

Spaces of Distributions of Besov and Triebel-Lizorkin Type for the Fourier-Bessel Transform

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In this paper we study new Besov and Triebel-Lizorkin spaces on the basis of the Fourier-Bessel transformation. © 2001 Academic Press

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1. INTRODUCTION

The Fourier-Bessel transform is defined by ([2])

$$h_{\mu}(f)(y) = \int_0^{\infty} (xy)^{-\mu} J_{\mu}(xy) f(x) x^{2\mu+1} dx \quad (y \in I = (0, \infty)),$$

where J_{μ} represents the Bessel function of the first kind and order $\mu \geq -1/2$ [15].

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We denote by L_μ^p ($1 \leq p < \infty$) the space of the measurable functions f defined on I such that

$$\|f\|_{p,\mu} = \left\{ \int_0^\infty |f(x)|^p x^{2\mu+1} dx \right\}^{1/p} < \infty,$$

and by L_μ^∞ the space of the essentially bounded measurable functions on I . We write L^p and $\|\cdot\|_p$ instead of L_μ^p and $\|\cdot\|_{p,\mu}$, respectively, when $\mu = -1/2$. It is known ([8, p. 997]) that h_μ is a continuous mapping from L_μ^1 into L_μ^∞ .

F. M. Cholewinski [4], D. T. Haimo [7], and I. I. Hirschman, Jr. [9] investigated a convolution operation associated with the Fourier-Bessel transform. To make the study of the Besov type spaces, the generalized convolution $\#$ plays an important role. If $f, g \in L_\mu^1$, the generalized convolution, $\#$ of f and g , is defined by

$$(f \# g)(x) = \int_0^\infty f(y)(\tau_x g)(y) d\gamma(y), \quad x \in I,$$

where $d\gamma(x) = \frac{x^{2\mu+1}}{2^\mu \Gamma(\mu+1)} dx$ and the generalized translation operator τ_x , is defined by

$$(\tau_x g)(y) = \int_0^\infty D_\mu(x, y, z) g(z) d\gamma(z)$$

being $D_\mu(x, y, z) = \frac{2^{3\mu-1} \Gamma(\mu+1)^2}{\sqrt{\pi} \Gamma(1/2 + \mu)} (xyz)^{-2\mu} A(x, y, z)^{2\mu-1} (x, y, z \in I)$, where $A(x, y, z)$ is the area of a triangle with sides x, y, z , if the triangle exists, and $A(x, y, z) = 0$ otherwise. We denote by Δ_μ to the Bessel differential operator $x^{-2\mu-1} D x^{2\mu+1} D$. It is well known that for a suitable function u ,

$$h_\mu(\Delta_\mu u)(x) = -x^2 h_\mu u(x). \quad (1.1)$$

In this paper, the work is realized in the following space of generalized functions \mathcal{H} , defined by G. Altenburg in [1], that consists of those smooth, complex-valued functions $f = f(x)$, $x \in I = (0, \infty)$, such that the quantities

$$\gamma_{m,k}(f) = \sup_{x \in I} |x^m (x^{-1} D)^k f(x)| \quad (m, k \in \mathbb{N})$$

are finite. \mathcal{H} is a Fréchet space equipped with the topology generated by the family of seminorms $\{\gamma_{m,k}\}_{(m,k) \in \mathbb{N} \times \mathbb{N}}$. The dual space of \mathcal{H} is denoted by \mathcal{H}' .

Our paper is devoted to the study of the spaces of Besov, Nikol'skij, and Triebel-Lizorkin type defined on the basis of the Fourier-Bessel transform

h_μ . Such a Besov type space $B_{p,q,\mu}^s$ was introduced by Altenburg in [2]. We first define the Nikol'skij and Triebel-Lizorkin spaces $b_{p,q,\mu}^s$ and $F_{p,q,\mu}^s$ and give a characterization of $B_{p,q,\mu}^s$ in terms of $b_{p,q,\mu}^s$ and an embedding theorem between $B_{p,q,\mu}^s$ and $F_{p,q,\mu}^s$. Then, we prove a one-to-one mapping property of the Hankel potentials spaces $W_\mu^{s,p}$ (given in [2], [5], or [10]) and of the Besov type spaces $B_{p,q,\mu}^s$ that together with certain relation of density and some properties of Rademacher functions ([12], [17]) are used to characterize $W_\mu^{s,p}$ in terms of the Triebel-Lizorkin spaces $F_{p,q,\mu}^s$; in particular, we characterize a Sobolev type space $L_\mu^{s,p}$ ([5]). Application is given to solve a differential equation involving the Bessel operator $\Delta_\mu = x^{-2\mu-1}Dx^{2\mu+1}D$.

