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Some Contributions to Stochastic Asymptotic Stability and Boundedness via Multiple Lyapunov Functions

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Most of the existing results on stochastic stability use a single Lyapunov function, but we shall instead use multiple Lyapunov functions in this paper to establish some sufficient criteria for locating the limit sets of solutions of stochastic differential equations. From them follow many useful results on stochastic asymptotic stability and boundedness, which enable us to construct the Lyapunov functions much more easily in applications. In particular, the well-known classical theorem on stochastic asymptotic stability is a special case of our more general results. These show clearly the power of our new results.

Key Words: Lyapunov functions; limit set; supermartingale convergence theorem; Itô's formula.

1. INTRODUCTION

Since Itô introduced his stochastic calculus about 50 years ago, the theory of stochastic differential equations has been developed very quickly. In particular, Lyapunov's second method has been developed to deal with stochastic stability by many authors, and we here only mention Arnold [1], Elworthy [4], Friedman and Pinsky [5], Has'minskii [7], Kolmanovskii and Myshkis [8], Kushner [9], Ladde and Lakshmikantham [10], Lakshmikantham *et al.* [11, 12], Mohammed [19], and the author himself [15–18] among others. Most of the existing results on stochastic stability use a single Lyapunov function, but we shall instead use multiple Lyapunov functions in this paper. We shall establish some sufficient criteria, in terms of multiple Lyapunov functions , for locating the limit sets of solutions of stochastic differential equations. These new results claim their originality is due to LaSalle [13] and generalize a key result

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obtained in the author's earlier work Mao [18]. Moreover, from them follow many useful results on stochastic asymptotic stability and boundedness, which enable us to construct the Lyapunov functions much more easily in applications. In particular, we shall show in Section 4 below that the well-known classical theorem on stochastic asymptotic stability is a special case of our more general results. These show clearly the power of our new results. It should also be pointed out that the idea of using two Lyapunov functions in the study of stability of ordinary differential equations can be found in Salvadori [20] and, more generally, the idea of using vector Lyapunov functions can be found in Lakshmikantham *et al.* [12].

2. MAIN RESULTS

Throughout this paper, unless otherwise specified, we let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t>0}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t>0}$ satisfying the usual conditions (i.e., it is right continuous and $\mathcal{F}_0$ contains all P-null sets). Let $B(t) = (B_1(t), \dots, B_m(t))^T$ be an m-dimensional Brownian motion defined on the probability space. Let $|\cdot|$ denote the Euclidean norm in R^n. If A is a vector or matrix, its transpose is denoted by A^T. If A is a matrix, its trace norm is denoted by $|A| = \sqrt{\text{trace}(A^T A)}$.

Consider a nonautonomous n-dimensional stochastic differential equation

$$dx(t) = f(x(t), t)\,dt + g(x(t), t)\,dB(t) \tag{2.1}$$

on $t \geq 0$ with initial value $x(0) = x_0 \in R^n$. Here f: $R^n \times R_+ \to R^n$ and g: $R^n \times R_+ \to R^{n\times m}$ are measurable functions. We assume that both f and g are sufficiently smooth for Eq. (2.1) to have a unique continuous solution on $t \geq 0$, which is denoted by $x(t; x_0)$ in this paper. For example, f and g satisfy the local Lipschitz condition and the linear growth condition. For other conditions please see Arnold [1] or Mao [17].

Let $C^{2,1}(R^n \times R_+; R_+)$ denote the family of all nonnegative functions $V(x, t)$ on $R^n \times R_+$ which are twice continuously differentiable in x and once in t. Define an operator L acting on $C^{2,1}(R^n \times R_+; R_+)$ functions by

$$LV(x, t) = V_t(x, t) + V_x(x, t)f(x, t) + \frac{1}{2}\text{trace}[g^T(x, t)V_{xx}(x, t)g(x, t)],$$

where

$$V_t(x, t) = \frac{\partial V(x, t)}{\partial t}, \qquad V_x(x, t) = \left(\frac{\partial V(x, t)}{\partial x_1}, \dots, \frac{\partial V(x, t)}{\partial x_n}\right),$$
$$V_{xx}(x, t) = \left(\frac{\partial^2 V(x, t)}{\partial x_i \partial x_j}\right)_{n\times n}.$$

Moreover, let $\mathscr{K}$ denote the class of continuous (strictly) increasing functions μ from R_+ to R_+ with $\mu(0) = 0$. Let $\mathscr{K}_\infty$ denote the class of functions μ in $\mathscr{K}$ with $\mu(r) \to \infty$ as $r \to \infty$. Functions in $\mathscr{K}$ and $\mathscr{K}_\infty$ are called class $\mathscr{K}$ and $\mathscr{K}_\infty$ functions, respectively. If $\mu \in \mathscr{K}$, its inverse function is denoted by μ^{-1} with domain $[0, \mu(\infty))$. We also denote by $L^1(R_+; R_+)$ the family of all functions γ: $R_+ \to R_+$ such that $\int_0^\infty \gamma(t)dt < \infty$ while we denote by $\mathscr{D}(R_+; R_+)$ the family of all continuous functions η: $R_+ \to R_+$ such that $\int_0^\infty \eta(t)dt = \infty$. If $K \subseteq R^n \times R_+$ and $H \subseteq R$, let $C(K; H)$ denote the family of all continuous mappings from K to H.

