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Holomorphic reproducing kernels for piecewise-smooth planar domains

Michael Bolt

Department of Mathematics, University of Michigan, Ann Arbor, MI 48109, USA

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Abstract

For smoothly bounded, multiply connected domains in the complex plane, S. Bell showed how the Kerzman–Stein method can be used to compute the Szegő kernel and Ahlfors map. In this paper, we present a modification of that method for domains that are piecewise-smooth. The procedure is based on a method of preliminary transformation and involves adding an explicit holomorphic function to the Cauchy kernel. In the last section, we show the effectiveness of using this method for the special case of a square.

Keywords: Reproducing kernels; Szegő kernel; Piecewise-smooth

1. Introduction

Kerzman and Stein found in [4] a method for computing the Szegő kernel and Riemann map for smooth, simply connected domains in the complex plane. Kerzman and Trummer then implemented this method in [5] and gave error estimates for the Riemann map for several example domains. Bell later extended the method in [1] to include multiply connected domains—he showed how the same basic techniques can be used in this situation to compute the Szegő kernel and Ahlfors map. In this paper we present one way to extend these methods to the case of domains with corners.

The Szegő kernel itself is one of the canonical domain functions, and is closely related to the Green's function of a domain. The Ahlfors map is a proper holomorphic map from

E-mail address: mbolt@umich.edu.

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the domain to the unit disc. It, too, plays a key role in solving problems that arise in electrostatics and fluid mechanics.

The aim of this paper is to construct holomorphic reproducing kernels that are suitable for the Kerzman–Stein method in the case of piecewise-smooth domains. For a smooth boundary, one uses the Cauchy kernel, which closely resembles the Szegő kernel as Kerzman and Stein showed. For a non-smooth boundary we here use preliminary transformations to construct a singular function that, when added to the Cauchy kernel, will more closely resemble the Szegő kernel. In the last section, we show the error when computing the Szegő kernel using our methods for the square.

2. Background

Let $\Omega \Subset \mathbb{C}$ be a multiply connected domain with piecewise-smooth boundary. Define $L^p(\partial\Omega)$ spaces using arclength measure, and for $f \in L^2(\partial\Omega)$ let

$$\mathcal{C}f(z) = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{f(w)\,dw}{w-z} \quad \text{for } z \in \Omega.$$

At first $\mathcal{C}f$ is defined on the interior of Ω, but by letting $z \to z_0 \in \partial\Omega$ it is defined almost everywhere on the boundary. In fact the limiting function is square-integrable and $\mathcal{C}$ extends to a bounded operator on $L^2(\partial\Omega)$.

The image of $\mathcal{C}$ is the space $H^2(\partial\Omega)$ of boundary values of holomorphic functions. Moreover, by the Cauchy integral formula, $\mathcal{C}$ reproduces $H^2(\partial\Omega)$. So $\mathcal{C}$ is a bounded projection $L^2(\partial\Omega) \to H^2(\partial\Omega)$. In the case of smooth boundary, Kerzman and Stein [4] showed that $\mathcal{C}$ is 'almost orthogonal,' in the sense that $\mathcal{A} = \mathcal{C} - \mathcal{C}^*$ is compact. Moreover, $\mathcal{C}$ and $\mathcal{A}$ can be used to write the Szegő projection, that is, the orthogonal projection to $H^2(\partial\Omega)$, as the composition $\mathcal{S} = \mathcal{C}(\mathcal{I} + \mathcal{A})^{-1}$.

An important application of this equation is that it leads to a Fredholm integral equation of the second kind,

$$S(z,\bar{a}) - \int_{\partial\Omega} A(z,w)S(w,\bar{a})\,ds_w = C^*(z,a) \quad \text{for } a \in \Omega,\ z \in \partial\Omega, \tag{1}$$

whose solution S is the Szegő kernel. One obtains the Szegő kernel by solving this equation numerically; one then obtains the Riemann map or Ahlfors map using identities involving the Szegő kernel. See [1,5], for instance. These techniques are further developed in [7,9,11].