2. NIKOL'SKIJ AND BESOV TYPE SPACES

In this section, we define Besov type spaces ([2]), Nikol'skij type spaces, and Triebel-Lizorkin type spaces and we demonstrate some relations among them.

DEFINITION 2.1. Let $s \in \mathbf{R}$, for $1 \leq p < \infty$ we define the sequence spaces l_p^s as

$$l_p^s = \left\{ \xi : \xi = (\xi_j)_{j=0}^\infty, \xi_j \text{ complex}, \|\xi\|_{l_p^s} = \left(\sum_{j=0}^\infty (2^{jsp} |\xi_j|^p) \right)^{1/p} < \infty \right\}. \quad (2.1)$$

and for $p = \infty$ we have

$$l_\infty^s = \left\{ \xi : \xi = (\xi_j)_{j=0}^\infty, \xi_j \text{ complex}, \|\xi\|_{l_\infty^s} = \sup_j 2^{js} |\xi_j| < \infty \right\}. \quad (2.2)$$

In the case of $s = 0$ we denote l_p^0 by l_p .

DEFINITION 2.2. Let Φ be the collection of all systems $\{\varphi_j(x)\}_{j=0}^\infty \subset \mathcal{H}$ with the following properties:

- (i) $\varphi_j(x) \in \mathcal{H}$, $h_\mu \varphi_j(x) \geq 0$ for $j = 0, 1, 2, 3, \dots$
- (ii) $\text{supp } h_\mu \varphi_j \subset \{x : \sqrt{2^{j-1}} - 1 \leq x \leq \sqrt{2^{j+1}} - 1\}$ for $j = 1, 2, 3, \dots$, and $\text{supp } h_\mu \varphi_0 \subset \{x : x \leq 1\}$.
- (iii) Exists a positive number c_1 such that

$$\left| (x^{-1}D)^k h_\mu \varphi_j(x) \right| \leq c_1 x^{-k} \quad (2.3)$$

for $j = 1, 2, \dots; 0 \leq k \leq [\mu] + 2$ and $x \in I$.

(iv) $\sum_{j=0}^{\infty} h_{\mu} \varphi_j(x) = 1$ for every $x \in I$.

Proceeding as in [14, pp. 171–172] we can see that $\Phi(I)$ is not empty.

The definitions of the Besov type spaces $B_{p,q,\mu}^s$ (see [2]) and Nikol'skij type spaces are the following.

DEFINITION 2.3. Let $1 < p < \infty$, $1 \leq q \leq \infty$, $\mu \geq -1/2$ and $s \in \mathbf{R}$. Then for any system of functions $\{\varphi_j\}_{j=0}^{\infty} \in \Phi$, the Besov type spaces are defined by

$$B_{p,q,\mu}^s = \left\{ f \in \mathcal{H}' : \|f\|_{B_{p,q,\mu}^s} = \left\| \{\varphi_j \# f\} \right\|_{l_q^s(L_{\mu}^p)} < \infty \right\}, \quad (2.4)$$

being

$$\|\cdot\|_{l_q^s(L_{\mu}^p)} = \|\|\cdot\|_{L_{\mu}^p}\|_{l_q^s} = \left(\sum_{j=0}^{\infty} \left(2^{sj} \|\cdot\|_{L_{\mu}^p} \right)^q \right)^{1/q}.$$