We can now formulate our first result, which is a stochastic version of the well-known LaSalle theorem (i.e., Theorem 1 of LaSalle [13] and the reader may refer to Hale and Lunel [16]) for locating limit sets of a system.

THEOREM 2.1. *Assume that there are functions* $V, U \in C^{2,1}(R^n \times R_+; R_+)$, $\gamma_1, \gamma_2 \in L^1(R_+; R_+)$, $\eta \in \mathscr{D}(R_+; R_+)$, *and* $\rho \in C(R_+; R_+)$, *such that*

$$LV(x, t) \le \gamma_1(t), \qquad LU(x, t) \le \gamma_2(t) \tag{2.2}$$

and

$$\gamma_1(t) - LV(x, t) + |V_x(x, t)g(x, t)|^2 \ge \eta(t)\rho(U(x, t)) \tag{2.3}$$

for all $(x, t) \in R^n \times R_+$. *Then* $D_\rho = \{u \ge 0\colon \rho(u) = 0\} \ne \emptyset$ (*as usual* $\emptyset$ *denotes the empty set*), *and*

$$\lim_{t\to\infty} U(x(t; x_0), t) \in D_\rho \qquad a.s. \tag{2.4}$$

for every $x_0 \in R^n$.

To prove this theorem let us present two useful lemmas. The first one is the nonnegative semimartingale convergence theorem established by Lipster and Shiryayev [14, Theorem 7, p. 139].

LEMMA 2.2. *Let* $A(t)$ *and* $U(t)$ *be two continuous adapted increasing processes on* $t \ge 0$ *with* $A(0) = U(0) = 0$ *a.s. Let* $M(t)$ *be a real-valued continuous local martingale with* $M(0) = 0$ *a.s. Let* ξ *be a nonnegative* $\mathscr{F}_0$-*measurable random variable such that* $E\xi < \infty$. *Define*

$$X(t) = \xi + A(t) - U(t) + M(t) \qquad \text{for } t \ge 0.$$

If $X(t)$ *is nonnegative, then*

$$\Big\{\lim_{t\to\infty} A(t) < \infty\Big\} \subset \Big\{\lim_{t\to\infty} X(t) < \infty\Big\} \cap \Big\{\lim_{t\to\infty} U(t) < \infty\Big\} \qquad a.s.,$$

where $B \subset D$ *a.s. means* $P(B \cap D^c) = 0$. *In particular, if* $\lim_{t\to\infty} A(t) < \infty$ *a.s., then for almost all* $\omega \in \Omega$

$$\lim_{t\to\infty} X(t, \omega) < \infty, \qquad \lim_{t\to\infty} U(t, \omega) < \infty$$

and

$$-\infty < \lim_{t\to\infty} M(t,\omega) < \infty.$$

LEMMA 2.3. *Assume that there are two functions* $V \in C^{2,1}(R^n \times R_+; R_+)$ *and* $\gamma \in L^1(R_+; R_+)$ *such that*

$$LV(x,t) \le \gamma(t) \tag{2.5}$$

for all $(x,t) \in R^n \times R_+$. *Then, for every* $x_0 \in R^n$ *the solution of Eq.* (1.1) *has the properties that*

$$\lim_{t\to\infty} V(x(t;x_0),t) < \infty \qquad a.s. \tag{2.6}$$

and

$$\int_0^\infty [\gamma(t) - LV(x(t;x_0),t) + |V_x(x(t;x_0),t)g(x(t;x_0),t)|^2]\,dt < \infty \qquad a.s. \tag{2.7}$$

Proof. Fix any initial value x_0 and write $x(t;x_0) = x(t)$ for simplicity. By Itô's formula

$$V(x(t),t) = V(x_0,0) + \int_0^t \gamma(s)\,ds - \int_0^t [\gamma(s) - LV(x(s),s)]\,ds + \int_0^t V_x(x(s),s)g(x(s),s)\,dB(s). \tag{2.8}$$

