

Large time behavior of bounded solutions to a parabolic system of chemotaxis in the whole space

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Abstract

We consider the Cauchy problem for a parabolic system of chemotaxis in \mathbb{R}^N ($N \geq 1$), and give the decay rates and asymptotic profiles of bounded solutions.

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1. Introduction

In this paper, we consider the following Cauchy problem:

$$(P) \quad \begin{cases} \partial_t u = \Delta u - \nabla \cdot (u \nabla v), & x \in \mathbb{R}^N, \ t > 0, \\ \partial_t v = \Delta v - v + u, & x \in \mathbb{R}^N, \ t > 0, \\ u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), & x \in \mathbb{R}^N. \end{cases}$$

This system is a mathematical model describing the motion of some species due to chemotaxis (see [8]), and $u(x, t)$ corresponds to the cell density of the species at place x and time t and $v(x, t)$ to the concentration of the chemical.

Concerning system (P), it is known that the finite-time blowup of nonnegative solutions can occur in the case of $N \geq 2$, but never occurs in the case of $N = 1$ (see [1,4–7,11], for example).

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For the large time behavior of bounded solutions of (P), the following was shown in [12]: Let $N \geq 2$ and $1 < p \leq \infty$. If the solution (u, v) of (P) on $\mathbb{R}^N \times [0, \infty)$ satisfies the condition

$$\sup_{t>0} (\|u(t)\|_q + \|v(t)\|_q) < \infty \quad \text{for } q = 1, \infty, \quad (1.1)$$

then

$$\sup_{t>0} (1+t)^{N(1-1/p)/2} \|u(t)\|_p < \infty, \quad (1.2)$$

$$\lim_{t \rightarrow \infty} t^{N(1-1/p)/2} \left\| u(t) - \int_{\mathbb{R}^N} u_0 dy G(t) \right\|_p = 0, \quad (1.3)$$

and v has the same decay rate and asymptotic profile as u . Here $\|\cdot\|_p$ is the usual $L^p(\mathbb{R}^N)$ -norm and $G = G(x, t)$ is the heat kernel, that is,

$$G(x, t) = (4\pi t)^{-N/2} \exp\left(-\frac{|x|^2}{4t}\right).$$

The result in [12] means that every bounded solution of (P) decays to zero as $t \rightarrow \infty$, and that the solution behaves like the heat kernel as the self-similar profile. We remark that a self-similar solution exists for a system replaced the second equation of (P) by $v_t = \Delta v + u$ (see [13]), but (P) does not have such a self-similar solution. For the large time behavior of a parabolic–elliptic system with degenerate diffusion modeling chemotaxis, we refer to [10]. The nature of self-similarity was observed for other nonlinear partial differential equations, for example, [2] for heat convection equations and [3] for the Navier–Stokes equation.

The purpose of the paper is to give the large time behavior of bounded solutions for $N = 1$, because the case $N = 1$ is excluded from [12], and is to improve the decay rates and asymptotic profiles of bounded solutions under the additional condition $\int_{\mathbb{R}^N} |y| |u_0| dy < \infty$ for $N \geq 1$.

2. Main theorems

In this paper, $W^{k,p}(\mathbb{R}^N)$ represents the usual Sobolev space and $\mathcal{B}(\mathbb{R}^N)$ the Banach space of all bounded and uniformly continuous functions on \mathbb{R}^N with the usual supremum norm. For simplicity, we use the notation

$$\partial_t = \frac{\partial}{\partial t}, \quad \partial_j = \frac{\partial}{\partial x_j}, \quad \partial_x^\alpha = \partial_1^{\alpha_1} \cdots \partial_N^{\alpha_N}, \quad \nabla = (\partial_1, \dots, \partial_N),$$

where $\alpha = (\alpha_1, \dots, \alpha_N)$ with $\alpha_j \in \mathbb{N} \cup \{0\}$ ($1 \leq j \leq N$).

For the initial functions u_0, v_0 , we always assume that

$$u_0, v_0, \partial_j v_0 \in L^1(\mathbb{R}^N) \cap \mathcal{B}(\mathbb{R}^N) \quad (1 \leq j \leq N).$$

To give the definition of solutions to (P), we define $e^{t\Delta} f$ by

$$e^{t\Delta} f(x) = \int_{\mathbb{R}^N} G(x-y, t) f(y) dy,$$

where $G(x, t)$ is the heat kernel.

Definition 2.1. A function (u, v) on $\mathbb{R}^N \times [0, T]$ ($0 < T < \infty$) is said to be a solution of (P) on $\mathbb{R}^N \times [0, T]$ if u and v satisfy

$$u, v, \partial_j v \in C([0, T]; L^1(\mathbb{R}^N)) \cap C([0, T]; \mathcal{B}(\mathbb{R}^N)) \quad (1 \leq j \leq N),$$

and for every $0 < t \leq T$,

$$u(t) = e^{t\Delta} u_0 - \int_0^t \nabla \cdot e^{(t-s)\Delta} (u \nabla v)(s) ds, \quad (2.1)$$

$$v(t) = e^{-t} e^{t\Delta} v_0 + \int_0^t e^{-(t-s)} e^{(t-s)\Delta} u(s) ds. \quad (2.2)$$

Also, (u, v) is said to be a solution of (P) on $\mathbb{R}^N \times [0, \infty)$ if (u, v) is a solution of (P) on $\mathbb{R}^N \times [0, T]$ for every $0 < T < \infty$.

Using regularity arguments for parabolic equations in [9], we see that if (u, v) is a solution of (P) on $\mathbb{R}^N \times [0, T]$, then (u, v) is a classical solution of (P) on $\mathbb{R}^N \times [0, T]$ and satisfies

$$u, v \in C((0, T); W^{2,p}(\mathbb{R}^N)) \cap C^1((0, T); L^p(\mathbb{R}^N))$$

for every $1 < p < \infty$, and

$$\partial_j u, \Delta v \in C((0, T); L^\infty(\mathbb{R}^N)) \quad (1 \leq j \leq N).$$

In the case of $N \geq 2$, it was shown in [12] that if $\|u_0\|_1, \|\nabla v_0\|_1, \|\nabla v_0\|_\infty$ are small enough, but $\|u_0\|_\infty$ not necessarily small, then the solution of (P) exists globally in time and is bounded.

Throughout this paper, $M, E, V, V(t)$ and $d(t; p)$ ($1 \leq p \leq \infty$) are specified as

$$\begin{aligned} M &= \int_{\mathbb{R}^N} u_0 dy, & E &= \int_{\mathbb{R}^N} y u_0 dy, \\ V &= \int_0^\infty \int_{\mathbb{R}^N} u \nabla v dy ds, & V(t) &= \int_0^t \int_{\mathbb{R}^N} u \nabla v dy ds, \\ d(t; p) &= \begin{cases} \frac{t^{(1-1/p)/2+1/2}}{\log t} & \text{if } N = 1, \\ t^{N(1-1/p)/2+1/2} & \text{if } N \geq 2. \end{cases} \end{aligned}$$

We first mention that the result in [12] is valid for $N \geq 1, 1 \leq p \leq \infty$.

Theorem 2.1. Let $N \geq 1, 1 \leq p \leq \infty$, and let (u, v) be the solution of (P) on $\mathbb{R}^N \times [0, \infty)$. If (1.1) is satisfied, then u satisfies (1.2) and (1.3), and v has the same decay rate and asymptotic profile as u .

Next, we give an improvement of Theorem 2.1 under the condition

$$\int_{\mathbb{R}^N} |y| |u_0| dy < \infty. \quad (2.3)$$

Theorem 2.2. Let (u, v) be the solution of (P) on $\mathbb{R}^N \times [0, \infty)$ satisfying (1.1), and let $1 \leq p \leq \infty$. Under (2.3), the following holds:

- (i) $\sup_{t \geq 2} d(t; p) \|u(t) - MG(t)\|_p < \infty$.
- (ii) In the case $N = 1$, $|V(t)| \leq \text{Const} \log(1+t)$ ($t > 0$) and

$$\lim_{t \rightarrow \infty} d(t; p) \|u(t) - \{MG(t) - (E + V(t)) \partial_x G(t)\}\|_p = 0.$$
- (iii) In the case $N \geq 2$, the integral $\int_0^\infty \int_{\mathbb{R}^N} u \nabla v \, dy \, ds$ converges and

$$\lim_{t \rightarrow \infty} d(t; p) \|u(t) - \{MG(t) - (E + V) \cdot \nabla G(t)\}\|_p = 0.$$
- (iv) v also has the same decay rate and asymptotic profile as u .