DEFINITION 2.4. For $s \in \mathbf{R}$, $1 < p < \infty$, $1 \leq q < \infty$, $\mu \geq -1/2$, we define

$$\begin{aligned} b_{p,q,\mu}^s &= \left\{ f : f \in \mathcal{H}', f =_{\mathcal{H}'} \sum_{i=0}^{\infty} a_i(x), \|\{a_i\}\|_{l_q^s(L_{\mu}^p)} \right. \\ &\quad \left. = \left(\sum_{i=0}^{\infty} \left(2^{si} \|a_i(x)\|_{L_{\mu}^p} \right)^q \right)^{1/q} < \infty \right\}, \end{aligned}$$

and for $q = \infty$, we set

$$\begin{aligned} b_{p,\infty,\mu}^s &= \left\{ f : f \in \mathcal{H}', f =_{\mathcal{H}'} \sum_{j=0}^{\infty} a_j(x), \|\{a_i\}\|_{l_{\infty}^s(L_{\mu}^p)} \right. \\ &\quad \left. = \sup_i 2^{si} \|a_i(x)\|_{L_{\mu}^p} < \infty \right\}, \end{aligned}$$

where $\text{supp } h_{\mu} a_i \subset \{\xi : \sqrt{2^{i-1}} - 1 \leq \xi \leq \sqrt{2^{i+1}} - 1\}$ for $i = 1, 2, \dots$ and $\text{supp } h_{\mu} a_0 \subset \{\xi : \xi \leq 1\}$.

By $f =_{\mathcal{H}'} \sum_{i=0}^{\infty} a_i(x)$ will be understood that $\sum_{i=0}^{\infty} a_i(x)$ converges in \mathcal{H}' to f . The norm of $b_{p,q,\mu}^s$ comes defined by

$$\|f\|_{b_{p,q,\mu}^s} = \inf_{f = \sum a_i} \|\{a_i\}\|_{l_q^s(L_{\mu}^p)}.$$

Now, we can state the following theorem.

THEOREM 2.1. *Let $\{\varphi_j\}_{j=0}^\infty \in \Phi$, $s \in \mathbf{R}$, $1 < p < \infty$, $1 \leq q \leq \infty$, and $\mu \geq -1/2$. Then $B_{p,q,\mu}^s \equiv b_{p,q,\mu}^s$.*

Proof. First, we prove that $B_{p,q,\mu}^s \subset b_{p,q,\mu}^s$.

Let $f \in B_{p,q,\mu}^s$. Given $\{\varphi_j\}_{j=0}^\infty \in \Phi$ we know that

$$\left(\sum_{j=0}^{\infty} h_\mu \varphi_j \right) (\xi) = 1.$$

Then

$$f = h_\mu h_\mu f = h_\mu \left(\sum_{j=0}^{\infty} h_\mu \varphi_j \cdot h_\mu f \right) =_{\mathcal{H}'} \sum_{j=0}^{\infty} h_\mu (h_\mu \varphi_j \cdot h_\mu f) = \sum_{j=0}^{\infty} \varphi_j \# f.$$

Therefore, taking $a_j = \varphi_j \# f$, we achieved

$$\|f\|_{b_{p,q,\mu}^s} \leq \| \{a_j\} \|_{l_q^s(L_\mu^p)} = \| \{ \varphi_j \# f \} \|_{l_q^s(L_\mu^p)} = \|f\|_{B_{p,q,\mu}^s}.$$

Then, we have $B_{p,q,\mu}^s \subset b_{p,q,\mu}^s$.

Now, we will see that $b_{p,q,\mu}^s \subset B_{p,q,\mu}^s$.

Let $f \in b_{p,q,\mu}^s$ and $f = \sum_{i=0}^\infty a_i(x)$ in the sense of the convergence in \mathcal{H}' .

Consider $\{\varphi_j\}_{j=0}^\infty \in \Phi$; then

$$(\varphi_j \# f)(x) =_{\mathcal{H}'} \sum_{i=0}^{\infty} (\varphi_j \# a_i)(x) = \sum_{i=j-1}^{j+1} (\varphi_j \# a_i)(x)$$

since $\varphi_j \# a_i = h_\mu (h_\mu \varphi_j \cdot h_\mu a_i) = 0$, for $i > j+1$ and $i < j-1$. Furthermore, if we define $\varphi_j = a_j = 0$ for $j < 0$, we have

$$\|f\|_{B_{p,q,\mu}^s} = \| \{ \varphi_j \# f \} \|_{l_q^s(L_\mu^p)} \leq \sum_{r=-1}^1 \| \{ \varphi_j \# a_{j+r} \} \|_{l_q^s(L_\mu^p)}. \quad (2.5)$$

On the other hand, applying [6, Corollary 1.2, p. 656] with $1 < p < \infty$ we get

$$\| \varphi_j \# a_{j+r} \|_{L_\mu^p} \leq c_1 \| a_{j+r} \|_{L_\mu^p} \quad (2.6)$$

being c_1 a suitable positive constant.