Noting that $\int_0^\infty \gamma(s)\,ds < \infty$ and $\gamma(s) - LV(x(s),s) \ge 0$, we can apply Lemma 2.2 to get the required (2.6) and, moreover,

$$\int_0^\infty [\gamma(s) - LV(x(s),s)]\,ds < \infty \qquad \text{a.s.} \tag{2.9}$$

and

$$-\infty < \lim_{t\to\infty} M(t) < \infty \qquad \text{a.s.,} \tag{2.10}$$

where

$$M(t) = \int_0^t V_x(x(s),s)g(x(s),s)\,dB(s).$$

For every integer $i \ge 1$, define a stopping time

$$\tau_i = \inf\{t \ge 0\colon |M(t)| \ge i\},$$

where here and throughout this paper we set $\inf \emptyset = \infty$. It is clear that τ_i is increasing. In particular, by (2.10), there is a subset Ω_1 of Ω with $P(\Omega_1) = 1$ such that for every $\omega \in \Omega_1$ there is an $i(\omega)$ such that

$$\tau_i(\omega) = \infty \qquad \text{for all } i \ge i(\omega). \tag{2.11}$$

On the other hand, we have, for any $t > 0$,

$$E\int_0^{t\wedge\tau_i} |V_x(x(s),s)g(x(s),s)|^2\,ds = E|M(t\wedge\tau_i)|^2 \le i^2.$$

Letting $t \to \infty$ yields

$$E\int_0^{\tau_i} |V_x(x(s),s)g(x(s),s)|^2\,ds \le i^2,$$

which implies that

$$\int_0^{\tau_i} |V_x(x(s),s)g(x(s),s)|^2\,ds < \infty \tag{2.12}$$

holds with probability 1. Hence there is another subset Ω_2 of Ω with $P(\Omega_2) = 1$ such that if $\omega \in \Omega_2$, (2.12) holds for every $i \ge 1$. Therefore, for any $\omega \in \Omega_1 \cup \Omega_2$, we have

$$\begin{aligned}&\int_0^{\infty} |V_x(x(s,\omega),s)g(x(s,\omega),s)|^2\,ds\\ &\quad = \int^{\tau_{i(\omega)}(\omega)} |V_x(x(s,\omega),s)g(x(s,\omega),s)|^2\,ds < \infty.\end{aligned}$$

Since $P(\Omega_1 \cup \Omega_2) = 1$, we must have

$$\int_0^{\infty} |V_x(x(s),s)g(x(s),s)|^2\,ds < \infty \qquad \text{a.s.}$$

This, together with (2.9), gives the other required result (2.7). The proof is therefore complete.

We can now easily prove Theorem 2.1.

Proof of Theorem 2.1. Fix any initial value x_0 and write $x(t;x_0) = x(t)$ as before. By Lemma 2.3, we observe that

$$\lim_{t\to\infty} U(x(t),t) < \infty \qquad \text{a.s.} \tag{2.13}$$

while

$$\int_0^{\infty} \eta(t)\rho(U(x(t),t))\,dt < \infty \qquad \text{a.s.} \tag{2.14}$$

Since $\rho \in C(R_+;R_+)$, it follows from (2.13) that

$$0 \le \lim_{t\to\infty} \rho(U(x(t),t)) = \rho\Big(\lim_{t\to\infty} U(x(t),t)\Big) < \infty \qquad \text{a.s.}$$

We now claim that

$$\lim_{t\to\infty} \rho(U(x(t),t)) = \rho\Big(\lim_{t\to\infty} U(x(t),t)\Big) = 0 \qquad \text{a.s.} \tag{2.15}$$

If this is false, there is some $\overline{\Omega} \subset \Omega$ with $P(\overline{\Omega}) > 0$ such that for any $\omega \in \overline{\Omega}$,

$$\lim_{t\to\infty} \rho(U(x(t,\omega),t)) > 0.$$

Hence, for any $\omega \in \overline{\Omega}$, one can find a pair of $\varepsilon(\omega) > 0$ and $T(\omega) > 0$ such that

$$\rho(U(x(t,\omega),t)) \geq \varepsilon(\omega) \qquad \text{whenever } t \geq T(\omega).$$

Consequently

$$\int_0^\infty \eta(t)\rho(U(x(t,\omega),t))\,dt \geq \varepsilon(\omega)\int_{T(\omega)}^\infty \eta(t)\,dt = \infty.$$

But this contradicts (2.14) so (2.15) must hold. It now follows from (2.15) immediately that $D_\rho \neq \emptyset$ and

$$\lim_{t\to\infty} U(x(t),t) \in D_\rho$$

as required. The proof is complete.

