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On the Characterization of the Fréchet Derivative with Respect to a Lipschitz Domain of the Acoustic Scattered Field

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In this paper, we consider the Fréchet differentiability with respect to a given domain of the solution of direct acoustic scattering problems with sound-soft, sound-hard, or lossy boundary conditions. We propose a new proof for characterizing the derivative of the acoustic scattered field as the solution of a particular direct scattering problem. This proof assumes that the boundary of the scatterer is only Lipschitzian and therefore extends this well-known characterization to obstacles with *rough* surfaces. Our result can be of practical interest because the characterization of the derivative of the acoustic scattered field has a great potential for the solution of inverse acoustic scattering problems by Newton-type methods. © 1999 Academic Press

Key Words: acoustics; scattering; Lipschitz domain; Fréchet differentiability; domain derivative.

1. INTRODUCTION

Recently we proved that the acoustic scattered field and its corresponding far-field pattern are continuously Fréchet differentiable with respect to the domain of a given obstacle [1]. We have established this result for acoustic scattering problems with a sound-soft obstacle, a sound-hard obstacle, or lossy boundary conditions and have assumed that the shape of the scatterer and the relevant perturbations are only Lipschitzian. Here, our aim is to propose a new and *simple* proof for characterizing the derivative with respect to an obstacle's domain of the acoustic scattered field as the solution of a particular direct scattering problem. Our proof assumes that the shape of the scatterer is only Lipschitzian, and that the

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admissible perturbations are $\mathscr{C}^1$-functions. It is based on the fact that the chain rule is also valid for infinite-dimensional spaces [1–3] and on the classical trace theorems [4–6]. Our result has a great potential for reducing the complexity of the solution of inverse scattering problems by iterative methods [7–9].

For a sound-soft scatterer, a similar result has previously been established by Kirsch [10], using a variational approach, and by Potthast [11], using an integral representation of the far-field pattern. However, both of these previous investigations have assumed $\mathscr{C}^2$ regularity for the boundary of the scatterer as well as for the relevant perturbations. Recently, Hettlich has extended this result for sound-soft scatterers that are also of class $\mathscr{C}^2$, but for perturbations of the domain that are only $\mathscr{C}^1$-functions. For both cases of a sound-hard obstacle and lossy boundary conditions, he has also proposed a characterization of the derivative with respect to the domain of the scattered field and has assumed for this purpose that the boundary of the domain is of class $\mathscr{C}^{2,1}$. However, it has been reported to us [19] that the characterization published in [9] contains an error. For the case of a sound-hard obstacle, Potthast has already proved in [13] the same characterization of the derivative with respect to the domain of the scattered field that we establish in this paper. However, his proof assumed more regularity for the boundary of the scatterer and the relevant perturbations than ours. Indeed, Potthast's proof requires that the boundary of the scatterer be of class $\mathscr{C}^{2,\alpha}$, and that the perturbations of the scatterer by $\mathscr{C}^2$-functions.

The remainder of this paper is organized as follows. In Section 2, we first specify the nomenclature and assumptions used in this work, then formulate the focus acoustic scattering problem [1, 14, 15]. In Section 3, we state as a theorem the characterization of the derivative of the scattered field with respect to the domain as the solution of a particular direct acoustic problem and deduce as a corollary a characterization of the Fréchet derivative of its far-field pattern. In Section 4 we prove the main result of this work.

2. PROBLEM STATEMENT

2.1. *Nomenclature and Assumptions*

Throughout this paper, we adopt the following nomenclature and assumptions

- Ω is a bounded domain of $\mathbb{R}^3$ representing an impenetrable obstacle.

- $\Omega^e = \mathbb{R}^3 \setminus \overline{\Omega}$ is the homogeneous isotropic medium in $\mathbb{R}^3$ where the obstacle is embedded.
- Γ is the boundary of Ω^e and is assumed to be Lipschitzian [5, 6, 16].
- $x = (x_1, x_2, x_3)$ is a point of $\mathbb{R}^3$, and $r = \|x\|_2$ is the distance from an origin point to x.
- "." denotes the scalar product in $\mathbb{R}^3$.
- $S^1 = \{x \in \mathbb{R}^3 \setminus \|x\|_2 = 1\}$ is the unit sphere in $\mathbb{R}^3$.
- ∇ is the gradient operator in $\mathbb{R}^3$.
- ∇ is the Laplacian operator in $\mathbb{R}^3$.
- ν is the outward normal to Γ and $\partial/\partial\nu$ is the normal derivative operator.
- k is a positive number representing the wavenumber of the incident wave.
- d is a vector of S^1 representing the normalized direction of the incident planar wave.
- $\mathscr{C}^{0,1}(\mathbb{R}^3)$ is the space of Lipschitzian functions in $\mathbb{R}^3$.
- $\mathscr{C}^1(\mathbb{R}^3)$ is the space of functions with continuous derivatives in $\mathbb{R}^3$.
- $\mathscr{D}(\Omega^e)$ is the standard space of infinitely differentiable functions with compact support in Ω^e.
- $\mathscr{D}'(\Omega^e)$ is the standard space of distributions.
- $L^2(\Omega^e)$ is the standard Lebesgue space and $H^1(\Omega^e)$ is the Sobolev space.
- $L^2_{\text{loc}}(\Omega^e)$ is the space of functions that are in $L^2(D)$ for any open bounded set D in Ω^e.
- $H_{\text{loc}}(\Delta, \Omega^e) = \{w; w \in L^2_{\text{loc}}(\Omega^e) \text{ and } \Delta w \in L^2_{\text{loc}}(\Omega^e)\}$.
- $H^1_{\text{loc}}(\Delta, \Omega^e) = \{w; w \in H^1_{\text{loc}}(\Omega^e) \text{ and } \Delta w \in L^2_{\text{loc}}(\Omega^e)\}$.
- $H^s(\Gamma)$ are the trace Sobolev spaces.
- $I: \mathbb{R}^3 \to \mathbb{R}^3$ is the identity mapping.
- $\theta: \mathbb{R}^3 \to \mathbb{R}^3$ denotes an admissible perturbation. θ is assumed to be in $(\mathscr{C}^1(\mathbb{R}^3))^3$ and to have a compact support. θ is also supposed to be small enough (for example, $\|\theta\|_{\mathscr{C}^1(\mathbb{R}^3))^3} < \frac{1}{2}$) such that $I + \theta$ is bijective in $\mathbb{R}^3$.
- $[\theta']$ is the Jacobian matrix of θ. Hence, $[\theta'] = (\partial\theta_l/\partial x_j)$.
- $[\theta']^T$ is the transpose Jacobian matrix of θ. Hence, $[\theta']^T = (\partial\theta_j/\partial x_l)$.
- $\Omega^e_\theta = (I + \theta)(\Omega^e)$ is an admissible perturbed configuration of the reference domain Ω^e. Note that $\Omega^e_0 = \Omega^e$.