Now, taking the norm of l_q^s in (2.6) it follows that

$$\| \{ \varphi_j \# a_{j+r} \} \|_{l_q^s(L_\mu^p)} \leq c_1 \| \{ a_{j+r} \} \|_{l_q^s(L_\mu^p)}. \quad (2.7)$$

Then from (2.5) we obtain

$$\begin{aligned} \|f\|_{B_{p,q,\mu}^s} &= \left\| \{\varphi_j \# f\} \right\|_{l_q^s(L_\mu^p)} \leq c_1 \sum_{r=-1}^1 \left\| \{a_{j+r}\} \right\|_{l_q^s(L_\mu^p)} \\ &\leq c_2 \left\| \{a_j\} \right\|_{l_q^s(L_\mu^p)}. \end{aligned} \quad (2.8)$$

Taking the infimum on the right-hand side of (2.8) we get

$$\|f\|_{B_{p,q,\mu}^s} \leq c_2 \cdot \|f\|_{b_{p,q,\mu}^s}.$$

Thus Theorem 2.1 is established.

Remark 2.1. Note that by Theorem 2.1, the spaces $B_{p,q,\mu}^s$ are independent of the functions $\{\varphi_j\}_{j=0}^\infty \in \Phi$.

We introduce new Triebel-Lizorkin type spaces as follows.

DEFINITION 2.5. Let $1 < p < \infty$, $1 \leq q \leq \infty$, $\mu \geq -1/2$ and $s \in \mathbf{R}$. Then, for any system of functions $\{\varphi_j\}_{j=0}^\infty \in \Phi$, the Triebel-Lizorkin type spaces are defined by

$$F_{p,q,\mu}^s = \left\{ f \in \mathcal{S}' : \|f\|_{F_{p,q,\mu}^s} = \left\| \{\varphi_j \# f\} \right\|_{L_\mu^p(l_q^s)} < \infty \right\}, \quad (2.9)$$

where

$$\|\cdot\|_{L_\mu^p(l_q^s)} = \|\|\cdot\|_{l_q^s}\|_{L_\mu^p} = \left\| \left(\sum_{j=0}^\infty (2^{sj}(\cdot))^q \right)^{1/q} \right\|_{L_\mu^p}.$$

THEOREM 2.2. Let $1 < p, q < \infty$, $\mu \geq -1/2$ and $s \in \mathbf{R}$, then

$$B_{p,\min\{p,q\},\mu}^s \subset F_{p,q,\mu}^s \subset B_{p,\max\{p,q\},\mu}^s, \quad (2.10)$$

where \subset means continuous embedding.

Proof. We must demonstrate that

$$B_{p,p,\mu}^s \subset F_{p,q,\mu}^s \subset B_{p,q,\mu}^s \quad (2.11)$$

for $p \leq q$, and

$$B_{p,q,\mu}^s \subset F_{p,q,\mu}^s \subset B_{p,p,\mu}^s \quad (2.12)$$

for $q \leq p$. We will use the monotony of the l_q^s spaces and the trivial equality $B_{p,p,\mu}^s \equiv F_{p,p,\mu}^s$.

First we will prove (2.11). Let $f \in F_{p,q,\mu}^s$ and $\{\varphi_j\}_{j=0}^\infty \in \Phi$,

$$\begin{aligned} \|f\|_{B_{p,q,\mu}^s} &= \|\{\varphi_j \# f\}\|_{l_q^s(L_\mu^p)} = \left(\sum_{j=0}^\infty \left(2^{sj} \|\{\varphi_j \# f\}\|_{L_\mu^p} \right)^q \right)^{1/q} \\ &= \left(\sum_{j=0}^\infty 2^{sjq} \left(\int_0^\infty |\varphi_j \# f|^p d\gamma(x) \right)^{q/p} \right)^{1/q} \\ &= \left\| \left\{ \int_0^\infty 2^{sjp} |\varphi_j \# f(x)|^p d\gamma(x) \right\} \right\|_{l_{q/p}^s}^{1/p}. \end{aligned}$$

