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Exact multiplicity results for a class of two-point boundary value problems ☆

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Abstract

This paper is concerned with the exact number of positive solutions for boundary value problems $(|y'|^{p-2}y')' + \lambda f(y) = 0$ and $y(-1) = y(1) = 0$, where $p > 1$ and $\lambda > 0$ is a positive parameter. We consider the case in which the nonlinearity f is positive on $(0, \infty)$ and $(p-1)f(u) - uf'(u)$ changes sign from negative to positive.

Keywords: Boundary value problem; p-Laplacian; Positive solution; Exact multiplicity

1. Introduction

This paper is concerned with the exact number of positive solutions of the one-dimensional p-Laplacian Dirichlet problem

$$\begin{cases} (|y'(t)|^{p-2}y'(t))' + \lambda f(y(t)) = 0, & t \in (-1, 1), \\ y(-1) = y(1) = 0, \end{cases} \tag{1.1}$$

where $p > 1$, $f \in C[0, \infty) \cap C^2(0, \infty)$ and $\lambda > 0$ is a positive parameter. The problem (1.1) arises in many different situations. Some results have been obtained under different assumptions on f and p, for instance, see [1–5,9–12,15–18].

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For (1.1) and when $p=2$, Laetsch [10] first gave an exact multiplicity result for general nonlinearity f under the assumptions that f satisfies $f(u)>0$ on $[0,\infty)$, $f(u)$ is a convex function of u, and $f(u)/u$ is eventually strictly increasing. He proved that there exists a $\lambda^*>0$ such that (1.1) has exactly two positive solutions for $0<\lambda<\lambda^*$, exactly one for $\lambda=\lambda^*$, and none for $\lambda>\lambda^*$. His method is based on a quadrature method (called the time map) and the convexity assumption of the nonlinearity f. Castro et al. [3] considered the case which f is monotonically increasing and concave ($f''<0$) with $f(0)<0$ and $f(u)>0$ for large u. For special nonlinearity f, Addou et al. [1], Sanchez and Ubilla [14] and Liu [11] obtained the exact multiplicity result for $f(u)=u^\alpha+u^\beta$ with $\alpha>p-1>\beta\geqslant 0$. Cheng [4] discussed the case of $p=2$ and $f(u)=u^\alpha-u^\beta$ with $\alpha>\beta>-1$.

Recently, for (1.1) with the general case of $p>1$, Karatson and Simon [8] investigated p-convex nonlinearity (a natural generalization of convexity). Wang and Yeh [17] obtained the exact result for nonlinearity f satisfying

(i) $f(0)\geqslant 0$ and $f(u)>0$ for $u>0$;

(ii) there exist positive numbers $A<B$ such that

$$\begin{cases} (p-1)f(u)-uf'(u)>0 & \text{on } (0,A),\\ (p-1)f(A)-Af'(A)=0, & \\ (p-1)f(u)-uf'(u)<0 & \text{on } (A,+\infty),\end{cases}$$

and

$$\int_0^B \big[(p-1)f(u)-uf'(u)\big]\,du = p\int_0^B f(u)\,du - Bf(B)=0;$$

(iii) $uf'(u)/f(u)>-1/(1+p)$ on $(0,A)$ and $uf'(u)/f(u)$ is increasing on (A,B).

In particular, f is positive on $(0,\infty)$ and $(p-1)f(u)-uf'(u)$ changes sign from positive to negative.

In the present paper our aim is to investigate the case which nonlinearity f is positive on $(0,\infty)$ and $(p-1)f(u)-uf'(u)$ changes sign from negative to positive, for example,

$$\begin{cases} y''+\lambda(y^\alpha e^{-y}+y)=0, & t\in(-1,1),\\ y(-1)=y(1)=0, \end{cases} \tag{1.2}$$

where $\alpha>1$. We note that $u^\alpha e^{-u}+u$ is convex–concave–convex on $(0,\infty)$. To the author's knowledge, it is difficult to obtain exact multiplicity result and remains to be considered for very large α, see Remark 4.3.

2. Main results

Assume that $p>1$, $f\in C[0,\infty)\cap C^2(0,\infty)$ and the following conditions are satisfied:

(A1) $f(0)=0$, $f(u)>0$ for $u>0$ and

$$\int_0^1 \big|uf'(u)\big|\,du<\infty, \qquad \int_0^1 \big|u^2f''(u)\big|\,du<\infty. \tag{2.1}$$

(A2) There exists a $U_1 \in (0, \infty)$ such that

$$g(u) < 0 \quad \text{for } 0 < u < U_1, \qquad g(U_1) = 0, \quad g(u) > 0 \quad \text{for } u > U_1, \tag{2.2}$$

and

$$0 < \int_0^{+\infty} ug(u)\,du \leqslant \infty, \tag{2.3}$$

where

$$g(u) = (p-1)f(u) - uf'(u), \quad u \in (0, \infty). \tag{2.4}$$

By (A2) it is clear that there exists a unique $U_2 \in (U_1, \infty)$ satisfying

$$\int_0^{U_2} ug(u)\,du = 0. \tag{2.5}$$

Since $(f(u)/u^{p-1})' = -g(u)/u^p$, we have that $f(u)/u^{p-1}$ is increasing on $(0, U_1)$ and decreasing on (U_1, ∞). Let

$$\lim_{u\to 0+} f(u)/u^{p-1} = L_1, \qquad \lim_{u\to +\infty} f(u)/u^{p-1} = L_2. \tag{2.6}$$