At a corner, the kernel $A(z,w)$ is unbounded, so Kerzman asked in [3] for how best to modify the procedure for domains with corners. The modification we present uses preliminary maps to construct a holomorphic function h, that, when added to the Cauchy kernel, gives a uniformly bounded kernel $A_h(z,w)$. The corresponding Fredholm equation can be readily solved using the Nyström method.

We point out that our method is simpler than the one Michel [6] used to study the Bergman theory for orthodomains. That modification depends on both the angles and the curvature at the corners, whereas our construction only depends on the angles at the corners and it applies to a larger class of domains. We also point out work of Thomas in [10]. He showed that the Nyström solution of Eq. (1) converges in the mean to the Szegő kernel, even for piecewise-smooth domains. The convergence is slow near corners, however, as will be evident for the case of a square.

3. Statement of results

We assume $\Omega \Subset \mathbb{C}$ is a multiply connected domain whose boundary is made of finitely many closed curves, each curve consisting of finitely many (smooth) arcs γ_j that have C^2 parameterizations $z_j : [t_{j-1}, t_j] \to \gamma_j$, $j = 1, \dots, n$, with

- $z_j(t_j) = z_{j+1}(t_j)$ for $j = 1, \dots, n-1$, and $z_n(t_n) = z_1(t_0)$,
- $\pi - \arg[z'_{j+1}(t_j)/z'_j(t_j)] = \pi/\alpha_j$ with $1 < \alpha_j < \infty$, and
- $|z'_j| \equiv 1$ on $[t_{j-1}, t_j]$.

The corners $P_j = z_j(t_j)$ have interior angles that measure π/α_j, which is between 0 and π. Let $T = T_w$ be the complex unit tangent vector at $w \in \partial\Omega$ provided w is not a corner. So $dw = T_w\, ds_w$. Here, and in what follows, we will often use subscripts to denote a function's argument, not its derivative, in order to reduce the number of parentheses.

A domain is piecewise-smooth if its boundary has parameterizations satisfying the above conditions. *We exclude corners with angle greater than* π because at such points the Szegő kernel is infinite, and our uniform estimates will fail.

Theorem 1. *If $\Omega \Subset \mathbb{C}$ is a multiply connected domain with piecewise-smooth boundary, then there is an explicit function $h \in O(\Omega \times \Omega)$, extending smoothly to $\bar{\Omega} \times \bar{\Omega} \setminus \{(P_j, P_j)\}$, so that the kernel*

$$C_h(z, w) = \frac{T_w}{2\pi i}\left[\frac{1}{w - z} + h(z, w)\right]$$

satisfies $A_h = C_h - C_h^* \in L^\infty(\partial\Omega \times \partial\Omega)$, where $C_h^*(z, w) = \overline{C_h(w, z)}$.

We identify an operator with its kernel via $\mathcal{A}f(z) = \int_{\partial\Omega} A(z, w) f(w)\, ds_w$, for instance. The operator $\mathcal{A}_h = \mathcal{C}_h - \mathcal{C}_h^*$ is then compact, and we have the following

Theorem 2. *With Ω and C_h as above, the Szegő projection can be written as the composition of bounded operators, $\mathcal{S} = \mathcal{C}_h(\mathcal{I} + \mathcal{A}_h)^{-1}$, and the Szegő kernel satisfies the Fredholm equation of the second kind,*

$$S(z, \bar{a}) - \int_{\partial\Omega} A_h(z, w) S(w, \bar{a})\, ds_w = C_h^*(z, a) \quad \textit{for } a \in \Omega,\ z \in \partial\Omega. \tag{2}$$

Proof. In fact, by Theorem 1, the operator $\mathcal{A}_h$ is Hilbert–Schmidt. Moreover, if $f \in O(\Omega) \cap C(\bar{\Omega})$, then by Stokes' theorem,

$$\int_{\partial\Omega} h(z,w) f(w)\, dw = \int\int_{\Omega} \frac{\partial}{\partial \bar{w}}\big[h(z,w) f(w)\big]\, d\bar{w}\, dw = 0 \quad \text{for } z \in \Omega,$$

as both h and f are holomorphic. So $\mathcal{C}_h$ reproduces holomorphic functions by the Cauchy integral formula. In fact, $\mathcal{C}_h$ projects to holomorphic functions, too, as h is holomorphic. So the theorem's first assertion follows from a fact about Hilbert spaces—if $\mathcal{C}$ is a densely defined projection on a Hilbert space and $\mathcal{A} = \mathcal{C} - \mathcal{C}^*$ is compact, then $\mathcal{C}$ is bounded, and the orthogonal projection can be written $\mathcal{C}(\mathcal{I} + \mathcal{A})^{-1}$.