In Theorem 2.1 two Lyapunov functions V and U are used but we can take a further step to use multiple Lyapunov functions. For this purpose we introduce a new notation $\overline{R}_+^k = \{u = (u_1, \ldots, u_k)\colon u_i \geq 0, 1 \leq i \leq k\}$

THEOREM 2.4. *Assume that there are functions* $V, U_1, \ldots, U_k \in C^{2,1}(R^n \times R_+; R_+)$, $\gamma_0, \gamma_1, \ldots, \gamma_k \in L^1(R_+; R_+)$, $\eta \in \mathscr{D}(R_+; R+)$, *and* $\bar\rho \in C(\overline{R}_+^k; R_+)$, *such that*

$$LV(x,t) \leq \gamma_0(t), \qquad LU_i(x,t) \leq \gamma_i(t)\ (1 \leq i \leq k) \tag{2.16}$$

and

$$\gamma_0(t) - LV(x,t) + |V_x(x,t)g(x,t)|^2 \geq \eta(t)\bar\rho(U_1(x,t), \ldots, U_k(x,t)) \tag{2.17}$$

for all $(x,t) \in R^n \times R_+$. *Then, for every* $x_0 \in R^n$,

$$\lim_{t\to\infty}(U_1(x(t;x_0),t), \ldots, U_k(x(t;x_0),t)) \in D_{\bar p} \qquad a.s., \tag{2.18}$$

where $D_{\bar p} = \{u \in \overline{R}_+^k\colon \bar\rho(u) = 0\} \neq \emptyset$.

This theorem can be proved in the same way as in the proof of Theorem 2.1. In the sequel we will only use two Lyapunov functions in order to simplify the statements of our new results, though they can be generalized by using multiple Lyapunov functions.

3. STOCHASTIC ASYMPTOTIC STABILITY AND BOUNDEDNESS

Let us demonstrate that the results obtained in the previous section can be used to establish useful criteria on stochastic asymptotic stability and boundedness.

THEOREM 3.1. *Let all the assumptions of Theorem* 2.1 *hold*.

(i) *If D_p is bounded and*

$$\liminf_{|x|\to\infty,\, t\to\infty} U(x,t) > \max\{u\colon u \in D_p\}, \tag{3.1}$$

then there is a constant $K > 0$ such that for every $x_0 \in R^n$,

$$\limsup_{t\to\infty} |x(t;x_0)| \le K \qquad a.s. \tag{3.2}$$

(ii) *If $D_\rho = \{0\}$ and for some $\mu \in \mathcal{K}$*,

$$\mu(|x|) \le U(x,t), \qquad (x,t) \in R^n \times R_+, \tag{3.3}$$

then for every $x_0 \in R^n$,

$$\lim_{t\to\infty} |x(t;x_0)| = 0 \qquad a.s. \tag{3.4}$$

Proof. (i) Let $K_1 = \max\{u\colon u \in D_\rho\}$. By condition (3.1), one can find a sufficiently small $\varepsilon > 0$ for which there is a pair of $K > 0$ and $T > 0$ such that

$$U(x,t) \ge K_1 + \varepsilon \qquad \text{for all } |x| \ge K \text{ and } t \ge T.$$

On the other hand, Theorem 2.1 shows that

$$\lim_{t\to\infty} U(x(t;x_0),t) \le K_1 \qquad \text{a.s.}$$

Hence, for almost all $\omega \in \Omega$, there is a $\overline{T}(\omega) \ge T$ such that

$$U(x(t,\omega;x_0),t) < K_1 + \varepsilon \qquad \text{whenever } t \ge \overline{T}(\omega).$$

Consequently we must have

$$|x(t,\omega;x_0)| < K \qquad \text{whenever } t \ge \overline{T}(\omega)$$

and the required assertion (3.2) follows.

(ii) Theorem 2.1, together with condition $D_p = \{0\}$, shows that

$$\lim_{t\to\infty} U(x(t;x_0),t) = 0 \qquad \text{a.s.}$$

Consequently, by (3.3)

$$\lim_{t\to\infty} \mu(|x(t;x_0)|) = 0 \qquad \text{a.s.}$$

This means that there is an $\overline{\Omega} \subset \Omega$ with $P(\overline{\Omega}) = 1$ such that

$$\lim_{t\to\infty} \mu(|x(t,w;x_0)|) = 0 \qquad \text{for all } \omega \in \overline{\Omega}. \tag{3.5}$$

It is enough to show that

$$\lim_{t\to\infty} |x(t,\omega;x_0)| = 0 \qquad \text{for all } \omega \in \overline{\Omega}. \tag{3.6}$$

If this is false, there is some $\bar{\omega} \in \overline{\Omega}$ such that

$$\limsup_{t\to\infty} |x(t,\bar{\omega};x_0)| > 0.$$

Hence one can find a sufficiently small number $\varepsilon > 0$ and a sequence $\{t_k\}_{k\geq 1}$ with $t_k \to \infty$ such that

$$|x(t_k,\bar{\omega};x_0)| \geq \varepsilon, \qquad k \geq 1.$$

Since μ is increasing, we have

$$\mu(|x(t_k,\bar{\omega};x_0)|) \geq \mu(\varepsilon), \qquad k \geq 1.$$

Thus

$$\limsup_{t\to\infty} \mu(|x(t,\bar{\omega};x_0)|) \geq \mu(\varepsilon) > 0$$

with contradicts (3.5) so (3.6) must hold. The proof is complete.

