

Periodic Solutions of Classical Hamiltonian Systems without Palais–Smale Condition

Guihua Fei

*Department of Mathematics and Statistics, University of Minnesota,
Duluth, Minnesota 55812*

and

Soon-Kyu Kim and Tixiang Wang

Department of Mathematics, University of Connecticut, Storrs, Connecticut 06269

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In this paper, we study the existence of periodic solutions for classical Hamiltonian systems without the Palais–Smale condition. We prove that the information of the potential function contained in a finite domain is sufficient for the existence of periodic solutions. Moreover, we establish the existence of infinitely many periodic solutions without any symmetric condition on the potential function V . © 2002 Elsevier Science (USA)

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1. INTRODUCTION AND MAIN RESULTS

In this paper, we consider the classical Hamiltonian system

$$\begin{aligned}\ddot{u} + V'(t, u(t)) &= 0, \\ u(0) - u(T) &= \dot{u}(0) - \dot{u}(T) = 0,\end{aligned}\tag{1.1}$$

where $T > 0$, N is a positive integer, $V: \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}$ is a function, and $V'(t, x)$ denotes its gradient with respect to the x variable. One can see [7] for the motivation for the study of the Hamiltonian system (1.1). The existence of periodic solutions of (1.1) has been studied by many authors

(cf. [1, 2, 6–12] and the references therein). By using variational methods, the existence of T -periodic solutions is equivalent to the existence of critical points of the action function f .

The purpose of this paper is to study the existence of periodic solutions of (1.1) in the case that the Palais–Smale condition may not be satisfied. In this paper, we denote by $x \cdot y$ and $|x|$ the usual inner product and norm in \mathbb{R}^N , respectively.

We assume that V satisfies the following conditions:

(V1) $V \in C^2(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$, $V(t+T, z) = V(t, z)$, $\forall (t, z) \in \mathbb{R} \times \mathbb{R}^N$.

(V2) There exists $M > 0$ such that

$$|V'(t, z)| \leq M, \quad \forall (t, z) \in \mathbb{R} \times \mathbb{R}^N.$$

Let $S_T = \mathbb{R}/T\mathbb{Z}$, $L_s(\mathbb{R}^N)$ be the set of symmetric $N \times N$ matrices, and

$$\gamma = \frac{2M(1+T)(4\pi^2 + T^2)}{4\pi^2}. \quad (1.2)$$

The main results of this paper are the following theorems.

THEOREM 1.1. *Suppose that V satisfies (V1), (V2), and*

(V3) *there exist $0 < c < d$ with $d - c > 2\gamma$ such that*

$$(V'(t, z) \cdot z) - \gamma|V'(t, z)| > 0, \quad \text{for } c \leq |z| \leq d.$$

Then the system (1.1) possesses a T -periodic solution. Moreover, suppose V satisfies

(V4) *there exists $B_0(t) \in C(S_T, L_s(\mathbb{R}^N))$ such that*

$$V'(t, z) = B_0(t)z + o(|z|) \quad \text{as } |z| \rightarrow 0.$$

Then the system (1.1) possesses a nontrivial T -periodic solution provided $N \notin [j_0, j_0 + v_0]$, where (j_0, v_0) are the Morse indices of $B_0(t)$ given by (2.2) (see Section 2).

THEOREM 1.2. *Suppose that V satisfies (V1), (V2), and*

(V5) *there exist $0 < a < b < c < d$ with*

$$b - a \geq 2\gamma, \quad d - c \geq 2\gamma,$$

such that

$$(V'(t, z) \cdot z) - \gamma|V'(t, z)| > 0, \quad \text{for } c \leq |z| \leq d;$$

$$(V'(t, z) \cdot z) + \gamma|V'(t, z)| < 0, \quad \text{for } a \leq |z| \leq b.$$

Then (1.1) has a T -periodic solution $x^(t)$ with*

$$a\sqrt{T} < \|x^*(t)\| < d\sqrt{T}.$$

THEOREM 1.3. Suppose that V satisfies (V1), (V2), and (V6) there exist $0 < a < b < c < d$ such that

$$\begin{aligned} (V'(t, z) \cdot z) &> 0, & \text{for } c \leq |z| \leq d; \\ (V'(t, z) \cdot z) &< 0, & \text{for } a \leq |z| \leq b. \end{aligned}$$