- Γ_θ is the boundary of Ω_θ^e and therefore is an admissible perturbation of the reference boundary Γ. Note that $\Gamma_0 = \Gamma$.
- $w_\theta = w(\theta, x)$ and $w = w(0, x)$.

Furthermore, in this paper we manipulate a function u_θ that is defined on an open set Ω_θ^e that varies with θ. Hence, this function cannot be differentiated with respect to θ in the classical sense. For this reason, we follow [2, 3] and adopt the following concept of a local derivative.

We say that $\theta \mapsto w_\theta$ is locally differentiable if for every open set D strictly included in Ω^e and strictly included in Ω_θ^e the restriction of w_θ to D is differentiable. We denote by $w' = (\partial w_\theta / \partial\theta)(0)h$ the local derivative of w_θ at $\theta = 0$ and in the direction h, where h is a vector field satisfying $h \in (\mathscr{C}^1(\mathbb{R}^3))^3$.

2.2. *Mathematical Formulation of the Problem*

The scattering of time-harmonic acoustic waves by an impenetrable obstacle embedded in a homogeneous medium can be formulated as the exterior boundary value problem (BVP) [1, 14, 15]

$$\begin{cases} \Delta u_\theta + k^2 u_\theta = 0 & \text{in } \Omega_\theta^e \\ B_\theta u_\theta = -B_\theta e^{ikx.d} & \text{on } \Gamma_\theta \\ \lim\limits_{r\to\infty} r\left(\dfrac{\partial u_\theta}{\partial r} - iku_\theta\right) = 0, & \end{cases} \tag{1}$$

where B_θ is a linear operator given by

$$B_\theta = aI + b\frac{\partial}{\partial \nu_\theta}. \tag{2}$$

Here, a and b are constants that do not vanish simultaneously, and ν_θ is the outward normal to Ω_θ^e. Since Γ_θ is Lipschitzian, it follows that the normal vector ν_θ is defined almost everywhere on Γ_θ [5, 6].

For a given θ, the operator B_θ defined in Eq. (2) allows a compact representation of all of the Dirichlet, Neumann, and lossy boundary conditions that are usually encountered in acoustic scattering problems. For $a = 1$ and $b = 0$, B_θ simplifies to the Dirichlet operator that is typically used for a sound-soft scatterer. For a sound-hard scatterer, the Neumann boundary condition is recovered by setting $a = 0$ and $b = 1$. Otherwise, B_θ reproduces the lossy boundary conditions [1, 14, 15].

In this paper, we also characterize the derivative of the acoustic far-field pattern. We remind the reader that the scattering amplitude $u_{\theta,\infty}$ of the acoustic scattered field u_θ that is the solution of BVP (1) is defined on the unit sphere S^1 and is obtained from the asymptotic behavior of u_θ [14, 15] as follows:

$$u_\theta(x) = \frac{e^{ikr}}{r}\left(u_{\theta,\infty}\left(\frac{x}{r}\right) + O\left(\frac{1}{r}\right)\right); \qquad r = \|x\|_2 \to +\infty. \tag{3}$$

3. THE MAIN RESULTS

Our aim is to prove that the local derivative of the acoustic scattered field with respect to the boundary of the scatterer is the solution of a boundary value problem that can be *viewed* as a particular direct acoustic scattering problem.

THEOREM. *Let u' be the local derivative at $\theta = 0$ and in a direction $h \in (\mathscr{C}^1(\mathbb{R}^3))^3$ of the solution u_θ of BVP* (1). *Then, u' is the solution of the boundary value problem*

$$\begin{cases} \Delta u' + k^2 u' = 0 & \text{in } \Omega^e \\ Bu' = F(u) & \text{on } \Gamma \\ \lim\limits_{r\to\infty} r\left(\dfrac{\partial u'}{\partial r} - iku'\right) = 0, & \end{cases} \tag{4}$$

where the function F depends on the expression of the boundary operator B.