Let us now establish another generalization of a classical result on the globally stochastically asymptotic stability.

THEOREM 3.2. *Assume that there are functions* $V, U \in C^{2,1}(R^n \times R_+; R_+)$, $\gamma_1, \gamma_2 \in L^1(R_+; R_+)$, $\eta \in \mathscr{D}(R_+; R_+)$, *and* $\mu_1, \mu_2, \mu_3 \in \mathscr{K}$, *such that*

$$LV(x,t) \leq \gamma_1(t), \quad LU(x,t) \leq \gamma_2(t), \mu_1(|x|) \leq U(x,t) \leq \mu_2(|x|) \tag{3.7}$$

and

$$\gamma_1(t) - LV(x,t) + |V_x(x,t)g(x,t)|^2 \geq \eta(t)\mu_3(|x|) \tag{3.8}$$

for all $(x,t) \in R^n \times R_+$. *Then, for every* $x_0 \in R^n$,

$$\lim_{t\to\infty} |x(t;x_0)| = 0 \qquad a.s. \tag{3.9}$$

Proof. Without loss of any generality we may assume that $\mu_2 \in \mathcal{K}_\infty$ (otherwise we can replace $\mu_2(u)$ by $\mu_2(u) + u$ which is in $\mathcal{K}_\infty$), and hence its inverse function $\mu_2^{-1} \in \mathcal{K}_\infty$. It follows from (3.7) that

$$\mu_2^{-1}(U(x, t)) \leq |x|.$$

Substituting this into (3.8) gives

$$\gamma_1(t) - LV(x, t) + |V_x(x, t)g(x, t)|^2 \geq \eta(t)\mu_3(\mu_2^{-1}(U(x, t))).$$

Let $\rho(\cdot) = \mu_3(\mu_2^{-1}(\cdot))$ which is in $\mathcal{K}$ whence $D_\rho = \{0\}$. The required assertion (3.9) now follows from part (ii) of Theorem 3.1. The proof is complete.

4. COMPARISONS WITH THE EXISTING RESULTS

Let us first recall the well-known classical result on the globally stochastically asymptotic stability (cf. Arnold [1], Has'minskii [7], or Kushner [9]).

THEOREM 4.1. *Assume that there are functions $V \in C^{2,1}(R^n \times R_+; R_+)$, $\mu_1, \mu_2 \in \mathcal{K}_\infty$ and $\mu_3 \in \mathcal{K}$ such that*

$$\mu_1(|x|) \leq V(x, t) \leq \mu_2(|x|) \tag{4.1}$$

and

$$LV(x, t) \leq -\mu_3(|x|) \tag{4.2}$$

for all $(x, t) \in R^n \times R_+$. Then, for every initial value $x_0 \in R^n$ the solution of Eq. (1.1) *has the property that*

$$\lim_{t\to\infty} |x(t; x_0)| = 0 \qquad a.s. \tag{4.3}$$

To see the important contributions of this paper, let us form a useful corollary that follows from Theorem 3.2 directly.

COROLLARY 4.2. *Assume that there are functions $V \in C^{2,1}(R^n \times R_+; R_+)$, $\mu_1, \mu_2, \mu_3 \in \mathcal{K}$ and $\gamma \in L^1(R_+; R_+)$ such that* (4.1) *is satisfied while* (4.2) *is replaced by the following weaker condition*

$$\begin{aligned} LV(x, t) \leq \gamma(t) \wedge [\gamma(t) + |V_x(x, t)g(x, t)|^2 - \mu_3(|x|)], \\ (x, t) \in R^n \times R_+. \end{aligned} \tag{4.4}$$

Then the conclusion of Theorem 4.1 *still holds.*

This corollary only requires that μ_1, μ_2 be class $\mathscr{K}$ functions but not class $\mathscr{K}_\infty$ functions. Of course this is a minor improvement. However, let us explain the significant features of this corollary. In the classical Theorem 4.1, LV is required to be negative definite. So far, to the best knowledge of the author, there is no paper that shows whether the asymptotic stability holds or not if this negative definiteness does not hold, but our Corollary 4.2 gives a positive answer. In fact, we see from condition (4.4) that LV may take positive values somewhere (see the example below for an explicit support) but yet the corollary shows that the equation may still be stochastically asymptotically stable. Moreover, condition (4.4) is equivalent to

$$LV(x,t) \le \gamma(t) \quad \text{and} \quad LV(x,t) \le \gamma(t) + |V_x(x,t)g(x,t)|^2 - \mu_3(|x|).$$

We hence see clearly that if (4.2) is satisfied, (4.4) must be satisfied but not conversely. It is the terms $\gamma(t)$ and $|V_x(x,t)g(x,t)|^2$ that make condition (4.4) able to be satisfied much more easily than condition (4.2). So Corollary 4.2 has already enabled us to construct the Lyapunov function more easily in application. Note furthermore that the term $|V_x(x,t)g(x,t)|^2$ is connected with the diffusion coefficient $g(x,t)$ so our result reveals the important role of noise in stabilizing the system. Stabilization by noise is a very interesting issue but we will not develop it further in this paper due to page limit and we only refer the reader to Arnold *et al.* [2, 3] and Mao [17]. Let us now use an example to illustrate the above features explicitly.

