u^p v^q}, & \\ v_t = D\Delta v + \frac{1}{u^r v^s}, & x \in \Omega,\ t > 0 \\ u(x,0) = u_0(x) > 0, & \\ v(x,0) = v_0(x) > 0, & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega,\ t > 0, \end{cases} \tag{1.1}$$

in a smooth, bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 1$. Here, the coefficients $d, D > 0$ are taken to be constant, and the exponents satisfy $p, s \geq 0$, $r, q > 0$. The novelty of this system is the singular nature of the reaction

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terms as $x \to \partial\Omega$. It is clear that one cannot expect a solution pair (u, v) to belong to $C^2(\overline{\Omega})$ for any $t > 0$. However, if one restricts the exponents appropriately, we can obtain weaker differentiability behaviour near the boundary. This will be made more precise in the following discussion.

Motivation to study system (1.1) stems from the so called *Lane-Emden* equation, which takes the form

$$-\Delta u = u^p,$$

in a ball $B_R(0)$, $R > 0$, subject to homogeneous Dirichlet boundary data. Here, the exponent p denotes the *polytropic index*. Radially symmetric solutions are used to describe the structure of spherically symmetric, polytropic fluid. Depending on the size of the index p, one obtains different applications to various physical processes. Classically, in the works by Lane [20] and Fowler [13] this model was introduced as a second order ordinary differential equation with care given to integer values of p. Analytic solutions were derived for some special cases. Although negative exponents were not originally considered, negative exponents relate to processes where work and heat flow simultaneously in or out of the system. These cases can become meaningful in some special circumstances, such as plasmas studied in astrophysics [17]. Motivated readers are directed to [5] and [17] for details.

These works further motivated the study of the so called Lane-Emden system, taking the form

$$\begin{cases} 0 = \Delta u + u^p v^q, \\ 0 = \Delta v + u^r v^s, \end{cases} \tag{1.2}$$

in either a bounded or unbounded domain.

In the case of a bounded domain, the authors in [18] discuss the existence and uniqueness of solutions when $p, s \le 0$ and $r, q > 0$. Furthermore, when $\Omega \equiv B_R(0)$, the solutions are shown to be radially symmetric. In [14] and [28], some existence, nonexistence and boundary behaviour results are derived for the case when $p, s \le 0$, $r, q < 0$. It is interesting that, depending on the size and relation of the exponents p, q, r, s, one can show that solutions belong to $C^{1,\tau}(\overline{\Omega})$, or merely $C^{\tau}(\overline{\Omega})$ from some $\tau \in (0, 1)$. In the latter case, it is then true that the gradient of the solution may blow up as one approaches the boundary of Ω, at least in the sense of L^∞. This yields the existence of a classical solution belonging to $C^2(\Omega) \cap C(\overline{\Omega})$ which may not be a weak solution in the sense that it does not belong to the Sobolev space $W_0^{1,p}(\Omega)$.

In the case of an unbounded domain, existence and non-existence results are obtained in the case when $p = s = 0$, $r, q > 0$. Whether a solution exists is formulated around whether you lie beneath or above a curve depending on r, q and the spatial dimension N. For example, in [24] it is shown that below the curve

$$\left\{ r, q > 0 \ : \ \frac{1}{r+1} + \frac{1}{q+1} = 1 - \frac{2}{N-2} \cdot \max\left(\frac{1}{r+1}, \frac{1}{q+1} \right) \right\}, \ N \ge 3,$$

there are no positive supersolutions to (1.2). Based on these results and other works, it has been conjectured that the critical curve separating existence and nonexistence on the whole space is defined by the *Sobolev hyperbola*:

$$\left\{ r, q > 0 \ : \ \frac{1}{r+1} + \frac{1}{q+1} = 1 - \frac{2}{N} \right\}.$$

For an extensive discussion on related results, readers are directed to [3] and the references therein.

Nonlinearities of the form u^p for $p < 0$ can also be seen as a limiting case in some models applicable to heterogeneous chemical catalyst kinetics of the form $u_t - d\Delta u = F(u, x)$. One example, the so called *Langmuir-Hinshelwood* model, has a nonlinearity of the form

$$F(u) := u^m \left(\frac{\varepsilon+1}{\varepsilon+u}\right)^{m+k},$$

for $m \geq 0$, $k > 0$ and $\varepsilon \ll 1$, see [1]. Another similar example features a nonlinearity of the form

$$F(u) := \frac{u^m}{\varepsilon + u^{m+k}},$$

which is applied to enzyme kinetics, see [2].

Nonlinearities of this form can also be applied to electrical phenomena. Suppose that $\Omega \subset \mathbb{R}^N$ is a region in space occupied by an electrical conductor. If $u(x,t)$ denotes the temperature of a point $x \in \Omega$ for some $t > 0$, and if $h(\sigma)$ represents the electrical resistivity, then the heat generation at any point $x \in \Omega$ is given by $E^2(x,t)/h(u)$, where E is the local voltage drop in Ω as a function of space and time. The temperature u then satisfies the equation $cu_t - d\Delta u = E^2/h(u)$, where c, d represent the specific heat and thermal conductivity of Ω, respectively. In general, $h(\cdot)$ is taken to be a nonnegative, increasing function of u with $h(0) = 0$. In particular, $h(u) = \alpha u^p$ for some constants $\alpha, p > 0$ satisfies such a condition. Readers are directed to [12] from which this description was taken.

Lastly, these singular nonlinearities often appear in the study of non-Newtonian fluids. First discussed in works such as [15,21], developed further in [26,4] and more recently in [23,29,25], authors derive a second order, nonlinear boundary value problem taking the form

$$\begin{cases} g''(y) = -yg^{-1/\kappa}(y), & y \in (0,1), \\ g'(0) = 0, & g(1) = 0, \end{cases} \tag{1.3}$$

for $\kappa > 0$. As proposed, $\kappa \in (0,1)$ corresponds to non-Newtonian fluid, $\kappa = 1$ to Newtonian fluid, and $\kappa > 1$ to dilatant fluid. Due to the singularity at $y = 0$, it is convenient to study problem (1.3) subject to boundary conditions without singularities, namely $g'(0) = 0$. However, such a simplification of the problem is not entirely necessary, at least from a theoretical point of view.

In this work, we discuss the existence and uniform boundedness of the time dependent counterpart to system (1.2) in a smooth, bounded domain subject to homogeneous Dirichlet boundary data. The mathematical difficulties of system (1.1) is the coupled form of the system as well as the singular nature of the nonlinearities as $x \to \partial\Omega$. Although the reaction terms can be seen to be quasimonotone non-increasing, due to the lack of growth estimates near 0, classical methods of sub/super solutions as applied to coupled parabolic systems are not applicable. To overcome this, we apply a functional method which allows us to obtain uniform bounds on the solutions to a perturbed system in relation to solutions of appropriate elliptic problems. For the remainder of this paper, we assume that the initial data $u_0(x), v_0(x) \in C_0^1(\overline{\Omega})$ and that there exists a constant $\varepsilon_0 > 0$ so that

$$u_0(x), v_0(x) \geq \varepsilon_0 \phi_1(x), \tag{1.4}$$

where $\phi_1(x)$ is the first eigenfunction of $-\Delta$ subject to homogeneous Dirichlet boundary data. Under this assumption, we have some control over u_0^{-1}, v_0^{-1} near the boundary, which allows us to obtain apriori estimates on the solution $(u_\varepsilon, v_\varepsilon)$ to the perturbed problem, independent of the parameter ε. Taking $\varepsilon \to 0$ up to subsequence yields the existence of a positive solution to problem (1.1). We call (u,v) a weak solution to problem (1.1) provided that $u, v \in L^2(0,T; W_0^{1,2}(\Omega)) \cap L^\infty(\overline{\Omega} \times (0,T))$,

$$\frac{1}{u^p v^q}, \frac{1}{u^r v^s} \in L^1(\Omega \times (0,T)), \tag{1.5}$$

and

$$\begin{cases} \int_\Omega u_0 \xi dx + \int_0^T \int_\Omega \left(u\xi_t - d\nabla u \nabla \xi + \frac{1}{u^p v^q}\xi \right) dxdt = 0, \\ \int_\Omega v_0 \xi dx + \int_0^T \int_\Omega \left(v\xi_t - D\nabla v \nabla \xi + \frac{1}{u^r v^s}\xi \right) dxdt = 0, \end{cases} \tag{1.6}$$

for all $\xi \in C^\infty(\overline{\Omega} \times (0,T))$ with $\xi(x,t) = 0$ on $\partial\Omega \times (0,T)$ and $\xi(x,T) = 0$ in Ω. We call (u,v) a classical solution to problem (1.1) if

$$u, v \in C^{2+\theta, 1+\theta/2}(\Omega \times (0,T)) \cap C^{1,0}(\overline{\Omega} \times [0,T))$$

for some $\theta \in (0,1)$ and satisfy (1.1) pointwise. Finally, we call a solution (u,v) global if the solution (u,v) exists for all $t \in (0,\infty)$, and globally bounded if $\|u\|_{L^\infty(\Omega)}, \|v\|_{L^\infty(\Omega)} < \infty$ uniformly for all $t \in (0,\infty)$.

1.2. Statement of results

The main results of this paper are the following.

Theorem 1. *Suppose $d = D$, $p, s \in [0,1)$ and $q, r \in (0,1)$ such that $p+q < 1$ and $r+s < 1$. Then, there exists at least one global weak solution (u,v) to problem (1.1). Furthermore, if $p+q < \frac{1}{N}$, u is a globally bounded classical solution. If $r+s < \frac{1}{N}$, v is a globally bounded classical solution.*

Theorem 2. *Suppose $d, D > 0$, $p, s \in [0,1)$ and $r, q \in (0,1)$ such that $p+q < 1/N$ and $r+q < 1/N$. Then there exists at least one globally bounded classical solution (u,v). Furthermore, there exists two positive constants δ, M such that*

$$\delta\phi_1(x) \le u(x,t), v(x,t) \le M\phi_1(x), \tag{1.7}$$

for all $t > 0$, where ϕ_1 is the first eigenfunction for $-\Delta$ in Ω subject to homogeneous Dirichlet boundary conditions (see (2.1)).