Then $0 \leqslant L_1 < \infty$ and $0 \leqslant L_2 < \infty$. Denote by

$$C_p = \left(\frac{p-1}{p}\right)^{1/p}, \tag{2.7}$$

$$W_1 = \begin{cases} \left(\frac{p}{L_1}\right)^{1/p} \int_0^1 \frac{dt}{(1-t^p)^{1/p}}, & L_1 > 0, \\ +\infty, & L_1 = 0, \end{cases} \tag{2.8}$$

and

$$W_2 = \begin{cases} \left(\frac{p}{L_2}\right)^{1/p} \int_0^1 \frac{dt}{(1-t^p)^{1/p}}\,dt, & L_2 > 0, \\ +\infty, & L_2 = 0. \end{cases} \tag{2.9}$$

Our main result for exact multiplicity of positive solutions of (1.1) is the following theorem.

Theorem 2.1. *Let $p > 1$ and $f \in C[0, \infty) \cap C^2(0, \infty)$ satisfy* (A1), (A2) *and*

(A3) $[rg(r)]^2 \geqslant [2g(r) + rg'(r)] \int_0^r ug(u)\,du$ *for* $r \in (U_1, U_2)$.

Put

$$\lambda_1 = \min\{[C_p W_1]^p,\ [C_p W_2]^p\}, \qquad \lambda_2 = \max\{[C_p W_1]^p,\ [C_p W_2]^p\}.$$

Then there exists $0 < \lambda^* < \lambda_1$ *such that* (1.1) *has*

(i) *no positive solution for* $0 < \lambda < \lambda^*$;
(ii) *exactly one positive solution for* $\lambda = \lambda^*$;
(iii) *exactly two for* $\lambda^* < \lambda < \lambda_1$;
(iv) *exactly one for* $\lambda_1 \leqslant \lambda < \lambda_2$ *when* $\lambda_1 < \lambda_2$;
(v) *no for* $\lambda \geqslant \lambda_2$ *when* $\lambda_2 < +\infty$.

The proof of this theorem will be conducted in the next section. Note that $g'(u) = (p-2)f'(u) - uf''(u)$ is permitted to change sign on (U_1, U_2) in (A3), see Remark 4.1. In particular, we have the following theorem. In Section 4, an interesting application of this theorem will be given.

Theorem 2.2. *Let $p > 1$ and $f \in C[0,\infty) \cap C^2(0,\infty)$ satisfy* (A1) *and* (A2). *Put*

$$g_0(r) = r^2 g(r), \qquad g_j(r) = r g'_{j-1}(r), \quad j = 1, 2, 3.$$

Assume that one of the following three conditions is satisfied:

(i) $g_1(r) \geqslant 0$ *on* (U_1, U_2);
(ii) $f \in C^3(0,\infty)$ *and there exists* $u_1 \in (U_1, U_2)$ *such that*

$$g_1(r) \geqslant 0 \quad \text{on } (U_1, u_1), \qquad g_1(r) \leqslant 0 \quad \text{on } (u_1, U_2),$$

and

$$g_2(r) \leqslant 0 \quad \text{on } (u_1, U_2);$$

(iii) $f \in C^4(0,\infty)$ *and there exist* $U_1 < u_1 < u_2 < U_2$ *such that*

$$g_1(r) \geqslant 0 \quad \text{on } (U_1, u_1), \qquad g_1(r) \leqslant 0 \quad \text{on } (u_1, U_2),$$

and

$$g_2(r) \leqslant 0 \quad \text{on } (u_1, u_2), \qquad g_3(r) \geqslant 0 \quad \text{on } (u_2, U_2).$$

Then (A3) *holds.*

Proof. Let

$$h_0(r) = r^2\left(\left[rg(r)\right]^2 - \left[2g(r) + rg'(r)\right]\int_0^r ug(u)\,du\right).$$

Then

$$h_0(r) = \left[g_0(r)\right]^2 - g_1(r)\int_0^r ug(u)\,du,$$

$$h_1(r) = rh'_0(r) = g_0(r)g_1(r) - g_2(r)\int_0^r ug(u)\,du,$$

and

$$h_2(r) = rh'_1(r) = \left[g_1(r)\right]^2 - g_3(r)\int_0^r ug(u)\,du.$$

By (A2) and (2.5) we have that

$$g_0(r) > 0 \quad \text{and} \quad \int_0^r ug(u)\,du < 0 \quad \text{for } r \in (U_1, U_2).$$

(1) If the condition (i) is satisfied, then it is clear that $h_0(r) \geqslant 0$ on (U_1, U_2).

(2) If the condition (ii) is satisfied, then we have that

$$h_0(r) \geqslant 0 \quad \text{on } (U_1, u_1),$$
$$h_0(U_2) = \left[g_0(U_2)\right]^2 \geqslant 0,$$

and

$$h_1(r) \leqslant 0 \quad \text{on } (u_1, U_2).$$

It follows that $h_0(r) \geqslant 0$ on (U_1, U_2).