Then, using Minkowski's inequality we obtain

$$\begin{aligned} \|f\|_{B_{p,q,\mu}^s} &\leq \left(\int_0^\infty \left\| \{ 2^{sjp} |\varphi_j \# f(x)|^p \} \right\|_{l_{q/p}^s} d\gamma(x) \right)^{1/p} \\ &= \left\| \left(\sum_{j=0}^\infty \left(2^{sj} |\varphi_j \# f| \right)^q \right)^{1/q} \right\|_{L_\mu^p} \\ &= \|\{\varphi_j \# f\}\|_{L_\mu^p(l_q^s)} = \|f\|_{F_{p,q,\mu}^s} \leq \|\{\varphi_j \# f\}\|_{L_\mu^p(l_p^s)} \\ &= \|\{\varphi_j \# f\}\|_{l_p^s(L_\mu^p)} = \|f\|_{B_{p,p,\mu}^s}. \end{aligned}$$

Now, we prove (2.12). For $f \in B_{p,q,\mu}^s$ we have

$$\begin{aligned} \|f\|_{B_{p,p,\mu}^s} &= \|\{\varphi_j \# f\}\|_{l_p^s(L_\mu^p)} = \|\{\varphi_j \# f\}\|_{l_p^s(L_\mu^p)} \\ &= \|\{\varphi_j \# f\}\|_{L_\mu^p(l_p^s)} \leq \|\{\varphi_j \# f\}\|_{L_\mu^p(l_q^s)} \\ &= \left\| \sum_{j=0}^\infty \left(2^{sj} |\varphi_j \# f| \right)^q \right\|_{L_\mu^{p/q}}^{1/q} \leq \left(\sum_{j=0}^\infty 2^{sjq} \|\{\varphi_j \# f(x)\}^q\|_{L_\mu^{p/q}} \right)^{1/p} \\ &= \|\{\varphi_j \# f\}\|_{l_q^s(L_\mu^p)} = \|f\|_{B_{p,q,\mu}^s}. \end{aligned}$$

3. A NEW CHARACTERIZATION OF THE HANKEL POTENTIALS SPACES

In this section we prove a lifting property, a classical equality for the Fourier transform ([14]), characterizing the Hankel potentials in terms of

the Triebel-Lizorkin spaces and also we see an application. For this we need to recall the definition of Hankel potentials given in [2, 5].

The definition of Hankel potentials H_μ^s of a function $u \in \mathcal{H}'$ of order $s, s \in \mathbf{R}, (\mu \geq -1/2)$ is

$$(H_\mu^s u)(x) = h_\mu \left((1 + \xi^2)^{-s/2} h_\mu(u) \right) (x).$$

For $s \in \mathbf{R}$ and $1 \leq p < \infty$ in ([2, Definition 5, p. 55], [5, Definition 2.3, p. 86]) we can see the Hankel potentials spaces

$$W_\mu^{s,p} = W_\mu^{s,p}(I) = \{ \phi \in \mathcal{H}' : H_\mu^{-s} \phi \in L_\mu^p(I) \}.$$

The norm in $W_\mu^{s,p}(I)$ is defined by

$$\|\phi\|_{s,p,\mu} = \|\phi\|_{W_\mu^{s,p}} = \|H_\mu^{-s} \phi\|_{p,\mu}.$$

Moreover, \mathcal{H} is dense in $W_\mu^{s,p}(I)$ (See [2, 5]).

Now, we see in the following theorem a lifting property.