In addition to (2.3), we consider the condition

$$\int_{\mathbb{R}^N} u_0 \, dy = 0. \quad (2.4)$$

Theorem 2.3. Let (u, v) be the solution of (P) on $\mathbb{R}^N \times [0, \infty)$ satisfying (1.1), and let $N \geq 1$, $1 \leq p \leq \infty$. Assume (2.3) and (2.4). Then the following holds:

- (i) $\sup_{t > 0} (1+t)^{N(1-1/p)/2+1/2} \|u(t)\|_p < \infty$, and $\int_0^\infty \int_{\mathbb{R}^N} u \nabla v \, dy \, ds$ converges and

$$\lim_{t \rightarrow \infty} t^{N(1-1/p)/2+1/2} \|u(t) + (E + V) \cdot \nabla G(t)\|_p = 0.$$
- (ii) v also has the same decay rate and asymptotic profile as u .

3. Preliminaries

We begin with mentioning $L^p - L^q$ estimates of $e^{t\Delta} f$, which are proved by Young's inequality for convolution.

Lemma 3.1. Let $1 \leq q \leq p \leq \infty$ and $\alpha \in (\mathbb{N} \cup \{0\})^N$. Then, for $f \in L^q(\mathbb{R}^N)$,

$$\|\partial_x^\alpha e^{t\Delta} f\|_p \leq C t^{-N(1/q-1/p)/2-|\alpha|/2} \|f\|_q, \quad (3.1)$$

where C is a positive constant depending on α, N, p, q .

Next lemma gives the large time behavior for $e^{t\Delta} f$ (see [2]).

Lemma 3.2. Let $N \geq 1$, $1 \leq p \leq \infty$ and $f \in L^1(\mathbb{R}^N)$. Then

$$\lim_{t \rightarrow \infty} t^{N(1-1/p)/2} \left\| e^{t\Delta} f - \int_{\mathbb{R}^N} f \, dy \, G(t) \right\|_p = 0.$$

Furthermore, under the condition $|x|f \in L^1(\mathbb{R}^N)$, it holds that

$$\sup_{t>0} t^{N(1-1/p)/2+1/2} \left\| e^{t\Delta} f - \int_{\mathbb{R}^N} f dy G(t) \right\|_p < \infty,$$

$$\lim_{t \rightarrow \infty} t^{N(1-1/p)/2+1/2} \left\| e^{t\Delta} f - \int_{\mathbb{R}^N} f dy G(t) + \int_{\mathbb{R}^N} y f dy \cdot \nabla G(t) \right\|_p = 0.$$

Lemma 3.3. Let $a > 0$ and $f \in L^1(\mathbb{R}^N)$. Then, for $s > 0$,

$$\int_{|y| \geq a} |e^{s\Delta} f(y)| dy \leq 2^{N/2} \|f\|_1 \exp\left(-\frac{a^2}{32s}\right) + \int_{|y| \geq a/2} |f(y)| dy. \quad (3.2)$$

Proof. By the definition of $e^{s\Delta} f$ and $\int_{\mathbb{R}^N} G(x, s) dx = 1$, we have

$$\begin{aligned} \int_{|y| \geq a} |e^{s\Delta} f(y)| dy &\leq \int_{|y| \geq a} \left\{ \int_{\mathbb{R}^N} G(y-z, s) |f(z)| dz \right\} dy \\ &= \int_{|z| \geq a/2} |f(z)| dz \int_{|y| \geq a} G(y-z, s) dy + \int_{|z| \leq a/2} |f(z)| dz \int_{|y| \geq a} G(y-z, s) dy \\ &\leq \int_{|z| \geq a/2} |f(z)| dz + (4\pi s)^{-N/2} \int_{|z| \leq a/2} |f(z)| dz \int_{|y| \geq a} \exp\left(-\frac{|y-z|^2}{4s}\right) dy \\ &= \int_{|z| \geq a/2} |f(z)| dz + L(s). \end{aligned}$$

Since $|y-z| \geq a/2$ for $|y| \geq a$, $|z| \leq a/2$, we get

$$\begin{aligned} L(s) &\leq \|f\|_1 (4\pi s)^{-N/2} \exp\left(-\frac{a^2}{32s}\right) \int_{\mathbb{R}^N} \exp\left(-\frac{|y-z|^2}{8s}\right) dy \\ &\leq 2^{N/2} \|f\|_1 \exp\left(-\frac{a^2}{32s}\right). \end{aligned}$$

Hence (3.2) is obtained. \square

Let $X = L^p(\mathbb{R}^N)$ for $1 \leq p < \infty$ or $X = \mathcal{B}(\mathbb{R}^N)$. The following lemma is needed to get decay estimates of ∇v .

Lemma 3.4. Define $v(t)$ by

$$v(t) = e^{-t} e^{t\Delta} v_0 + \int_0^t e^{-(t-s)} e^{(t-s)\Delta} u(s) ds \quad (t > 0), \quad v(0) = v_0.$$

Then the following holds:

- (i) If $v_0, \partial_j v_0 \in X$ and $u \in C([0, \infty); X)$, then $v, \partial_j v \in C([0, \infty); X)$.

(ii) Let $1 \leq q \leq p \leq \infty$, $1/q - 1/p < 1/N$. If $v_0, |\nabla v_0| \in L^p(\mathbb{R}^N)$ and $u \in C([0, \infty); L^q(\mathbb{R}^N))$, then

$$\|\nabla v(t)\|_p \leq e^{-t} \|\nabla v_0\|_p + C\Gamma(\beta) \sup_{0 < s < t} \|u(s)\|_q, \quad (3.3)$$

$$\|\nabla v(t)\|_p \leq e^{-t} \|\nabla v_0\|_p + Ct^{\beta-1} e^{-t/2} \sup_{0 < s < t/2} \|u(s)\|_q + C\Gamma(\beta) \sup_{t/2 < s < t} \|u(s)\|_q. \quad (3.4)$$

Furthermore, under the additional condition $|\nabla u| \in C((0, \infty); L^q(\mathbb{R}^N))$,

$$\|\nabla v(t)\|_p \leq e^{-t} \|\nabla v_0\|_p + Ct^{\beta-1} e^{-t/2} \sup_{0 < s < t/2} \|u(s)\|_q + C\Gamma(\gamma) \sup_{t/2 < s < t} \|\nabla u(s)\|_q. \quad (3.5)$$

Here C is a positive constant depending only on p and q , $\Gamma(z)$ is the gamma function and $\beta = 1/2 - N(1/q - 1/p)/2$, $\gamma = \beta + 1/2$.

See Lemma 2.2 of [12] for (i) and Lemma 2.3 of [12] for (3.3) and (3.4). The proof of (3.5) is the same as (3.4).

4. Some decay estimates of solutions to (P)

In this section, we consider the solution (u, v) of (P) satisfying (1.1), and give some L^p -estimates for the solution. Firstly we note that using (1.1) and (3.3) implies

$$\sup_{t>0} \|\nabla v(t)\|_p < \infty \quad \text{for every } 1 \leq p \leq \infty.$$

In the case of $N \geq 2$, the following decay estimate of u is obtained in [12], and that of ∇v by making use of (3.4).

Lemma 4.1. *Let $N \geq 2$, $1 < p \leq \infty$. Then*

$$\sup_{t>0} (1+t)^{N(1-1/p)/2} \|u(t)\|_p < \infty,$$

$$\sup_{t>0} (1+t)^{N(1-1/p)/2} \|\nabla v(t)\|_p < \infty.$$

In the case of $N = 1$, the decay estimates of $u, \nabla v$ are given as follows.