In the first result, we are able to obtain a globally bounded solution (u,v) under the assumption that $d = D$. In this case, we are able to obtain uniform (in time) bounds on the quantity $\phi_1 u^{-\alpha} v^{-\beta}$ for some $\alpha, \beta > 0$ satisfying $\alpha + \beta \le 1$. This allows us to then obtain uniform upper bounds on u and v, which in turn provide us useful estimates on the nonlinear reaction terms. This method is an adaptation and improvement of the methods used in [9]. In fact, Theorem 1 can be seen as a generalization of the results presented in [9] when one takes $p = s = 0$ and $d = D$ and a constant function in the numerator of the nonlinearities.

In contrast to the first case, the uniform bound on $\phi_1 u^{-\alpha} v^{-\beta}$ is not easily obtained when $d \neq D$. Despite this, we are still able to prove the existence of at least one globally bounded solution (u,v) satisfying (1.7) by first proving the existence of a global classical solution, and then improving the bound to hold uniformly for all $t \in (0,\infty)$.

The remainder of this paper will be organized as follows. In section 2, we present some relevant preliminary results necessary to prove the main results. In section 3, we assume without loss of generality that $d = D = 1$ and apply the functional method to obtain the existence of globally bounded solutions to system (1.1). Then, in section 4, we obtain time dependent bounds which allow us to prove the existence of global solutions. We then improve this bound to show that solutions are, in fact, globally bounded.

2. Preliminary results

Before proving the main results, we highlight some important preliminary results necessary for the subsequent proofs. For the remainder of the paper, we denote by (λ_1, ϕ_1) the first eigenvalue/eigenfunction pair of

$$\begin{cases} 0 = \Delta\phi(x) + \lambda\phi(x), & x \in \Omega, 0 \\ 0 = \phi(x), & x \in \partial\Omega. \end{cases} \tag{2.1}$$

By the smoothness of Ω, it is standard that such a pair exists with ϕ positive in Ω belonging to $C^2(\overline{\Omega})$. Additionally, we have that $\lambda_1 > 0$ and $\frac{\partial\phi_1}{\partial\mathbf{n}} < 0$ on $\partial\Omega$, where $\mathbf{n}$ is the outward facing unit normal vector to $\partial\Omega$. Denote by $\psi(x)$ the solution to the following problem:

$$\begin{cases} 0 = \Delta\psi(x) + \psi^{-\sigma}(x), & x \in \Omega, \\ 0 = \psi(x), & x \in \partial\Omega, \end{cases} \tag{2.2}$$

for any $\sigma \in (0, 1)$. By [11], a unique solution belonging to $C^2(\Omega) \cap C^\alpha(\overline{\Omega})$ for some $\alpha \in (0, 1)$ exists, and for $\sigma > -1$, Gui & Lin [16] proved that $\psi(x) \in C^{1,1-\sigma}(\overline{\Omega})$. In particular, this means that there exist constants $\gamma_0, \gamma_1 > 0$ such that

$$\gamma_0\phi_1(x) \le \psi(x) \le \gamma_1\phi_1(x). \tag{2.3}$$

This will prove useful, as we are able to relate $\phi_1(x)$ to $\psi(x)$ near the boundary by the smoothness of $\partial\Omega$. Without loss of generality, we may assume that $\gamma_0 \equiv 1$.

The following result is standard when applying the functional method. A proof is found in [7].

Lemma 3 *(Generalized Young's inequality). Suppose $u(x), v(x), g(x) > 0$. For any indices $p_1, q_1, q_2, \alpha_1, \beta_1, \beta_2, \theta_1$ satisfying $\theta_1 < p_1 < \alpha_1$ (not necessarily positive), and given any constant $c > 0$, we have that*

$$\frac{u^{p_1}}{v^{q_1}g^{q_2}} \le c\frac{u^{\alpha_1}}{v^{\beta_1}g^{\beta_2}} + c^{-(p_1-\theta_1)/(\alpha_1-p_1)}\frac{u^{\theta_1}}{v^{\eta_1}g^{\eta_2}} \ ,$$

where

$$\eta_1 = [q_1(\alpha_1 - \theta_1) - \beta_1(p_1 - \theta_1)](\alpha_1 - p_1)^{-1},$$
$$\eta_2 = [q_2(\alpha_1 - \theta_1) - \beta_2(p_1 - \theta_1)](\alpha_1 - p_1)^{-1} \ .$$

Let $(u, v) \in [C^{1,1}(\overline{\Omega} \times [0, T))]^2$ be positive solutions in Ω satisfying

$$\begin{cases} u_t = d\Delta u + F(x, t), \\ v_t = D\Delta v + G(x, t), & x \in \Omega,\ t > 0, \\ u = v = 0, & x \in \partial\Omega, \end{cases} \tag{2.4}$$

where $F, G \in C(\Omega \times [0, T])$ and $d, D > 0$. By Hopf's boundary lemma, since $u, v = 0$ on $\partial\Omega$, we have that $\partial v/\partial\mathbf{n} < 0$ and $\partial u/\partial\mathbf{n} < 0$ on $\partial\Omega$, where $\mathbf{n}$ is as defined previously. The first inequality considered will be useful in obtaining lower bounds for the solutions u and v.

Lemma 4. *Let ϕ solve either (2.1) or (2.2) and u be a solution of (2.4). For any $n > 2$, we have that*

$$\frac{d}{dt}\int_\Omega \frac{\phi^{n+2}}{u^n}dx \le -dn\int_\Omega \frac{\phi^{n+1}}{u^n}\Delta\phi dx - n\int_\Omega \frac{\phi^{n+2}}{u^{n+1}}F(x, t)dx. \tag{2.5}$$

In particular, this implies that

$$\frac{d}{dt}\int\limits_\Omega \frac{\phi_1^{n+2}}{u^n}dx \le d\lambda_1 n\int\limits_\Omega \frac{\phi_1^{n+2}}{u^n}dx - n\int\limits_\Omega \frac{\phi_1^{n+2}}{u^{n+1}}F(x,t)dx, \tag{2.6}$$

and

$$\frac{d}{dt}\int\limits_\Omega \frac{\psi^{n+2}}{u^n}dx \le dn\int\limits_\Omega \frac{\psi^{n+1-\sigma}}{u^n}\psi dx - n\int\limits_\Omega \frac{\psi^{n+2}}{u^{n+1}}F(x,t)dx, \tag{2.7}$$

when $\phi = \phi_1$ or $\phi = \psi$, respectively.

Proof. From Lemma 2.2 in [10], one can obtain (2.7) for the case of $\phi = \psi$. The same method allows one to obtain (2.5) when $\psi = \phi_1$, and (2.6) follows after substituting $\Delta\phi_1$, completing the proof. □

The next lemma allows one to obtain uniform in time bounds to (1.1) when the diffusion coefficients are the same.

Lemma 5. *Let ϕ_1 be the solution of (2.1) and u, v be solutions of (2.4) for $d = D$. Then, for any $\alpha, \beta > 0$ satisfying $\alpha + \beta \le 1$, it is true that*

$$\begin{aligned}\frac{d}{dt}\int\limits_\Omega \frac{\phi_1^{n+2}}{u^{\alpha n}v^{\beta n}} \le &-\alpha n\int\limits_\Omega \frac{\phi_1^{n+2}}{u^{\alpha n+1}v^{\beta n}}F(x,t)dx - \beta n\int\limits_\Omega \frac{\phi_1^{n+2}}{u^{\alpha n}v^{\beta n+1}}G(x,t)dx\\ &+ \lambda_1(n+2)\int\limits_\Omega \frac{\phi_1^{n+2}}{u^{\alpha n}v^{\beta n}}dx,\end{aligned}$$

for any $n > 0$.

Proof. See Lemma 2.3 in [9]. □

The following lemma allows us to obtain uniform (in ε) upper bounds on the perturbed solutions to problem (1.1). The key is to treat the perturbed quantity $u + \varepsilon$ separately from the solution u itself.

Lemma 6. *Suppose that u is a solution of (2.4) and let ψ be the solution of (2.2). For any $\varepsilon > 0$, define $w_\varepsilon = u + \varepsilon$. Then, for any $\alpha, \beta \in (0, 1)$ satisfying $\alpha + \beta \le 1$, we have*

$$\begin{aligned}\frac{d}{dt}\int\limits_\Omega \frac{u^n}{{w_\varepsilon}^{\alpha n}\psi^{\beta n-2}}dx \le &\, n\int\limits_\Omega \frac{u^{n-1}}{{w_\varepsilon}^{\alpha n}\psi^{\beta n-2}}F(x,t)dx - \alpha n\int\limits_\Omega \frac{u^n}{{w_\varepsilon}^{\alpha n+1}\psi^{\beta n-2}}F(x,t)dx\\ &- d(\beta n-2)\int\limits_\Omega \frac{u^n}{{w_\varepsilon}^{\alpha n}\psi^{\beta n-1+\sigma}}dx,\end{aligned}$$

for all $n > 2$.

Proof. See Lemma 2.5 in [8]. A proof is also provided in the appendix of [9]. □

This final inequality is what will allow us to prove the uniform boundedness of solutions to (1.1) when $d \ne D$.