Equation (2) follows by writing $\mathcal{S}(\mathcal{I} + \mathcal{A}_h) = \mathcal{C}_h$, taking adjoints, and applying an approximate identity. By the Fredholm alternative, its solution is unique. □

4. Construction of the singular function

The function h will be a sum $\sum_j h_j$, where each h_j is holomorphic on $\Omega \times \Omega$, and smooth on $\bar{\Omega} \times \bar{\Omega}$ except at (P_j, P_j). Since h_j comes from each P_j, we drop subscripts for the remainder of this section and let P refer to any of the P_j, and h the corresponding h_j.

At corner P, construct two circles that are exterior to Ω so that one circle is tangent to $\partial\Omega$ at P in each direction. These circles intersect at P with angle π/α and they have a second point of intersection P'. We allow for the case that lines are circles (with infinite radii), so possibly $P' = \infty$. If $P' \neq \infty$, the map $w \to (w - P)/(w - P')$ sends Ω injectively into a wedge with angle π/α, and the map

$$u(w) = \begin{cases} [P'(w-P)/(P'-w)]^{\alpha}, & P' \neq \infty, \\ (w-P)^{\alpha}, & P' = \infty, \end{cases} \tag{3}$$

sends Ω injectively into a halfplane. Furthermore, u straightens the corner at P so that the boundary of $\Omega' = u(\Omega)$ is at least C^1-smooth near $u(P)$. Define

$$h(z,w) = \frac{\sqrt{u'(w)}\sqrt{u'(z)}}{u(w) - u(z)} - \frac{1}{w - z}, \quad z \neq w \in \bar{\Omega},$$

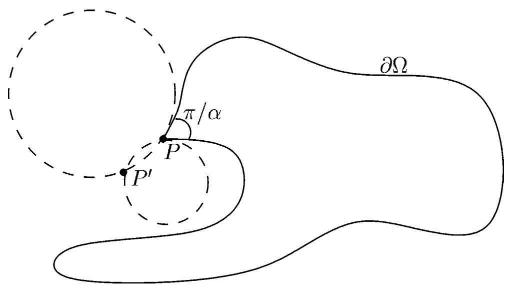


Fig. 1. Construction of the singular function.

so that h is holomorphic in both $w, z \in \Omega$. The square roots make sense since u is injective. Moreover, since $\alpha > 1$, it follows that $h(z, w) = -1/(w - z)$ if w or z equals P, so h is unbounded for w, z near P. Elsewhere, $u' \neq 0$ and h extends smoothly to the boundary.

The function

$$\frac{T_w}{2\pi i} \frac{\sqrt{u'(w)}\sqrt{u'(z)}}{u(w) - u(z)}$$

is the pullback of the Cauchy kernel from $u(\Omega)$ under the H^2-preserving isometry $L^2(\partial\Omega) \ni (f \circ u) \cdot \sqrt{u'} \leftarrow f \in L^2(\partial\Omega')$. Moreover, since $u(\Omega)$ is smoother at $u(P)$ than is Ω at P, it should more closely resemble the Szegő kernel at P than does the Cauchy kernel for Ω. So as defined, h gives a reasonable correction to the Cauchy kernel for Ω at P.

5. Proof of Theorem 1

Back to the earlier notation we let $h = \sum_j h_j$, where each h_j corresponds to a corner P_j, and is constructed as in the previous section. We must show that

$$\begin{aligned} C_h(z, w) - \overline{C_h(w, z)} = {} & \frac{T_w}{2\pi i}\left[\frac{1}{w - z} + \sum_j h_j(z, w)\right] \\ & - \frac{\bar{T}_z}{2\pi i}\left[\frac{1}{\bar{w} - \bar{z}} - \sum_j \overline{h_j(w, z)}\right] \end{aligned}$$