EXAMPLE 4.3. Let us consider a two-dimensional equation

$$d\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} -x_1(t) + x_1(t)x_2(t) + 2e^{-t} \\ -x_1^2(t) - 2x_2(t) \end{bmatrix} dt + \begin{bmatrix} e^{-t} dB_1(t) \\ 2x_2(t) dB_2(t) \end{bmatrix}, \tag{4.5}$$

where $(B_1(t), B_2(t))$ is a two-dimensional Brownian motion. Let $V(x) = |x|^2$ so condition (4.1) is satisfied with $\mu_1(r) = \mu_2(r) = r^2$. Compute

$$\begin{aligned} LV(x,t) &= 2x_1(-x_1 + x_1x_2 + 2e^{-t}) + 2x_2(-x_1^2 - 2x_2) + e^{-2t} + 4x_2^2 \\ &= e^{-2t} + 4x_1e^{-t} - 2x_1^2 \le 5e^{-2t} - x_1^2 \le 5e^{-2t} \end{aligned}$$

which is not negative definite even $LV(x,t) > 0$ for some (x,t), e.g., when $x_1 = 0.5$ and $t = 1$. Hence we cannot apply the classical Theorem 4.1 using the Lyapunov function above to deduce the asymptotic stability in probability 1. However, we can apply Corollary 4.2. In fact, compute

$$|V_x(x,t)g(x,t)|^2 = 4x_1^2e^{-2t} + 16x_2^4$$

so

$$5e^{-2t} - LV(x,t) + |V_x(x,t)g(x,t)|^2 \ge x_1^2 + 16x_2^4 \ge \mu_3(|x|),$$

where

$$\mu_3(r) = \min_{|x|\geq r}\left[x_1^2 + 16x_2^4\right] \qquad \text{for } r \geq 0,$$

which is in $\mathscr{K}$ clearly. Thus

$$LV(x,t) \leq 5e^{-2t} + |V_x(x,t)g(x,t)|^2 - \mu_3(|x|)$$

and condition (4.4) is satisfied. We can therefore conclude by our new Corollary 4.2 that the solution of Eq. (4.5) starting from anywhere in R^2 will tend to zero asymptotically with probability one.

In the previous sections we use two or more Lyapunov functions while most authors use usually a single one. To show the advantage of using two or more Lyapunov functions , let us obtain a useful corollary by letting $U = V$ and $\gamma_1 = \gamma_2 = \gamma$ in Theorem 2.1.

COROLLARY 4.4. *Assume that there are functions $V \in C^{2,1}(R^n \times R_+; R_+)$, $\gamma \in L^1(R_+; R_+)$, $\eta \in \mathscr{D}(R_+; R_+)$, and $\rho \in C(R_+; R_+)$, such that*

$$\begin{aligned} &LV(x,t) \leq \gamma(t) \quad \textit{and} \\ &\gamma(t) - LV(x,t) + |V_x(x,t)g(x,t)|^2 \geq \eta(t)\rho(V(x,t)) \end{aligned} \tag{4.6}$$

for all $(x,t) \in R^n \times R_+$. Then, for every $x_0 \in R^n$,

$$\lim_{t\to\infty} V(x(t;x_0),t) \in D_\rho \quad \textit{a.s.},$$

where D_ρ is the same as defined in Theorem 2.1.

This corollary is a generalization of Theorem 2.6 of the author's earlier work Mao [18]. In fact, if we let $\eta \equiv 1$ and replace (4.6) by the stronger condition

$$LV(x,t) \leq \gamma(t) - \rho(V(x,t)), \tag{4.7}$$

then Corollary 4.4 reduces to Theorem 2.6 of Mao [18]. Moreover, comparing Theorem 2.1 with Corollary 4.4 we observe that both V and U in Theorem 2.1 satisfy less restricted conditions, which gives us more flexibility to construct them in applications. This will be illustrated by Example 6.1 below but let us first discuss linear stochastic differential equations.