Lemma 7. *Let ψ solve (2.2) for any $\sigma \in (0,1)$ and u be a solution of (2.4). For any $n > 2$, it is true that*

$$\frac{d}{dt}\int\limits_{\Omega} \frac{u^n}{\psi^{n-2}}dx \leq n\int\limits_{\Omega} \frac{u^{n-1}}{\psi^{n-2}}F(x,t)dx - dn\int\limits_{\Omega} \frac{u^n}{\psi^{n-1+\sigma}}dx. \tag{2.8}$$

Proof. First, note that since our solutions ψ, u are smooth enough (i.e. $C^1(\overline{\Omega})$), the quantity u/ψ is well defined up to the boundary of Ω. Next, differentiating with respect to t and integrating by parts, we have that

$$\begin{aligned}\frac{d}{dt}\int\limits_{\Omega} \frac{u^n}{\psi^{n-2}}dx &= dn\int\limits_{\Omega} \frac{u^{n-1}}{\psi^{n-2}}\Delta u dx + n\int\limits_{\Omega} \frac{u^{n-1}}{\psi^{n-2}}F(x,t)dx \\ &= -dn\int\limits_{\Omega} \nabla\left(\frac{u^{n-1}}{\psi^{n-2}}\right)\nabla u dx + n\int\limits_{\Omega} \frac{u^{n-1}}{\psi^{n-2}}F(x,t)dx \\ &= -dn(n-1)\int\limits_{\Omega} \frac{u^{n-2}}{\psi^{n-2}}\left|\nabla u\right|^2 dx + dn(n-2)\int\limits_{\Omega} \frac{u^{n-1}}{\psi^{n-1}}\nabla u\nabla\psi dx \\ &\quad + n\int\limits_{\Omega} \frac{u^{n-1}}{\psi^{n-2}}F(x,t)dx.\end{aligned} \tag{2.9}$$

Continuing, we use the following identity:

$$\psi^2\left|\nabla u\right|^2 = \left|\psi\nabla u - u\nabla\psi\right|^2 + 2u\psi\nabla u\nabla\psi - u^2\left|\nabla\psi\right|^2. \tag{2.10}$$

We then see that (2.9) can be written instead as

$$\begin{aligned}\frac{d}{dt}\int\limits_{\Omega} \frac{u^n}{\psi^{n-2}}dx = &-dn(n-1)\int\limits_{\Omega} \frac{u^{n-2}}{\psi^n}\left|\psi\nabla u - u\nabla\psi\right|^2 dx + n\int\limits_{\Omega} \frac{u^{n-1}}{\psi^{n-2}}F(x,t)dx \\ &- dn^2\int\limits_{\Omega} \frac{u^{n-1}}{\psi^{n-1}}\nabla\psi\nabla u dx + dn(n-1)\int\limits_{\Omega} \frac{u^n}{\psi^n}\left|\nabla\psi\right|^2 dx.\end{aligned} \tag{2.11}$$

If we integrate by parts on the third term of (2.11), we find

$$\begin{aligned}-dn^2\int\limits_{\Omega} \frac{u^{n-1}}{\psi^{n-1}}\nabla u\nabla\psi dx &= -dn\int\limits_{\Omega} \frac{\nabla\psi}{\psi^{n-1}}\nabla\left(u^n\right)dx \\ &= dn\int\limits_{\Omega} \nabla\left(\frac{\nabla\psi}{\psi^{n-1}}\right)u^n dx \\ &= dn\int\limits_{\Omega} \frac{u^n}{\psi^{n-1}}\Delta\psi dx - dn(n-1)\int\limits_{\Omega} \frac{u^n}{\psi^n}\left|\nabla\psi\right|^2 dx.\end{aligned} \tag{2.12}$$

Combining the above with (2.11) and throwing away the leading negative term gives us that

$$\frac{d}{dt}\int\limits_{\Omega} \frac{u^n}{\psi^{n-2}}dx \leq n\int\limits_{\Omega} \frac{u^{n-1}}{\psi^{n-2}}F(x,t)dx + dn\int\limits_{\Omega} \frac{u^n}{\psi^{n-1}}\Delta\psi dx. \tag{2.13}$$

Substituting $\Delta\psi$ completes the proof. $\square$

With these results we are now equipped to prove Theorem 1 and Theorem 2.

3. Proof of Theorem 1

To begin, we perturb system (1.1). By rescaling spatial variables, we may assume without loss of generality that $d = D = 1$. We then have

$$\begin{cases} u_t = \Delta u + \frac{1}{(u+\varepsilon)^p(v+\varepsilon)^q}, & \\ v_t = \Delta v + \frac{1}{(u+\varepsilon)^r(v+\varepsilon)^s}, & x \in \Omega,\ t > 0, \\ u(x,0) = u_0(x) > 0, & \\ v(x,0) = v_0(x) > 0, & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega,\ t > 0. \end{cases} \tag{3.1}$$

By standard theory, a classical solution exists for each $\varepsilon > 0$. (See [19] or [27], for example). Denote the solution by $(u_\varepsilon, v_\varepsilon)$. To start, let $w_\varepsilon = u_\varepsilon + \varepsilon$ and $z_\varepsilon = v_\varepsilon + \varepsilon$. Lemma 5 with u replaced by w_ε and v replaced by z_ε gives us

$$\begin{aligned} \frac{d}{dt}\int_\Omega \frac{\phi^{n+2}}{w_\varepsilon^{\alpha n} z_\varepsilon^{\beta n}}dx \le \lambda_1(n+2)\int_\Omega \frac{\phi^{n+2}}{w_\varepsilon^{\alpha n} z_\varepsilon^{\beta n}}dx - \alpha n\int_\Omega \frac{\phi^{n+2}}{w_\varepsilon^{\alpha n+p+1} z_\varepsilon^{\beta n+q}}dx \\ - \beta n\int_\Omega \frac{\phi^{n+2}}{w_\varepsilon^{\alpha n+r} z_\varepsilon^{\beta n+s+1}}dx\ . \end{aligned} \tag{3.2}$$

For any $\delta > 0$ and $n \ge 2$, we may apply Lemma 3 to the first term of (3.2) with

$$p_1 = \alpha n, \quad \alpha_1 = \alpha n + p + 1, \quad \theta_1 = \alpha n - \delta(p+1),$$
$$q_1 = \beta n, \quad \beta_1 = \beta n + q$$

to obtain

$$\begin{aligned} \frac{\lambda_1(n+2)}{w_\varepsilon^{\alpha n} z_\varepsilon^{\beta n}} &= \lambda_1(n+2)\frac{(w_\varepsilon^{-1})^{\alpha n}}{z_\varepsilon^{\beta n}} \\ &\le \alpha n\frac{(w_\varepsilon^{-1})^{\alpha n+p+1}}{z_\varepsilon^{\beta n+q}} + \lambda_1(n+2)\left(\frac{\alpha n}{\lambda_1(n+2)}\right)^{-\delta}\frac{(w_\varepsilon^{-1})^{\alpha n-\delta(p+1)}}{z_\varepsilon^{\beta n-q\delta}}\ , \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} \lambda_1(n+2)\left(\frac{\alpha n}{\lambda_1(n+2)}\right)^{-\delta} &\le \lambda_1(n+2)\left(\frac{2\lambda_1}{\alpha}\right)^{\delta} \\ &\equiv c_1(n)\ . \end{aligned} \tag{3.4}$$

Combining (3.3) and (3.4) with (3.2) leaves us with

$$\begin{aligned} \frac{d}{dt}\int_\Omega \frac{\phi^{n+2}}{w_\varepsilon^{\alpha n} z_\varepsilon^{\beta n}}dx \le c_1(n)\int_\Omega \frac{\phi^{n+2}}{w_\varepsilon^{\beta n-\delta(p+1)} z_\varepsilon^{\beta n-q\delta}}dx \\ - \beta n\int_\Omega \frac{\phi^{n+2}}{w_\varepsilon^{\alpha n+r} z_\varepsilon^{\beta n+s+1}}dx\ . \end{aligned} \tag{3.5}$$

If we apply Lemma 3 to the first term of (3.5) with

$$p_1 = \alpha n - \delta(p+1), \quad \alpha_1 = \alpha n + r, \quad \theta_1 = 0,$$
$$q_1 = \beta n - q\delta, \quad \beta_1 = \beta n + s + 1$$

we find

$$\begin{aligned} \frac{c_1(n)}{w_\varepsilon^{\alpha n - \delta(p+1)} z_\varepsilon^{\beta n - q\delta}} &= c_1(n) \frac{(w_\varepsilon^{-1})^{\alpha n - \delta(p+1)}}{z_\varepsilon^{\beta n - q\delta}} \\ &\leq \beta n \frac{(w_\varepsilon^{-1})^{\alpha n + r}}{z_\varepsilon^{\beta n + s + 1}} \\ &\quad + c_1(n) \left(\frac{\beta n}{c_1(n)} \right)^{-\frac{(\alpha n - \delta(p+1))}{(r + \delta(p+1))}} \frac{1}{z_\varepsilon^{\eta_1}} \ , \end{aligned} \tag{3.6}$$

where

$$\begin{aligned} \eta_1 &= \frac{[(\beta n - q\delta)(\alpha n + r) - (\beta n + s + 1)(\alpha n - \delta(p+1))]}{r + \delta(p+1)} \\ &= \frac{[\beta n r - \alpha n(s+1) - \delta(\alpha n q - \beta n(p+1) + rq - (s+1)(p+1))]}{r + \delta(p+1)} \ , \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} c_1(n) \left(\frac{\beta n}{c_1(n)} \right)^{-\frac{(\alpha n - \delta(p+1))}{(r + \delta(p+1))}} &\leq \lambda_1(n+2) \left(\frac{2\lambda_1}{\alpha} \right)^{\delta} \left(\frac{(2\lambda_1)^{\delta+1}}{\alpha^\delta \beta} \right)^{\frac{(\alpha n - \delta(p+1))}{(r + \delta(p+1))}} \\ &\equiv c_2(n) \ . \end{aligned} \tag{3.8}$$

Combining (3.6)-(3.8) with (3.5) then yields

$$\frac{d}{dt} \int\limits_\Omega \frac{\phi^{n+2}}{w_\varepsilon^{\alpha n} z_\varepsilon^{\beta n}} dx \leq c_2(n) \int\limits_\Omega \frac{\phi^{n+2}}{z_\varepsilon^{\eta_1}} dx. \tag{3.9}$$

Setting $\eta_1 = 0$ and solving for δ yields

$$\delta = \frac{n(\alpha(s+1) - \beta r)}{n(\beta(p+1) - \alpha q) + (s+1)(p+1) - rq} \ . \tag{3.10}$$