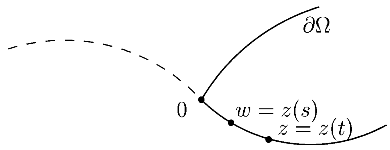
is bounded for $z, w \in \partial\Omega$, and for this we need only consider w and z close to one another. Recall that h_j is unbounded only for both w and z near P_j; moreover, where $\partial\Omega$ is smooth, $T_w(w - z)^{-1} - \bar{T}_z(\bar{w} - \bar{z})^{-1}$ is bounded. So, dropping subscripts, we need only verify the boundedness of

$$T_w\left[\frac{1}{w - z} + h(z, w)\right] - \bar{T}_z\left[\frac{1}{\bar{w} - \bar{z}} - \overline{h(w, z)}\right] = \frac{T_w\sqrt{u'_w}\sqrt{u'_z}}{u_w - u_z} - \frac{\overline{T_z\sqrt{u'_w}\sqrt{u'_z}}}{\bar{u}_w - \bar{u}_z},$$

where $w, z \in \partial\Omega$ are both near a corner P which has interior angle π/α, and u is given by Eq. (3). For simplicity assume $P = 0$ and replace P' by P.

We use the convention that $A \approx B$ means there is a constant c so that $c|A| \leqslant |B| \leqslant c^{-1}|A|$, and $A \lesssim B$ means there is a constant c so that $|A| \leqslant c^{-1}|B|$. The constants may depend on up to two derivatives of the parameterization, and can therefore be controlled by the maximum curvature of the boundary. We reiterate the convention that *subscripts will often be used to indicate a function's argument*; primed notation will be used to indicate the derivative.

Case 1. Suppose first that w and z are on the same side of the corner and that the common side is parametrized by $t \to z_t = z(t)$ with $z_0 = 0$, $z'_0 = e^{i\theta}$ and $|z'_t| \equiv 1$. We assume that z_t is smooth up to the corner, but it parameterizes $\partial\Omega$ only on one side. We set $w = z_s$ and $z = z_t$, where $s, t \geqslant 0$.



The estimates are essentially the same for $P=\infty$, so we consider the case $P\neq\infty$. We have first,

$$u(w)-u(z)=\left(\frac{Pz_s}{P-z_s}\right)^{\alpha}-\left(\frac{Pz_t}{P-z_t}\right)^{\alpha}=\int_t^s \frac{\alpha P^{\alpha+1}z_r^{\alpha-1}z_r'}{(P-z_r)^{\alpha+1}}\,dr.$$

Then using $z_r=e^{i\theta}r+O(r^2)$ and $z_r'=e^{i\theta}+O(r)$ we have

$$\begin{aligned}u(w)-u(z)&=\alpha\int_t^s r^{\alpha-1}e^{i\theta\alpha}+O(r^{\alpha})\,dr\\&=e^{i\theta\alpha}(s^{\alpha}-t^{\alpha})+O\big((s^{\alpha}+t^{\alpha})(s-t)\big)\approx s^{\alpha}-t^{\alpha}.\end{aligned}\tag{4}$$

Next,

$$\begin{aligned}&T_w\sqrt{u_w'}\sqrt{u_z'}\cdot\big[\overline{u(w)}-\overline{u(z)}\big]\\&\quad=z_s'\sqrt{\frac{\alpha P^{\alpha+1}z_s^{\alpha-1}}{(P-z_s)^{\alpha+1}}}\sqrt{\frac{\alpha P^{\alpha+1}z_t^{\alpha-1}}{(P-z_t)^{\alpha+1}}}\int_t^s\frac{\alpha\overline{P^{\alpha+1}z_r^{\alpha-1}z_r'}}{(\overline{P-z_r})^{\alpha+1}}\,dr\\&\quad=(st)^{(\alpha-1)/2}z_s'\sqrt{\frac{P^{\alpha+1}(z_s/s)^{\alpha-1}}{(P-z_s)^{\alpha+1}}}\sqrt{\frac{P^{\alpha+1}(z_t/t)^{\alpha-1}}{(P-z_t)^{\alpha+1}}}\\&\qquad\times\int_t^s\alpha^2r^{\alpha-1}\frac{\overline{P^{\alpha+1}(z_r/r)^{\alpha-1}}}{(\overline{P-z_r})^{\alpha+1}}\bar z_r'\,dr,\end{aligned}$$