Lemma 4.2. *Let $N = 1$, $1 < p \leq \infty$ and $0 < \beta \leq 1/2$. Then*

$$\|u(t)\|_p \leq C_1 t^{-(1-1/p)/2+\beta} \quad (t \geq 1), \quad (4.1)$$

$$\|\partial_x v(t)\|_p \leq C_2 e^{-t/2} + C_1 t^{-(1-1/p)/2+\beta} \quad (t \geq 1), \quad (4.2)$$

where C_1 is a positive constant such that $C_1 \rightarrow \infty$ as $\beta \rightarrow +0$, and C_2 is a positive constant independent of β .

Proof. (4.1) is proved by using an argument similar to that in the proof of Lemma 3.2 of [12]. (4.2) follows from (3.4) and (4.1). \square

The following proposition is an improvement of Lemmas 4.1 and 4.2, and a key one to proving the theorems in Section 2.

Proposition 4.1. *Let $N \geq 1$, $1 < p \leq \infty$. Then the following holds:*

$$\sup_{t \geq 1} t^{N(1-1/p)/2+1/2} \|\nabla u(t)\|_p < \infty, \quad (4.3)$$

$$\sup_{t > 0} (1+t)^{N(1-1/p)/2+1/2} \|\nabla v(t)\|_p < \infty, \quad (4.4)$$

$$\sup_{t > 0} (1+t)^{N(1-1/p)/2} \|u(t)\|_p < \infty. \quad (4.5)$$

Proof. Since (4.5) for $N \geq 2$ follows from Lemma 4.1, we show it for $N = 1$, and (4.3), (4.4) for $N \geq 1$. The proof is divided into two steps.

Step 1. Fix $t \geq 4$ and $\varepsilon \in (0, 1/2)$. Take β such that

$$0 < \beta \leq \frac{1}{2} \quad \text{if } N = 1, \quad \beta = 0 \quad \text{if } N \geq 2.$$

With β , Lemmas 4.1 and 4.2 read as follows: For $N \geq 1$, $1 < p \leq \infty$,

$$\sup_{t > 0} t^{N(1-1/p)/2-\beta} (\|u(t)\|_p + \|\nabla v(t)\|_p) < \infty. \quad (4.6)$$

To estimate $\|\nabla u(t)\|_p$, we use (2.1) to get

$$\begin{aligned} \nabla u(t) &= \nabla e^{t\Delta} u_0 - \int_0^{(1-\varepsilon)t} \nabla \nabla \cdot e^{(t-s)\Delta} (u \nabla v)(s) ds - \int_{(1-\varepsilon)t}^t \nabla e^{(t-s)\Delta} (\nabla u \cdot \nabla v)(s) ds \\ &\quad - \int_{(1-\varepsilon)t}^t \nabla e^{(t-s)\Delta} (u \Delta v)(s) ds \\ &= \nabla e^{t\Delta} u_0 - I_1^\varepsilon(t) - I_2^\varepsilon(t) - I_3^\varepsilon(t). \end{aligned}$$

Using (3.1), (4.6) and $\sup_{t > 0} \|\nabla v(t)\|_1 < \infty$, we have

$$\begin{aligned} \|I_1^\varepsilon(t)\|_p &\leq C \int_0^{(1-\varepsilon)t} (t-s)^{-N(1-1/p)/2-1} \|u(s)\|_\infty \|\nabla v(s)\|_1 ds \\ &\leq C \int_0^{(1-\varepsilon)t} (t-s)^{-N(1-1/p)/2-1} (1+s)^{-N/2+\beta} ds \\ &\leq C \varepsilon^{-N(1-1/p)/2-1} t^{-N(1-1/p)/2-1} \int_0^{(1-\varepsilon)t} (1+s)^{-N/2+\beta} ds \end{aligned}$$

$$\leq C\varepsilon^{-N(1-1/p)/2-1}t^{-N(1-1/p)/2-1} \times \begin{cases} t^{1/2+\beta} & \text{if } N = 1, \\ \log t & \text{if } N = 2, \\ 1 & \text{if } N \geq 3, \end{cases}$$

$$\|I_2^\varepsilon(t)\|_p \leq C \int_{(1-\varepsilon)t}^t (t-s)^{-1/2} \|\nabla u(s)\|_p \|\nabla v(s)\|_\infty ds$$

$$\leq C \int_{(1-\varepsilon)t}^t (t-s)^{-1/2} s^{-N/2+\beta} \|\nabla u(s)\|_p ds,$$

$$\|I_3^\varepsilon(t)\|_p \leq C \int_{(1-\varepsilon)t}^t (t-s)^{-1/2} s^{-N/2+\beta} \|\Delta v(s)\|_p ds.$$

Summing up these yields that

$$\begin{aligned} \|\nabla u(t)\|_p &\leq \|\nabla e^{t\Delta} u_0\|_p + C\varepsilon^{-N(1-1/p)/2-1}t^{-N(1-1/p)/2-1/2+\beta} \\ &\quad + C \int_{(1-\varepsilon)t}^t (t-s)^{-1/2} s^{-N/2+\beta} \|\nabla u(s)\|_p ds \\ &\quad + C \int_{(1-\varepsilon)t}^t (t-s)^{-1/2} s^{-N/2+\beta} \|\Delta v(s)\|_p ds. \end{aligned} \quad (4.7)$$

Define $F(t)$ by

$$F(t) = \sup_{1 \leq s \leq t} s^{N(1-1/p)/2+1/2} \|\nabla u(s)\|_p.$$

Then the third term on the right-hand side of (4.7) is estimated as

$$\begin{aligned} &\int_{(1-\varepsilon)t}^t (t-s)^{-1/2} s^{-N/2+\beta} \|\nabla u(s)\|_p ds \\ &\leq F(t) \int_{(1-\varepsilon)t}^t (t-s)^{-1/2} s^{-N/2+\beta-N(1-1/p)/2-1/2} ds \\ &\leq C\varepsilon^{1/2} t^{-N(1-1/p)/2-N/2+\beta} F(t). \end{aligned} \quad (4.8)$$

We next estimate $\|\Delta v(s)\|_p$ on the right-hand side of (4.7). Fix $s \geq 2$. We use (2.2) to get

$$\begin{aligned} \Delta v(s) &= e^{-s} \nabla \cdot e^{s\Delta} \nabla v_0 + \int_0^{(1-\varepsilon)s} e^{-(s-\xi)} \Delta e^{(s-\xi)\Delta} u(\xi) d\xi \\ &\quad + \int_{(1-\varepsilon)s}^s e^{-(s-\xi)} \nabla \cdot e^{(s-\xi)\Delta} \nabla u(\xi) d\xi \\ &= e^{-s} \nabla \cdot e^{s\Delta} \nabla v_0 + J_1^\varepsilon(s) + J_2^\varepsilon(s). \end{aligned}$$

Using (3.1) gives

$$\begin{aligned}\|J_1^\varepsilon(s)\|_p &\leq C \int_0^{(1-\varepsilon)s} e^{-(s-\xi)} (s-\xi)^{-N(1-1/p)/2-1} \|u(\xi)\|_1 d\xi \\ &\leq C e^{-\varepsilon s} \varepsilon^{-N(1-1/p)/2-1} s^{-N(1-1/p)/2} \\ &\leq C \varepsilon^{-N(1-1/p)/2-3/2} s^{-N(1-1/p)/2-1/2}.\end{aligned}$$

By calculations similar to those of (4.8), we have

$$\begin{aligned}\|J_2^\varepsilon(s)\|_p &\leq C \int_{(1-\varepsilon)s}^s e^{-(s-\xi)} (s-\xi)^{-1/2} \|\nabla u(\xi)\|_p d\xi \\ &\leq F(s) \int_{(1-\varepsilon)s}^s e^{-(s-\xi)} (s-\xi)^{-1/2} \xi^{-N(1-1/p)/2-1/2} d\xi \\ &\leq F(s) \{(1-\varepsilon)s\}^{-N(1-1/p)/2-1/2} \int_{(1-\varepsilon)s}^s e^{-(s-\xi)} (s-\xi)^{-1/2} d\xi \\ &\leq C s^{-N(1-1/p)/2-1/2} F(s).\end{aligned}$$

