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Bounds for blow-up time in nonlinear parabolic problems

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Abstract

A first order differential inequality technique is used on suitably defined auxiliary functions to determine lower bounds for blow-up time in initial-boundary value problems for parabolic equations of the form

$$u_t = \operatorname{div}\big(\rho(u)\operatorname{grad} u\big) + f(u)$$

if blow-up occurs. In addition, conditions which ensure that blow-up occurs or does not occur are presented.

Keywords: Blow-up; Bounds on blow-up time; Nonlinear parabolic problems

1. Introduction

In a recent paper [10], Payne and Schaefer considered an initial-boundary value problem for the semilinear heat equation

$$u_t = \Delta u + f(u) \tag{1.1}$$

in a bounded domain $\Omega \subset \mathbb{R}^3$ under a homogeneous Dirichlet boundary condition and a prescribed nonnegative initial condition. The nonnegative nonlinear function f was subject to suitable constraints which permitted, for instance, $f(s) = s^p$, $p > 1$, and $f(s) = 2(\cosh \gamma s - 1)$, $\gamma > 0$, and provided that the solution may blow up in finite time. In [10], a differential inequality technique was used to determine a lower bound for blow-up time if blow-up occurs. A second method based on a comparison principle was also presented in [10].

There is a vast literature on global existence or nonexistence and the blow-up in finite time of solutions to semilinear and nonlinear parabolic and hyperbolic equations and systems. We refer the reader to [1–8,10,11] and the many references cited therein. These papers deal with the questions of blow-up in finite time, blow-up rate, blow-up set, global existence, asymptotic behavior and so forth as well as a variety of methods used to study these questions.

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In this work we extend the results in [10] to more general parabolic equations where the Laplacian is replaced by an elliptic differential operator in divergence form, i.e.,

$$u_t = \operatorname{div}\bigl(\rho(u)\operatorname{grad} u\bigr) + f(u). \tag{1.2}$$

In Section 2 we impose conditions on f and a condition which relates the nonlinearities ρ and f (see (2.2)) and determine a lower bound on blow-up time if blow-up occurs as well as determine when blow-up cannot occur. We then consider an "easier to verify" set of conditions on functions ρ and f in Section 3 (see (3.1)) for which analogous results may be determined. We then present a sufficient condition in Section 4 which guarantees that blow-up will occur in finite time as well as an upper bound for the blow-up time. These results are obtained by a differential inequality technique on suitably defined auxiliary functions. A forthcoming paper will treat the nonlinear differential equation where the function ρ in (1.2) depends on the square of the gradient of the solution.

2. General case

Let Ω be a bounded domain in $\mathbb{R}^3$ with sufficiently smooth boundary $\partial\Omega$. We consider the nonlinear initial-boundary value problem

$$\begin{aligned} &u_t = \bigl(\rho(u)u_{,i}\bigr)_{,i} + f(u) && \text{in } \Omega \times (0, t^*),\\ &u(x,t) = 0 && \text{on } \partial\Omega \times (0, t^*),\\ &u(x,0) = g(x) \geqslant 0 && \text{in } \Omega, \end{aligned} \tag{2.1}$$

where u_t denotes the partial derivative of u with respect to t, the comma i denotes partial differentiation with respect to x_i, $i = 1, 2, 3$, and we use the summation convention on any pair of repeated indexes. The functions ρ and f satisfy the following conditions:

$$\text{(i)}\quad f(0) = 0, \qquad f(s) > 0 \quad \text{for } s > 0,$$

$$\text{(ii)}\quad \int_s^\infty \frac{d\eta}{f(\eta)} \quad \text{is bounded for all } s \geqslant s_0 > 0,$$

there exist positive constants $n \geqslant 2$ and β such that

$$\text{(iii)}\quad f(s)\left(\int_s^\infty \frac{d\eta}{f(\eta)}\right)^{n+1} \to \infty \quad \text{as } s \to 0^+,$$

$$\text{(iv)}\quad f'(s)\int_s^\infty \frac{d\eta}{f(\eta)} \leqslant n + 1 - \beta \quad \text{for } s \geqslant 0,$$

and ρ and f are related by

$$\text{(v)}\quad \rho(s) \geqslant K\left(\int_s^\infty \frac{d\eta}{f(\eta)}\right)^{-\gamma} \quad \text{for } s > 0, \tag{2.2}$$

where γ and K are positive constants. In addition, we assume that g satisfies the compatibility condition $g(x) = 0$ for $x \in \partial\Omega$. It follows from the parabolic maximum principle that u is nonnegative in x for $t \in [0, t^*)$.

For certain classes of functions $\rho(u)$ and $f(u)$, it is known that the solution of (2.1) can fail to exist globally only if it blows up at some finite time (e.g., when $\rho(u) = 1$ see [1,6]). Whether the solution blows up depends on the form of $f(u)$, the smoothness and form of the positive function $\rho(u)$, the initial data $g(x)$, and the geometry of the domain. We assume the existence of a positive solution that is classical except for the possibility that it might blow up at some finite time t^*. If blow-up occurs, we aim to find a lower bound on the blow-up time; otherwise we seek conditions which ensure that the solution does not blow up.