and after switching z and w, and conjugating,

$$\begin{aligned}\overline{T_z\sqrt{u_w'}\sqrt{u_z'}}\cdot\big[u(w)-u(z)\big]&=(st)^{(\alpha-1)/2}\bar z_t'\overline{\sqrt{\frac{P^{\alpha+1}(z_s/s)^{\alpha-1}}{(P-z_s)^{\alpha+1}}}}\,\overline{\sqrt{\frac{P^{\alpha+1}(z_t/t)^{\alpha-1}}{(P-z_t)^{\alpha+1}}}}\\&\qquad\times\int_t^s\alpha^2r^{\alpha-1}\frac{P^{\alpha+1}(z_r/r)^{\alpha-1}}{(P-z_r)^{\alpha+1}}z_r'\,dr.\end{aligned}$$

Then, since

$$\begin{aligned}&z_s'\sqrt{\frac{P^{\alpha+1}(z_s/s)^{\alpha-1}}{(P-z_s)^{\alpha+1}}}\sqrt{\frac{P^{\alpha+1}(z_t/t)^{\alpha-1}}{(P-z_t)^{\alpha+1}}}\,\frac{\overline{P^{\alpha+1}(z_r/r)^{\alpha-1}}}{(\overline{P-z_r})^{\alpha+1}}\bar z_r'\\&\quad-\bar z_t'\overline{\sqrt{\frac{P^{\alpha+1}(z_s/s)^{\alpha-1}}{(P-z_s)^{\alpha+1}}}}\,\overline{\sqrt{\frac{P^{\alpha+1}(z_t/t)^{\alpha-1}}{(P-z_t)^{\alpha+1}}}}\,\frac{P^{\alpha+1}(z_r/r)^{\alpha-1}}{(P-z_r)^{\alpha+1}}z_r'\end{aligned}$$

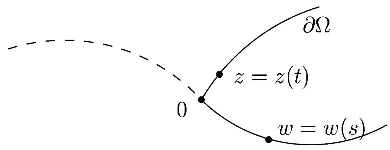
vanishes at $s=t=r$, and since the square roots are bounded away from 0, it follows that

$$\begin{aligned} &T_w\sqrt{u'_w}\sqrt{u'_z}\cdot\left[\overline{u(w)}-\overline{u(z)}\right]-\overline{T_z\sqrt{u'_w}\sqrt{u'_z}}\cdot\left[u(w)-u(z)\right] \\ &\quad\lesssim (st)^{(\alpha-1)/2}(s-t)\int_t^s r^{\alpha-1}\,dr \lesssim (st)^{(\alpha-1)/2}(s-t)(s^\alpha-t^\alpha). \end{aligned} \tag{5}$$

(In this estimate, the implied constant depends on two derivatives of z.) Then, putting together (4) and (5), we have

$$\frac{T_w\sqrt{u'_w}\sqrt{u'_z}}{u(w)-u(z)}-\frac{\overline{T_z\sqrt{u'_w}\sqrt{u'_z}}}{\overline{u(w)}-\overline{u(z)}}\lesssim\frac{(st)^{(\alpha-1)/2}(s-t)(s^\alpha-t^\alpha)}{(s^\alpha-t^\alpha)^2}\lesssim 1.$$

Case 2. Suppose, then, that w and z are on opposite sides of the corner, and the two sides are parameterized by w_s with $w_0=0$, $w'_0=e^{i\theta}$, and $|w'_s|\equiv 1$, and by z_t with $z_0=0$, $z'_0=e^{i(\theta+\pi/\alpha)}$, and $|z'_t|\equiv 1$. We assume that w_s and z_t are smooth up to the corner, and we set $w=w_s$ and $z=z_t$ for $s,t\geqslant 0$. Then $T_w=w'_s$ and $T_z=-z'_t$.