For a sound-soft scatterer, B is the Dirichlet operator (see Eq. (2) with $a = 1$ and $b = 0$), and

$$F(u) = F_D(u),$$

where

$$F_D(u) = -\sum_{j=1}^{3} h_j \frac{\partial}{\partial x_j}(u + e^{ikx.d}) \qquad \text{on } \Gamma\ a.e. \tag{5}$$

For a sound-hard scatterer, B is the Neumann operator (see Eq. (2) with $a = 0$ and $b = 1$), and

$$F(u) = F_N(u),$$

where

$$F_N(u) = -\sum_{\{l,j\}=1}^{3} \frac{\partial^2(u + e^{ikx.d})}{\partial x_l \partial x_j} h_j \nu_l + \sum_{\{l,j\}=1}^{3} \frac{\partial}{\partial x_l}(u + e^{ikx.d}) \frac{\partial h_l}{\partial x_j} \nu_j \qquad \text{on } \Gamma \; a.e. \tag{6}$$

For a lossy boundary condition, B is a Robin-type operator (see Eq. (2) with $a \neq 0$ and $b \neq 0$), and

$$F(u) = F_R(u),$$

where

$$F_R(u) = aF_D(u) + a\left(\sum_{\{l,j\}=1}^{3} \frac{\partial h_l}{\partial x_j} \nu_j \nu_l\right)(u - f) + bF_N(u) \qquad \text{on } \Gamma \; a.e. \tag{7}$$

From the above theorem and the fact that the mapping $\theta \to u_{\theta,\infty}$, from $(\mathscr{C}^1(\mathbb{R}^3))^3$ to $\mathscr{C}^m(S^1)$, is continuously Fréchet differentiable at $\theta = 0$ in the direction $h \in (\mathscr{C}^1(\mathbb{R}^3))^3$ (see Theorem 4.2 in [1]), and from the uniqueness of the acoustic far-field pattern [14], we deduce the following characterization of the derivative of the far-field pattern $(\partial u_{\theta,\infty}/\partial\theta)(0)h$.

COROLLARY. *Let u'_∞ be the far-field pattern of the solution u' of BVP* (4), *and let $(\partial u_{\theta,\infty}/\partial\theta)(0)h$ be the derivative at $\theta = 0$ and in a direction h of the far-field pattern $u_{\theta,\infty}$ of the solution u_θ of BVP* (1). *We have*

$$\frac{\partial u_{\theta,\infty}}{\partial\theta}(0)h = u'_\infty. \tag{8}$$

4. PROOF OF THE THEOREM

We prove our theorem in five steps, each formulated as a lemma. The first lemma states that the local derivative u' at $\theta = 0$ and in a direction $h \in (\mathscr{C}^1(\mathbb{R}^3))^3$ of the solution of BVP (1) is a solution of the Helmholtz equation.

LEMMA 1. *The local derivative u' of the acoustic scattered field at $\theta = 0$ and in a direction $h \in (\mathscr{C}^1(\mathbb{R}^3))^3$ satisfies the following equation*:

$$\Delta u' + k^2 u' = 0 \qquad \text{in } \Omega^e. \tag{9}$$

Proof of Lemma 1. Let

$$A = \Delta + k^2 I \tag{10}$$

and define

$$\phi_\theta = A u_\theta , \tag{11}$$

where u_θ is the acoustic scattered field that is the solution of BVP (1). We have

$$\phi_\theta = 0 \qquad \text{in } \Omega_\theta^e, \tag{12}$$

and thus,

$$\phi_\theta \circ (I + \theta) = 0 \qquad \text{in } \Omega^e. \tag{13}$$

Furthermore, since A is a linear and continuous operator from $H^1_{\text{loc}}(\Omega^e)$ into $\mathscr{D}'(\Omega^e)$, A is differentiable at least in the distribution sense, i.e., $v \mapsto \langle Av, \psi \rangle$ is differentiable for each $\psi \in \mathscr{D}(\Omega^e)$, and

$$\frac{\partial A}{\partial v} = A. \tag{14}$$

Therefore, it follows from the differentiability of $\theta \mapsto u_\theta \circ (I + \theta)$ and $\theta \mapsto u_\theta$ ([1, Theorem 3.3 and Corollary 3.4]) that $\theta \mapsto \phi_\theta \circ (I + \theta)$ (resp. $\theta \mapsto \phi_\theta$) is continuously Fréchet differentiable (resp. locally continuously Fréchet differentiable)—at least in the distribution sense—at $\theta = 0$ and in a direction $h \in (\mathscr{C}^1(\mathbb{R}^3))^3$. Moreover, for every *admissible* perturbation θ, i.e., $\theta \in (\mathscr{C}^1(\mathbb{R}^3))^3$ and θ in the neighborhood of 0, their derivatives satisfy [2, 3]:

$$\frac{\partial \phi_\theta}{\partial \theta}(0)h = \frac{\partial}{\partial \theta}\big(\phi_\theta \circ (I + \theta)\big)(0)h - h.\nabla\phi \qquad \text{in } \Omega^e. \tag{15}$$

From Eqs. (11–13) and (15), we deduce that

$$\frac{\partial A}{\partial u}\frac{\partial u_\theta}{\partial \theta}(0)h = 0 \qquad \text{in } \Omega^e, \tag{16}$$

and from Eqs. (10, 14) and (16) it follows that

$$Au' = \Delta u' + k^2 u' = 0 \qquad \text{in } \Omega^e, \tag{17}$$

which proves Lemma 1.