From this, we see that we can ensure that

$$\alpha(s+1) - \beta r > 0, \tag{3.11}$$

$$\beta(p+1) - \alpha q \geq 0, \tag{3.12}$$

so that $\delta > 0$. Define $2^* \equiv q + r$. If we choose $\alpha = \frac{r}{2^*}$, $\beta = \frac{q}{2^*}$ so that $\alpha + \beta = 1$, we see that under the hypotheses of the theorem,

$$\alpha(s+1) - \beta r = \frac{r}{2^*}(s + 1 - q) > 0,$$
$$\beta(p+1) - \alpha q = \frac{q}{2^*}(p + 1 - r) > 0,$$

and so (3.11), (3.12) are satisfied. Putting all of this together, (3.11) becomes

$$\frac{d}{dt}\int_\Omega \frac{\phi^{n+2}}{w_\varepsilon^{\alpha n} z_\varepsilon^{\beta n}}dx \le c_2(n)\int_\Omega \phi^{n+2}dx. \tag{3.13}$$

Integrating (3.13) from 0 to t, we then arrive at

$$\int_\Omega \frac{\phi^{n+2}}{w_\varepsilon^{\alpha n} z_\varepsilon^{\beta n}}dx \le c_2(n)t\int_\Omega \phi^{n+2}dx + \int_\Omega \frac{\phi^{n+2}}{u_0^{\alpha n} v_0^{\beta n}}dx.$$

Extracting nth roots and taking $n \to \infty$, we obtain

$$\begin{aligned}\frac{\phi(x)}{w_\varepsilon^{\frac{r}{2^*}}(x,t) z_\varepsilon^{\frac{q}{2^*}}(x,t)} &\le \left\| \frac{\phi}{w_\varepsilon^{\frac{r}{2^*}} z_\varepsilon^{\frac{q}{2^*}}} \right\|_\infty \\ &\le \max\left\{ m_1 \left\|\phi\right\|_\infty, \varepsilon_0^{-1}\right\} \\ &\equiv M_1 < \infty, \end{aligned} \tag{3.14}$$

where ε_0 is as defined in (1.4) and

$$m_1 = \left(\frac{(2\lambda_1)^{\delta+1}}{\alpha^\delta \beta}\right)^{\frac{\alpha}{r+\delta(p+1)}},$$

independent of ε and t. We now obtain upper bounds for the solution u_ε, independent of ε and t. Referring to Lemma 6, we replace u with u_ε and see that

$$\begin{aligned}\frac{d}{dt}\int_\Omega \frac{u_\varepsilon^n}{w_\varepsilon^{\alpha n}\psi^{\beta n-2}}dx \le{}& n\int_\Omega \frac{u_\varepsilon^{n-1}}{w_\varepsilon^{\alpha n+p} z_\varepsilon^q \psi^{\beta n-2}}dx - \alpha n\int_\Omega \frac{u_\varepsilon^n}{w_\varepsilon^{\alpha n+p+1} z_\varepsilon^q \psi^{\beta n-2}}dx \\ &- (\beta n-2)\int_\Omega \frac{u_\varepsilon^n}{w_\varepsilon^{\alpha n}\psi^{\beta n-1+\sigma}}dx\ . \end{aligned} \tag{3.15}$$

Note that these α, β are new parameters to be determined, independent of the previous choices $r/2^*$ and $q/2^*$. Now, for any $\delta_1 > 1$ we may apply Lemma 3 to first term of (3.15) with

$$p_1 = n-1, \quad \alpha_1 = n, \quad \theta_1 = n-\delta_1,$$
$$q_1 = \alpha n + p, \quad \beta_1 = \alpha n + p + 1,$$

to find that

$$\frac{u_\varepsilon^{n-1}}{w_\varepsilon^{\alpha n+p}} \le \alpha\frac{u_\varepsilon^n}{w_\varepsilon^{\alpha n+p+1}} + \alpha^{-(\delta_1-1)}\frac{u_\varepsilon^{n-\delta_1}}{w_\varepsilon^{\alpha n+p+1-\delta_1}}\ ,$$

in which case (3.15) becomes

$$\begin{aligned}\frac{d}{dt}\int_\Omega \frac{u_\varepsilon^n}{w_\varepsilon^{\alpha n}\psi^{\beta n-2}}dx \le{}& \frac{n}{\alpha^{\delta_1-1}}\int_\Omega \frac{u_\varepsilon^{n-\delta_1}}{w_\varepsilon^{\alpha n+p+1-\delta_1} z_\varepsilon^q \psi^{\beta n-2}}dx \\ &- (\beta n-2)\int_\Omega \frac{u_\varepsilon^n}{w_\varepsilon^{\alpha n}\psi^{\beta n-1+\sigma}}dx\ . \end{aligned} \tag{3.16}$$

If we now apply the uniform bound (3.14) to (3.16), we can then remove z_ε^q from the denominator of the positive term above as follows:

$$\frac{u_\varepsilon^{n-\delta_1}}{w_\varepsilon^{\alpha n+p+1-\delta_1} z_\varepsilon^q \psi^{\beta n-2}} = \frac{u_\varepsilon^{n-\delta_1}}{w_\varepsilon^{\alpha n+p+1-\delta_1-r}\psi^{\beta n-2+2^*}} \left(\frac{\phi}{w_\varepsilon^{\frac{r}{2^*}} z_\varepsilon^{\frac{q}{2^*}}}\right)^{2^*} \left(\frac{\psi}{\phi}\right)^{2^*}$$
$$\leq \gamma_1^{2^*} M_1^{2^*} \frac{u_\varepsilon^{n-\delta_1}}{w_\varepsilon^{\alpha n+p+1-r-\delta_1}\psi^{\beta n-2+2^*}} \ . \tag{3.17}$$

If we pair the above with Lemma 3 with

$$p_1 = n-\delta_1, \quad \alpha_1 = n, \quad \theta_1 = 0,$$
$$q_1 = \alpha n + p + 1 - r - \delta_1, \quad \beta_1 = \alpha n,$$
$$q_2 = \beta n - 2 + 2^*, \quad \beta_2 = \beta n - 1 + \sigma,$$

we then obtain

$$\frac{M_1^{2^*}\gamma_1^{2^*} n}{\alpha^{\delta_1-1}} \frac{u_\varepsilon^{n-\delta_1}}{w_\varepsilon^{\alpha n+p+1-r-\delta_1}\psi^{\beta n-2+2^*}}$$
$$\leq (\beta n-2)\frac{u_\varepsilon^n}{w_\varepsilon^{\alpha n}\psi^{\beta n-1+\sigma}} + \left(\frac{M_1^{2^*}\gamma_1^{2^*} n}{\alpha^{\delta_1-1}}\right)\left(\frac{(\beta n-2)\alpha^{\delta_1-1}}{M_1^{2^*}\gamma_1^{2^*} n}\right)^{-\frac{n}{\delta_1}+1} \frac{1}{w_\varepsilon^{\eta_1}\psi^{\eta_2}}$$
$$\leq (\beta n-2)\frac{u_\varepsilon^n}{w_\varepsilon^{\alpha n}\psi^{\beta n-1+\sigma}} + c_4(n)\frac{1}{w_\varepsilon^{\eta_1}\psi^{\eta_2}} \ , \tag{3.18}$$

where

$$\eta_1 = \frac{[(\alpha n+p+1-r-\delta_1)(n) - (\alpha n)(n-\delta_1)]}{\delta_1}$$
$$= \frac{n(p+1-r) - n\delta_1(1-\alpha)}{\delta_1} \ ,$$
$$\eta_2 = \frac{[(\beta n-2+2^*)(n) - (\beta n-1+\sigma)(n-\delta_1)]}{\delta_1}$$
$$= \frac{-n(1+\sigma-2^*-\delta_1\beta)}{\delta_1} - (1-\sigma) \ , \tag{3.19}$$

and

$$\left(\frac{M_1^{2^*}\gamma_1^{2^*} n}{\alpha^{\delta_1-1}}\right)\left(\frac{(\beta n-2)\alpha^{\delta_1-1}}{M_1^{2^*}\gamma_1^{2^*} n}\right)^{-\frac{n}{\delta_1}+1} \leq (\beta n-2)\left(\frac{2M_1^{2^*}\gamma_1^{2^*}}{\alpha^{\delta_1-1}\beta}\right)^{\frac{n}{\delta_1}}$$
$$\equiv c_4(n). \tag{3.20}$$

With these computations, (3.16) then becomes

$$\frac{d}{dt}\int\limits_\Omega \frac{u_\varepsilon^n}{w_\varepsilon^{\alpha n}\psi^{\beta n-2}}dx \leq c_4(n)\int\limits_\Omega \frac{1}{w_\varepsilon^{\eta_1}\psi^{\eta_2}}dx. \tag{3.21}$$

Again, setting $\eta_1 = 0$ we find

$$\delta_1 = \frac{p+1-r}{1-\alpha}. \tag{3.22}$$

Recall that we require $\delta_1 > 1$. Given that $p+1-r > 0$, $\delta_1 > 1$ for any $\alpha \in (r, 1)$. (3.21) may then be written as

$$\frac{d}{dt}\int\limits_{\Omega} \frac{u_\varepsilon^n}{w_\varepsilon^{\alpha n}\psi^{\beta n-2}}dx \le c_4(n)\int\limits_{\Omega} \frac{1}{\psi^{\eta_2}}dx. \tag{3.23}$$