(3) For the condition (iii), we have that

$$h_0(r) \geqslant 0 \quad \text{on } (U_1, u_1), \qquad h_0(U_2) = \left[g_0(U_2)\right]^2 \geqslant 0,$$
$$h_1(r) \leqslant 0 \quad \text{on } (u_1, u_2), \qquad h_1(U_2) = g_0(U_2)g_1(U_2) \leqslant 0,$$

and

$$h_2(r) \geqslant 0 \quad \text{on } (u_2, U_2).$$

It follows that $h_1(r) \leqslant 0$ on (u_1, U_2) and $h_0(r) \geqslant 0$ on (U_1, U_2).

The proof of Theorem 2.2 is complete. □

Remark 2.1. Let $p > 1$ and $f \in C[0,\infty) \cap C^3(0,\infty)$ satisfy (A1) and (A2). If there exists $r_1 \in (U_1, U_2)$ such that $g_1(r) \geqslant 0$ on (U_1, r_1) and $g_2(r) \leqslant 0$ on (r_1, U_2), then by Theorem 2.2 we have that (A3) holds.

Remark 2.2. It follows from (A2) that $-\int_0^r ug(u)\,du$ is positive and decreasing with respect to $r \in (U_1, U_2)$. By

$$\left(\frac{r^2 g(r)}{-\int_0^r ug(u)\,du}\right)' = \frac{r([rg(r)]^2 - [2g(r) + rg'(r)]\int_0^r ug(u)\,du)}{(-\int_0^r ug(u)\,du)^2}$$

we have that (A3) holds if and only if

$$\eta(r) = \frac{r^2 g(r)}{-\int_0^r ug(u)\,du}$$

is nondecreasing on (U_1, U_2).

3. Proof of Theorem 2.1

The time map formula which we apply to study the p-Laplacian Dirichlet problem (1.1) takes the form as follows:

$$T(r) = \int_0^r \left(\int_v^r f(u)\,du\right)^{-1/p} dv, \quad r \in (0,\infty). \tag{3.1}$$

First, we state the following lemma without proof (cf. [2,6–8,13,15,17]).

Lemma 3.1. *Let $r > 0$. Then* (1.1) *has a positive solution satisfying $r = \max_{t\in[-1,1]} y(t)$ if and only if $\lambda = [C_p T(r)]^p$, and in this case the solution is unique.*

Thus to study the exact number of positive solutions of (1.1) is equivalent to studying the shape of the time map $T(r)$ on $(0,\infty)$. In the following lemmas, we summarize some basic results required for our investigations.

Lemma 3.2. $\lim_{r\to 0+} T(r) = W_1$, *and* $\lim_{r\to +\infty} T(r) = W_2$.

Proof. It follows that

$$T(r) = \int_0^1 \left(\int_t^1 \frac{f(rs)}{(rs)^{p-1}} s^{p-1}\, ds \right)^{-1/p} dt.$$

If $L_1 = 0$, then it is clear that $\lim_{r\to 0+} T(r) = +\infty = W_1$. If $L_1 > 0$, then the dominated convergence theorem implies that

$$\lim_{r\to 0+} T(r) = \left(\frac{p}{L_1}\right)^{1/p} \int_0^1 \left(\int_t^1 p s^{p-1}\, ds \right)^{-1/p} dt = W_1.$$

Finally, $\lim_{r\to +\infty} T(r) = W_2$ can be obtained by [13, Corollary 2.6]. □

Now, for $r > 0$ and $t \in [0,1]$, denote by

$$D_0(r,t) = r^{-p} \int_{rt}^r f(u)\, du, \qquad D_1(r,t) = r^{-p} \int_{rt}^r u f'(u)\, du, \tag{3.2}$$

and

$$D_2(r,t) = r^{-p} \int_{rt}^r u^2 f''(u)\, du. \tag{3.3}$$

It is clear that

$$(p-1) D_0(r,t) - D_1(r,t) = r^{-p} \int_{rt}^r g(u)\, du. \tag{3.4}$$

Let

$$F(r) = \int_0^1 \frac{(p-1) D_0(r,t) - D_1(r,t)}{[D_0(r,t)]^{1+1/p}}\, dt, \tag{3.5}$$

$$G_1(r) = \int_0^1 \frac{-(p-1)^2 D_0(r,t) + (2p-3) D_1(r,t) - D_2(r,t)}{[D_0(r,t)]^{1+1/p}}\, dt, \tag{3.6}$$

and

$$G_2(r) = \int_0^1 \frac{[(p-1)D_0(r,t) - D_1(r,t)]^2}{[D_0(r,t)]^{2+1/p}}\,dt. \tag{3.7}$$

Remark 3.1. Fix $r > 0$. It is clear that $D_0(r,t) > 0$ for $0 \leqslant t < 1$. Since

$$\lim_{t\to 1} \frac{D_0(r,t)}{1-t} = r^{-p+1} f(r), \qquad \lim_{t\to 1} \frac{D_1(r,t)}{D_0(r,t)} = \frac{rf'(r)}{f(r)}, \qquad \lim_{t\to 1} \frac{D_2(r,t)}{D_0(r,t)} = \frac{r^2 f''(r)}{f(r)},$$

we have that

$$\int_0^1 \frac{1}{[D_0(r,t)]^{1/p}}\,dt < \infty, \qquad \sup_{t\in(0,1)} \left|\frac{D_1(r,t)}{D_0(r,t)}\right| < \infty, \qquad \sup_{t\in(0,1)} \left|\frac{D_2(r,t)}{D_0(r,t)}\right| < \infty.$$

It follows from $p > 1$ that $F(r)$, $G_1(r)$ and $G_2(r)$ are well defined for $r \in (0,\infty)$.