The second lemma considers the case of the sound-soft scatterer, i.e., the case where B_θ is the Dirichlet operator (see Eq. (2) with $a = 1$ and $b = 0$), and states the boundary equation satisfied by u'. This result has already been established by Kirsch in [10] and by Potthast in [11], using, however, the assumption that the boundary of the scatterer as well as its perturbations are of class $\mathscr{C}^2$.

LEMMA 2. *The local derivative of the sound-soft scattered field u', at $\theta = 0$ and in a direction $h \in (\mathscr{C}^1(\mathbb{R}^3))^3$, belongs to $H^{-1/2}(\Gamma)$ and satisfies the following boundary equation*:

$$u' = -\sum_{j=1}^{3} h_j \frac{\partial}{\partial x_j}(u + e^{ikx.d}) \quad on\ \Gamma\ a.e. \tag{18}$$

Proof of Lemma 2. Since the scattered field u is in $H^1_{\text{loc}}(\Delta, \Omega^e)$ [17, 18], its local derivative u' is at least in $L^2_{\text{loc}}(\Omega^e)$ [1–3]. From Lemma 1, we deduce that u' is in fact in $H_{\text{loc}}(\Delta, \Omega^e_R)$. Therefore, it follows from the classical trace theorems [4–6] that the trace of u' on the boundary Γ can be defined in $H^{-1/2}(\Gamma)$.

Define

$$f = -e^{ikx.d}. \tag{19}$$

For every perturbation $\theta \in (\mathscr{C}^1(\mathbb{R}^3))^3$, the sound-soft scattered field u_θ that is the solution of BVP (1) satisfies

$$u_\theta - f = 0 \qquad \text{on } \Gamma_\theta. \tag{20}$$

Hence, for every perturbation $\theta \in (\mathscr{C}^1(\mathbb{R}^3))^3$ we have

$$(u_\theta - f) \circ (I + \theta) = 0 \qquad \text{on } \Gamma. \tag{21}$$

We now introduce the following auxiliary function Φ_θ that is defined in the whole domain Ω^e_θ by

$$\Phi_\theta = u_\theta - f \qquad \text{in } \Omega^e_\theta. \tag{22}$$

Since Φ_θ and $\Phi_\theta \circ (I + \theta)$ are differentiable ([1, Theorem 3.3 and Corollary 3.4]), one can apply the classical rule of derivation [2, 3] for every *admissible* perturbation θ, i.e., $\theta \in (\mathscr{C}^1(\mathbb{R}^3))^3$ and θ in the neighborhood of 0, and obtain

$$\frac{\partial \Phi_\theta}{\partial \theta}(0)h = \frac{\partial}{\partial \theta}(\Phi_\theta \circ (I + \theta))(0)h - h.\nabla\Phi \qquad \text{in } \Omega^e. \tag{23}$$

Thus, for every *admissible* perturbation θ, u' can be written as

$$u' = \frac{\partial}{\partial\theta}(\Phi_\theta \circ (I + \theta))(0)h - h.\nabla(u - f) \qquad \text{in } \Omega^e. \tag{24}$$

Furthermore, since u satisfies the Helmholtz equation, we deduce that

$$\Delta(\nabla u) = -k^2 \nabla u \qquad \text{in } \Omega^e. \tag{25}$$

Therefore, given that $\nabla(u - f) \in (L^2_{\text{loc}}(\Omega^e))^3$, it follows from Eq. (25) that $\nabla(u - f) \in (H_{\text{loc}}(\Delta, \Omega^e))^3$. Since u' is also in $H_{\text{loc}}(\Delta, \Omega^e)$, we deduce that the trace of u' (resp. $\nabla(u - f)$) exists and belongs to $H^{-1/2}(\Gamma)$ (resp. $(H^{-1/2}(\Gamma))^3$). Furthermore, since the trace operator is linear and continuous from $H_{\text{loc}}(\Delta, \Omega^e)$ into $H^{-1/2}(\Gamma)$ [4–6], we deduce that Eq. (24) is also valid on Γ. Hence, using Eq. (20), we can write u' as

$$u' = -h.\nabla(u - f) \qquad \text{in } \Gamma, \tag{26}$$

which concludes the proof of Lemma 2.

Remark 1. Since $\partial u/\partial\nu \in H^{-1/2}(\Gamma)$, we deduce from Eq. (20) that the gradient of $u - f$ is proportional to ν on Γ [2, 3]—that is,

$$\nabla(u - f) = \nu(\nu.\nabla(u - f)) \qquad \text{on } \Gamma. \tag{27}$$

Hence

$$\nabla(u - f) = \nu\frac{\partial}{\partial\nu}(u - f) \qquad \text{on } \Gamma. \tag{28}$$

From (26) and (28) it follows that

$$u' = -h.\nu\frac{\partial}{\partial\nu}(u - f) \qquad \text{on } \Gamma, \tag{29}$$

which is the classical form [10, 11] for expressing the boundary condition satisfied by the derivative u' of the acoustic sound-soft scattered field.

Remark 2. One can observe that Lemmas 1 and 2 remain valid even if the perturbations θ of the scatterer are assumed to be only Lipschitzian functions. The $\mathscr{C}^1$-assumption on the perturbations is needed only to characterize the boundary condition of the derivative of the scattered field in the case of the sound-hard scatterer and the lossy boundary condition.

The third lemma considers the case of the sound-hard scatterer; i.e., B_θ is the Neumann operator (see Eq. (2) with $a = 0$ and $b = 1$) and specifies the boundary equation satisfied by u'. This result has been established by

Potthast in [13], assuming that the boundary of the scatterer is $\mathscr{C}^{2,\alpha}$ and its perturbations are $\mathscr{C}^2$. To the best of our knowledge, the result obtained by Hettlich [12] for this case is erroneous.