Hence, for $s \geq 2$ we get

$$\begin{aligned}\|\Delta v(s)\|_p &\leq C e^{-s} \|\nabla v_0\|_p + C \varepsilon^{-N(1-1/p)/2-3/2} s^{-N(1-1/p)/2-1/2} \\ &\quad + C s^{-N(1-1/p)/2-1/2} F(s).\end{aligned}\tag{4.9}$$

Substituting (4.8) and (4.9) into the right-hand side of (4.7) yields that

$$t^{N(1-1/p)/2+1/2} \|\nabla u(t)\|_p \leq C + C \varepsilon^{-N(1-1/p)/2-1} t^\beta + C \varepsilon^{1/2} t^\beta + C \varepsilon^{1/2} t^\beta F(t),$$

where a positive constant C is taken as $C > 1$. Take ε such that

$$\varepsilon = \frac{1}{4C^2} t^{-2\beta}.$$

By $C > 1$, we see that

$$0 < \varepsilon = \frac{1}{4C^2} t^{-2\beta} = \frac{1}{2} \cdot \frac{1}{2C^2} < \frac{1}{2},$$

and hence

$$t^{N(1-1/p)/2+1/2} \|\nabla u(t)\|_p \leq C + C t^{(N+2-N/p)\beta} + \frac{1}{2} F(t),$$

from which it follows that

$$F(t) \leq C + C t^{(N+2-N/p)\beta}.$$

Hence, for $t \geq 4$ we get

$$\|\nabla u(t)\|_p \leq C t^{-N(1-1/p)/2-1/2} + C t^{-N(1-1/p)/2-1/2+(N+2-N/p)\beta}.\tag{4.10}$$

For $N \geq 2$, (4.3) follows from (4.10) because of $\beta = 0$, and then (4.4) by (3.5).

For $N = 1$, take β again as follows:

$$0 < \beta \leq \frac{p}{2(3p-1)} \quad \text{if } 1 < p < \infty, \quad 0 < \beta \leq \frac{1}{6} \quad \text{if } p = \infty.$$

Then $0 < (3 - 1/p)\beta \leq 1/2$. Replacing $(3 - 1/p)\beta$ with β in (4.10), for $0 < \beta \leq 1/2$ we get

$$\|\partial_x u(t)\|_p \leq C t^{-(1-1/p)/2-1/2+\beta} \quad (t \geq 4),$$

which together with (3.5) gives

$$\|\partial_x v(t)\|_p \leq C(1+t)^{-(1-1/p)/2-1/2+\beta} \quad (t > 0). \quad (4.11)$$

Step 2. Let $N = 1$. We firstly prove (4.5). Let $1 < p \leq \infty$ and fix $t \geq 2$. We put

$$\begin{aligned} I(t) &= \int_0^t \partial_x e^{(t-s)\Delta} (u \partial_x v)(s) ds \\ &= \int_0^{t/2} \partial_x e^{(t-s)\Delta} (u \partial_x v)(s) ds + \int_{t/2}^t \partial_x e^{(t-s)\Delta} (u \partial_x v)(s) ds \\ &= I_1(t) + I_2(t). \end{aligned}$$

By using (3.1), (4.1) and (4.11), $I_1(t)$ is estimated as follows:

$$\begin{aligned} \|I_1(t)\|_p &\leq C \int_0^{t/2} (t-s)^{-(1-1/p)/2-1/2} \|u(s)\|_2 \|\partial_x v(s)\|_2 ds \\ &\leq C t^{-(1-1/p)/2-1/2} \int_0^{t/2} (1+s)^{-1+2\beta} ds \\ &\leq C t^{-(1-1/p)/2-1/2+2\beta}. \end{aligned}$$

Taking r such that $1 < r < p$, $1/2 - (1/r - 1/p)/2 > 0$, we have

$$\begin{aligned} \|I_2(t)\|_p &\leq C \int_{t/2}^t (t-s)^{-(1/r-1/p)/2-1/2} \|u(s)\|_\infty \|\partial_x v(s)\|_r ds \\ &\leq C \int_{t/2}^t (t-s)^{-(1/r-1/p)/2-1/2} s^{-(1-1/r)/2-1+2\beta} ds \\ &\leq C t^{-(1-1/p)/2-1/2+2\beta}. \end{aligned}$$

Hence

$$\|I(t)\|_p \leq C t^{-(1-1/p)/2-1/2+2\beta},$$

from which it follows that

$$\|u(t)\|_p \leq \|e^{t\Delta} u_0\|_p + \|I(t)\|_p \leq C t^{-(1-1/p)/2} + C t^{-(1-1/p)/2-1/2+2\beta}.$$

Taking β in the inequality above such as $0 < \beta \leq 1/4$, we obtain

$$\|u(t)\|_p \leq C t^{-(1-1/p)/2} \quad (t \geq 2).$$

Thus, we have (4.5) for $N = 1$.

We next show (4.3) and (4.4). It follows from (3.4) and (4.5) that

$$\|\partial_x v(t)\|_p \leq C(1+t)^{-(1-1/p)/2} \quad (t > 0)$$

for $1 < p \leq \infty$. Repeating arguments in Step 1 as $\beta = 0$, we have (4.3) and then (4.4) by (3.5). Thus, the proof of Proposition 4.1 is complete. \square

5. Proof of Theorem 2.1

In this section, we give the proof of Theorem 2.1. We write (2.1) as

$$u(t) = e^{t\Delta} u_0 - I(t), \quad I(t) = \int_0^t \nabla \cdot e^{(t-s)\Delta} (u \nabla v)(s) ds. \quad (5.1)$$

We begin with the following proposition.

Proposition 5.1. *Let $1 \leq p \leq \infty$. Then*

$$\sup_{t \geq 2} d(t; p) \|I(t)\|_p < \infty. \quad (5.2)$$

Proof. Fix $t \geq 2$. We divide $I(t)$ into two parts.

$$\begin{aligned} I(t) &= \int_0^{t/2} \nabla \cdot e^{(t-s)\Delta} (u \nabla v)(s) ds + \int_{t/2}^t \nabla \cdot e^{(t-s)\Delta} (u \nabla v)(s) ds \\ &= I_1(t) + I_2(t). \end{aligned}$$

For $1 \leq p \leq \infty$, using (3.1) and Proposition 4.1 gives

$$\begin{aligned} \|I_1(t)\|_p &\leq C \int_0^{t/2} (t-s)^{-N(1-1/p)/2-1/2} \|u(s)\|_2 \|\nabla v(s)\|_2 ds \\ &\leq C t^{-N(1-1/p)/2-1/2} \int_0^{t/2} (1+s)^{-N/2-1/2} ds \\ &\leq C t^{-N(1-1/p)/2-1/2} \times \begin{cases} \log t & \text{if } N = 1, \\ 1 & \text{if } N \geq 2. \end{cases} \end{aligned}$$

For $1 < p \leq \infty$, take r such that $1 < r < p \leq \infty$, $1/2 - N(1/r - 1/p)/2 > 0$. Then

$$\begin{aligned}
\|I_2(t)\|_p &\leq C \int_{t/2}^t (t-s)^{-N(1/r-1/p)/2-1/2} \|u(s)\|_\infty \|\nabla v(s)\|_r ds \\
&\leq C \int_{t/2}^t (t-s)^{-N(1/r-1/p)/2-1/2} s^{-N/2-N(1-1/r)/2-1/2} ds \\
&\leq Ct^{-N(1-1/p)/2-N/2}.
\end{aligned}$$

For $p = 1$, we have

$$\begin{aligned}
\|I_2(t)\|_1 &\leq C \int_{t/2}^t (t-s)^{-1/2} \|u(s)\|_2 \|\nabla v(s)\|_2 ds \\
&\leq C \int_{t/2}^t (t-s)^{-1/2} s^{-N/2-1/2} ds \leq Ct^{-N/2}.
\end{aligned}$$

Hence, putting together these estimates implies (5.2). Thus the proof is complete. \square