Then there exists a $\lambda_0 > 0$ such that for $0 < \lambda \leq \lambda_0$ the system

$$\begin{aligned} \ddot{u} + \lambda V'(t, u(t)) &= 0, \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) &= 0 \end{aligned} \tag{1.3}$$

has a T -periodic solution $x^*(t)$ with

$$a\sqrt{T} \leq \|x^*(t)\| \leq d\sqrt{T}.$$

THEOREM 1.4. Suppose that V satisfies (V1), (V2), and

(V7) there exist $\alpha > 0$, $\tau > 0$, $a_n > 0$, and $b_n > 0$ such that

$$\begin{aligned} a_n + \tau < b_n, \quad b_n + \tau < a_{n+1}; \\ \frac{|V'(t, z)|}{|V'(t, z) \cdot z|} &\leq \alpha, \quad (V'(t, z) \cdot z) > 0, & \text{for } b_n \leq |z| \leq b_n + \tau; \\ \frac{|V'(t, z)|}{|V'(t, z) \cdot z|} &\leq \alpha, \quad (V'(t, z) \cdot z) < 0, & \text{for } a_n \leq |z| \leq a_n + \tau. \end{aligned}$$

Then there exists a $\lambda_0 > 0$ such that for $0 < \lambda \leq \lambda_0$ the system (1.3) has infinitely many T -periodic solutions.

Our results are different from previous results in three ways.

1. Many results in the literature require that the action function f satisfies the Palais–Smale (PS) condition or its variation. In order to verify the (PS) condition, one usually assumes that V satisfies the Landsman–Lazer-type condition

$$V(t, z) \rightarrow \pm\infty \quad \text{as } |z| \rightarrow \infty$$

or some conditions about the behavior of V at infinity. In our results, the action function f may not satisfy the (PS) condition or its variation.

2. Theorem 1.2 shows that the information of the potential function V contained in the “outer boundary” and the “inner boundary” of a finite domain is sufficient for the existence of T -periodic solutions. We don’t need any information about the behavior of V at infinity.

3. Theorem 1.4 shows that one can get the existence of infinitely many T -periodic solutions without any symmetric conditions about V .

In Section 4, we give some examples which are not covered by previous results.

To prove our results, we first use Galerkin approximation to restrict the action function f on finite-dimensional space E_n , $f_n = f|_{E_n}$. Then we construct the isolating block for f_n on E_n . By using Conley index theory [3], we get a critical point x_n for each f_n . Since x_n is bounded by a constant which is independent on n , we can prove that $\{x_n\}$ has a subsequence which converges to a critical point of f .

2. PRELIMINARIES

Let $E_T = W^{1,2}(S_T, \mathbb{R}^N)$. Recall that E is a Hilbert space with norm $\|\cdot\|$ and inner product $\langle \cdot, \cdot \rangle$ and that E_T consists of those $z(t)$ in $L^2(S_T, \mathbb{R}^N)$ whose Fourier series

$$z(t) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{2\pi}{T}nt\right) + b_n \sin\left(\frac{2\pi}{T}nt\right) \right)$$

satisfies

$$\|z\|^2 = T|a_0|^2 + \frac{T}{2} \sum_{n=1}^{\infty} \left[\left(\frac{2\pi}{T}n \right)^2 + 1 \right] (|a_n|^2 + |b_n|^2) < \infty,$$

where $a_j, b_j \in \mathbb{R}^N$.