LEMMA 3. *Let u' be the local derivative of the sound-hard scattered field, at $\theta = 0$ and in a direction $h \in (\mathscr{C}^2(\mathbb{R}^3))^3$. Then, the trace on Γ of its normal derivative $\partial u'/\partial \nu$ belongs to $H^{-3/2}(\Gamma)$ and satisfies the following boundary equation*:

$$\frac{\partial u'}{\partial \nu} = -\sum_{\{l,j\}=1}^{3} \frac{\partial^2 (u + e^{ikx.d})}{\partial x_l \partial x_j} h_j \nu_l + \sum_{\{l,j\}=1}^{3} \frac{\partial}{\partial x_l}(u + e^{ikx.d}) \frac{\partial h_l}{\partial x_j} \nu_j \qquad \textit{on } \Gamma \ a.e. \tag{30}$$

Proof of Lemma 3. We have already proved (see proof of Lemma 2) that u' belongs to $H_{\text{loc}}(\Delta, \Omega^e)$. Therefore, it follows from the trace theorems [4–6] that $\partial u'/\partial \nu$, the trace of the normal derivative of u', exists and belongs to $H^{-3/2}(\Gamma)$.

Furthermore, for every perturbation $\theta \in (\mathscr{C}^1(\mathbb{R}^3))^3$, the sound-hard scattered field u_θ that is the solution of BVP (1) satisfies

$$\nu_\theta . \nabla u_\theta = \nu_\theta . \nabla f \qquad \text{on } \Gamma_\theta \ a.e., \tag{31}$$

where f is given in Eq. (19). Hence,

$$(\nu_\theta . \nabla u_\theta) \circ (I + \theta) = (\nu_\theta . \nabla f) \circ (I + \theta) \qquad \text{on } \Gamma \ a.e.; \tag{32}$$

i.e.,

$$\nu_\theta \circ (I + \theta).(\nabla u_\theta) \circ (I + \theta) = \nu_\theta \circ (I + \theta).(\nabla f) \circ (I + \theta) \qquad \text{on } a.e. \tag{33}$$

With the use of the chain rule [2, 3], Eq. (33) becomes

$$\nu_\theta \circ (I + \theta).P(\theta)\nabla(u_\theta \circ (I + \theta)) = \nu_\theta \circ (I + \theta).P(\theta)\nabla(f \circ (I + \theta)) \qquad \text{on } \Gamma \ a.e., \tag{34}$$

where the matrix $P(\theta)$ is given by

$$P(\theta) = [(I + \theta)']^{-T} \qquad \text{in } \mathbb{R}^3. \tag{35}$$

Moreover [2, Lemma 4.8], we have

$$\nu_\theta \circ (I+\theta) = \frac{1}{\|P(\theta)\nu\|_2} P(\theta)\nu \qquad \text{on } \Gamma \; a.e., \tag{36}$$

and therefore from Eqs. (34) and (36) it follows that

$$P(\theta)\nu . P(\theta)\nabla\big(u_\theta \circ (I+\theta)\big) = P(\theta)\nu . P(\theta)\nabla\big(f \circ (I+\theta)\big) \qquad \text{on } \Gamma \; a.e., \tag{37}$$

which implies that for every perturbation $\theta \in (\mathscr{C}^1(\mathbb{R}^3))^3$, the sound-hard scattered field u_θ solution of BVP (1) satisfies

$$P(\theta)\nu . P(\theta)\nabla\big((u_\theta - f)\circ(I+\theta)\big) = 0 \qquad \text{on } \Gamma \; a.e. \tag{38}$$

We now introduce the following auxiliary function Ψ_θ defined in the whole domain Ω^e by

$$\Psi_\theta = P(\theta)\tilde{\nu} . P(\theta)\nabla\big((u_\theta - f)\circ(I+\theta)\big) \qquad \text{in } \Omega^e, \tag{39}$$

where $\tilde{\nu}$ is an extension of the normal vector ν into $(L^\infty(\Omega^e))^3$. Hence, for every perturbation $\theta \in (\mathscr{C}^1(\mathbb{R}^3))^3$ we have

$$\Psi_\theta = 0 \qquad \text{on } \Gamma \; a.e. \tag{40}$$

First, we show that $\theta \mapsto \Psi_\theta$ is differentiable—at least in the distribution sense— at $\theta = 0$ in a direction $h \in (\mathscr{C}^1(\mathbb{R}^3))^3$. Indeed, this result is a consequence of the two following considerations:

- The differentiability of $\theta \mapsto u_\theta \circ (I+\theta)$ [1, Theorem 3.3] and $\theta \mapsto f \circ (I+\theta)$ at $\theta = 0$, and the fact that ∇ is a linear and continuous operator, allows us to affirm that $\theta \mapsto \nabla((u_\theta - f)\circ(I+\theta))$ is differentiable—at least in the distribution sense—at $\theta = 0$ in a direction $h \in (\mathscr{C}^2(\mathbb{R}^3))^3$. We also have

$$\frac{\partial}{\partial\theta}\big(\nabla((u_\theta - f)\circ(I+\theta))\big)(0)h = \nabla\left(\frac{\partial}{\partial\theta}\big((u_\theta - f)\circ(I+\theta)\big)(0)h\right) \qquad \text{in } \Omega^e. \tag{41}$$