It is well known (see the first Lemma of section 3 in [21], for example) that the right hand side of (3.23) is finite if and only if $\eta_2 < 1$. Notice that this follows from the inequality found in (2.3), and so the result as stated in [21] is true for ψ only when $\sigma > -1$. Explicitly, this means that we require

$$-1 < \frac{n}{\delta_1}\left(1+\sigma-2^*-\beta\delta_1\right)+1-\sigma, \tag{3.24}$$

but as n gets large, this can only be true if $1+\sigma-2^*-\beta\delta_1 > 0$. Recall that $2^* = q+r < 2$ and choose $\alpha \in (r,1)$ with $\beta = 1-\alpha$. Then for σ sufficiently close to 1, $1+\sigma-2^* > 0$. It is easy to then see that

$$\begin{aligned}
1+\sigma-2^*-\beta\delta_1 &\ge 1+\sigma-q-r-(p+1-r)\\
&= \sigma-(p+q)\\
&> 0,
\end{aligned}$$

which is true for σ sufficiently close to 1, since $p+q<1$ by hypothesis. Consequently, (3.24) is true. With this, we may now solve (3.23) to find that

$$\begin{aligned}
\int\limits_{\Omega} \frac{u_\varepsilon^n}{w_\varepsilon^{\alpha n}\psi^{\beta n-2}}dx &\le c_4(n)t\int\limits_{\Omega} \psi^{\frac{n}{\delta_1}(1+\sigma-2^*-\beta\delta_1)+1-\sigma}dx + \int\limits_{\Omega} \frac{u_0^n}{(u_0+\varepsilon)^{\alpha n}\psi^{\beta n-2}}dx\\
&\le c_4(n)t\int\limits_{\Omega} \psi^{\frac{n}{\delta_1}(1+\sigma-2^*-\beta\delta_1)+1-\sigma}dx + \int\limits_{\Omega} \frac{u_0^{n(1-\alpha)}}{\psi^{\beta n-2}}dx,
\end{aligned} \tag{3.25}$$

for any $t > 0$. Note that since $u_0 \in C_0^1(\overline{\Omega})$ and $\sigma \in (0,1)$, the ratio u_0/ψ is well defined up to the boundary $\partial\Omega$. Extracting nth roots and letting $n \to \infty$, we arrive at

$$\begin{aligned}
\frac{u_\varepsilon(x,t)}{w_\varepsilon^\alpha(x,t)\psi^\beta(x)} &\le \left\|\frac{u_\varepsilon}{w_\varepsilon^\alpha\psi^\beta}\right\|_\infty\\
&\le \max\left\{m_2\left\|\psi^{(\sigma-p-q)/\delta_1}\right\|_\infty, \left\|\frac{u_0}{\psi}\right\|_\infty\right\}\\
&\equiv M_2 < \infty,
\end{aligned} \tag{3.26}$$

where

$$m_2 = \left(\frac{2M_1^{2^*}\gamma_1^{2^*}}{\alpha^{\delta_1-1}\beta}\right)^{\frac{1}{\delta_1}},$$

independent of ε and t. Consequently, u_ε is uniformly bounded. We now state the analogous result for v_ε. Many of the technical details will be omitted as they are essentially the same as those done for u_ε. Notice that in line (3.17), we have control over a term involving $w_\varepsilon^{-r/2^*}$. In the same way we removed z_ε^{-q} for the equation for u_ε, we are able to remove w_ε^{-r} in the equation for v_ε. Hence, we find that

$$\frac{d}{dt}\int_{\Omega}\frac{v_\varepsilon^n}{z_\varepsilon^{\alpha n}\psi^{\beta n-2}}dx \le c_5(n)\int_{\Omega}\psi^{\frac{n}{\delta_2}(\sigma-r-s)+1-\sigma}dx, \tag{3.27}$$

where

$$\delta_2 = \frac{s+1-q}{1-\alpha},$$
$$c_5(n) = (\beta n-2)\left(\frac{2M_1^{2^*}\gamma_1^{2^*}}{\alpha^{\delta_2-1}\beta}\right)^{\frac{n}{\delta_2}},$$

where we now choose $\alpha \in (q,1)$ so that $\delta_2 > 1$. Integrating, and again using the fact that $v_0 \in C_0^1(\overline{\Omega})$, we obtain

$$\int_{\Omega}\frac{v_\varepsilon^n}{z_\varepsilon^{\alpha n}\psi^{\beta n-2}}dx \le c_5(n)t\int_{\Omega}\psi^{\frac{n}{\delta_2}(\sigma-r-s)+1-\sigma}dx + \int_{\Omega}\frac{v_0^{(1-\alpha)n}}{\psi^{\beta n}}dx, \tag{3.28}$$

for any $t > 0$. By the same reasoning, we may choose $\beta = 1-\alpha$ and σ sufficiently close to 1 so that the exponent on ψ on the right hand side of (3.27) is positive. Extracting nth roots and letting $n \to \infty$ yields

$$\begin{aligned}\frac{v_\varepsilon(x,t)}{z_\varepsilon^\alpha(x,t)\psi^\beta(x)} &\le \left\|\frac{v_\varepsilon}{z_\varepsilon^\alpha\psi^\beta}\right\|_\infty \\ &\le \max\left\{m_3\left\|\psi^{(\sigma-r-s)/\delta_2}\right\|_\infty, \left\|\frac{v_0}{\psi}\right\|_\infty\right\} \\ &\equiv M_3 < \infty,\end{aligned} \tag{3.29}$$

where

$$m_3 = \left(\frac{2M_1^{2^*}\gamma_1^{2^*}}{\alpha^{\delta_2-1}\beta}\right)^{\frac{1}{\delta_2}}.$$

Thus, v_ε is uniformly bounded.

The next step is to obtain uniform lower bounds on the solutions $(u_\varepsilon, v_\varepsilon)$. To see this, we apply Lemma 4 with $\phi = \phi_1$ and u replaced by w_ε to see that

$$\frac{d}{dt}\int_{\Omega}\frac{\phi_1^{n+2}}{w_\varepsilon^n}dx \le \lambda_1 n\int_{\Omega}\frac{\phi_1^{n+2}}{w_\varepsilon^n}dx - n\int_{\Omega}\frac{\phi_1^{n+2}}{w_\varepsilon^{n+p+1}z_\varepsilon^q}dx. \tag{3.30}$$

We then apply Lemma (3) to the first term of (3.30) with

$$p_1 = n, \quad \alpha_1 = n+p+1, \quad \theta_1 = 0,$$
$$q_1 = 0, \quad \beta_1 = q,$$

to obtain

$$\lambda_1(w_\varepsilon^{-1})^n \le \frac{(w_\varepsilon^{-1})^{n+p+1}}{z_\varepsilon^q} + \lambda_1\left(\frac{1}{\lambda_1}\right)^{-\frac{n}{p+1}}\frac{1}{z_\varepsilon^{-\frac{qn}{p+1}}}.$$

(3.30) then becomes

$$\frac{d}{dt}\int_{\Omega}\frac{\phi_1^{n+2}}{w_\varepsilon^n}dx \leq n\lambda_1^{\frac{n+p+1}{p+1}}\int_{\Omega}\phi_1^{n+2}z_\varepsilon^{\frac{nq}{p+1}}dx. \tag{3.31}$$

Since $z_\varepsilon = v_\varepsilon + \varepsilon$ is uniformly bounded in ε and t by (3.29), we may again integrate and extract nth roots to obtain

$$\begin{aligned}\frac{\phi_1(x)}{w_\varepsilon(x,t)} &\leq \left\|\frac{\phi_1}{w_\varepsilon}\right\|_\infty \\ &\equiv M_4 < \infty.\end{aligned} \tag{3.32}$$

Rearranging, we see that $w_\varepsilon \geq \phi_1 M_4^{-1}$, independent of ε and t. Repeating this process for v_ε gives us

$$\begin{aligned}\frac{\phi_1(x)}{z_\varepsilon(x,t)} &\leq \left\|\frac{\phi_1}{v_\varepsilon}\right\|_\infty \\ &\equiv M_5 < \infty.\end{aligned} \tag{3.33}$$

Thus, $z_\varepsilon(x,t) \geq M_5^{-1}\phi_1(x)$, independent of ε and t.

We now use these estimates to show that the nonlinear terms of our system are uniformly bounded in $L^k(\Omega)$ for some $k > 1$. To see this, for any $k_1 \in (1, \frac{1}{p+q})$ it is true that

$$\begin{aligned}\int_{\Omega}\left|\frac{1}{w_\varepsilon^p z_\varepsilon^q}\right|^{k_1}dx &\leq M_6\int_{\Omega}\phi_1^{-k_1(p+q)}dx \\ &< \infty.\end{aligned} \tag{3.34}$$

Similarly, for $k_2 \in (1, \frac{1}{r+s})$

$$\begin{aligned}\int_{\Omega}\left|\frac{1}{w_\varepsilon^r z_\varepsilon^s}\right|^{k_2}dx &\leq M_7\int_{\Omega}\phi_1^{-k_2(r+s)}dx \\ &< \infty.\end{aligned} \tag{3.35}$$

Hence, both reaction terms belong to $L^k(\Omega)$ for some $k > 1$. By standard L^p-theory of parabolic equations, we find that $(u_\varepsilon, v_\varepsilon)$ are uniformly bounded in $[W_{k_1}^{2,1}(\Omega \times (0,\infty))] \times [W_{k_2}^{2,1}(\Omega \times (0,\infty))]$. Consequently, a subsequence $(u_{\varepsilon_i}, v_{\varepsilon_i})$ can be extracted which converges to a weak solution (u, v) of system (1.1).

Finally, if $N < (p+q)^{-1}$, we can choose $k_1 > N$ so that (3.34) remains true. By the Sobolev embedding theorem, up to a subsequence which we denote still by ε, we have that $u_\varepsilon \to u \in C^{1+\kappa,(1+\kappa)/2}(\overline{\Omega} \times [0,T))$ as $\varepsilon \searrow 0$, for some $\kappa \in (0,1)$, for any $T > 0$. If we then fix $\Omega' \subset\subset \Omega$ and define $Q'_T = \Omega' \times (t_*, t^*)$ for any $0 < t_* < t^* < \infty$, it is easy to see that our nonlinear term $u^{-p}v^{-q} \in C^{\kappa,\kappa/2}(\overline{Q'}_T)$ for some $\kappa \in (0,1)$. By classical theory of parabolic equations (see Theorem 8.3.7 in [27], for example), we then have that $u \in C^{2+\kappa,1+\kappa/2}(\overline{Q'}_T)$ for any such Q'_T, and so $u \in C^{2+\kappa,1+\kappa/2}(Q_T)$. Consequently, since $T > 0$ was arbitrary, $u \in C^{2+\kappa,1+\kappa/2}(\Omega \times (0,\infty)) \cap C^{1+\kappa,(1+\kappa)/2}(\overline{\Omega} \times [0,\infty))$ is a classical solution to (1.1). A similar result holds for the solution v. This completes the proof.