For a given continuous T -periodic and symmetric $N \times N$ matrix function $A(t) \in C(S_T, L_s(\mathbb{R}^N))$, we define a linear operator $K \in \mathcal{L}(E)$ by extending the bilinear forms

$$\langle Kx, y \rangle = \int_0^T (x(t) + A(t)x(t), y(t)) dt \quad (2.1)$$

on E_T . Then K is self-adjoint and compact (see [7]). Let $E_T^0 = \ker(id - K)$, E_T^+ be the positive definite subspace of $id - K$, and E_T^- be the negative definite subspace of $id - K$. Then

$$E_T = E_T^0 \oplus E_T^+ \oplus E_T^-, \quad \dim(E_T^0 \oplus E_T^-) < +\infty.$$

We define

$$j(A, T) = \dim E_T^- \quad \text{and} \quad v(A, T) = \dim E_T^0. \quad (2.2)$$

If $A(t) = A \in L_s(\mathbb{R}^N)$ is a constant matrix, by direct calculation, for $z(t) \in E$ we have

$$\begin{aligned} (id - K)z(t) = & -Aa_0 + \sum_{n=1}^{\infty} \frac{4\pi^2 n^2}{4\pi^2 n^2 + T^2} \left(Id - \frac{T^2}{4\pi^2 n^2} A \right) \\ & \times \left(a_n \cos\left(\frac{2\pi}{T}nt\right) + b_n \sin\left(\frac{2\pi}{T}nt\right) \right), \end{aligned}$$

and the eigenvalues of $id - K$ are given by

$$\begin{aligned}\lambda_{0,k} &= -\alpha_k, & 1 \leq k \leq N, \\ \lambda_{n,k} &= \frac{4\pi^2 n^2}{4\pi^2 n^2 + T^2} \left(1 - \frac{T^2}{4\pi^2 n^2} \alpha_k \right), & n \in \mathbb{N}, \quad 1 \leq k \leq N,\end{aligned}\quad (2.3)$$

where Id is the identity $N \times N$ matrix and $\{\alpha_k\}_{1 \leq k \leq N}$ are the eigenvalues of A .

For $m \geq 1$, let

$$\begin{aligned}E_0 &= \mathbb{R}^N, \\ E_m &= \left\{ z(t) \in E_T : z(t) = a_0 + \sum_{n=1}^m \left(a_n \cos\left(\frac{2\pi}{T} nt\right) + b_n \sin\left(\frac{2\pi}{T} nt\right) \right) \right\},\end{aligned}$$

and P_m be the orthogonal projection from E_T to E_m . Then let $\{P_m\}$ be an approximation scheme with respect to the operator $id - K$; i.e.,

$$(id - K)P_m = P_m(id - K),$$

$$P_m x \rightarrow x \text{ as } m \rightarrow \infty \text{ for any } x \in E_T.$$

Let $\eta: (\mathbb{R}^n \times \mathbb{R}) \rightarrow \mathbb{R}$ be the flow on \mathbb{R}^n . Let $D \subset \mathbb{R}^n$ be a closed set and $x \in \partial D$ be a boundary point. Then x is called a strict egress (strict ingress, bounce-off, respectively) point of D if there are $c, d > 0$ such that for $0 < t \leq c$: $\eta(x, t) \notin D$ ($\eta(x, t) \in \text{int}(D)$, $\eta(x, t) \notin D$, respectively) and for $0 < -t \leq d$: $\eta(x, t) \in \text{int}(D)$ ($\eta(x, t) \notin D$, $\eta(x, t) \notin D$, respectively). We use D^e (D^i , D^b , respectively) to denote the set of strict egress (strict ingress, bounce-off) points of the closed set D . Let $D^- = D^e \cup D^b$.

A closed set $D \subset \mathbb{R}^n$ is called an isolating block if $\partial D = D^e \cup D^i \cup D^b$ and $D^- = D^e \cup D^b$ is closed.

Let $D \subset \mathbb{R}^n$ be a bounded isolating block under the flow η . We define

$$I(\eta, D) = \sum_{k \geq 0} r^k(D, D^-) t^k, \quad (2.4)$$

where

$$r^k(D, D^-) = \text{rank}(H_k(D, D^-))$$

is the rank of the k th homology group $H_k(D, D^-)$.

Let $f: \mathbb{R}^n \rightarrow \mathbb{R} \in C^2$. η is the gradient flow generated by

$$\frac{dx(t)}{dt} = -f'(x(t)).$$

Let $D \subset \mathbb{R}^n$ be a bounded isolating block under the flow η . Using the algebraic topology theory and the results in [3], one can prove the following theorem [4].