Furthermore, using the classical derivation rule [2, 3], we have

$$\frac{\partial}{\partial\theta}\big((u_\theta - f)\circ(I+\theta)\big)(0)h = \frac{\partial u_\theta}{\partial\theta}(0)h + h.\nabla(u - f) \qquad \text{in } \Omega^e. \tag{42}$$

Therefore, from (41) and (42) it follows that

$$\frac{\partial}{\partial\theta}\big(\nabla((u_\theta - f)\circ(I+\theta))\big)(0)h = \nabla u' + \nabla(h.\nabla(u-f)) \qquad \text{in } \Omega^e. \tag{43}$$

- $\theta \mapsto P(\theta)$ is differentiable at $\theta = 0$ in a direction $h \in (\mathscr{C}^1(\mathbb{R}^3))^3$, from $(\mathscr{C}^1(\mathbb{R}^3))^3$ into $(\mathscr{C}^0(\mathbb{R}^3))^3$, and we have [2, 3]

$$\frac{\partial P(\theta)}{\partial\theta}(0)h = -[h']^T \qquad \text{in } \Omega^e. \tag{44}$$

Hence, using the chain rule we deduce from Eq. (39) that the derivative of Ψ_θ at $\theta = 0$ in a direction $h \in (\mathscr{C}^1(\mathbb{R}^3))^3$ satisfies

$$\begin{aligned}\frac{\partial\Psi_\theta}{\partial\theta}(0)h = {} & \frac{\partial P(\theta)}{\partial\theta}(0)h\tilde{\nu}.P(0)\nabla(u-f)\\ & + P(0)\tilde{\nu}.\frac{\partial P(\theta)}{\partial\theta}(0)h\nabla(u-f)\\ & + P(0)\tilde{\nu}.P(0)\frac{\partial}{\partial\theta}\big(\nabla((u_\theta - f)\circ(I+\theta))\big)(0)h \qquad \text{in } \Omega^e.\end{aligned} \tag{45}$$

From Eqs. (43)–(45) it follows that

$$\begin{aligned}\frac{\partial\Psi_\theta}{\partial\theta}(0)h = {} & -[h']^T\tilde{\nu}.\nabla(u-f) - \tilde{\nu}.[h']^T\nabla(u-f)\\ & + \tilde{\nu}.\nabla u' + \tilde{\nu}.\nabla(h.\nabla(u-f)) \qquad \text{in } \Omega^e.\end{aligned} \tag{46}$$

Furthermore, since

$$\nabla(h.\nabla(u-f)) = [h']\nabla(u-f) + \nabla(\nabla(u-f))h \qquad \text{in } \Omega^e, \tag{47}$$

we deduce from Eqs. (46) and (47) that

$$\begin{aligned}\frac{\partial\Psi_\theta}{\partial\theta}(0)h = {} & -[h']^T\tilde{\nu}.\nabla(u-f) - \tilde{\nu}.[h']^T\nabla(u-f)\\ & + \tilde{\nu}.\nabla u' + \tilde{\nu}.[h']\nabla(u-f) + \tilde{\nu}.\nabla(\nabla(u-f))h \qquad \text{in } \Omega^e.\end{aligned} \tag{48}$$

Hence, using the fact that

$$\tilde{\nu}.[h']\nabla(u-f) = [h']^T\tilde{\nu}.\nabla(u-f) \qquad \text{in } \Omega^e \tag{49}$$

and

$$\tilde{\nu}.\nabla(\nabla(u-f))h = \nabla(\nabla(u-f))\tilde{\nu}.h \qquad \text{in } \Omega^e, \tag{50}$$

Eq. (48) becomes

$$\begin{aligned}\frac{\partial\Psi_\theta}{\partial\theta}(0)h = &-[h']\tilde{\nu}.\nabla(u-f) + \tilde{\nu}.\nabla u' \\ &+ \tilde{\nu}.\nabla(\nabla(u-f))h \qquad \text{in } \Omega^e.\end{aligned} \tag{51}$$

Next, we show that the trace of $(\partial\Psi_\theta/\partial\theta)(0)h$ exists in $H^{-3/2}(\Gamma)$. Indeed, this property results from the three following considerations:

- Since $u \in H^1_{\mathrm{loc}}(\Delta, \Omega^e)$, the trace of its normal derivative is in $H^{-1/2}(\Gamma)$, i.e., $\nu.\nabla u \in H^{-1/2}(\Gamma)$. Then, $\nu.\nabla(u-f) \in H^{-1/2}(\Gamma)$. Furthermore, for every $h \in \mathscr{C}^1(\mathbb{R}^3)$, the coefficients of the matrix $[h']$ are in $\mathscr{C}^0(\mathbb{R}^3)$. It follows that $[h']\nu.\nabla(u-f) \in H^{-1/2}(\Gamma)$. Hence, the embedding theorems [5, 6] allow us to deduce that

$$[h']\nu.\nabla(u-f) \in H^{-3/2}(\Gamma). \tag{52}$$

- We have already shown that the trace of the normal derivative of u' belongs to $H^{-3/2}(\Gamma)$, i.e.,

$$\nu.\nabla u' \in H^{-3/2}(\Gamma). \tag{53}$$

- From Eq. (25), we deduce that ∇u belongs to $(H_{\mathrm{loc}}(\Delta, \Omega^e))^3$. Hence, using the trace theorems [4–6], we deduce that the trace of the normal derivative of $\nabla(u-f)$ exists in $(H^{-3/2}(\Gamma))^3$. Therefore for every $h \in \mathscr{C}^1(\mathbb{R}^3)$, we have

$$\nabla(\nabla(u-f))\nu.h \in H^{-3/2}(\Gamma). \tag{54}$$

Hence, Eq. (39) is also valid on Γ, and one can deduce from Eqs. (40) and (51) that

$$-[h']\nu\cdot\nabla(u-f) + \nu.\nabla u' + \nabla(\nabla(u-f))\nu.h = 0 \qquad \text{on } \Gamma, \tag{55}$$

which concludes the proof of Lemma 3.