4. Proof of Theorem 2

We now prove the result stated in Theorem 2. For this proof, we are able to show an inequality of the form

$$X \leq 1 + X^{\alpha}, \quad X \geq 0, \tag{4.1}$$

for some $\alpha \in (0,1)$, which implies that X is bounded. More precisely, we can show that (4.1) is true for $X = \left\| \frac{\cdot}{\psi_2} \right\|_{L^\infty(Q_T)}$. Here, $\cdot = u$ or v, where (u,v) is a global classical solution to (1.1) in $Q_T = \Omega \times (0,T)$ for arbitrary $T > 0$ and ψ_2 is a solution to (2.2) for a particular choice of $\sigma \in (0,1)$.

First, we prove the existence of a global classical solution in Q_T when $d \neq D$. Considering the same perturbed system (3.1) with $d \neq D$, standard parabolic theory still guarantees the existence of a classical solution $(u_\varepsilon, v_\varepsilon)$, for each $\varepsilon > 0$. Then, we notice that for any $d > 0$, Lemma 4 gives us

$$\frac{d}{dt}\int_\Omega \frac{\phi_1^{n+2}}{w_\varepsilon^n}dx \leq \lambda_1 dn \int_\Omega \frac{\phi_1^{n+2}}{w_\varepsilon^n}dx - n\int_\Omega \frac{\phi_1^{n+2}}{w_\varepsilon^{n+p+1}z_\varepsilon^q}dx \tag{4.2}$$

$$\leq \lambda_1 dn \int_\Omega \frac{\phi_1^{n+2}}{w_\varepsilon^n}dx, \tag{4.3}$$

and so by Gronwall's inequality and the assumption that $\varepsilon_0\phi(x) \leq u_0(x)$, we find

$$\int_\Omega \frac{\phi_1^{n+2}}{w_\varepsilon^n}dx \leq e^{n\lambda_1 dt}\int_\Omega \frac{\phi_1^{n+2}}{u_0^n}dx$$

$$\leq \varepsilon_0^{-n}e^{n\lambda_1 dt}\int_\Omega \phi_1^2 dx, \tag{4.4}$$

and so extracting nth roots yields

$$\phi_1(x) \leq \varepsilon_0^{-1}e^{\lambda_1 dt}w_\varepsilon(x,t), \tag{4.5}$$

for all $\varepsilon > 0$ and any $t > 0$. The same procedure gives us that

$$\phi_1(x) \leq \varepsilon_0^{-1}e^{\lambda_1 Dt}z_\varepsilon(x,t). \tag{4.6}$$

Next, by Lemma 7 with u replaced by u_ε we have that for any $\sigma \in (0,1)$,

$$\frac{d}{dt}\int_\Omega \frac{u_\varepsilon^n}{\psi^{n-2}}dx \leq n\int_\Omega \frac{u_\varepsilon^{n-p-1}}{\psi^{n-2}z_\varepsilon^q}dx - dn\int_\Omega \frac{u_\varepsilon^n}{\psi^{n-1+\sigma}}dx. \tag{4.7}$$

Using (4.6) and (2.3), we estimate (4.7) as follows:

$$\frac{d}{dt}\int_\Omega \frac{u_\varepsilon^n}{\psi^{n-2}}dx \leq n\varepsilon_0^{-q}e^{\lambda_1 Dqt}\int_\Omega \frac{u_\varepsilon^{n-p-1}}{\psi^{n-2}\phi_1^q}dx - dn\int_\Omega \frac{u_\varepsilon^n}{\psi^{n-1+\sigma}}dx$$

$$\leq n\gamma_1^q\varepsilon_0^{-q}e^{\lambda_1 Dqt}\int_\Omega \frac{u_\varepsilon^{n-p-1}}{\psi^{n-2+q}}dx - dn\int_\Omega \frac{u_\varepsilon^n}{\psi^{n-1+\sigma}}dx. \tag{4.8}$$

We can apply Lemma 3 to the first term on the right hand side of (4.8) with

$$p_1 = n - p - 1, \quad q_1 = n - 2 + q,$$
$$\alpha_1 = n, \quad \beta_1 = n - 1 + \sigma,$$
$$\theta_1 = 0,$$

to find that

$$\frac{u_\varepsilon^{n-p-1}}{\psi^{n-2+q}} \le \frac{d}{\gamma_1^q \varepsilon_0^{-q} e^{\lambda_1 Dqt}} \frac{u_\varepsilon^n}{\psi^{n-1+\sigma}} + \left(\frac{d}{\gamma_1^q \varepsilon_0^{-q} e^{\lambda_1 Dqt}}\right)^{-(n-p-1)/(p+1)} \frac{1}{\psi^{\eta_1}}, \tag{4.9}$$

where

$$\begin{aligned}\eta_1 &= [(n-2+q)n - (n-1+\sigma)(n-p-1)]/(p+1) \\ &= -\frac{n(\sigma - p - q) + (1-\sigma)}{p+1}.\end{aligned}$$

Notice that we may choose σ sufficiently close to 1 so that $\eta_1 < 0$. Consequently, (4.8) becomes

$$\begin{aligned}\frac{d}{dt}\int_\Omega \frac{u_\varepsilon^n}{\psi^{n-2}}dx &\le n\gamma_1^q \varepsilon_0^{-q} e^{\lambda_1 Dqt}\left(\frac{\gamma_1^q e^{\lambda_1 Dqt}}{\varepsilon_0^q d}\right)^{(n-p-1)/(p+1)} \int_\Omega \psi^{-\eta_1} dx \\ &= dn\left(\frac{\gamma_1^q e^{\lambda_1 Dqt}}{\varepsilon_0^q d}\right)^{n/(p+1)} \int_\Omega \psi^{-\eta_1} dx.\end{aligned} \tag{4.10}$$

Integrating (4.10) from $0 \to t$, extracting nth roots and sending $n \to \infty$ yields

$$\begin{aligned}\frac{u_\varepsilon}{\psi} &\le \left\|\frac{u_\varepsilon}{\psi}\right\|_{L^\infty(\Omega)} \\ &\le \left\|\frac{u_0}{\psi}\right\|_{L^\infty(\Omega)} + \left(\frac{\gamma_1^q e^{\lambda_1 Dqt}}{\varepsilon_0^q d}\right)^{1/(p+1)} \left\|\psi^{(\sigma-p-q)/(p+1)}\right\|_{L^\infty(\Omega)}.\end{aligned} \tag{4.11}$$

Hence, $\frac{u_\varepsilon}{\psi}$ is uniformly bounded for any $\varepsilon > 0$ in Q_T, for any $T > 0$. The same procedure applied to $\frac{v_\varepsilon}{\psi}$ also gives us

$$\begin{aligned}\frac{v_\varepsilon}{\psi} &\le \left\|\frac{v_\varepsilon}{\psi}\right\|_{L^\infty(\Omega)} \\ &\le \left\|\frac{v_0}{\psi}\right\|_{L^\infty(\Omega)} + \left(\frac{\gamma_1^r e^{\lambda_1 drt}}{\varepsilon_0^r D}\right)^{1/(s+1)} \left\|\psi^{(\sigma-r-s)/(s+1)}\right\|_{L^\infty(\Omega)},\end{aligned} \tag{4.12}$$

where we may again choose σ sufficiently close to 1 so that $\sigma - r - s > 0$. Hence, $\frac{v_\varepsilon}{\psi}$ is uniformly bounded for any $\varepsilon > 0$ in Q_T, for any $T > 0$. Using these bounds, we can see that (3.34) and (3.35) are true in Q_T, and so we have that $(u_\varepsilon + \varepsilon, v_\varepsilon + \varepsilon)$ and the nonlinear terms appearing in (3.1) are uniformly bounded in $L^\xi(Q_T)$ for some $\xi > 1$. More precisely, by standard parabolic L^p-theory we have that $(u_\varepsilon, v_\varepsilon)$ is uniformly bounded in $W^{2,1}_{\xi_1}[\Omega \times (0,T)] \times W^{2,1}_{\xi_2}[\Omega \times (0,T)]$ for any $(\xi_1, \xi_2) \in \left(1, \frac{1}{p+q}\right) \times \left(1, \frac{1}{r+s}\right)$. Hence, there exists a subsequence such that $(u_\varepsilon, v_\varepsilon) \to (u, v)$, where (u, v) is a weak solution to our original problem. Since

$p+q < \frac{1}{N}$, $r+s < \frac{1}{N}$, Sobolev embedding ensures that in fact $(u,v) \in \left[C^1(\overline{\Omega})\right]^2$. We now show that in fact (u,v) are uniformly bounded for all $t \in (0,\infty)$.