THEOREM 3.1. *Let $\sigma, s \in \mathbf{R}, \mu \geq -1/2, 1 < p < \infty$, and $1 \leq q \leq \infty$. Then H_μ^σ is a linear bounded one-to-one operator from $W_\mu^{s,p}$ onto $W_\mu^{s+\sigma,p}$ and from $B_{p,q,\mu}^s$ onto $B_{p,q,\mu}^{s+\sigma}$.*

Proof. Consider $\{\varphi_j\}_{j=0}^\infty \in \Phi$. We define $\{\psi_j\}_{j=0}^\infty$ as follows

$$\psi_j = \varphi_j \# h_\mu \left((1 + x^2)^{\sigma/2} 2^{j\sigma} \right).$$

A straightforward manipulation leads to $\{\psi_j\}_{j=0}^\infty \in \Phi$. Thus

$$\begin{aligned} H_\mu^\sigma f \# \psi_j &= h_\mu (h_\mu \psi_j \cdot h_\mu H_\mu^\sigma f) = h_\mu \left(h_\mu \psi_j \cdot (1 + \xi^2)^{-\sigma/2} h_\mu f \right) \\ &= h_\mu \left(2^{j\sigma} h_\mu \varphi_j \cdot h_\mu f \right) = 2^{j\sigma} f \# \varphi_j. \end{aligned}$$

Now, the result follows immediately as in [14, pp. 180–181].

By [2, Satz 1, p. 57], Theorem 2.1 and Theorem 2.2 we obtain the following theorem.

THEOREM 3.2. *Let $s \in \mathbf{R}, 1 < p < \infty, 1 \leq q < \infty, \mu \geq -1/2$. Then \mathcal{H} is dense in $B_{p,q,\mu}^s$, and if $1 < p, q < \infty, \mathcal{H}$ is dense in $F_{p,q,\mu}^s$.*

To obtain a new characterization of the Hankel potentials spaces, we need to prove in advance the following lemma.

LEMMA 3.1. *Let $s \in \mathbf{R}$, let $\{\varphi_j\}_{j=0}^\infty \in \Phi$ and let $\{r_j\}_{j=0}^\infty$ be the Rademacher functions (see [12, p. 104] or [17, Chapter V, Theorem 8.4, p. 213]). Then, for every p with $1 < p < \infty$ and for all $t \in (0, 1)$, there are some constants A_i ,*

$i = 1, 2$, such that

$$\|h_\mu(m_i h_\mu f)\|_{p, \mu} \leq A_i \|f\|_{p, \mu},$$

being

$$m_1(x) = \sum_{j=0}^{\infty} 2^{js} r_j(t) (1+x^2)^{-s/2} h_\mu \varphi_j(x)$$

and

$$m_2(x) = \left(\sum_{j=0}^{\infty} (h_\mu \varphi_j)^2(x) \right)^{-1}.$$

Proof. We can see without difficulty that m_i satisfies

$$|(x^{-1}D)^k m_i(x)| \leq A_i x^{-k},$$

for $k = 0, 1, \dots, [\mu] + 2$ and $i = 1, 2$. Then applying [6, Corollary 1.2, p. 656] we obtain the desired result.

Next, we prove that for particular cases the Triebel-Lizorkin type spaces are reduced to the Hankel potentials spaces.

THEOREM 3.3. *If $s \in \mathbf{R}$, $\mu \geq -1/2$ and $1 < p < \infty$ we have*

$$F_{p,2,\mu}^s(I) = W_{\mu}^{s,p}(I),$$

where $\|f\|_{W_{\mu}^{s,p}(I)}$ is an equivalent norm in $F_{p,2,\mu}^s(I)$.

Proof. We shall see that there exist c_1, c_2 positive constants such that

$$c_1 \|f\|_{s,p,\mu} \leq \left\| \left(\sum_{j=0}^{\infty} 2^{2js} |\varphi_j \# f|^2 \right)^{1/2} \right\|_{p,\mu} \leq c_2 \|f\|_{s,p,\mu}. \quad (3.1)$$

By Theorem 3.2 we know that \mathcal{H} is dense in $F_{p,2,\mu}^s$ for $1 < p < \infty$. Then, it is not difficult to see that the functions $f \in L_{\mu}^p$ with $\text{supp } h_{\mu} f$ compact are dense both in $W_{\mu}^{s,p}$ and in $F_{p,2,\mu}^s$, for $1 < p < \infty$. Therefore it is enough to prove (3.1) for functions of this type. Note that in this case the infinite sum in (3.1) is actually finite.

Initially, we will prove the estimate on the right-hand side.