Lemma 3.3. *T and F are two differentiable functions, and for any $r \in (0,\infty)$*

$$T(r) = \int_0^1 \frac{dt}{[D_0(r,t)]^{1/p}}, \tag{3.8}$$

$$prT'(r) = F(r), \tag{3.9}$$

$$prF'(r) = pG_1(r) + (1+p)G_2(r). \tag{3.10}$$

Proof. (3.8) can be obtained by (3.1), immediately. From

$$\begin{aligned}
\frac{\partial D_0(r,t)}{\partial r} &= -pr^{-p-1} \int_{rt}^{r} f(u)\,du + r^{-p}\big[f(r) - tf(rt)\big] \\
&= -pr^{-p-1} \int_{rt}^{r} f(u)\,du + r^{-p-1}\big[rf(r) - (rt)f(rt)\big] \\
&= -pr^{-p-1} \int_{rt}^{r} f(u)\,du + r^{-p-1} \int_{rt}^{r} \big[uf(u)\big]'\,du \\
&= r^{-p-1}\left((1-p)\int_{rt}^{r} f(u)\,du + \int_{rt}^{r} uf'(u)\,du\right)
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial D_1(r,t)}{\partial r} &= -pr^{-p-1} \int_{rt}^{r} uf'(u)\,du + r^{-p}\big[rf'(r) - t^2 rf'(rt)\big] \\
&= -pr^{-p-1} \int_{rt}^{r} uf'(u)\,du + r^{-p-1}\big[r^2 f'(r) - (rt)^2 f'(rt)\big]
\end{aligned}$$

$$= -pr^{-p-1}\int_{rt}^{r} uf'(u)\,du + r^{-p-1}\int_{rt}^{r}\left[u^2 f'(u)\right]'du$$
$$= r^{-p-1}\left((2-p)\int_{rt}^{r} uf'(u)\,du + \int_{rt}^{r} u^2 f''(u)\,du\right)$$

we have that

$$\frac{\partial D_0(r,t)}{\partial r} = \frac{(1-p)D_0(r,t) + D_1(r,t)}{r}, \tag{3.11}$$

and

$$\frac{\partial D_1(r,t)}{\partial r} = \frac{(2-p)D_1(r,t) + D_2(r,t)}{r}. \tag{3.12}$$

Combine (3.8) and (3.11) to obtain

$$T'(r) = -\frac{1}{p}\int_0^1 \frac{1}{[D_0(r,t)]^{1+1/p}}\,\frac{(1-p)D_0(r,t) + D_1(r,t)}{r}\,dt = \frac{F(r)}{pr}. \tag{3.13}$$

It follows from (3.11) and (3.12) that

$$\begin{aligned}\frac{\partial}{\partial r}\left(\frac{(p-1)D_0 - D_1}{D_0^{1+1/p}}\right) &= -\frac{(1+p)[(1-p)D_0 + D_1][(p-1)D_0 - D_1]}{prD_0^{2+1/p}}\\ &\quad + \frac{(p-1)[(1-p)D_0 + D_1] - [(2-p)D_1 + D_2]}{rD_0^{1+1/p}}\\ &= \frac{(1+p)[(p-1)D_0 - D_1]^2}{prD_0^{2+1/p}}\\ &\quad + \frac{-(p-1)^2 D_0 + (2p-3)D_1 - D_2}{rD_0^{1+1/p}}.\end{aligned}$$

This implies (3.10). □

Lemma 3.4. $G_2(r) > 0$ *for* $r > 0$.

Proof. Fix $r > 0$. From (3.4) and (A2) we have that $(p-1)D_0(r,t) - D_1(r,t)$ is not identically zero on any open subinterval of $(0,1)$ with respect to t. Thus, this lemma is clear. □

Lemma 3.5. $F(r) < 0$ *for* $r \in (0, U_1]$.

Proof. From (A2) and (3.4) we have that $(p-1)D_0(r,t) - D_1(r,t) < 0$ for $r \in (0, U_1]$ and $t \in (0,1)$. Thus, the lemma follows. □

Lemma 3.6. $F(r) > 0$ *for* $r \in [U_2, \infty)$.

Proof. Fix $r \in [U_2, \infty)$. We distinguish two cases.

Case (i). Assume that $\int_0^r g(u)\,du \geqslant 0$. It follows from (A2) that

(i-i) if $\int_0^{rt} g(u)\,du \geqslant 0$, then $rt > U_1$ and $\int_{rt}^{r} g(u)\,du > 0$;

(i-ii) if $\int_0^{rt} g(u)\,du < 0$, then

$$\int_{rt}^{r} g(u)\,du = \int_0^r g(u)\,du - \int_0^{rt} g(u)\,du \geqslant -\int_0^{rt} g(u)\,du > 0.$$

By (3.4) we have that $(p-1)D_0(r,t) - D_1(r,t) > 0$ for $t \in (0,1)$. In particular, $F(r) > 0$.