The fourth lemma considers the case of the lossy boundary condition, i.e., B_θ is a Robin-type operator (see Eq. (2) with $a \neq 0$ and $b \neq 0$), and states the boundary equation satisfied by u'.

LEMMA 4. *Let u' be the local derivative of the acoustic scattered field, at $\theta = 0$ and in a direction $h \in (\mathscr{C}^1(\mathbb{R}^3))^3$ for a scatterer with a lossy boundary condition. Then, Bu' is in $H^{-3/2}(\Gamma)$ and satisfies the boundary equation*

$$Bu' = aF_D(u) + a\left(\sum_{\{l,j\}=1}^{3} \frac{\partial h_l}{\partial x_j}\nu_j\nu_l\right)(u - f) + bF_N(u) \qquad \text{on } \Gamma \ a.e., \tag{56}$$

where f is given in Eq. (19), *$F_D(u)$ is given in Eq.* (5), *and $F_N(u)$ is given in Eq.* (6).

Proof of Lemma 4. For the same reasons as in the proofs of Lemmas 2 and 3, the derivative u' of the acoustic scattered field that is the solution of BVP (1) with a Robin-type boundary condition belongs to $H_{\text{loc}}(\Delta, \Omega^e)$. It follows that $(u', \partial u'/\partial\nu) \in H^{-1/2}(\Gamma) \times H^{-3/2}(\Gamma)$. Since the operator B is a linear combination of the trace and the normal derivative operators, using the embedding theorems [5, 6], we conclude that $Bu' \in H^{-3/2}(\Gamma)$. Furthermore, for every perturbation $\theta \in (\mathscr{C}^1(\mathbb{R}^3))^3$, the acoustic scattered field u_θ that is the solution of BVP (1) with a lossy boundary condition satisfies

$$B_\theta u_\theta = B_\theta f \qquad \text{on } \Gamma_\theta \ a.e., \tag{57}$$

where the operator B_θ is given in Eq. (2), and the function f is given in Eq. (19). It follows that

$$(B_\theta u_\theta)\circ(I + \theta) = (B_\theta f)\circ(I + \theta) \qquad \text{on } \Gamma \ a.e., \tag{58}$$

i.e.,

$$\begin{aligned} a(u_\theta - f)\circ(I + \theta) &+ b\nu_\theta \circ (I + \theta) \\ &.(\nabla(u_\theta - f))\circ(I + \theta) = 0 \qquad \text{on } \Gamma \ a.e. \end{aligned} \tag{59}$$

Hence, as in Eq. (34), we deduce that

$$\begin{aligned} &a(u_\theta - f)\circ(I + \theta) \\ &\quad + b\nu_\theta \circ(I + \theta).P(\theta)\nabla((u_\theta - f)\circ(I + \theta)) = 0 \qquad \text{on } \Gamma \ a.e., \end{aligned} \tag{60}$$

where the matrix $P(\theta)$ is given in Eq. (35). Moreover, from Eqs. (36) and (60) we deduce that

$$\begin{aligned} &a\|P(\theta)\nu\|_2(u_\theta - f)\circ(I + \theta) \\ &\quad + bP(\theta)\nu.P(\theta)\nabla((u_\theta - f)\circ(I + \theta)) = 0 \qquad \text{on } \Gamma \ a.e. \end{aligned} \tag{61}$$

As in the proofs of Lemmas 2 and 3, we now introduce the following auxiliary function $\widetilde{\Psi_\theta}$ defined in the whole domain Ω^e by

$$\widetilde{\Psi_\theta} = a \| P(\theta) \tilde{\nu} \|_2 \Phi_\theta \circ (I + \theta) + b \Psi_\theta \qquad \text{in } \Omega^e, \tag{62}$$

where $\tilde{\nu}$ is an extension of the normal vector ν into $(L^\infty(\Omega^e))^3$, Φ_θ is given in Eq. (22), and Ψ_θ is given in Eq. (39). Hence, for every perturbation $\theta \in (\mathscr{C}^1(\mathbb{R}^3))^3$, from Eqs. (61) and (62) it follows that

$$\widetilde{\Psi_\theta} = 0 \qquad \text{on } \Gamma. \tag{63}$$

Furthermore, the function $\theta \mapsto \widetilde{\Psi_\theta}$ is Fréchet differentiable—at least in the distribution sense—at $\theta = 0$ and in a direction $h \in (\mathscr{C}^1(\mathbb{R}^3))^3$. This property is a consequence of the three following observations:

- The function $\theta \mapsto \|P(\theta)\tilde{\nu}\|_2$ is Fréchet differentiable, at $\theta = 0$ and in a direction $h \in (\mathscr{C}^1(\mathbb{R}^3))^3$, from $(\mathscr{C}^1(\mathbb{R}^3))^3$ into $L^\infty(\Omega^e)$. Furthermore, because of Eq. (44), one can easily check that for every *admissible* perturbation θ we have

$$\frac{\partial}{\partial \theta} (\| P(\theta) \tilde{\nu} \|_2)(0) h = \frac{-1}{\| P(0) \tilde{\nu} \|_2} [h']^T \tilde{\nu}.\tilde{\nu} \qquad \text{in } \Omega^e. \tag{64}$$