Proof of Theorem 2.1. Let $1 \leq p \leq \infty$. (1.2) follows from (1.1) and (4.5). To obtain the decay rate for $\|v(t)\|_p$, we use (2.2) to get

$$\begin{aligned}
v(t) &= e^{-t} e^{t\Delta} v_0 + \int_0^{t/2} e^{-(t-s)} e^{(t-s)\Delta} u(s) ds + \int_{t/2}^t e^{-(t-s)} e^{(t-s)\Delta} u(s) ds \\
&= e^{-t} e^{t\Delta} v_0 + K_1(t) + K_2(t).
\end{aligned} \tag{5.3}$$

Using $\sup_{t>0} \|u(t)\|_p < \infty$ and (1.2) yields that

$$\begin{aligned}
\|K_1(t)\|_p &\leq C \int_0^{t/2} e^{-(t-s)} \|u(s)\|_p ds \leq C \int_0^{t/2} e^{-(t-s)} ds \leq Ce^{-t/2}, \\
\|K_2(t)\|_p &\leq C \int_{t/2}^t e^{-(t-s)} \|u(s)\|_p ds \leq C \int_{t/2}^t e^{-(t-s)} (1+s)^{-N(1-1/p)/2} ds \\
&\leq C(1+t)^{-N(1-1/p)/2} \int_{t/2}^t e^{-(t-s)} ds \leq C(1+t)^{-N(1-1/p)/2}.
\end{aligned}$$

Hence we obtain

$$\sup_{t>0} (1+t)^{N(1-1/p)/2} \|v(t)\|_p < \infty.$$

Next, we show the asymptotic profile of u . By Lemma 3.2,

$$\lim_{t \rightarrow \infty} t^{N(1-1/p)/2} \|e^{t\Delta} u_0 - MG(t)\|_p = 0,$$

and by (5.2),

$$\lim_{t \rightarrow \infty} t^{N(1-1/p)/2} \|I(t)\|_p = 0.$$

Since $u(t) - MG(t) = \{e^{t\Delta}u_0 - MG(t)\} - I(t)$, we have

$$\lim_{t \rightarrow \infty} t^{N(1-1/p)/2} \|u(t) - MG(t)\|_p = 0.$$

We prove the asymptotic profile of v . As shown in the proof of Theorem 1.1 of [12], $v(t) - MG(t)$ is represented as

$$v(t) - MG(t) = \{e^{t\Delta}u_0 - MG(t)\} + e^{-t}e^{t\Delta}(v_0 - u_0) - J(t), \quad (5.4)$$

where

$$J(t) = \int_0^t e^{-(t-s)} e^{(t-s)\Delta} I(s) ds.$$

We claim

$$\sup_{t \geq 2} d(t; p) \|J(t)\|_p < \infty. \quad (5.5)$$

In order to show (5.5), for $t \geq 2$ we divide $J(t)$ into two parts.

$$\begin{aligned} J(t) &= \int_0^{t/2} e^{-(t-s)} e^{(t-s)\Delta} I(s) ds + \int_{t/2}^t e^{-(t-s)} e^{(t-s)\Delta} I(s) ds \\ &= J_1(t) + J_2(t). \end{aligned}$$

Since (5.2) implies that for $s > 0$,

$$\|I(s)\|_p \leq C(1+s)^{-N(1-1/p)/2-1/2} \begin{cases} \log(2+s) & \text{if } N = 1, \\ 1 & \text{if } N \geq 2, \end{cases}$$

in the case of $N = 1$ we have

$$\begin{aligned} \|J_1(t)\|_p &\leq C \int_0^{t/2} e^{-(t-s)} (t-s)^{-(1-1/p)/2} \|I(s)\|_1 ds \\ &\leq Ce^{-t/4} \int_0^{t/2} e^{-(t-s)/2} (t-s)^{-(1-1/p)/2} (1+s)^{-1/2} \log(2+s) ds \\ &\leq Ce^{-t/4}, \end{aligned}$$

$$\begin{aligned} \|J_2(t)\|_p &\leq C \int_{t/2}^t e^{-(t-s)} \|I(s)\|_p ds \\ &\leq C \int_{t/2}^t e^{-(t-s)} (1+s)^{-(1-1/p)/2-1/2} \log(2+s) ds \\ &\leq C(1+t)^{-(1-1/p)/2-1/2} \log(2+t). \end{aligned}$$

In the case of $N \geq 2$, we next have

$$\begin{aligned}\|J_1(t)\|_p &\leq C \int_0^{t/2} e^{-(t-s)} (t-s)^{-N(1-1/p)/2} \|I(s)\|_1 ds \leq C e^{-t/4}, \\ \|J_2(t)\|_p &\leq C \int_{t/2}^t e^{-(t-s)} \|I(s)\|_p ds \leq C \int_{t/2}^t e^{-(t-s)} s^{-N(1-1/p)/2-1/2} ds \\ &\leq C t^{-N(1-1/p)/2-1/2}.\end{aligned}$$

Therefore, for $t \geq 2$ we get

$$\|J(t)\|_p \leq C t^{-N(1-1/p)/2-1/2} \times \begin{cases} \log t & \text{if } N = 1, \\ 1 & \text{if } N \geq 2, \end{cases}$$

which implies (5.5).

By (5.5), we easily see that $\lim_{t \rightarrow \infty} t^{N(1-1/p)/2} \|J(t)\|_p = 0$. Hence it follows from (5.4) that

$$\lim_{t \rightarrow \infty} t^{N(1-1/p)/2} \|v(t) - MG(t)\|_p = 0.$$

Thus, the proof of Theorem 2.1 is complete. \square

6. Proof of Theorem 2.2

To prove Theorem 2.2, we firstly show the following proposition.

Proposition 6.1. *Let $N \geq 1$, $1 \leq p \leq \infty$. Then*

$$\lim_{t \rightarrow \infty} d(t; p) \|I(t) - V(t) \cdot \nabla G(t)\|_p = 0. \quad (6.1)$$

Proof. Take an arbitrary number $\varepsilon \in (0, 1)$ and fix it. We divide $I(t) - V(t) \cdot \nabla G(t)$ as follows:

$$\begin{aligned}I(t) - V(t) \cdot \nabla G(t) &= \int_0^t \int_{\mathbb{R}^N} \nabla G(x-y, t-s) \cdot u(y, s) \nabla v(y, s) dy ds \\ &\quad - \int_0^t \int_{\mathbb{R}^N} \nabla G(x, t) \cdot u(y, s) \nabla v(y, s) dy ds \\ &= K_1^\varepsilon(t) + K_2^\varepsilon(t) + K_3^\varepsilon(t) + K_4^\varepsilon(t) + K_5^\varepsilon(t),\end{aligned}$$

where

$$\begin{aligned}K_1^\varepsilon(t) &= \int_{\varepsilon t/2}^t \int_{\mathbb{R}^N} \nabla G(x-y, t-s) \cdot u(y, s) \nabla v(y, s) dy ds, \\ K_2^\varepsilon(t) &= - \int_{\varepsilon t/2}^t \int_{\mathbb{R}^N} \nabla G(x, t) \cdot u(y, s) \nabla v(y, s) dy ds,\end{aligned}$$

$$K_3^\varepsilon(t) = \int_0^{\varepsilon t/2} \int_{|y| \geq \varepsilon \sqrt{t}} \nabla G(x-y, t-s) \cdot u(y, s) \nabla v(y, s) dy ds,$$

$$K_4^\varepsilon(t) = - \int_0^{\varepsilon t/2} \int_{|y| \geq \varepsilon \sqrt{t}} \nabla G(x, t) \cdot u(y, s) \nabla v(y, s) dy ds,$$

$$K_5^\varepsilon(t) = \int_0^{\varepsilon t/2} \int_{|y| \leq \varepsilon \sqrt{t}} \{ \nabla G(x-y, t-s) - \nabla G(x, t) \} \cdot u(y, s) \nabla v(y, s) dy ds.$$