Case (ii). Assume that $\int_0^r g(u)\,du < 0$. By $r \geqslant U_2 > U_1$ and (A2) we have that there exists a unique $t_0 \in (0,1)$ such that

$$(p-1)D_0(r,t) - D_1(r,t) = r^{-p}\int_{rt}^{r} g(u)\,du < 0 \quad \text{for } t \in (0,t_0),$$

and

$$(p-1)D_0(r,t) - D_1(r,t) = r^{-p}\int_{rt}^{r} g(u)\,du > 0 \quad \text{for } t \in (t_0,1).$$

Since $D_0(r,t)$ is positive and decreasing with respect to $t \in (0,1)$, it follows that

$$\begin{aligned} F(r) &> \frac{1}{[D_0(r,t_0)]^{1+1/p}}\int_0^{t_0}\big[(p-1)D_0(r,t) - D_1(r,t)\big]\,dt \\ &\quad + \frac{1}{[D_0(r,t_0)]^{1+1/p}}\int_{t_0}^{1}\big[(p-1)D_0(r,t) - D_1(r,t)\big]\,dt \\ &= \frac{1}{[D_0(r,t_0)]^{1+1/p}}\int_0^{1}\big[(p-1)D_0(r,t) - D_1(r,t)\big]\,dt. \end{aligned}$$

By (3.4) and $r \geqslant U_2$ we have that

$$\int_0^1 \big[(p-1)D_0(r,t) - D_1(r,t)\big]\,dt = r^{-p}\int_0^1 dt\int_{rt}^{r} g(u)\,du = r^{-1-p}\int_0^r ug(u)\,du \geqslant 0.$$

Thus, $F(r) > 0$ follows.

The proof is complete. □

In the following part, denote by

$$\eta(r) = \frac{r^2 g(r)}{-\int_0^r ug(u)\,du}, \quad r \in (0,U_2), \tag{3.14}$$

$$\phi(r,v) = rg(r) - vg(v) + \big(\eta(r)+1\big)\int_v^r g(t)\,dt, \quad r \in (0,U_2),\ v \in (0,r], \tag{3.15}$$

and

$$\psi(r,v)=r^2g(r)-v^2g(v)+\eta(r)\int_v^r tg(t)\,dt,\quad r\in(0,U_2),\ v\in(0,r]. \tag{3.16}$$

It follows from (A2), (A3) and Remark 2.2 that

$$\eta(U_1)=0,\eta'(r)\geqslant 0\quad\text{and}\quad\eta(r)\geqslant 0\quad\text{for } r\in(U_1,U_2). \tag{3.17}$$

Lemma 3.7. *$\psi(r,v)\geqslant 0$ for $r\in(U_1,U_2)$ and $v\in(0,r)$.*

Proof. Fix $r\in(U_1,U_2)$. Since

$$\begin{aligned}\frac{\partial\psi(r,v)}{\partial v}&=-\bigl[2vg(v)+v^2g'(v)\bigr]-\eta(r)vg(v)\\&=-\bigl[2vg(v)+v^2g'(v)\bigr]-\eta(v)vg(v)+\bigl[\eta(v)-\eta(r)\bigr]vg(v),\end{aligned}$$

it follows that

$$\frac{\partial\psi(r,v)}{\partial v}=v\frac{[2g(v)+vg'(v)]\int_0^v ug(u)\,du-[vg(v)]^2}{-\int_0^v ug(u)\,du}+\bigl[\eta(v)-\eta(r)\bigr]vg(v).$$

By (A2), (2.5) and (3.17) we have that

$$\int_0^v ug(u)\,du<0,\qquad g(v)>0\quad\text{and}\quad\eta(v)-\eta(r)\leqslant 0\quad\text{for } v\in(U_1,r).$$

Thus, (A3) implies that

$$\frac{\partial\psi(r,v)}{\partial v}\leqslant 0\quad\text{for } v\in(U_1,r),$$

and

$$\psi(r,v)\geqslant\psi(r,r)=0\quad\text{for } v\in[U_1,r). \tag{3.18}$$

For $v\in(0,U_1)$, since $g(v)<0$, $\int_0^v ug(u)\,du<0$ and $\eta(r)\geqslant 0$, we have that

$$\psi(r,v)\geqslant r^2g(r)+\eta(r)\int_0^r ug(u)\,du=0\quad\text{for } v\in(0,U_1). \tag{3.19}$$

The proof is complete. □

Lemma 3.8. *$k(r,v)=\int_v^r\phi(r,u)\,du\geqslant 0$ for $r\in(U_1,U_2)$ and $v\in[0,r]$.*

Proof. Fix $r\in(U_1,U_2)$. From

$$\begin{aligned}k(r,v)&=\int_v^r\bigl[rg(r)-ug(u)\bigr]\,du+\bigl(\eta(r)+1\bigr)\int_v^r du\int_u^r g(t)\,dt\\&=\int_v^r\bigl[rg(r)-ug(u)\bigr]\,du+\bigl(\eta(r)+1\bigr)\int_v^r(t-v)g(t)\,dt\end{aligned}$$

$$= (r-v)rg(r) + \eta(r)\int_v^r ug(u)\,du - \big(\eta(r)+1\big)v\int_v^r g(t)\,dt$$
$$= \psi(r,v) - v\phi(r,v)$$

and

$$\frac{\partial k(r,v)}{\partial v} = -\phi(r,v)$$

we have that

$$k(r,v) - v\frac{\partial k(r,v)}{\partial v} = \psi(r,v) \quad \text{for } v \in (0,r). \tag{3.20}$$

Since $k(r,r)=0$ and

$$k(r,0) = r^2 g(r) + \eta(r)\int_0^r ug(u)\,du = 0,$$

by Lemma 3.7 we have that $k(r,v) \geqslant 0$ for $v \in [0,r]$. $\square$

Lemma 3.9. $G_1(r) + [p+\eta(r)+1]F(r) \geqslant 0$ *for* $r \in (U_1,U_2)$.