We first estimate

$$\begin{aligned} u(w)-u(z)&=\left[\frac{Pw_s}{P-w_s}\right]^\alpha-\left[\frac{Pz_t}{P-z_t}\right]^\alpha=w_s^\alpha(1+\cdots)^\alpha-z_t^\alpha(1+\cdots)^\alpha \\ &\approx (e^{i\theta}s)^\alpha-(e^{i(\theta+\pi/\alpha)}t)^\alpha=e^{i\theta\alpha}(s^\alpha+t^\alpha). \end{aligned} \tag{6}$$

Next,

$$\begin{aligned} &T_w\sqrt{u'_w}\sqrt{u'_z}\cdot\overline{u(w)}-\overline{T_z\sqrt{u'_w}\sqrt{u'_z}}\cdot u(w) \\ &\quad= w'_s\sqrt{\frac{\alpha P^{\alpha+1}w_s^{\alpha-1}}{(P-w_s)^{\alpha+1}}}\sqrt{\frac{\alpha P^{\alpha+1}z_t^{\alpha-1}}{(P-z_t)^{\alpha+1}}}\,\overline{\left(\frac{Pw_s}{P-w_s}\right)^\alpha} \\ &\qquad+z'_t\,\overline{\sqrt{\frac{\alpha P^{\alpha+1}w_s^{\alpha-1}}{(P-w_s)^{\alpha+1}}}}\,\overline{\sqrt{\frac{\alpha P^{\alpha+1}z_t^{\alpha-1}}{(P-z_t)^{\alpha+1}}}}\left(\frac{Pw_s}{P-w_s}\right)^\alpha \\ &\quad=\alpha\cdot s^\alpha(st)^{(\alpha-1)/2}\left[w'_s\sqrt{\frac{P^{\alpha+1}(w_s/s)^{\alpha-1}}{(P-w_s)^{\alpha+1}}}\sqrt{\frac{P^{\alpha+1}(z_t/t)^{\alpha-1}}{(P-z_t)^{\alpha+1}}}\,\overline{\left(\frac{Pw_s/s}{P-w_s}\right)^\alpha}\right. \\ &\qquad\left.+z'_t\,\overline{\sqrt{\frac{P^{\alpha+1}(w_s/s)^{\alpha-1}}{(P-w_s)^{\alpha+1}}}}\,\overline{\sqrt{\frac{P^{\alpha+1}(z_t/t)^{\alpha-1}}{(P-z_t)^{\alpha+1}}}}\left(\frac{Pw_s/s}{P-w_s}\right)^\alpha\right]. \end{aligned}$$

Then what is in brackets is differentiable in s and t, the square roots are again bounded away from 0, and at $s=t=0$ it equals

$$\begin{aligned}
&e^{i\theta}e^{i\theta(\alpha-1)/2}e^{i(\theta+\pi/\alpha)(\alpha-1)/2}e^{-i\theta\alpha}+e^{-i(\theta+\pi/\alpha)}e^{-i\theta(\alpha-1)/2}e^{-i(\theta+\pi/\alpha)(\alpha-1)/2}e^{i\theta\alpha}\\
&\quad=e^{i(\pi/\alpha)(\alpha-1)/2}\big[e^{i\theta}e^{i\theta(\alpha-1)}e^{-i\theta\alpha}+e^{-i(\theta+\pi/\alpha)}e^{-i(\theta+\pi/\alpha)(\alpha-1)}e^{i\theta\alpha}\big]\\
&\quad=e^{i(\pi/\alpha)(\alpha-1)/2}\big[1+e^{-i(\theta+\pi/\alpha)\alpha}e^{i\theta\alpha}\big]=0.
\end{aligned}$$

From this we conclude that

$$T_w\sqrt{u'_w}\sqrt{u'_z}\cdot\overline{u(w)}-\overline{T_z\sqrt{u'_w}\sqrt{u'_z}}\cdot u(w)\lesssim s^\alpha(st)^{(\alpha-1)/2}\cdot(s+t). \tag{7}$$

(The implied constant again depends on two derivatives of z and w.) By interchanging w and z, then conjugating, we also conclude that

$$-T_w\sqrt{u'_w}\sqrt{u'_z}\cdot\overline{u(z)}+\overline{T_z\sqrt{u'_w}\sqrt{u'_z}}\cdot u(z)\lesssim t^\alpha(st)^{(\alpha-1)/2}(s+t). \tag{8}$$