Let $f \in W_{\mu}^{s,p}$ then $f = G_s \# g$; i.e.,

$$h_{\mu} f(\xi) = (1 + \xi^2)^{-s/2} h_{\mu} g(\xi).$$

Applying Lemma 3.1 we have for all $t \in (0, 1)$

$$\left\| \sum_{j=0}^{\infty} r_j(t) 2^{js} \varphi_j \# f \right\|_{p, \mu} \leq A_1 \|g\|_{p, \mu} = A_1 \|f\|_{s, p, \mu}.$$

Thus, it follows that

$$\int_0^1 \left\| \sum_{j=0}^{\infty} r_j(t) 2^{js} \varphi_j \# f \right\|_{p, \mu} dt \leq A_1 \|f\|_{s, p, \mu}. \quad (3.2)$$

Using the right-hand inequality of ([12, p. 104] or [17, Chapter V, Theorem 8.4, p. 213]) with $p = 1$ and the Minkowski's inequality we obtain

$$\begin{aligned} \left\| \left(\sum_{j=0}^{\infty} |2^{js} \varphi_j \# f|^2 \right)^{1/2} \right\|_{p, \mu} &\leq c \left\| \int_0^1 \left| \sum_{j=0}^{\infty} r_j(t) 2^{js} \varphi_j \# f \right| dt \right\|_{p, \mu} \\ &\leq c \int_0^1 \left\| \sum_{j=0}^{\infty} r_j(t) 2^{js} \varphi_j \# f \right\|_{p, \mu} dt. \end{aligned}$$

Now, by (3.2) we have

$$\left\| \left(\sum_{j=0}^{\infty} |2^{js} \varphi_j \# f|^2 \right)^{1/2} \right\|_{p, \mu} \leq c \|f\|_{s, p, \mu},$$

being c a suitable positive constant.

Therefore we achieve that $f \in F_{p, 2, \mu}^s$.

Now, we will see the converse inequality. For this we will use duality. Let $f \in F_{p, 2, \mu}^s$ and

$$k = h_{\mu} \left(\sum_{j=0}^{\infty} (h_{\mu} \varphi_j)^2 \cdot h_{\mu} g \right). \quad (3.3)$$

Applying Lemma 3.1 with $m_2(x) = (\sum_{j=0}^{\infty} (h_{\mu} \varphi_j)^2)^{-1}$ we obtain

$$\begin{aligned} \|g\|_{p, \mu} &= \left\| h_{\mu} \left(\left(\sum_{j=0}^{\infty} (h_{\mu} \varphi_j)^2 \right)^{-1} h_{\mu} h_{\mu} \left(\sum_{j=0}^{\infty} (h_{\mu} \varphi_j)^2 \cdot h_{\mu} g \right) \right) \right\|_{p, \mu} \\ &\leq A_2 \left\| h_{\mu} \left(\sum_{j=0}^{\infty} (h_{\mu} \varphi_j)^2 \cdot h_{\mu} g \right) \right\| = A_2 \|k\|_{p, \mu}. \end{aligned} \quad (3.4)$$

Now, consider $u \in L^q_\mu(I)$ to be a function such that $\|u\|_{q,\mu} = 1$, $\text{supp } h_\mu u$ is compact ($1/p + 1/q = 1$), and

$$\int_0^\infty u(x)k(x) d\gamma(x) \geq \frac{1}{2}\|k\|_{p,\mu}. \quad (3.5)$$

Let w a function defined by $h_\mu w(\xi) = (1 + \xi^2)^{s/2} h_\mu u(\xi)$; i.e., $u = G_s \# w$ and $f = G_s \# g$, as above, so that $h_\mu f \cdot h_\mu w = h_\mu g \cdot h_\mu u$. Then from (3.3)–(3.5) we obtain

$$\begin{aligned} \|f\|_{s,p,\mu} &= \|g\|_{p,\mu} \leq c \int_0^\infty u(x)k(x) d\gamma(x) = c \int_0^\infty h_\mu u(\xi) h_\mu k(\xi) d\gamma(\xi) \\ &= c \int_0^\infty h_\mu u(\xi) \sum_{j=0}^\infty (h_\mu \varphi_j)^2 h_\mu g(\xi) d\gamma(\xi) \\ &= c \int_0^\infty \sum_{j=0}^\infty (2^{js} h_\mu f(\xi) (h_\mu \varphi_j)(\xi)) \\ &\quad \times (2^{-js} (h_\mu w)(\xi) (h_\mu \varphi_j)(\xi)) d\gamma(\xi), \end{aligned}$$