Proof. It follows from (3.2)–(3.6) that

$$G_1(r) + pF(r) = \int_0^1 \frac{(p-1)D_0(r,t) + (p-3)D_1(r,t) - D_2(r,t)}{[D_0(r,t)]^{1+1/p}}\,dt \tag{3.21}$$

and

$$F(r) = \int_0^1 \frac{r^{-p}\int_{rt}^r g(u)\,du}{[D_0(r,t)]^{1+1/p}}\,dt. \tag{3.22}$$

By $[ug(u)]' = (p-1)f(u) + (p-3)uf'(u) - u^2 f''(u)$ we have that

$$(p-1)D_0(r,t) + (p-3)D_1(r,t) - D_2(r,t) = r^{-p}\big[rg(r) - rtg(rt)\big]. \tag{3.23}$$

Combine (3.21)–(3.23) to obtain

$$G_1(r) + \big[p+\eta(r)+1\big]F(r) = r^{-p}\int_0^1 \frac{\phi(r,rt)}{[D_0(r,t)]^{1+1/p}}\,dt. \tag{3.24}$$

By Lemma 3.8 we have that

$$\int_s^1 \phi(r,rt)\,dt \geqslant 0 \quad \text{for } r \in (U_1,U_2) \text{ and } s \in [0,1]. \tag{3.25}$$

Suppose to the contrary that there exists an $r \in (U_1,U_2)$ such that

$$G_1(r) + \big[p+\eta(r)+1\big]F(r) < 0.$$

Then it follows from (3.24) and the dominated convergence theorem that there exists $\varepsilon > 0$ with

$$\int_0^1 \frac{\phi(r, rt)}{[D_0(r,t)+\varepsilon]^{1+1/p}}\,dt < 0.$$

By the second mean value theorem of integral we may imply that there exists $\xi \in [0,1]$ satisfying

$$\int_0^1 \frac{\phi(r, rt)}{[D_0(r,t)+\varepsilon]^{1+1/p}}\,dt = \frac{1}{[D_0(r,1)+\varepsilon]^{1+1/p}}\int_\xi^1 \phi(r,rt)\,dt = \frac{1}{\varepsilon^{1+1/p}}\int_\xi^1 \phi(r,rt)\,dt,$$

which contradicts (3.25).

The proof is complete. $\square$

Lemma 3.10. *There exists $r^* \in (U_1, U_2)$ such that $F(r) < 0$ for $r \in (0, r^*)$ and $F(r) > 0$ for $r \in (r^*, \infty)$.*

Proof. It follows from Lemmas 3.5 and 3.6 that there exists $r^* \in (U_1, U_2)$ such that $F(r^*) = 0$ and $F(r) > 0$ for $r \in (r^*, \infty)$.

By Lemmas 3.3, 3.4 and 3.9 we can obtain that

$$pr^*F'(r^*) = p\big[G_1(r^*) + \big(p + \eta(r^*) + 1\big)F(r^*)\big] + (1+p)G_2(r^*) > 0. \tag{3.26}$$

This implies that there exists $r_0 \in (0, r^*)$ such that $F(r) < 0$ for $r \in (r_0, r^*)$. Denote by

$$r_1 = \inf\big\{r_0 \in (0, r^*)\colon\ F(r) < 0 \text{ for } r \in (r_0, r^*)\big\}.$$

Suppose to the contrary that $r_1 > 0$. It follows $F(r_1) = 0$. By Lemmas 3.3–3.6 and 3.9 we can also obtain that $r_1 \in (U_1, U_2)$ and

$$pr_1F'(r_1) = p\big[G_1(r_1) + \big(p + \eta(r_1) + 1\big)F(r_1)\big] + (1+p)G_2(r_1) > 0,$$

which contradicts $F(r) < 0$ for $r \in (r_1, r^*)$ and $F(r_1) = 0$. It follows that $r_1 = 0$. Thus, this lemma holds. $\square$

Now, we give the proof of Theorem 2.1.

Proof of Theorem 2.1. It follows from Lemmas 3.3 and 3.10 that there exists $r^* \in (U_1, U_2)$ such that $T'(r) < 0$ for $r \in (0, r^*)$ and $T'(r) > 0$ for $r > r^*$. Let $\lambda^* = [C_pT(r^*)]^p$, then by Lemmas 3.1 and 3.2 we have the results of Theorem 2.1, immediately. $\square$

4. An application of Theorem 2.2

Consider the general case of (1.2)

$$\begin{cases} (|y'|^{p-2}y')' + \lambda(y^\alpha e^{-y} + \beta y^{p-1}) = 0, & t \in (-1, 1), \\ y(-1) = y(1) = 0, \end{cases} \tag{4.1}$$

where $p > 1$, $\alpha > p - 1$ and $\beta \geqslant 0$.