THEOREM 2.1. *If $I(\eta, D) \neq 0$, then there is at least one critical point of f in D . Moreover, suppose that there are a symmetric matrix L on \mathbb{R}^n , $d > 0$, and $r > 0$ such that*

$$B_{2r} \subset \text{int}(D);$$

$$\|f'(x) - Lx\|/\|x\| \rightarrow 0 \quad \text{as } \|x\| \rightarrow 0;$$

$$\|f''(x) - L\| < \frac{1}{2}d, \quad x \in \text{int}(B_{2r});$$

where $B_{2r} = \{x \in \mathbb{R}^n: \|x\| \leq 2r\}$. If $I(\eta, D) \neq 0$ and $I(\eta, D)$ does not contain the terms with the form t^s for

$$\dim M_d^-(L) \leq s \leq \dim M_d^-(L) + \dim M_d^0(L),$$

then f has at least one critical point out of B_r , where $M_d^+(\cdot)$, $M_d^-(\cdot)$, and $M_d^0(\cdot)$ denote the eigenspace corresponding to the eigenvalue λ belonging to $[d, +\infty)$, $(-\infty, -d]$, and $(-d, d)$ respectively.

3. THE EXISTENCE OF PERIODIC SOLUTIONS

In this paper, for any bounded self-adjoint linear operator L , we denote by $M^+(L)$, $M^-(L)$, and $M^0(L)$ the positive definite, negative definite, and null subspaces of L , respectively. We also denote $L^\# = (L|_{\text{Im}L})^{-1}$.

Let $E_T = W^{1,2}(S_T, \mathbb{R}^N)$, and let K_∞ and K_0 be the operators defined by (2.1), corresponding to $B_\infty = 0$ and $B_0(t)$, respectively. We also denote by (j_0, v_0) and (j_∞, v_∞) the Morse indices of $B_0(t)$ and $B_\infty = 0$, respectively, given by (2.2). Then by (2.3) we have

$$j_\infty = 0, \quad v_\infty = N, \quad \ker(id - K_\infty) = \mathbb{R}^N; \quad (3.1)$$

$$\|(id - K_\infty)^\#\| = \frac{(4\pi^2 + T^2)}{4\pi^2}. \quad (3.2)$$

For any $z \in E_T$, we define

$$g(z) = \int_0^T V(t, z) dt, \quad (3.3)$$

$$f(z) = \frac{1}{2} \langle (id - K_\infty)z, z \rangle - g(z).$$

By (V1) we know that $f \in C^2(E_T, \mathbb{R})$ and the critical points of f in E_T are T -periodic solutions of (1.1).

Let $\{E_n\}$ and $\{P_m\}$ be the Galerkin approximation scheme given in Section 2.

The Proof of Theorem 1.1. We carry out the proof in several steps.

Step 1. For $m \geq 0$,

$$f_m(z) = \frac{1}{2} \langle (id - K_\infty)z, z \rangle - g(z) \quad \text{for } z \in E_m.$$

Since $\{P_m\}$ is the approximation scheme with respect to the operator $id - K_\infty$, by (3.1) we have

$$\begin{aligned} M^+(P_m(id - K_\infty)P_m) &= P_m M^+(id - K_\infty), \\ M^0(P_m(id - K_\infty)P_m) &= \mathbb{R}^N, \quad M^-(P_m(id - K_\infty)P_m) = 0. \end{aligned} \quad (3.4)$$

For $m \geq 1$, let

$$V_m^* = \{z \in M^*(P_m(id - K_\infty)P_m) : \|z\| \leq r_*\}, \quad \text{where } * = +, 0.$$

We want to show that there are $r_+, r_0 > 0$, which do not depend on m , such that

$$D_m = V_m^+ \times V_m^0$$

is an isolating block of the gradient flow η generated by

$$\frac{dz}{dt} = -P_m(id - K_\infty)P_m z + P_m g'(z)$$

in E_m .