for certain constants $c > 0$. Hence, by Plancharel's formula and the Cauchy and Hölder inequalities we get

$$\begin{aligned} \|f\|_{s,p,\mu} &= c \int_0^\infty \sum_{j=0}^\infty (2^{js} (\varphi_j \# f)(x)) (2^{-js} (\varphi_j \# w)(x)) d\gamma(x) \\ &\leq c \int_0^\infty \left(\sum_{j=0}^\infty (2^{js} (\varphi_j \# f)(x))^2 \right)^{1/2} \\ &\quad \times \left(\sum_{j=0}^\infty (2^{-js} (\varphi_j \# w)(x))^2 \right)^{1/2} d\gamma(x) \\ &\leq c \left\| \left(\sum_{j=0}^\infty 2^{2js} |\varphi_j \# f|^2 \right)^{1/2} \right\|_{p,\mu} \cdot \left\| \left(\sum_{j=0}^\infty 2^{-2js} |\varphi_j \# w|^2 \right)^{1/2} \right\|_{q,\mu}. \quad (3.6) \end{aligned}$$

Then by the right-hand side of inequality of (3.1) we achieve

$$\left\| \left(\sum_{j=0}^\infty 2^{-2js} |\varphi_j \# w|^2 \right)^{1/2} \right\|_{q,\mu} \leq c_2 \|w\|_{-s,q,\mu} = c_2 \|u\|_{q,\mu} = c_2. \quad (3.7)$$

Therefore, combining (3.6) and (3.7) the proof is finished.

As a consequence of Theorem 3.3 we obtain the following results.

COROLLARY 3.1. *If $s \in \mathbf{N}$ and $1 < p < \infty$ then $F_{p,2,\mu}^{2s}(I) = L_{\mu}^{s,p}(I)$, where*

$$L_{\mu}^{s,p}(I) = \left\{ T \in \mathcal{H}' : T \in L_{\mu}^{1,loc} \text{ and } \Delta_{\mu}^j T \in L_{\mu}^p, 0 \leq j \leq s \right\}.$$

Proof. It follows from Theorem 3.3 and of the equality (known for the Fourier transform as Calderón's theorem) given in [5]; i.e., $L_{\mu}^{s,p} \equiv W_{\mu}^{2s,p}$ for $1 < p < \infty$.

Moreover by Theorem 3.3 and using Theorem 2.2 we can obtain similar results to those previous obtained in [13, Theorem 15] and [12, p. 155, Theorem 5] for the Fourier transform. Namely, for $s \in \mathbf{R}$, we have

$$B_{p,2,\mu}^s \subset W_{\mu}^{s,p} \subset B_{p,p,\mu}^s, \quad 2 \leq p < \infty, \quad (3.8)$$

$$B_{p,p,\mu}^s \subset W_{\mu}^{s,p} \subset B_{p,2,\mu}^s, \quad 1 < p \leq 2. \quad (3.9)$$

Finally, we give an application to solve a differential equation.

THEOREM 3.4. *Let $f \in B_{p,q,\mu}^s$. Then there exists $u \in \mathcal{H}'$ such that*

$$(E - \Delta_{\mu})^m u = f,$$

where E is the identity operator and $m \in \mathbf{N} \setminus \{0\}$.

Proof. Consider $f \in B_{p,q,\mu}^s$. We want to obtain a distribution $u \in \mathcal{H}'$ such that

$$(E - \Delta_{\mu})^m u = f. \quad (3.10)$$

Applying the Fourier-Bessel transform and by (1.1) we obtain

$$(1 + \xi^2)^k h_{\mu} u = h_{\mu} f \text{ in } \mathcal{H}'.$$

Then $u = h_{\mu}(1 + \xi^2)^{-m} h_{\mu} f = H_{\mu}^{2m} f$ and by Theorem 3.1 we have that the solution of (3.10) as $u \in B_{p,q,\mu}^{s+2m}$.

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