By (1.2) and (4.4), $\|K_1^\varepsilon(t)\|_p$ is estimated as follows:

$$\begin{aligned} \|K_1^\varepsilon(t)\|_p &\leq C \int_{\varepsilon t/2}^t (t-s)^{-1/2} \|u(s)\|_p \|\nabla v(s)\|_\infty ds \\ &\leq C \int_{\varepsilon t/2}^t (t-s)^{-1/2} s^{-N(1-1/p)/2-N/2-1/2} ds \\ &\leq C \varepsilon^{-N(1-1/p)/2-N/2-1/2} t^{-N(1-1/p)/2-N/2}. \end{aligned}$$

Next,

$$\begin{aligned} \|K_2^\varepsilon(t)\|_p &\leq \int_{\varepsilon t/2}^t \int_{\mathbb{R}^N} |u(y, s)| |\nabla v(y, s)| dy ds \|\nabla G(t)\|_p \\ &\leq C t^{-N(1-1/p)/2-1/2} \int_{\varepsilon t/2}^t \|u(s)\|_2 \|\nabla v(s)\|_2 ds \\ &\leq C t^{-N(1-1/p)/2-1/2} \int_{\varepsilon t/2}^t s^{-N/2-1/2} ds \\ &\leq C \varepsilon^{-N/2-1/2} t^{-N(1-1/p)/2-N/2}, \\ \|K_4^\varepsilon(t)\|_p &\leq C t^{-N(1-1/p)/2-1/2} \int_0^{\varepsilon t/2} \int_{|y| \geq \varepsilon \sqrt{t}} |u(y, s)| |\nabla v(y, s)| dy ds. \end{aligned}$$

Using Minkowski's inequality, we get

$$\|K_3^\varepsilon(t)\|_p \leq \int_0^{\varepsilon t/2} \int_{|y| \geq \varepsilon \sqrt{t}} \|\nabla G(\cdot - y, t-s)\|_p |u(y, s)| |\nabla v(y, s)| dy ds$$

$$\leq C t^{-N(1-1/p)/2-1/2} \int_0^{\varepsilon t/2} \int_{|y| \geq \varepsilon \sqrt{t}} |u(y, s)| |\nabla v(y, s)| dy ds.$$

Similarly,

$$\begin{aligned} \|K_5^\varepsilon(t)\|_p &\leq \int_0^{\varepsilon t/2} \int_{|y| \leq \varepsilon \sqrt{t}} A(y, t, s) |u(y, s)| |\nabla v(y, s)| dy ds \\ &\leq \left\{ \sup_{0 \leq s \leq \varepsilon t/2, |y| \leq \varepsilon \sqrt{t}} A(y, t, s) \right\} B(t), \end{aligned}$$

where

$$\begin{aligned} A(y, t, s) &= \|\nabla G(\cdot - y, t - s) - \nabla G(\cdot, t)\|_p, \\ B(t) &= \int_0^{\varepsilon t/2} \int_{|y| \leq \varepsilon \sqrt{t}} |u(y, s)| |\nabla v(y, s)| dy ds. \end{aligned}$$

Calculating directly gives

$$A(y, t, s) \leq C t^{-N(1-1/p)/2-1} |y| + C(t-s)^{-N(1-1/p)/2-3/2} |s|.$$

Then

$$\sup_{0 \leq s \leq \varepsilon t/2, |y| \leq \varepsilon \sqrt{t}} A(y, t, s) \leq C \varepsilon t^{-N(1-1/p)/2-1/2}.$$

Making use of (4.4) and (4.5) yields that for $t \geq 2$,

$$\begin{aligned} B(t) &\leq \int_0^t \|u(s)\|_2 \|\nabla v(s)\|_2 ds \leq C \int_0^t (1+s)^{-N/2-1/2} ds \\ &\leq C \times \begin{cases} \log t & \text{if } N = 1, \\ 1 & \text{if } N \geq 2. \end{cases} \end{aligned}$$

Therefore, for $t \geq 2$ we have

$$\|K_5^\varepsilon(t)\|_p \leq \frac{C\varepsilon}{d(t; p)}.$$

Summing up these estimates on $\|K_k^\varepsilon(t)\|_p$ ($1 \leq k \leq 5$) yields that

$$\begin{aligned} &\|I(t) - V(t) \cdot \nabla G(t)\|_p \\ &\leq C \varepsilon^{-N(1-1/p)/2-N/2-1/2} (1 + \varepsilon^{N(1-1/p)/2}) t^{-N(1-1/p)/2-N/2} \\ &\quad + C t^{-N(1-1/p)/2-1/2} \int_0^{\varepsilon t/2} \int_{|y| \geq \varepsilon \sqrt{t}} |u(y, s)| |\nabla v(y, s)| dy ds + \frac{C\varepsilon}{d(t; p)} \end{aligned}$$

for $t \geq 2$. Hence

$$d(t; p) \|I(t) - V(t) \cdot \nabla G(t)\|_p \leq C_\varepsilon C(t) + CD(t) + C\varepsilon \quad (t \geq 2), \quad (6.2)$$

where

$$C_\varepsilon = C\varepsilon^{-N(1-1/p)/2 - N/2 - 1/2} (1 + \varepsilon^{N(1-1/p)/2}),$$

$$C(t) = \begin{cases} \frac{1}{\log t} & \text{if } N = 1, \\ t^{-N/2 + 1/2} & \text{if } N \geq 2, \end{cases}$$

$$D(t) = \int_0^{\varepsilon t/2} \int_{|y| \geq \varepsilon \sqrt{t}} |u(y, s)| |\nabla v(y, s)| dy ds \times \begin{cases} \frac{1}{\log t} & \text{if } N = 1, \\ 1 & \text{if } N \geq 2. \end{cases}$$

We show

$$\lim_{t \rightarrow \infty} D(t) = 0.$$

Consider the case $N = 1$. By (4.4), we have

$$\begin{aligned} D(t) &\leq \frac{1}{\log t} \int_0^{\varepsilon t/2} \|\partial_y v(s)\|_\infty ds \int_{|y| \geq \varepsilon \sqrt{t}} |u(y, s)| dy \\ &\leq \frac{C}{\log t} \int_0^{\varepsilon t/2} (1+s)^{-1} ds \int_{|y| \geq \varepsilon \sqrt{t}} |u(y, s)| dy. \end{aligned}$$

It follows from (5.1) that

$$\int_{|y| \geq \varepsilon \sqrt{t}} |u(y, s)| dy \leq \int_{|y| \geq \varepsilon \sqrt{t}} |e^{s\Delta} u_0(y)| dy + \int_{|y| \geq \varepsilon \sqrt{t}} |I(y, s)| dy.$$

Applying Lemma 3.3 as $a = \varepsilon \sqrt{t}$ gives

$$\int_{|y| \geq \varepsilon \sqrt{t}} |e^{s\Delta} u_0(y)| dy \leq C \exp\left(-\frac{\varepsilon^2 t}{32s}\right) + \int_{|y| \geq \varepsilon \sqrt{t}/2} |u_0(y)| dy.$$

Using (5.2) yields that

$$\int_{|y| \geq \varepsilon \sqrt{t}} |I(y, s)| dy \leq \|I(s)\|_1 \leq C(1+s)^{-1/2} \log(2+s).$$

Hence

$$\begin{aligned} D(t) &\leq \frac{C}{\log t} \int_0^{\varepsilon t/2} (1+s)^{-1} \exp\left(-\frac{\varepsilon^2 t}{32s}\right) ds + \frac{C}{\log t} \int_{|y| \geq \varepsilon \sqrt{t}/2} |u_0(y)| dy \int_0^{\varepsilon t/2} (1+s)^{-1} ds \\ &\quad + \frac{C}{\log t} \int_0^{\varepsilon t/2} (1+s)^{-3/2} \log(2+s) ds \end{aligned}$$

$$\leq \frac{C}{\log t} \int_{\varepsilon/16}^{\infty} \frac{e^{-\xi}}{\xi^2} d\xi + C \int_{|y| \geq \varepsilon \sqrt{t}/2} |u_0(y)| dy + \frac{C}{\log t},$$

which implies $\lim_{t \rightarrow \infty} D(t) = 0$.

Next, consider the case $N \geq 2$. Using (4.4) and (4.5) yields that

$$\int_0^\infty \int_{\mathbb{R}^N} |u(y, s)| |\nabla v(y, s)| dy ds \leq \int_0^\infty \|u(s)\|_2 \|\nabla v(s)\|_2 ds \leq C \int_0^\infty (1+s)^{-N/2-1/2} ds < \infty.$$

Hence, $\lim_{t \rightarrow \infty} D(t) = 0$.