By (V2) we know that

$$\|g'(z)\| \leq \sqrt{T}M.$$

For $z = z^+ + z^0 \in \partial V_m^+ \times V_m^0$,

$$\begin{aligned} \left\langle \frac{dz}{dt}, z^+ \right\rangle \Big|_{t=0} &= -\langle P_m(id - K_\infty)P_m z, z^+ \rangle + \langle P_m g'(z), z^+ \rangle \\ &= -\langle (id - K_\infty)z^+, z^+ \rangle + \langle g'(z), z^+ \rangle \\ &\leq -\|(id - K_\infty)^\# \|^{-1} \|z^+\|^2 + \sqrt{T}M \|z^+\| < 0, \end{aligned}$$

if $\|z^+\| > \sqrt{T}M \|(id - K_\infty)^\# \|$. Therefore we can choose

$$r_+ = 2\sqrt{T}M \|(id - K_\infty)^\# \|. \quad (3.5)$$

For $z = z^+ + z^0 \in V_m^+ \times \partial V_m^0$,

$$\begin{aligned} \left\langle \frac{dz}{dt}, z^0 \right\rangle \Big|_{t=0} &= -\langle P_m(id - K_\infty)P_m z, z^0 \rangle + \langle P_m g'(z), z^0 \rangle \\ &= \langle g'(z), z^0 \rangle = \int_0^T (V'(t, z) \cdot z^0) dt. \end{aligned} \quad (3.6)$$

Note that, for any $u \in E_T$,

$$\|u\|_\infty \leq \left(\frac{\sqrt{T} + T\sqrt{T}}{T}\right)\|u\|.$$

By (3.2), (3.4), (3.5), and (1.2) we have

$$|z(t)| \geq |z^0| - |z^+| \geq |z^0| - (1 + T)2M\|(id - K_\infty)^\# \| \geq |z^0| - \gamma \tag{3.7}$$

and

$$|z(t)| \leq |z^0| + |z^+| \leq |z^0| + \gamma. \tag{3.8}$$

Note that $\|z^0\| = \sqrt{T}|z^0|$. We can choose

$$r_0 = (c + \gamma)\sqrt{T}. \tag{3.9}$$

Combining (3.7)–(3.9) and (V3), we have

$$c < |z(t)| < d.$$

Therefore, by (3.2), (3.5), (3.6), (1.2), and (V3), we have

$$\begin{aligned} \left\langle \frac{dz}{dt}, z^0 \right\rangle \Big|_{t=0} &= \int_0^T (V'(t, z) \cdot z^0) \, dt \\ &= \int_0^T (V'(t, z) \cdot z) \, dt - \int_0^T (V'(t, z) \cdot z^+) \, dt \\ &\geq \int_0^T [(V'(t, z) \cdot z) - \gamma|V'(t, z)|] \, dt > 0. \end{aligned}$$

Thus D_m is an isolating block of the gradient flow η and

$$D_m^- = (V_m^+ \times \partial V_m^0).$$

Then

$$I(\eta, D_m) = t^N. \tag{3.10}$$

By Theorem 2.1, there exists a critical point x_m of f_m in D_m . This implies that

$$\|x_m\| \leq r_+ + r_0;$$

i.e., $\{x_m\}$ is bounded. By a standard argument (cf. [7]), $\{x_m\}$ has a subsequence which converges to x^* in E_T , and x^* is a critical point of f .

Step 2. Suppose V satisfies (V4). Let $d = \frac{1}{4}\|(id - K_0)^\# \|^{-1}$. Then there exists $m_1 > 0$ such that for $m \geq m_1$,

$$\dim M_d^-(P_m(id - K_0)P_m) = j_0, \quad \dim M_d^0(P_m(id - K_0)P_m) = v_0, \quad (3.11)$$

where $M_d^+(\cdot)$, $M_d^-(\cdot)$, and $M_d^0(\cdot)$ denote the eigenspace corresponding to the eigenvalue λ belonging to $[d, +\infty)$, $(-\infty, -d]$, and $(-d, d)$ respectively.