By Lemma 4 with $\phi = \psi_1$, we have that

$$\frac{d}{dt}\int_\Omega \frac{\psi_1^{n+2}}{u^n}dx \le dn\int_\Omega \frac{\psi_1^{n+1-\sigma_1}}{u^n}dx - n\int_\Omega \frac{\psi_1^{n+2}}{u^{n+p+1}v^q}dx, \tag{4.13}$$

where $\psi_1 \in C^2(\Omega)\cap C^1(\overline{\Omega})$ solves (2.2) for some $\sigma_1 \in (0,1)$ to be chosen. Notice that the regularity of u, v, ψ_1 imply that the integrals in (4.13) are well defined. Applying Lemma 3 to the first term on the right hand side of (4.13) with

$$\begin{aligned} p_1 = n+1-\sigma_1, \quad q_1 = n, \quad q_2 = 0,\\ \alpha_1 = n+2, \quad \beta_1 = n+p+1, \quad \beta_2 = q, \end{aligned}$$

leaves us with

$$\frac{\psi_1^{n+1-\sigma_1}}{u^n} \le \left(\frac{1}{d}\right)\frac{\psi_1^{n+2}}{u^{n+p+1}v^q} + \left(\frac{1}{d}\right)^{-\frac{(p_1-\theta_1)}{1+\sigma_1}}\frac{\psi_1^{\theta_1}}{u^{\eta_1}v^{\eta_2}}, \tag{4.14}$$

where

$$\begin{aligned}\eta_1 &= \left[n(n+2-\theta_1) - (n+p+1)(n+1-\sigma_1-\theta_1)\right]/(1+\sigma_1),\\ \eta_2 &= -q(n+1-\sigma_1-\theta_1)/(1+\sigma_1).\end{aligned}$$

If we set $\eta_1 = 0$ and solve for θ_1, we see that

$$\begin{aligned} &0 = n(\sigma_1 - p) - (p+1)(1-\sigma_1) + \theta_1(p+1)\\ \iff &\theta_1 = \left((p+1)(1-\sigma_1) - n(\sigma_1-p)\right)/(p+1)\\ &\quad = (1-\sigma_1) - \frac{n(\sigma_1-p)}{p+1}, \end{aligned} \tag{4.15}$$

and so

$$\begin{aligned}\eta_2 &= -q\left(n+1-\sigma_1-(1-\sigma_1)+\frac{n(\sigma_1-p)}{p+1}\right)\\ &= -\frac{qn}{p+1}.\end{aligned}$$

Also note that

$$p_1 - \theta_1 = \frac{n(1+\sigma_1)}{p+1}.$$

Hence, (4.13) becomes

$$\begin{aligned}\frac{d}{dt}\int_\Omega \frac{\psi_1^{n+2}}{u^n}dx &\le dn\left(\frac{1}{d}\right)^{-n/(p+1)}\int_\Omega \frac{v^{nq/(p+1)}}{\psi_1^{-\theta_1}}dx\\ &= d^{(n+p+1)/(p+1)}n\int_\Omega \frac{v^{nq/(p+1)}}{\psi_1^{-\theta_1}}dx.\end{aligned} \tag{4.16}$$

Notice that the right hand side of (4.16) is well defined whenever

$$\frac{nq}{p+1} + \theta_1 = \frac{n}{p+1}(p+q-\sigma_1) + (1-\sigma_1) > 0,$$

which is true if we set $\sigma_1 = \min\{p+q, r+s\} < 1$. If we integrate both sides of (4.16) from $0 \to t$, we then have

$$\int_\Omega \frac{\psi_1^{n+2}}{u^n} dx \le \int_\Omega \frac{\psi_1^{n+2}}{u_0^n} dx + d^{(n+p+1)/(p+1)} n \int_0^t \int_\Omega \frac{v^{nq/(p+1)}}{\psi_1^{-\theta_1}} dx dt. \tag{4.17}$$

Extracting nth roots and taking $n \to \infty$ yields

$$\begin{aligned} \frac{\psi_1(x)}{u(x,t)} &\le \left\| \frac{\psi_1}{u} \right\|_{L^\infty(\Omega)} \\ &\le \max\left\{ \left\| \frac{\psi_1}{u_0} \right\|_{L^\infty(\Omega)}, \; d^{1/(p+1)} \left\| \frac{v^{q/(p+1)}}{\psi_1^{(\sigma_1-p)/(p+1)}} \right\|_{L^\infty(Q_T)} \right\} \\ &\le \varepsilon_0 + d^{1/(p+1)} \left\| \frac{v^{q/(p+1)}}{\psi_1^{(\sigma_1-p)/(p+1)}} \right\|_{L^\infty(Q_T)}, \end{aligned} \tag{4.18}$$

where ε_0 is defined in (1.4). This implies that

$$\left\| \frac{\psi_1}{u} \right\|_{L^\infty(Q_T)} \le \varepsilon_0 + d^{1/(p+1)} \left\| \frac{v^{q/(p+1)}}{\psi_1^{(\sigma_1-p)/(p+1)}} \right\|_{L^\infty(Q_T)}. \tag{4.19}$$

We now apply the same procedure for $v(x,t)$. By Lemma 4,

$$\frac{d}{dt} \int_\Omega \frac{\psi_1^{n+2}}{v^n} dx \le Dn \int_\Omega \frac{\psi_1^{n+1-\sigma_1}}{v^n} dx - n \int_\Omega \frac{\psi_1^{n+2}}{v^{n+s+1} u^r} dx. \tag{4.20}$$

Applying Lemma 3 with

$$p_1 = n+1-\sigma_1, \quad q_1 = n, \quad q_2 = 0,$$
$$\alpha_1 = n+2, \quad \beta_1 = n+s+1, \quad \beta_2 = r,$$

we can see that

$$\frac{\psi_1^{n+1-\sigma_1}}{v^n} \le \left(\frac{1}{D}\right) \frac{\psi_1^{n+2}}{v^{n+s+1} u^r} + \left(\frac{1}{D}\right)^{-(p_1-\theta_1)/(1+\sigma_1)} \frac{\psi_1^{\theta_1}}{v^{\eta_1} u^{\eta_2}}, \tag{4.21}$$

where

$$\begin{aligned} \eta_1 &= [n(n+2-\theta_1) - (n+s+1)(n+1-\sigma_1-\theta_1)]/(1+\sigma_1) \\ &= [n(\sigma_1 - s) - (s+1)(1-\sigma_1-\theta_1)]/(1+\sigma_1), \\ \eta_2 &= -r(1-\sigma_1-\theta_1)/(1+\sigma_1). \end{aligned}$$

Setting $\eta_1 = 0$ and repeating the process in lines (4.15)–(4.19) one can obtain

$$\left\|\frac{\psi_1}{v}\right\|_{L^\infty(Q_T)} \leq \varepsilon_0 + D^{1/(s+1)} \left\|\frac{u^{r/(s+1)}}{\psi_1^{(\sigma_1-s)/(s+1)}}\right\|_{L^\infty(Q_T)}. \tag{4.22}$$

We now obtain upper bounds for u and v. Let $\psi_2 \in C^2(\Omega) \cap C^1(\overline{\Omega})$ solve (2.2) for some $\sigma_2 \in (0,1)$ to be chosen. Then, by Lemma 7 we have that for any $n > 2$,

$$\frac{d}{dt}\int_\Omega \frac{u^n}{\psi_2^{n-2}}dx \leq n\int_\Omega \frac{u^{n-p-1}}{\psi_2^{n-2}v^q}dx - dn\int_\Omega \frac{u^n}{\psi_2^{n-1+\sigma_2}}dx. \tag{4.23}$$

Applying Lemma 3 to the first term on the right hand side of (4.23) with

$$p_1 = n-p-1, \quad q_1 = n-2, \quad q_2 = q,$$
$$\alpha_1 = n, \quad \beta_1 = n-1+\sigma_2, \quad \beta_2 = 0, \quad \theta_1 = 0,$$

gives us that

$$\frac{u^{n-p-1}}{\psi_2^{n-2}v^q} \leq d\frac{u^n}{\psi_2^{n-1+\sigma_2}} + d^{-(n-p-1)/(p+1)}\frac{1}{\psi_2^{\eta_1}v^{\eta_2}},$$

where

$$\begin{aligned}\eta_1 &= \left[(n-2)n - (n-1+\sigma_2)(n-p-1)\right]/(p+1)\\ &= \frac{n(p-\sigma_2)}{p+1} - (1-\sigma_2),\\ \eta_2 &= \frac{nq}{p+1}.\end{aligned}$$

Hence, (4.23) becomes

$$\frac{d}{dt}\int_\Omega \frac{u^n}{\psi_2^{n-2}}dx \leq n\left(\frac{1}{d}\right)^{(n-p-1)/(p+1)}\int_\Omega \frac{\psi_2^{-\eta_1}}{v^{nq/(p+1)}}dx. \tag{4.24}$$

Notice that the integral on the right hand side of (4.24) is well defined since

$$-\eta_1 - \frac{nq}{p+1} = \frac{n(\sigma_2-p-q)}{p+1} + (1-\sigma_2) > 0$$

if we choose $\sigma_2 = \max\{p+q, r+s\}$. Integrating (4.24) from $0 \to t$, extracting nth roots and sending $n \to \infty$ then yields

$$\begin{aligned}\frac{u(x,t)}{\psi_2(x)} &\leq \left\|\frac{u}{\psi_2}\right\|_{L^\infty(\Omega)}\\ &\leq \max\left\{\left\|\frac{u_0}{\psi_2}\right\|_{L^\infty(\Omega)}, \left(\frac{1}{d}\right)^{1/(p+1)}\left\|\frac{\psi_2^{(\sigma_2-p)/(p+1)}}{v^{q/(p+1)}}\right\|_{L^\infty(Q_T)}\right\}\\ &\leq \left\|\frac{u_0}{\psi_2}\right\|_{L^\infty(\Omega)} + \left(\frac{1}{d}\right)^{1/(p+1)}\left\|\frac{\psi_2^{(\sigma_2-p)/(p+1)}}{v^{q/(p+1)}}\right\|_{L^\infty(Q_T)},\end{aligned} \tag{4.25}$$

for all $(x,t) \in \Omega \times (0,T)$. Hence,

$$\left\| \frac{u}{\psi_2} \right\|_{L^\infty(Q_T)} \leq \left\| \frac{u_0}{\psi_2} \right\|_{L^\infty(\Omega)} + \left(\frac{1}{d} \right)^{1/(p+1)} \left\| \frac{\psi_2^{(\sigma_2-p)/(p+1)}}{v^{q/(p+1)}} \right\|_{L^\infty(Q_T)}. \tag{4.26}$$

Similarly, we have that

$$\left\| \frac{v}{\psi_2} \right\|_{L^\infty(Q_T)} \leq \left\| \frac{v_0}{\psi_2} \right\|_{L^\infty(\Omega)} + \left(\frac{1}{D} \right)^{1/(s+1)} \left\| \frac{\psi_2^{(\sigma_2-s)/(s+1)}}{u^{r/(s+1)}} \right\|_{L^\infty(Q_T)}. \tag{4.27}$$