It is clear that (A1) and (A2) are satisfied with

$$g(u) = (p-1)f(u) - uf'(u) = u^\alpha e^{-u}\big(u - (\alpha - p + 1)\big), \tag{4.2}$$

$$U_1 = (\alpha - p + 1), \tag{4.3}$$

and

$$W_1 = W_2 = \begin{cases} \left(\frac{p}{\beta}\right)^{1/p} \int_0^1 \frac{dt}{(1-t^p)^{1/p}}, & \beta > 0, \\ +\infty, & \beta = 0. \end{cases} \tag{4.4}$$

In this section, we discuss the assumption (A3) for (4.1). By Theorem 2.2 and $ug(u) = (p+1)uf(u) - (u^2 f(u))'$, the following theorem can be obtained.

Theorem 4.1. *Assume that $p > 1$, $\alpha > p - 1$ and $\beta \geqslant 0$. If there exists $\mu > U_1$ such that*

$$\int_0^\mu ug(u)\,du = (p+1)\int_0^\mu u^{\alpha+1}e^{-u}\,du - \mu^{\alpha+2}e^{-\mu} \geqslant 0, \tag{4.5}$$

and

$$K_1(\mu) \geqslant 0 \quad \text{or} \quad K_2(\mu) \leqslant 0 \quad \text{or} \quad K_3(\mu) \geqslant 0, \tag{4.6}$$

then (A3) *holds for* (4.1)*, where $K_0(u) = (u - U_1)$ and*

$$K_j(u) = \big[(\alpha+2) - u\big]K_{j-1}(u) + uK'_{j-1}(u), \quad j = 1, 2, 3. \tag{4.7}$$

Proof. It follows that

$$g_0(u) = u^2 g(u) = u^{\alpha+2}e^{-u}K_0(u),$$
$$g_j(u) = ug'_{j-1}(u) = u^{\alpha+2}e^{-u}K_j(u),$$

and

$$K_j(0) = -(\alpha+2)^j U_1, \qquad \lim_{u\to+\infty} \frac{K_j(u)}{u^{j+1}} = (-1)^j, \tag{4.8}$$

$j = 1, 2, 3$. Since K_1 is a 2-order polynomial function and $K_1(U_1) > 0$, by (4.8) we have that K_1 can be written in the form of

$$K_1(u) = -(u - B_{1,1})(u - B_{1,2}), \tag{4.9}$$

where

$$0 < B_{1,1} < U_1 < B_{1,2}, \tag{4.10}$$

and

$$B_{1,2} < u \quad \Longleftrightarrow \quad U_1 < u \quad \text{and} \quad K_1(u) < 0. \tag{4.11}$$

By $K_2(B_{1,1}) > 0$, $K_2(B_{1,2}) < 0$ and (4.8), we can obtain that

$$K_2(u) = (u - B_{2,1})(u - B_{2,2})(u - B_{2,3}), \tag{4.12}$$

where

$$0 < B_{2,1} < B_{1,1} < B_{2,2} < B_{1,2} < B_{2,3}, \tag{4.13}$$

and

$$B_{2,3} < u \iff B_{1,2} < u \quad \text{and} \quad K_2(u) > 0. \tag{4.14}$$

Similarly, $K_3(B_{2,1}) > 0$, $K_3(B_{2,2}) < 0$, $K_3(B_{2,3}) > 0$ and (4.8) imply

$$K_3(u) = -(u - B_{3,1})(u - B_{3,2})(u - B_{3,3})(u - B_{3,4}), \tag{4.15}$$

where

$$0 < B_{3,1} < B_{2,1} < B_{3,2} < B_{2,2} < B_{3,3} < B_{2,3} < B_{3,4}, \tag{4.16}$$

and

$$B_{3,4} < u \iff B_{2,3} < u \quad \text{and} \quad K_3(u) < 0. \tag{4.17}$$

Now, we have obtained

$$g_1(u) > 0 \quad \text{on } (U_1, B_{1,2}), \qquad g_1(u) < 0 \quad \text{on } (B_{1,2}, \infty), \tag{4.18}$$

$$g_2(u) < 0 \quad \text{on } (B_{1,2}, B_{2,3}), \qquad g_2(u) > 0 \quad \text{on } (B_{2,3}, \infty), \tag{4.19}$$

$$g_3(u) > 0 \quad \text{on } (B_{2,3}, B_{3,4}), \qquad g_3(u) < 0 \quad \text{on } (B_{3,4}, \infty). \tag{4.20}$$

Finally, $U_2 \leqslant \mu \leqslant B_{3,4}$ follows from (4.5), (4.11), (4.14) and (4.17). Thus, Theorem 2.2 implies that (A3) holds. □

In the following examples and remarks, much of the computation is calculated by the symbolic manipulator *Mathematica 4.0*.