In fact, since $\dim[M^0(id - K_0) \oplus M^-(id - K_0)] < +\infty$ and K_0 is compact, there exists $m_1 > 0$ such that for $m \geq m_1$,

$$\dim P_m M^0(id - K_0) = \dim M^0(id - K_0),$$

$$\dim P_m M^-(id - K_0) = \dim M^-(id - K_0),$$

$$\|P_m K_0(id - P_m) + (id - P_m)K_0\| \leq \frac{1}{3}d,$$

and for any $z \in M^0(id - K_0) \oplus M^-(id - K_0)$,

$$\|(id - P_m)z\| \leq \frac{1}{3}d\|z\|.$$

For $u \in M^-(id - K_0)$, $v \in M^0(id - K_0)$, and $w \in M^+(id - K_0)$, we have

$$\begin{aligned} \langle P_m(id - K_0)P_m(P_mu), P_mu \rangle &= \langle (id - K_0)u, u \rangle - \langle (id - P_m)u, u \rangle \\ &\quad + \langle (P_m K_0(id - P_m) + (id - P_m)K_0)u, u \rangle \\ &\leq -4d\|u\|^2 + \frac{1}{3}d\|u\|^2 + \frac{1}{3}d\|u\|^2 \\ &< -3d\|u\|^2, \end{aligned}$$

$$\begin{aligned} |\langle P_m(id - K_0)P_m(P_mv), P_mv \rangle| &= |\langle (id - K_0)v, v \rangle - \langle (id - P_m)v, v \rangle \\ &\quad + \langle (P_m K_0(id - P_m) + (id - P_m)K_0)v, v \rangle| \\ &\leq \frac{1}{3}d\|v\| + \frac{1}{3}d\|v\|, \end{aligned}$$

$$\begin{aligned} \langle P_m(id - K_0)P_m(P_mw), P_mw \rangle &= \langle (id - K_0)w, w \rangle - \langle (id - P_m)w, w \rangle \\ &\quad + \langle (P_m K_0(id - P_m) + (id - P_m)K_0)w, w \rangle \\ &\geq 4d\|w\|^2 - \frac{1}{3}d\|w\|^2 - \frac{1}{3}d\|w\|^2 \\ &> 3d\|w\|^2. \end{aligned}$$

Therefore we have

$$\dim M_d^0(P_m(id - K_0)P_m) \geq \dim P_m M^0(id - K_0),$$

$$\dim M_d^-(P_m(id - K_0)P_m) \geq \dim P_m M^-(id - K_0),$$

$$\dim M_d^+(P_m(id - K_0)P_m) \geq \dim P_m M^+(id - K_0),$$

which imply (3.11).

Step 3. For $m \geq m_1$, we consider

$$f_m(z) = \frac{1}{2} \langle (id - K_0)z, z \rangle - \psi_0(z)$$

on E_m , where

$$\psi_0(z) = \int_0^T \left(V(t, z) - \frac{1}{2} (B_0(t)z, z) \right) dt.$$

Since V satisfies (V1)–(V4), using the same arguments as those in the proof of Lemma 3.1 in [5] we have

$$\begin{aligned} \frac{\|f'(z) - (id - K_0)z\|}{\|z\|} &\rightarrow 0 & \text{as } \|z\| \rightarrow 0, \\ \|f''(z) - (id - K_0)\| &\rightarrow 0 & \text{as } \|z\| \rightarrow 0. \end{aligned} \quad (3.12)$$

Note that $f'_m(z) = P_m(id - K_0)z - P_m\psi'_0(z)$; we then have

$$\frac{\|f'_m(z) - P_m(id - K_0)P_m z\|}{\|z\|} \leq \frac{\|f'(z) - (id - K_0)z\|}{\|z\|} \rightarrow 0$$

as $\|z\| \rightarrow 0$ and $z \in E_m$. By (3.12) there exists $r > 0$ such that

$$\|f''(z) - (id - K_0)\| < \frac{1}{2}d \quad \text{for } z \in B_{2r} = \{z \in E_T: \|z\| \leq 2r\}.$$

Hence we have

$$\|f''_m(z) - P_m(id - K_0)P_m\| \leq \|f''(z) - (id - K_0)\| < \frac{1}{2}d \quad \text{for } z \in B_{2r} \cap E_m.$$

Since $N \notin [j_0, j_0 + v_0]$, by (3.11) and Theorem 2.1 there exists a critical point x_m of f_m in D_m with $\|x_m\| \geq r$, where r is a constant which does not depend on m .