We now proceed as follows: since $\sigma_1, \sigma_2 \in (0,1)$, by Theorem 2.1 in [16] there exist positive constants a_1, a_2 such that

$$a_1\psi_1(x) \leq \psi_2(x) \leq a_2\psi_1(x), \quad x \in \overline{\Omega}. \tag{4.28}$$

We can use this fact to further estimate the bound obtained for u/ψ_2 in line (4.26):

$$\begin{aligned} \left\| \frac{u}{\psi_2} \right\|_{L^\infty(Q_T)} &\leq \left\| \frac{u_0}{\psi_2} \right\|_{L^\infty(\Omega)} + \left(\frac{1}{d} \right)^{1/(p+1)} \left\| \frac{\psi_2^{(\sigma_2-p)/(p+1)}}{v^{q/(p+1)}} \right\|_{L^\infty(Q_T)} \\ &= \left\| \frac{u_0}{\psi_2} \right\|_{L^\infty(\Omega)} + \left(\frac{1}{d} \right)^{1/(p+1)} \left\| \left(\frac{\psi_1}{v} \right)^{q/(p+1)} \left(\frac{\psi_2^{\sigma_2-p}}{\psi_1^q} \right)^{1/(p+1)} \right\|_{L^\infty(Q_T)} \\ &\leq \left\| \frac{u_0}{\psi_2} \right\|_{L^\infty(\Omega)} + M_1 \left\| \frac{\psi_1}{v} \right\|_{L^\infty(Q_T)}^{q/(p+1)}, \end{aligned} \tag{4.29}$$

where

$$M_1 = \left(\frac{a_2^q}{d} \right)^{1/(p+1)} \|\psi_2\|_{L^\infty(\Omega)}^{(\sigma_2-p-q)/(p+1)}.$$

We now apply the bound obtained for ψ_1/v in line (4.22) to (4.29) to see that in fact

$$\begin{aligned} \left\| \frac{u}{\psi_2} \right\|_{L^\infty(Q_T)} - \left\| \frac{u_0}{\psi_2} \right\|_{L^\infty(\Omega)} &\leq M_1 \left\| \frac{\psi_1}{v} \right\|_{L^\infty(Q_T)}^{q/(p+1)} \\ &\leq M_1 \left(\varepsilon_0 + D^{1/(s+1)} \left\| \frac{u^{r/(s+1)}}{\psi_1^{(\sigma_1-s)/(s+1)}} \right\|_{L^\infty(Q_T)} \right)^{q/(p+1)} \\ &\leq M_1 \left(\varepsilon_0 + D^{1/(s+1)} \left\| \frac{u}{\psi_1} \right\|_{L^\infty(Q_T)}^{r/(s+1)} \|\psi_1\|_{L^\infty(\Omega)}^{(r+s-\sigma_1)/(s+1)} \right)^{q/(p+1)} \\ &\leq M_1 \max\left\{ 1, \varepsilon_0 + D^{1/(s+1)} \left\| \frac{u}{\psi_1} \right\|_{L^\infty(Q_T)}^{r/(s+1)} \|\psi_1\|_{L^\infty(\Omega)}^{(r+s-\sigma_1)/(s+1)} \right\}, \end{aligned} \tag{4.30}$$

where the last inequality holds since $q/(p+1) < 1$. Assuming this maximum is not 1 (otherwise we are done), we apply (4.28) once more to conclude that

$$\left\|\frac{u}{\psi_2}\right\|_{L^\infty(Q_T)} \leq M_2 + M_3 \left\|\frac{u}{\psi_1}\right\|_{L^\infty(Q_T)}^{r/(s+1)}$$
$$\leq M_2 + M_3 a_2^{r/(s+1)} \left\|\frac{u}{\psi_2}\right\|_{L^\infty(Q_T)}^{r/(s+1)}, \tag{4.31}$$

where

$$M_2 = \left\|\frac{u_0}{\psi_2}\right\|_{L^\infty(\Omega)} + \varepsilon_0 M_1, \qquad M_3 = M_1 D^{1/(s+1)} \|\psi_1\|_{L^\infty(\Omega)}^{(r+s-\sigma_1)/(s+1)}.$$

Consequently, since $r/(s+1) < 1$, $u(x,t)/\psi_2(x)$ is uniformly bounded for all $t > 0$. Note that (4.31) is of the form described in (4.1) with the inclusion of some finite constants and $\alpha = r/(s+1)$.

We may now repeat this procedure for v. From line (4.27) and (4.28) we have that

$$\left\|\frac{v}{\psi_2}\right\|_{L^\infty(Q_T)} \leq \left\|\frac{v_0}{\psi_2}\right\|_{L^\infty(\Omega)} + \left(\frac{1}{D}\right)^{1/(s+1)} \left\|\frac{\psi_2^{(\sigma_2-s)/(s+1)}}{u^{r/(s+1)}}\right\|_{L^\infty(Q_T)}$$
$$\leq \left\|\frac{v_0}{\psi_2}\right\|_{L^\infty(\Omega)} + M_4 \left\|\frac{\psi_1}{u}\right\|_{L^\infty(Q_T)}^{r/(s+1)}, \tag{4.32}$$

where

$$M_4 = \left(\frac{a_2^r}{D}\right)^{1/(s+1)} \|\psi_2\|_{L^\infty(\Omega)}^{(\sigma_2-r-s)/(s+1)}$$

Using the bound for ψ_1/u found in line (4.19) and repeating the estimates found in line (4.30), we arrive at

$$\left\|\frac{v}{\psi_2}\right\|_{L^\infty(Q_T)} \leq M_5 + M_6 \left\|\frac{v}{\psi_1}\right\|_{L^\infty(Q_T)}^{q/(p+1)}$$
$$\leq M_5 + M_6 a_2^{q/(p+1)} \left\|\frac{v}{\psi_2}\right\|_{L^\infty(Q_T)}^{q/(p+1)}, \tag{4.33}$$

where

$$M_5 = \left\|\frac{v_0}{\psi_2}\right\|_{L^\infty(\Omega)} + \varepsilon_0 M_4 \qquad M_6 = M_3 d^{1/(p+1)} \|\psi_1\|_{L^\infty(\Omega)}^{(p+q-\sigma_1)/(p+1)}.$$

As argued previously, since $q/(p+1) < 1$, we conclude that $v(x,t)/\psi_2(x)$ is uniformly bounded for all $t > 0$.

Finally, if we refer to line (4.16), we see that $\psi_1(x)/u(x,t)$ is uniformly bounded for all $t > 0$ since the integral on the right hand side of (4.16) is uniformly bounded for all $t > 0$ as v/ψ_1 is uniformly bounded. Similarly, we see that $\psi_1(x)/v(x,t)$ is uniformly bounded for all $t > 0$ since u/ψ_1 is uniformly bounded. One may then simply use a bound of the form found in line (4.28) in order to compare ϕ_1 to ψ_1 and ψ_2. This yields the final bound asserted in the statement of the theorem, completing the prof.

5. Discussion

In this paper, we have discussed the existence, regularity and boundedness of solutions to a singular parabolic system related to numerous biological, physical and chemical phenomena. When $d = D$, under appropriate assumptions on the exponents p, q, r, s as well as reasonable compatibility conditions on the initial data $u_0(x), v_0(x)$, we are able to obtain the existence of at least one globally bounded weak solution (u, v). Using the Sobolev embedding, we are then able to obtain higher regularity of the resultant weak solutions. In the case of $d \neq D$, under an assumption on the exponents that ensures the existence of at least one global classical solution (u, v), we are able to prove that in fact this solution is uniformly bounded from above and below by a constant multiplied by the first eigenfunction of $-\Delta$ subject to homogeneous Dirichlet boundary data in Ω. These results generalize previous results found in [9] in a significant way.

From these results, one can see that the functional method is a very powerful tool in proving the existence of solutions to parabolic systems with difficult to treat nonlinearities. Originally motivated by the techniques involved in the treatment of homogeneous Neumann problems as in [22] and [6], this method has since been adapted to treat problems featuring homogeneous Dirichlet boundary data as well. In this case, the reaction terms grow singular as one approaches the boundary $\partial\Omega$. In such cases, standard methods, such as sub/super solutions or variational techniques, are insufficient tools to prove the existence of solutions. However, there are drawbacks to the methods used in this paper as well. The most noticeable drawback is the necessary use of L^p theory of parabolic equations paired with Sobolev embeddings. These techniques give a restriction on the size of the exponent(s) in that they cannot exceed N^{-1}. In the case of elliptic equations or systems, it is possible to allow the exponents to leave the interval $(0, 1)$. In these cases, solutions still exist, though they are no longer linear up to the boundary, e.g. $u \notin C^1(\overline{\Omega})$. However, it is possible to show that the solution $u \in C^\alpha(\overline{\Omega})$ for some $\alpha \in (0, 1)$, and furthermore, α decreases as the exponents in the singular reaction terms increase. This implies that the size of the exponents on the singular terms have an intimate relation with the steepness of the solution near the boundary. It is interesting that in such cases, one can find a classical solution $u \in C^2(\Omega) \cap C^\alpha(\overline{\Omega})$ for some $\alpha \in (0, 1)$ that is *not* a weak solution. It is reasonable to assume that similar behaviour is expected in the parabolic case, but it does not appear to be an easy generalization to be made concrete. To be more precise, such a generalization would require moving away from parabolic L^p theory and the use of Sobolev embeddings.

Accordingly, future works include relaxing conditions on the exponents p, q, r, s. Such relaxations will give a more complete picture on the behaviour of the solutions, as well as the steep boundary layers they may form. Furthermore, relaxing these conditions may allow one to prove the finite blowup time of solutions, see the recent paper by Chen and Xu [10] for example, which may correspond to the nonexistence criteria described in papers such as [14]. As a final note, it is easy to see that the methods presented here can also include the case when functions $f(x), g(x)$ appear in the numerator of the nonlinearities with conditions similar to those found in [9].

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