- As in the proof of Lemma 2, the function $\theta \mapsto \Phi_\theta \circ (I + \theta)$ is Fréchet differentiable—at least in the distribution sense—at $\theta = 0$ and in a direction $h \in (\mathscr{C}^1(\mathbb{R}^3))^3$, and its derivative satisfies (see Eq. (24))

$$\frac{\partial}{\partial \theta} (\Phi_\theta \circ (I + \theta))(0) h = u' + h.\nabla\Phi \qquad \text{in } \Omega^e. \tag{65}$$

- As in the proof of Lemma 3, the function $\theta \mapsto \Psi_\theta$ is Fréchet differentiable—at least in the distribution sense—at $\theta = 0$ and in a direction $h \in (\mathscr{C}^1(\mathbb{R}^3))^3$, and for every *admissible* perturbation θ we have (see Eq. (51))

$$\begin{aligned} \frac{\partial \Psi_\theta}{\partial \theta}(0) h = {} & -[h'] \tilde{\nu}.\nabla(u - f) + \tilde{\nu}.\nabla u' \\ & + \tilde{\nu}.\nabla(\nabla(u - f)) h \qquad \text{in } \Omega^e. \end{aligned} \tag{66}$$

Therefore, using the chain rule, it follows from Eq. (62) and Eqs. (64)–(66) that the derivative of $\widetilde{\Psi_\theta}$ at $\theta = 0$ and in a direction $h \in (\mathscr{C}^1(\mathbb{R}^3))^3$

satisfies

$$\frac{\partial\tilde{\Psi}_\theta}{\partial\theta}(0)h = a\left\{\frac{-1}{\|P(0)\,\tilde{\nu}\|_2}[h']^T\tilde{\nu}.\tilde{\nu}(u-f)\right.$$
$$\left.+\|P(0)\,\tilde{\nu}\|_2(u' + h.\nabla(u-f))\right\}$$
$$+\, b\{-[h']\tilde{\nu}.\nabla(u-f)$$
$$+\,\tilde{\nu}.\nabla u' + \tilde{\nu}.\nabla(\nabla(u-f))h\} \qquad \text{in } \Omega^e. \qquad (67)$$

The arguments used to prove Lemmas 2 and 3 allow us to affirm that the trace of $(\partial\tilde{\Psi}_\theta/\partial\theta)(0)h$ exists in $H^{-3/2}(\Gamma)$ and that Eq. (67) is also valid on Γ. Therefore, using Eq. (63) we obtain

$$a\{-[h']^T\nu.\nu(u-f) + u' + h.\nabla(u-f)\}$$
$$+\, b\{-[h']\nu.\nabla(u-f) + \nu.\nabla u'$$
$$+\,\nu.\nabla(\nabla(u-f))h\} = 0 \qquad \text{on } \Gamma\ a.e. \qquad (68)$$

Hence

$$Bu' = a[h']^T\nu/\nu(u-f) - ah.\nabla(u-f)$$
$$+\, b[h']\nu.\nabla(u-f) - b\nabla(\nabla(u-f))\,\nu.h \qquad \text{on } \Gamma\ a.e., \quad (69)$$

and then Lemma 4 is proved.

The fifth lemma states that the derivative with respect to an obstacle's domain of the acoustic scattered field satisfies the outgoing radiation condition.

LEMMA 5. *The local derivative u' at $\theta = 0$ and in a direction $h \in (\mathscr{C}^1(\mathbb{R}^3))^3$ of the acoustic scattered field u_θ that is the solution of the BVP* (1) *satisfies the following outgoing radiation condition*:

$$\lim_{r\to\infty} r\left(\frac{\partial u'}{\partial r} - iku'\right) = 0.$$

Proof of Lemma 5. This lemma results from the asymptotic behavior of the scattered field u_θ [14, 15]. Indeed, one can easily check that for r large enough, Eq. (3) can be rewritten as

$$u_\theta(x) = \frac{e^{ikr}}{r}\left(u_{\theta,\infty}\left(\frac{x}{r}\right) + M_\theta O\left(\frac{1}{r}\right)\right), \qquad (70)$$

where M_θ is a function that is continuously Fréchet differentiable at $\theta = 0$ and in a direction $h \in (\mathscr{C}^1(\mathbb{R}^3))^3$ because u_θ and $u_{\theta,\infty}$ are differentiable [1, Corollary 3.4 and Theorem 4.2]. Differentiating the above expression at $\theta = 0$ and in a direction h leads to

$$u'(x) = \frac{e^{ikr}}{r}\left(\frac{\partial u_{\theta,\infty}}{\partial\theta}(0)h\left(\frac{x}{r}\right) + O\left(\frac{1}{r}\right)\right). \tag{71}$$

It follows that

$$r\left(\frac{\partial u'}{\partial r} - iku'\right) = -\frac{e^{ikr}}{r}\left(\frac{\partial u_{\theta,\infty}}{\partial\theta}(0)h + O\left(\frac{1}{r}\right)\right) + e^{Ikr}\left(O\left(\frac{1}{r^2}\right)\right), \tag{72}$$

which proves Lemma 5.

Remark 3. The corollary is an immediate consequence of Eq. (71) and the uniqueness of the far-field pattern of the acoustic scattered field [14].

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