Note that $x_m \in D_m$ implies that

$$\|x_m\| \leq r_+ + r_0;$$

i.e. $\{x_m\}$ is bounded. By a standard argument (cf. [7]), $\{x_m\}$ has a subsequence which converges to x^* in E_T , $\|x^*\| \geq r$, and x^* is a critical point of f . ■

Proof of Theorem 1.2. According to the proof of Theorem 1.1, we can choose

$$\begin{aligned} r_+ &= 2\sqrt{T}M \frac{(4\pi^2 + T^2)}{4\pi^2}, \\ r_{in} &= (a + \gamma)\sqrt{T}, \quad r_{out} = (c + \gamma)\sqrt{T}. \end{aligned}$$

Let

$$C_m = \{z \in M^0(P_m(id - K_\infty)P_m): r_{in} \leq \|z\| \leq r_{out}\},$$

$$D_m = V_m^+ \times C_m.$$

By Step 1 in the proof of Theorem 1.1, we have

$$\left\langle \frac{dz}{dt}, z^+ \right\rangle \Big|_{t=0} < 0, \quad \text{for } z = z^+ + z^0 \in \partial V_m^+ \times C_m,$$

and for $z = z^+ + z^0 \in V_m^+ \times \partial C_m$ with $\|z^0\| = r_{out}$,

$$\left\langle \frac{dz}{dt}, z^0 \right\rangle \Big|_{t=0} > 0.$$

For $z = z^+ + z^0 \in V_m^+ \times \partial C_m$ with $\|z^0\| = r_{in}$, by using arguments similar to (3.7)–(3.9) we have

$$a < |z(t)| < b.$$

Therefore, by (3.2) and (V5), we have

$$\begin{aligned} \left\langle \frac{dz}{dt}, z^0 \right\rangle \Big|_{t=0} &= \int_0^T (V'(t, z) \cdot z^0) dt \\ &= \int_0^T (V'(t, z) \cdot z) dt - \int_0^T (V'(t, z) \cdot z^+) dt \\ &\leq \int_0^T (V'(t, z) \cdot z) dt + (1 + T)2M\|(id - K_\infty)^\# \| \\ &\quad \times \int_0^T |V'(t, z)| dt \\ &\leq \int_0^T [(V'(t, z) \cdot z) + \gamma|V'(t, z)|] dt \\ &< 0. \end{aligned}$$

Thus $D_m = V_m^+ \times C_m$ is an isolating block of the gradient flow η and

$$D_m^- = (V_m^+ \times \partial C_m).$$

By direct computation, we know that the homology group

$$H_{N-1}(D_m, D_m^-) \cong H_{N-1}(C_m, \partial C_m) \neq 0;$$

i.e.,

$$I(\eta, D_m) \neq 0.$$

Theorem 2.1 implies that there exists a critical point x_m of f_m in D_m with

$$r_{in} \leq \|x_m\| \leq r_{out} + r_+.$$

By a standard argument, $\{x_m\}$ has a subsequence which converges to x^* in E_T , and x^* is a critical point of f . Moreover,

$$a\sqrt{T} < r_{in} \leq \|x^*\| \leq r_{out} + r_+ < d\sqrt{T}.$$

■

Proof of Theorem 1.3. According to (1.3), (V2), and the proofs of Theorem 1.1 and Theorem 1.2, in this case

$$|\lambda V'(t, z)| \leq \lambda M, \quad \gamma = \frac{\lambda 2M(1+T)(4\pi^2 + T^2)}{4\pi^2},$$

and $\gamma \rightarrow 0$ as $\lambda \rightarrow 0$.