5. LINEAR STOCHASTIC DIFFERENTIAL EQUATIONS

Consider an n-dimensional linear stochastic differential equation

$$dx(t) = Ax(t)dt + \sum_{i=1}^{m} C_i x(t) dB_i(t). \tag{5.1}$$

Let Q be a positive-definite symmetric $n \times n$-matrix and define $V(x, t) = V(x) = x^T Q_x$. It is known that

$$\lambda_{\min}(Q)|x|^2 \le x^T Qx \le \lambda_{\max}(Q)|x|^2,$$

where $\lambda_{\min}(Q)$ and $\lambda_{\max}(Q)$ denote the smallest and largest eigenvalue of Q (so $0 < \lambda_{\min}(Q) \le \lambda_{\max}(Q)$). It is also easy to compute

$$LV(x) = x^T Hx, \quad \text{where } H = QA + A^T Q + \sum_{i=1}^{m} C_i^T QC_i,$$

and

$$|V_x(x)g(x)|^2 = \sum_{i=1}^{m} \left| x^T \left(QC_i + C_i^T Q \right) x \right|^2,$$

where $g(x) = C_1 x, \ldots, C_m x)$. If we impose a hypothesis that

$$H \text{ is a non-positive definite,} \tag{5.2}$$

then $LV(x) \le 0$. Set

$$G_0 = \{x \in R^n\colon\ x^T Hx = 0\} \quad \text{and}$$
$$G_i = \{x \in R^n\colon\ x^T \left(QC_i + C_i^T Q \right) x = 0\} \ (i \ge 1)$$

and impose another hypothesis that

$$\bigcap_{i=0}^{m} G_i = \{0\}. \tag{5.3}$$

Then

$$-LV(x) + |V_x(x)g(x)|^2 > 0 \quad \text{if and only if } x \ne 0.$$

Define

$$\mu_3(r) = \min_{|x| \ge r} [-LV(x) + |V_x(x)g(x)|^2] \quad \text{for } r \ge 0.$$

Clearly, $\mu_3(\cdot) \in \mathscr{K}$ and, moreover,

$$LV(x) \le |V_x(x)g(x)|^2 - \mu_3(|x|).$$

By Corollary 4.2, we therefore obtain a useful result for linear stochastic differential equations.

COROLLARY 5.1. *If there is a positive-definite symmetric $n \times n$-matrix Q such that* (5.2) *and* (5.3) *hold, then all the solutions of the linear equation* (5.1) *will tend to zero with probability* 1.

Let us now discuss an example to illustrate this result, especially to explain how to verify hypothesis (5.3).

EXAMPLE 5.2. Consider a two-dimensional linear stochastic differential equation

$$dx(t) = Ax(t)dt + C_1 x(t)dB_1(t), \tag{5.4}$$

where $B_1(t)$ is a scalar Brownian motion,

$$A = \begin{bmatrix} -6.5 & -3 \\ 1 & -2 \end{bmatrix} \quad \text{and} \quad C_1 \begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix}.$$

Let Q be the identity matrix. Then

$$H = A + A^T + C_1^T C_1 = -\begin{bmatrix} 8 & 4 \\ 4 & 2 \end{bmatrix},$$

which is non-positive definite. It is easy to show that the eigenvalues of H are $\lambda_1 = 0$ and $\lambda_2 = 10$ and the corresponding normalized eigenvectors are $\theta_1 = (-1, 2)^T/\sqrt{5}$ and $\theta_2 = (2, 1)^T/\sqrt{5}$. Note also that any $x \in R^2$ can be represented as $x = \alpha_1\theta_1 + \alpha_2\theta_2$ for some real numbers α_1, α_2 and hence

$$x^T Hx = (\alpha_1\theta_1 + \alpha_2\theta_2)^T(\alpha_1\lambda_1\theta_1 + \alpha_2\lambda_2\theta_2) = 10\alpha_2^2$$

which is zero if and only if $\alpha_2 = 0$. Therefore

$$G_0 = \{x \in R^2\colon x = \alpha_1\theta_1 \text{ for } \alpha_1 \in R\} = \{x \in R^2\colon x_2 = -2x_1\},$$

which is a line in R^2 through the origin with slope -2. On the other hand,

$$C_1 + C_1^T = \begin{bmatrix} 2 & -3 \\ -3 & 2 \end{bmatrix}$$

has the eigenvalues $\lambda_1 = 5$ and $\lambda_2 = -1$ and their corresponding normalized eigenvectors $\varphi_1 = (1, -1)^T/\sqrt{2}$ and $\varphi_2 = (1, 1)^T/\sqrt{2}$. Representing any $x \in R^2$ as $x = \beta_1\varphi_1 + \beta_2\varphi_2$ for some real numbers β_1, β_2 we have

$$x^T(C_1 + C_1^T)x = (\beta_1\varphi_1 + \beta_2\varphi_2)^T(\beta_1\gamma_1\varphi_1 + \beta_2\gamma_2\varphi_2) = 5\beta_1^2 - \beta_2^2$$

which equals zero if and only if $\beta_2 = \pm\sqrt{5}\beta_1$. Hence

$$\begin{aligned} G_1 &= \{x \in R^2\colon x = \beta_1\varphi_1 \pm \sqrt{5}\beta_1\varphi_2 \text{ for } \beta_1 \in R\} \\ &= \{x \in R^2\colon x = \beta(1 \pm \sqrt{5}, -1 \pm \sqrt{5})^T \text{ for } \beta \in R\} \\ &= \{x \in R^2\colon x_2 = x_1(\sqrt{5} \pm 1)^2/4\}, \end{aligned}$$

which are two lines in R^2 through the origin with slopes $(\sqrt{5} - 1)^2/4$ and $(\sqrt{5} + 1)^2/4$, respectively. We therefore see that $G_0 \cap G_1 = \{0\}$ and can conclude that all the solutions of Eq. (5.4) will tend to the origin with probability 1.