Putting together (6)–(8), we have

$$\frac{T_w\sqrt{u'_w}\sqrt{u'_z}}{u(w)-u(z)}-\frac{\overline{T_z\sqrt{u'_w}\sqrt{u'_z}}}{\overline{u(w)}-\overline{u(z)}}\lesssim\frac{(s^\alpha+t^\alpha)(st)^{(\alpha-1)/2}(s+t)}{(s^\alpha+t^\alpha)^2}\lesssim 1. \quad\square$$

Notice that it is not the case that $C_h(z,w)-\overline{C_h(w,z)}$ is continuous at the corner. For suppose $w=z_s$ and $z=z_t$ lie to the same side of the corner, and take $t=\lambda s$ for some $\lambda>0$. Then

$$\frac{(st)^{(\alpha-1)/2}}{s^{\alpha-1}+t^{\alpha-1}}=\frac{\lambda^{(\alpha-1)/2}}{1+\lambda^{\alpha-1}},$$

which assumes values from 0 to 0.5 depending on λ. So for w and z close to the corner, the kernel $C_h(z,w)-\overline{C_h(w,z)}$ assumes a range of values, and is therefore discontinuous.

6. Example: the square

In this section, we illustrate for the square how quickly the Nyström solution of Eq. (2) converges to the Szegő kernel. We use a square with side length 1 and with center at a. Let $t\to z(t)$ parameterize the boundary, with $t\in[0,4]$ the arclength parameter, and $z(0)$ one of the corners. We show the error in computing the function $t\to S(z_t,\bar a)$ for three situations. First, we compute the Szegő kernel using the unmodified Cauchy kernel; this is Kerzman and Trummer's method in [5]. Then we compute the kernel using preliminary maps that are squaring maps—these maps completely straighten the boundary near the corners. Finally, we compute it using maps that do not depend on the curvature near the corners. This better illustrates how the method should work for general piecewise smooth domains.

For comparison, we also compute the Szegő kernel using the Riemann map. In particular, the function $s=s(z)$ that maps the square biholomorphically to the unit disc, so $s(a)=0$ and $s'(a)>0$, is given by $s(z)=\psi\circ\mathrm{sn}(z-1/2)$, where $\mathrm{sn}\,z$ is the Jacobian elliptic function (see Nehari [8, p. 280]), and ψ is a linear transformation that ensures normalization. The Szegő kernel is then $S(z,\bar a)=\sqrt{s'(z)\overline{s'(a)}}/2\pi$ (see Bell [2, p. 92]).

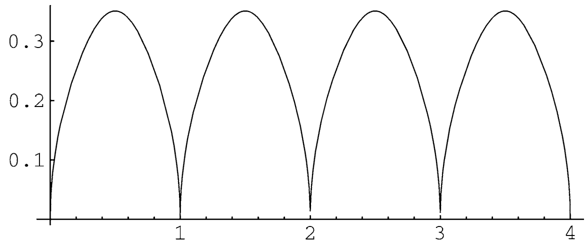


Fig. 2. Graph of $|S(z_t, \bar{a})|$ for the square.

The reason we use the square is because of this alternate solution for the Szegő kernel, with which we can determine the error for any approximate solution. The actual size of the Szegő kernel according to this method is shown in Fig. 2. Notice that the kernel vanishes at the corners.

For our approximate solutions, recall the Nyström method. Given an equation

$$f(t) + \int_0^L K(t,s) f(s)\,ds = g(t),$$

and $n > 0$, define collocation points $0 < t_j < L$ for $1 \leqslant j \leqslant n$ according to $t_j = (j - 1/2)L/n$. At these points, define a function f_n as the solution of the matrix equation

$$f_n(t_j) + \frac{L}{n}\sum_{k=1}^{n} K(t_j, t_k) f_n(t_k) = g(t_j).$$

Then define f_n on $[0, L]$ by interpolating its values at the t_j,

$$f_n(t) = g(t) - \frac{L}{n}\sum_{k=1}^{n} K(t, t_k) f_n(t_k).$$

Of course, this function does not solve the problem exactly, but there are convergence results for $f_n \to f$ depending on the smoothness of K.