By (V1) and (V6) there exists $\beta > 0$ such that

$$\begin{aligned} (V'(t, z) \cdot z) &\geq \beta, & \text{for } c \leq |z| \leq d; \\ (V'(t, z) \cdot z) &\leq -\beta, & \text{for } a \leq |z| \leq b. \end{aligned}$$

Then there exists $\lambda_0 > 0$ such that for $0 < \lambda \leq \lambda_0$,

$$2\gamma < \min\{d - c, b - a\};$$

$$\begin{aligned} (\lambda V'(t, z) \cdot z) - \gamma |\lambda V'(t, z)| &\geq \lambda\beta - \gamma\lambda M > 0, & \text{for } c \leq |z| \leq d; \\ (\lambda V'(t, z) \cdot z) + \gamma |\lambda V'(t, z)| &\leq -\lambda\beta + \gamma\lambda M < 0, & \text{for } a \leq |z| \leq b. \end{aligned}$$

By Theorem 1.2 we get the conclusion. ■

Proof of Theorem 1.4. Note that in this case

$$\gamma = \frac{\lambda 2M(1+T)(4\pi^2 + T^2)}{4\pi^2},$$

and $\gamma \rightarrow 0$ as $\lambda \rightarrow 0$. Let $\lambda_0 > 0$ be small enough such that for $0 < \lambda < \lambda_0$,

$$\gamma \leq \min\left\{\frac{\tau}{2}, \frac{1}{2\alpha}\right\}.$$

Then the condition (V5) holds for each $n \geq 1$ with $0 < a_n < a_n + \tau < b_n < b_n + \tau$. By Theorem 1.2, for each $n \geq 1$, we have a distinct T -periodic solution. ■

4. EXAMPLES

EXAMPLE 4.1. Let $T = 1$, $V \in C^2(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$, and $V(t+1, z) = V(t, z)$. Moreover,

$$\begin{aligned} V(t, x) &= -\frac{1}{2}|x|^2, & \text{if } |x| < 1; \\ V(t, x) &= \arctan(|x|^2), & \text{if } 320 \leq |x| \leq 640; \\ |V'(t, x)| &\leq 10, & \text{for } (t, x) \in \mathbb{R} \times \mathbb{R}^N. \end{aligned}$$

In this example, $B_0(t) = -Id$, which means that $(j_0, v_0) = (0, 0)$. Now we have

$$\gamma = \frac{10(4\pi^2 + 1)}{\pi^2}.$$

It is easy to verify that (V3) holds. Theorem 1.1 gives us a nontrivial 1-periodic solution of the system (1.1).

EXAMPLE 4.2. Let $T = 1$, $V \in C^2(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$, and $V(t + 1, z) = V(t, z)$. Moreover,

$$\begin{aligned} V(t, x) &= -\frac{1}{200}|x|^2, & \text{if } 100 < |x| < 200; \\ V(t, x) &= \arctan(|x|^2), & \text{if } 320 \leq |x| \leq 640; \\ |V'(t, x)| &\leq 10, & \text{for } (t, x) \in \mathbb{R} \times \mathbb{R}^N. \end{aligned}$$

In this case,

$$\gamma = \frac{10(4\pi^2 + 1)}{\pi^2}.$$

By Theorem 1.2, there exists a 1-periodic solution $x^*(t)$ with the $W^{1,2}$ -norm in $(100, 640)$.

EXAMPLE 4.3. Let $T = 1$, $V \in C^2(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$, and $V(t + 1, z) = V(t, z)$. Moreover, for $n \geq 1$,

$$\begin{aligned} V(t, x) &= \cos\left(\frac{\pi}{4}|x|\right), & \text{if } |x| \in [8n + 1, 8n + 2] \cup [8n + 5, 8n + 6]; \\ |V'(t, x)| &\leq 10, & \text{for } (t, x) \in \mathbb{R} \times \mathbb{R}^N. \end{aligned}$$

In this case,

$$\gamma = \frac{10(4\pi^2 + 1)}{\pi^2}, \quad \tau = 1, \quad \alpha = 1.$$

By Theorem 1.4, we can choose $\lambda_0 = 1/100$ such that, for $0 < \lambda \leq \lambda_0$, there exist infinitely many 1-periodic solutions for the system (1.3).

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