6. FURTHER EXAMPLES

Although two examples have been discussed before to illustrate our theory, let us discuss two more examples in this section. In the following examples whenever we use g we mean the diffusion coefficient of the equations discussed. The first example not only illustrates the advantage of using two Lyapunov functions but also shows how our theory can be applied to deal with partial asymptotic stability (i.e., only a part (or subset) of the components tends to zero).

EXAMPLE 6.1. Let $B(t) = (B_1(t), B_2(t))$ be a two-dimensional Brownian motion. Consider a two-dimensional stochastic differential equation

$$\begin{cases} dx_1(t) = -x_2^2(t)\sin(x_1(t))\cos^2 t dt + e^{-t}\cos(x_2(t))dB_1(t), \\ dx_2(t) = -x_2(t)\cos^2(x_1(t))\cos^2 t dt + e^{-t}\cos(x_1(t))dB_2(t). \end{cases} \tag{6.1}$$

Let

$$V(x,t) = V(x_1, x_2) = 1 - \cos x_1 + \frac{1}{2}x_2^2 \quad \text{and} \quad U(x,t) = U(x_2) = x_2^2.$$

Compute

$$\begin{aligned} LV(x,t) &= -x_2^2\sin^2(x_1)\cos^2 t + \frac{1}{2}e^{-2t}\cos(x_1)\cos^2(x_2) \\ &\quad - x_2^2\cos^2(x_1)\cos^2 t + \frac{1}{2}e^{-2t}\cos^2(x_1) \\ &\le e^{-2t} - x_2^2\cos^2 t \le e^{-2t}, \end{aligned}$$

and hence

$$e^{-2t} - LV(x,t) + |V_x(x,t)g(x,t)|^2 \ge x_2^2\cos^2 t = U(x_2)\cos^2 t,$$

while

$$LU(x,t) = -2x_2^2\cos^2(x_1)\cos^2 t + e^{-2t}\cos^2(x_1) \le e^{-2t}.$$

Theorem 2.1 shows that for any given initial value $x_0 \in R^2$ the solution of Eq. (6.1) has the property

$$\lim_{t\to\infty} x_2(t; x_0) = 0 \quad \text{a.s.} \tag{6.2}$$

Moreover, by Lemma 2.3,

$$\lim_{t\to\infty}\left[1 - \cos(x_1(t;x_0)) + \frac{1}{2}x_2^2(t;x_0)\right] < \infty \quad \text{a.s.}$$

This, together with (6.2), implies that

$$\lim_{t\to\infty}\cos(x_1(t;x_0)) \text{ exists almost surely.}$$

It then follows easily that

$$\lim_{t\to\infty} x_1(t;x_0) \text{ exists and is finite almost surely.} \tag{6.3}$$

EXAMPLE 6.2. Let $B(t)$ be a scalar Brownian motion. Consider a two-dimensional stochastic equation

$$\begin{cases} dx_1(t) = -\sin^2 t[x_1(t) - x_2(t)]dt + \sqrt{2}\sin t[|x(t)| \wedge 1]dB(t), \\ dx_2(t) = -\sin^2 t[x_1(t) + x_2(t)]dt + e^{-t}\cos(|x(t)|)dB(t), \end{cases} \tag{6.4}$$

where $x = (x_1, x_2)$ of course. Let $V(x, t) = V(x) = |x|^2$ and compute

$$\begin{aligned} LV(x, t) &= -2\sin^2 t[|x|^2 + (|x|^2 \wedge 1)] + e^{-2t}\cos^2(|x|) \\ &\le e^{-2t} - 2\sin^2 t[(|x|^2 \vee 1) - 1] \le e^{-2t} \end{aligned} \tag{6.5}$$

while

$$e^{-2t} - LV(x, t) + |V_x(x, t)g(x, t)|^2 \ge 2\sin^2 t[(|x|^2 \vee 1) - 1].$$

We can therefore conclude by Theorem 3.1 that for any initial value $x_0 \in R^2$ the solution of Eq. (6.4) has the property

$$\lim_{t\to\infty} |x(t; x_0)| \le 1 \quad \text{a.s.} \tag{6.6}$$

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