6.1. Unmodified Cauchy kernel

This is the method Kerzman and Trummer used in [5]. We find f_n for $n = 32$, where

$$g(t) = \frac{1}{2\pi i}\frac{\bar{z}'_t}{\bar{a} - \bar{z}_t} \quad \text{and} \quad K(t,s) = -\frac{1}{2\pi i}\left[\frac{z'_s}{z_s - z_t} - \frac{\bar{z}'_t}{\bar{z}_s - \bar{z}_t}\right].$$

In Fig. 3, we show the error $|S_n(z_t, \bar{a}) - S(z_t, \bar{a})|$ as a function of t, where $S_n(z_t, \bar{a})$ is the approximate solution given by the Nyström method. The spikes correspond to the corners, where in fact, the Szegő kernel is zero. For larger values of n these spikes seem to become narrower, but they remain tall.

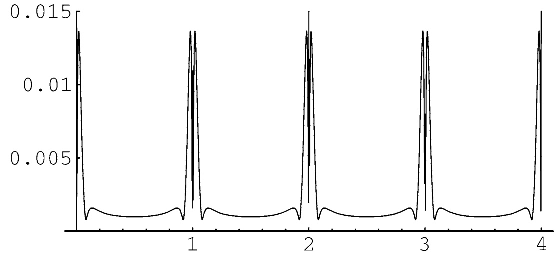


Fig. 3. Error in computing $S(z_t, \bar{a})$ using the unmodified kernel ($n = 32$).

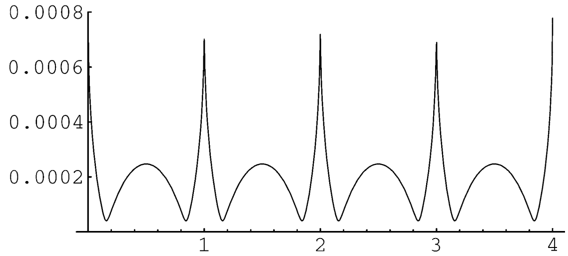


Fig. 4. Error in computing $S(z_t, \bar{a})$ using the first modified kernel ($n = 16$).

6.2. *First modified Cauchy kernel*

We find f_n for $n = 16$, using a modified Cauchy kernel based on squaring maps at the corners. In particular, let $u_j(w) = (w - P_j)^2$ for $j = 1, 2, 3, 4$, and

$$h_j(z, w) = \frac{\sqrt{u'_w}\sqrt{u'_z}}{u_w - u_z} - \frac{1}{w - z},$$

as in Section 3. Set

$$C_h(z, w) = \frac{T_w}{2\pi i}\left[\frac{1}{w - z} + \sum_{j=1,2,3,4} h_j(z, w)\right].$$

We then use

$$g(t) = \overline{C_h(a, z_t)} \quad \text{and} \quad K(t, s) = -C_h(z_t, z_s) + \overline{C_h(z_s, z_t)}.$$

In Fig. 4, we show the error $|S_n(z_t, \bar{a}) - S(z_t, \bar{a})|$, where $S_n(z_t, \bar{a})$ is the solution given by the Nyström method. There is significant improvement, especially at the corners.

6.3. *Second modified Cauchy kernel*

Here we use maps $u_j(w) = [P'_j(w - P_j)/(P'_j - w)]^2$, where P'_j lies outside the square, a distance $\sqrt{2}$ from P_j along the line extending from the square's center. The use of these maps better reflects the situation when one uses maps that do not depend on the curvature near the corner. We show the error in Fig. 5. Evidently, the error at the corners is compa-

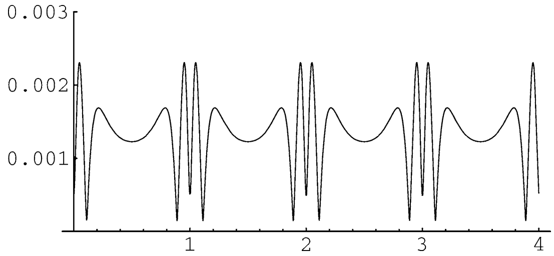


Fig. 5. Error in computing $S(z_t, \bar{a})$ using the second modified kernel ($n = 16$).

rable to the previous situation, but the error at the smooth points is comparable to the first situation.

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