Example 4.1. It follows that

$$K_2(\alpha + 7) = 45 + 18p - \alpha(15 + p)$$

and

$$K_3(\alpha + 7) = 118 - 3\alpha^2 - 27p + \alpha(103 + 14p) \stackrel{\text{def}}{=} \Phi(\alpha).$$

By $\Phi''(\alpha) < 0$,

$$\Phi(p - 1) = 12 + 68p + 11p^2 > 0,$$

and

$$\Phi\left(\frac{45 + 18p}{15 + p}\right) = \frac{25(3600 + 1380p + 184p^2 + 9p^3)}{(15 + p)^2} > 0,$$

we have that if $\alpha > p - 1$ and $K_2(\alpha + 7) \geqslant 0$ then $K_3(\alpha + 7) > 0$, i.e.,

$$K_2(\alpha + 7) < 0 \quad \text{or} \quad K_3(\alpha + 7) > 0 \quad \text{for all } \alpha > p - 1.$$

Thus, Theorem 4.1 implies (A3) if

$$\int_0^{\alpha+7} ug(u)\,du = (p + 1)\int_0^{\alpha+7} u^{\alpha+1}e^{-u}\,du - (\alpha + 7)^{\alpha+2}e^{-(\alpha+7)} \geqslant 0.$$

Since $(u^{\alpha+1}e^{-u})' < 0$ on $(\alpha + 1, \infty)$, we have

$$(p + 1)\int_0^{\alpha+7} u^{\alpha+1}e^{-u}\,du > (p + 1)\int_{\alpha+1}^{\alpha+7} u^{\alpha+1}e^{-u}\,du > 6(p + 1)(\alpha + 7)^{\alpha+1}e^{-(\alpha+7)}.$$

It follows that (A3) holds if $0 < p - 1 < \alpha \leqslant 6p - 1$ and $\beta \geqslant 0$.

Example 4.2. For $p = 2$, $\alpha = 10^2$ and $\beta \geqslant 0$, let $\mu = 113$. Then

$$(p+1)\int_0^\mu u^{\alpha+1}e^{-u}\,du - \mu^{\alpha+2}e^{-\mu} = 2.53503 \times 10^{159},$$

and

$$K_1(\mu) = -41, \qquad K_2(\mu) = -2261.$$

By Theorem 4.1 we have that (A3) holds.

Remark 4.1. For Example 4.2, if $\mu = 112$ then

$$(p+1)\int_0^\mu u^{\alpha+1}e^{-u}\,du - \mu^{\alpha+2}e^{-\mu} = -2.08942 \times 10^{158},$$

$K_1(\mu) = -18$ and $K_2(\mu) = -2284$. By (4.10), (4.11) and (4.14) we have

$$0 < B_{1,1} < U_1 < B_{1,2} < 112 < U_2 < 113 < B_{2,3}.$$

It follows that $g_1(u) = ug_0'(u) = u^2(2g(u) + ug'(u))$ changes sign on (U_1, U_2), and the condition (ii) of Theorem 2.2 is satisfied. In particular, by $g_1(U_1) > 0 > g_1(U_2)$, $g(U_1) = 0$, and $g(U_2) > 0$ we have $g'(U_1) > 0$ and $g'(U_2) < 0$. It follows that $g'(u)$ changes sign on (U_1, U_2).

Example 4.3. For $p = 2$, $\alpha = 10^4$ and $\beta \geqslant 0$, let $\mu = 10232$. Then

$$(p+1)\int_0^\mu u^{\alpha+1}e^{-u}\,du - \mu^{\alpha+2}e^{-\mu} = 4.80629528 \times 10^{35661},$$

and

$$K_1(\mu) = -43358, \qquad K_2(\mu) = 5245165, \qquad K_3(\mu) = 110390664.$$

By Theorem 4.1 we have that (A3) holds.

Remark 4.2. For Example 4.3, if $\mu = 10231$ then

$$(p+1)\int_0^\mu u^{\alpha+1}e^{-u}\,du - \mu^{\alpha+2}e^{-\mu} = -1.448940255 \times 10^{35662},$$

and

$$K_1(\mu) = -42897, \qquad K_2(\mu) = 5117153, \qquad K_3(\mu) = 130731728.$$

By the proof of Theorem 4.1 we have

$$0 < B_{1,1} < U_1 < B_{1,2} < B_{2,3} < 10231 < U_2 < 10232 < B_{3,4}.$$

It follows that condition (iii) of Theorem 2.2 is satisfied.

Remark 4.3. By the proof of Theorem 4.1 we know that $\mu > U_1$ satisfies (4.5) and (4.6) if and only if $U_2 \leqslant \mu \leqslant B_{3,4}$. Reversely, if there exists a $\mu > U_1$ such that

$$\int_0^\mu ug(u)\,du = (p+1)\int_0^\mu u^{\alpha+1}e^{-u}\,du - \mu^{\alpha+2}e^{-\mu} < 0,$$

$K_1(\mu) < 0$, $K_2(\mu) > 0$, and $K_3(\mu) < 0$, then $B_{3,4} < \mu < U_2$ and (4.18)–(4.20) imply that (A3) cannot be checked by Theorem 2.2. For example, let $p = 2$ and $\alpha = 10^5$. If $\mu = 100800$, then

$$(p+1)\int_0^\mu u^{\alpha+1}e^{-u}\,du - \mu^{\alpha+2}e^{-\mu} = -6.586400759 \times 10^{456578},$$

$K_1(\mu) = -538398$, $K_2(\mu) = 268563204$ and $K_3(\mu) = -51984713592$.

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