By $C(t) \rightarrow 0$, $D(t) \rightarrow 0$ ($t \rightarrow \infty$), it follows from (6.2) that

$$\limsup_{t \rightarrow \infty} d(t; p) \|I(t) - V(t) \cdot \nabla G(t)\|_p \leq C\varepsilon.$$

Since ε is arbitrarily small, we obtain (6.1). Therefore, the proof of Proposition 6.1 is complete. \square

Proof of Theorem 2.2. Let $1 \leq p \leq \infty$. By Lemma 3.2,

$$\sup_{t \geq 2} t^{N(1-1/p)/2+1/2} \|e^{t\Delta} u_0 - MG(t)\|_p < \infty, \quad (6.3)$$

which together with (5.1) and (5.2) yields that

$$\sup_{t \geq 2} d(t; p) \|u(t) - MG(t)\|_p < \infty.$$

The representation (5.4) of $v(t) - MG(t)$ together with (5.5) and (6.3) implies that

$$\sup_{t \geq 2} d(t; p) \|v(t) - MG(t)\|_p < \infty.$$

Next, we show the asymptotic profile of u . Note that

$$\begin{aligned} u(t) - \{MG(t) - (E + V(t)) \cdot \nabla G(t)\} \\ = \{e^{t\Delta} u_0 - MG(t) + E \cdot \nabla G(t)\} - \{I(t) - V(t) \cdot \nabla G(t)\}. \end{aligned}$$

By Lemma 3.2,

$$\lim_{t \rightarrow \infty} t^{N(1-1/p)/2+1/2} \|e^{t\Delta} u_0 - MG(t) + E \cdot \nabla G(t)\|_p = 0.$$

Hence it follows from Proposition 6.1 that

$$\lim_{t \rightarrow \infty} d(t; p) \|u(t) - \{MG(t) - (E + V(t)) \cdot \nabla G(t)\}\|_p = 0. \quad (6.4)$$

Using (4.4) and (4.5), we see that for the case $N = 1$,

$$|V(t)| \leq C \log(1+t) \quad (t \geq 1),$$

and that for the case $N \geq 2$, the integral $V = \int_0^\infty \int_{\mathbb{R}^N} u \nabla v dy ds$ converges. Therefore, noting $\|\nabla G(t)\|_p \leq C t^{-N(1-1/p)/2-1/2}$, we can replace $V(t)$ in (6.4) with V for the case $N \geq 2$ and obtain the assertion (iii) of Theorem 2.2.

To prove the asymptotic profile of v , we claim the following representation of v :

$$\begin{aligned} v(t) &= \{MG(t) - (E + V(t)) \cdot \nabla G(t)\} \\ &= e^{-t} \{e^{t\Delta} v_0 - MG(t) + E \cdot \nabla G(t)\} + P(t) \cdot \nabla G(t) + Q(t), \end{aligned} \quad (6.5)$$

where

$$\begin{aligned} P(t) &= \int_0^t \int_{\mathbb{R}^N} e^{-(t-s)} u \nabla v \, dy \, ds, \\ Q(t) &= \int_0^t e^{-(t-s)} e^{(t-s)\Delta} \{u(s) - MG(s) + (E + V(s)) \cdot \nabla G(s)\} \, ds. \end{aligned}$$

In fact, the integral expression (2.2) of v yields that

$$\begin{aligned} v(t) &= e^{-t} e^{t\Delta} v_0 + M \int_0^t e^{-(t-s)} e^{(t-s)\Delta} G(s) \, ds \\ &\quad - \int_0^t e^{-(t-s)} e^{(t-s)\Delta} (E + V(s)) \cdot \nabla G(s) \, ds + Q(t). \end{aligned} \quad (6.6)$$

Using

$$\begin{aligned} \int_{\mathbb{R}^N} G(x-y, t-s) G(y, s) \, dy &= G(x, t), \\ \int_{\mathbb{R}^N} G(x-y, t-s) \nabla G(y, s) \, dy &= \nabla G(x, t), \end{aligned}$$

we have

$$M \int_0^t e^{-(t-s)} e^{(t-s)\Delta} G(s) \, ds = M(1 - e^{-t}) G(t), \quad (6.7)$$

$$\begin{aligned} \int_0^t e^{-(t-s)} e^{(t-s)\Delta} (E + V(s)) \cdot \nabla G(s) \, ds \\ = (1 - e^{-t}) E \cdot \nabla G(t) + V(t) \cdot \nabla G(t) - P(t) \cdot \nabla G(t). \end{aligned} \quad (6.8)$$

Substituting (6.7) and (6.8) into (6.6) gives (6.5).

We show

$$\lim_{t \rightarrow \infty} t^{N(1-1/p)/2+1/2} \|P(t) \cdot \nabla G(t)\|_p = 0. \quad (6.9)$$

To show this, we estimate $P(t)$ as

$$|P(t)| \leq \int_0^t e^{-(t-s)} \|u(s)\|_2 \|\nabla v(s)\|_2 \, ds,$$

and then, by (4.4) and (4.5),

$$\|u(s)\|_2 \|\nabla v(s)\|_2 \leq C(1+s)^{-N/2-1/2} \rightarrow 0 \quad (s \rightarrow \infty).$$

Hence, $P(t) \rightarrow 0$ ($t \rightarrow \infty$). Therefore, by $t^{N(1-1/p)/2+1/2} \|\nabla G(t)\|_p \leq C$, we have (6.9).

Next, we show

$$\lim_{t \rightarrow \infty} d(t; p) \|Q(t)\|_p = 0. \quad (6.10)$$

To prove this, we divide $Q(t)$ into two parts.

$$\begin{aligned} Q(t) &= \int_0^{t/2} e^{-(t-s)} e^{(t-s)\Delta} R(s) ds + \int_{t/2}^t e^{-(t-s)} e^{(t-s)\Delta} R(s) ds \\ &= Q_1(t) + Q_2(t), \end{aligned}$$

where

$$R(s) = u(s) - MG(s) + (E + V(s)) \cdot \nabla G(s).$$

Let $t \geq 2$. We then see that

$$\begin{aligned} \|Q_1(t)\|_p &\leq C \int_0^{t/2} e^{-(t-s)} (t-s)^{-N(1-1/p)/2} \|R(s)\|_1 ds \\ &\leq C e^{-t/2} t^{-N(1-1/p)/2} \int_0^{t/2} \|R(s)\|_1 ds. \end{aligned}$$

Using $\|G(s)\|_1 = 1$ and $\|\nabla G(s)\|_1 \leq C s^{-1/2}$ gives

$$\|R(s)\|_1 \leq C + C s^{-1/2} \times \begin{cases} \log(2+s) & \text{if } N = 1, \\ 1 & \text{if } N \geq 2. \end{cases}$$

Hence

$$d(t; p) \|Q_1(t)\|_p \leq C e^{-t/4} \quad (t \geq 2).$$

Next,

$$\|Q_2(t)\|_p \leq C \int_{t/2}^t e^{-(t-s)} \frac{1}{d(s; p)} ds \sup_{t/2 \leq s \leq t} d(s; p) \|R(s)\|_p.$$

In the case of $N = 1$, we have

$$\begin{aligned} \int_{t/2}^t e^{-(t-s)} \frac{1}{d(s; p)} ds &= \int_{t/2}^t e^{-(t-s)} s^{-(1-1/p)/2-1/2} \log s ds \\ &\leq C t^{-(1-1/p)/2-1/2} \log t \int_{t/2}^t e^{-(t-s)} ds \leq \frac{C}{d(t; p)}. \end{aligned}$$

In the case of $N \geq 2$, we have

$$\int_{t/2}^t e^{-(t-s)} \frac{1}{d(s; p)} ds = \int_{t/2}^t e^{-(t-s)} s^{-N(1-1/p)/2-1/2} ds \leq \frac{C}{d(t; p)}.$$

Hence

$$d(t; p) \|Q_2(t)\|_p \leq C \sup_{t/2 \leq s \leq t} d(s; p) \|R(s)\|_p.$$

Therefore

$$d(t; p) \|Q(t)\|_p \leq C e^{-t/4} + C \sup_{t/2 \leq s \leq t} d(s; p) \|R(s)\|_p,$$

which together with (6.4) implies (6.10).

Applying Lemma 3.2, (6.9) and (6.10) to (6.5), we have

$$\lim_{t \rightarrow \infty} d(t; p) \|v(t) - \{MG(t) - (E + V(t)) \cdot \nabla G(t)\}\|_p = 0.$$

As mentioned above, for the case of $N \geq 2$ we can replace $V(t)$ with V . Therefore the proof of Theorem 2.2 is complete. \square

7. Proof of Theorem 2.3

Theorem 2.3 for $N \geq 2$ follows from Theorem 2.2. Hence we prove the theorem only for $N = 1$.

Let $N = 1$ and $1 \leq p \leq \infty$. We repeat arguments similar to those in Sections 5 and 6, and use the same notation as in the sections. By $M = 0$, the assertion (i) of Theorem 2.2 implies

$$\|u(t)\|_p \leq C t^{-(1-1/p)/2-1/2} \log t \quad (t \geq 2).$$

Then the term $\|I_1(t)\|_p$ in Proposition 5.1 is estimated as

$$\|I_1(t)\|_p \leq C \int_0^{t/2} (t-s)^{-(1-1/p)/2-1/2} (2+s)^{-3/2} \log(2+s) ds \leq C t^{-(1-1/p)/2-1/2}.$$

By this estimate and the estimate of $\|I_2(t)\|_p$ in the proof of Proposition 5.1, we have

$$\sup_{t \geq 2} t^{(1-1/p)/2+1/2} \|I(t)\|_p < \infty.$$

Lemma 3.2 implies $\sup_{t \geq 2} t^{(1-1/p)/2+1/2} \|e^{t\Delta} u_0\|_p < \infty$ because of $M = 0$. Hence it follows from (5.1) that

$$\sup_{t \geq 2} t^{(1-1/p)/2+1/2} \|u(t)\|_p < \infty. \quad (7.1)$$

Next, applying (7.1) to $K_2(t)$ in (5.3), we have

$$\sup_{t \geq 2} t^{(1-1/p)/2+1/2} \|v(t)\|_p < \infty.$$

By using (4.4) and (7.1), calculations similar to those in the proof of Proposition 6.1 give the following estimates of $\|K_k^\varepsilon(t)\|_p$ ($1 \leq k \leq 5$).

$$\begin{aligned}
\|K_1^\varepsilon(t)\|_p &\leq C\varepsilon^{-(1-1/p)/2-3/2}t^{-(1-1/p)/2-1}, \\
\|K_2^\varepsilon(t)\|_p &\leq C\varepsilon^{-3/2}t^{-(1-1/p)/2-1}, \\
\|K_3^\varepsilon(t)\|_p + \|K_4^\varepsilon(t)\|_p &\leq Ct^{-(1-1/p)/2-1/2} \int_0^{\varepsilon t/2} \int_{|y|\geq \varepsilon\sqrt{t}} |u(y,s)| |\partial_x v(y,s)| dy ds, \\
\|K_5^\varepsilon(t)\|_p &\leq C\varepsilon t^{-(1-1/p)/2-1/2} \int_0^{\varepsilon t/2} \int_{|y|\leq \varepsilon\sqrt{t}} |u(y,s)| |\partial_x v(y,s)| dy ds.
\end{aligned}$$

Note that by (4.4) and (7.1),

$$\begin{aligned}
\int_0^\infty \int_{\mathbb{R}} |u(y,s)| |\partial_x v(y,s)| dy ds &\leq \int_0^\infty \|u(s)\|_2 \|\partial_x v(s)\|_2 ds \\
&\leq C \int_0^\infty (1+s)^{-3/2} ds < \infty.
\end{aligned} \tag{7.2}$$

Then

$$\|K_5^\varepsilon(t)\|_p \leq C\varepsilon t^{-(1-1/p)/2-1/2}.$$

Hence

$$\begin{aligned}
t^{(1-1/p)/2+1/2} \|I(t) - V(t)\partial_x G(t)\|_p \\
\leq C\varepsilon^{-(1-1/p)/2-3/2}t^{-1/2} + C \int_0^{\varepsilon t/2} \int_{|y|\geq \varepsilon\sqrt{t}} |u\partial_x v| dy ds + C\varepsilon.
\end{aligned}$$

Since (7.2) implies

$$\lim_{t \rightarrow \infty} \int_0^{\varepsilon t/2} \int_{|y|\geq \varepsilon\sqrt{t}} |u\partial_x v| dy ds = 0,$$

we have

$$\limsup_{t \rightarrow \infty} t^{(1-1/p)/2+1/2} \|I(t) - V(t)\partial_x G(t)\|_p \leq C\varepsilon.$$

Therefore we obtain

$$\lim_{t \rightarrow \infty} t^{(1-1/p)/2+1/2} \|I(t) - V(t)\partial_x G(t)\|_p = 0,$$

because ε is arbitrarily small.

Since $V(t)$ converges to V as $t \rightarrow \infty$ because of (7.2), we have

$$\lim_{t \rightarrow \infty} t^{(1-1/p)/2+1/2} \|I(t) - V\partial_x G(t)\|_p = 0,$$

and obtain the conclusion of Theorem 2.3 by the same argument as in the proof of Theorem 2.2. Thus the proof of Theorem 2.3 is complete.

References

- [1] S. Childress, J.K. Percus, Nonlinear aspects of chemotaxis, *Math. Biosci.* 56 (1981) 217–237.
- [2] M. Escobedo, E. Zuazua, Large time behavior for convection–diffusion equation in \mathbb{R}^n , *J. Funct. Anal.* 100 (1991) 119–161.
- [3] Y. Fujigaki, T. Miyakawa, Asymptotic profiles of the nonstationary incompressible Navier–Stokes flow in the whole space, *SIAM J. Math. Anal.* 33 (2001) 523–544.
- [4] H. Gajewski, K. Zacharias, Global behavior of a reaction–diffusion system modelling chemotaxis, *Math. Nachr.* 195 (1998) 77–114.
- [5] M.A. Herrero, J.J.L. Velázquez, A blow-up mechanism for a chemotactic model, *Ann. Sc. Norm. Super. Pisa XXIV* (1997) 633–683.
- [6] D. Horstmann, From 1970 until present: The Keller–Segel model in chemotaxis and its consequences, I, *Jahresber. Deutsch. Math.-Verein.* 105 (2003) 103–165.
- [7] W. Jäger, S. Luckhaus, On explosions of solutions to a system of partial differential equations modelling chemotaxis, *Trans. Amer. Math. Soc.* 329 (1992) 819–824.
- [8] E.F. Keller, L.A. Segel, Initiation of slime mold aggregation viewed as an instability, *J. Theoret. Biol.* 26 (1970) 399–415.
- [9] O.A. Ladyzhenskaya, V.A. Solonnikov, N.N. Ural'tseva, *Linear and Quasi-Linear Equations of Parabolic Type*, Nauka, Moscow, 1967; English translation: Amer. Math. Soc., Providence, RI, 1968.
- [10] S. Luckhaus, Y. Sugiyama, Large time behavior in super-critical cases to degenerate Keller–Segel system, *M2AN Math. Model. Numer. Anal.* 40 (2006) 597–621.
- [11] T. Nagai, T. Senba, K. Yoshida, Application of the Trudinger–Moser inequality to a parabolic system of chemotaxis, *Funkcial. Ekvac.* 40 (1997) 411–433.
- [12] T. Nagai, R. Sukuinn, M. Umesako, Decay properties and asymptotic profiles of bounded solutions to a parabolic system of chemotaxis in R^N , *Funkcial. Ekvac.* 46 (2003) 383–407.
- [13] Y. Naito, T. Suzuki, Y. Yoshida, Self-similar solutions to a parabolic system modeling chemotaxis, *J. Differential Equations* 184 (2002) 386–421.