We define the auxiliary function

$$\varphi(t)=\int_\Omega v^n\big(u(x,t)\big)\,dx,\qquad v(s)=\left(\int_s^\infty\frac{d\eta}{f(\eta)}\right)^{-1},\tag{2.3}$$

and compute

$$\begin{aligned}\varphi'(t)&=n\int_\Omega v^{n-1}v^2\big[f(u)\big]^{-1}u_t\,dx=n\int_\Omega v^{n+1}\left[1+\frac{(\rho(u)u_{,i})_{,i}}{f(u)}\right]dx\\&=n\int_\Omega v^{n+1}\,dx+n\int_\Omega\left[\left(\frac{v^{n+1}}{f(u)}\rho(u)u_{,i}\right)_{,i}-\frac{(n+1)v^{n+2}}{[f(u)]^2}\rho(u)u_{,i}u_{,i}+\frac{v^{n+1}}{[f(u)]^2}f'(u)\rho(u)u_{,i}u_{,i}\right]dx\\&\leqslant n\int_\Omega v^{n+1}\,dx-n\beta\int_\Omega\frac{v^{n+2}}{[f(u)]^2}\rho(u)u_{,i}u_{,i}\,dx,\end{aligned}$$

where we used the divergence theorem, the Dirichlet condition on u, and (2.2)(iv). We now use (2.2)(v) and

$$\big|\nabla v^{\frac{n+\gamma}{2}}\big|^2=\left(\frac{n+\gamma}{2}\right)^2v^{n+\gamma+2}\frac{1}{[f(u)]^2}|\nabla u|^2,$$

where ∇ is the gradient operator, to obtain

$$\varphi'(t)\leqslant n\int_\Omega v^{n+1}\,dx-\frac{4nK\beta}{(n+\gamma)^2}\int_\Omega\big|\nabla v^{\frac{n+\gamma}{2}}\big|^2\,dx.\tag{2.4}$$

We will now consider two cases for γ in (2.2)(v), namely, $0<\gamma<1$ and $\gamma>1$.

In the first case, $0<\gamma<1$, we let

$$\alpha=\frac{n+\gamma}{2}\tag{2.5}$$

and obtain by Hölder's inequality

$$\int_\Omega v^{n+1}\,dx\leqslant\left(\int_\Omega v^{4\alpha}\,dx\right)^{\frac14}\left(\int_\Omega v^{\frac43(n+1-\alpha)}\,dx\right)^{\frac34}.\tag{2.6}$$

Using $q=6$ and $p=2$ in the Sobolev inequality [12]

$$\left(\int_\Omega|w|^q\,dx\right)^{\frac1q}\leqslant C\left(\int_\Omega|\nabla w|^p\,dx\right)^{\frac1p},\tag{2.7}$$

where

$$C=4^{\frac13}\cdot3^{-\frac12}\cdot\pi^{-\frac23},\tag{2.8}$$

we have by Schwarz's inequality and (2.7)

$$\int_\Omega v^{4\alpha}\,dx\leqslant\left(\int_\Omega v^{2\alpha}\,dx\right)^{\frac12}\left(\int_\Omega(v^\alpha)^6\,dx\right)^{\frac12}\leqslant C^3\left(\int_\Omega v^{2\alpha}\,dx\right)^{\frac12}\left(\int_\Omega\big|\nabla v^\alpha\big|^2\,dx\right)^{\frac32}.\tag{2.9}$$

We note that an estimate for the optimal constant in an inequality like (2.9) is given by Levine in [9]. Upon substitution in (2.6), we have

$$\int_\Omega v^{n+1}\,dx\leqslant C^{\frac34}\left(\int_\Omega v^{2\alpha}\,dx\right)^{\frac18}\left(\int_\Omega\big|\nabla v^\alpha\big|^2\,dx\right)^{\frac38}\left(\int_\Omega v^{\frac43\sigma}\,dx\right)^{\frac34},\tag{2.10}$$

where we let $\sigma = n + 1 - \alpha$. We now use the Rayleigh principle

$$\int_\Omega v^{2\alpha}\,dx \leqslant \frac{1}{\lambda_1}\int_\Omega \left|\nabla v^\alpha\right|^2 dx, \tag{2.11}$$

where λ_1 is the first eigenvalue in the fixed membrane problem

$$\Delta w + \lambda w = 0, \quad w > 0 \text{ in } \Omega,\ w = 0 \text{ on } \partial\Omega, \tag{2.12}$$

and Hölder's inequality

$$\int_\Omega v^{\frac{4}{3}\sigma}\,dx \leqslant \left(\int_\Omega v^n\,dx\right)^{\frac{4\sigma}{3n}} |\Omega|^{1-\frac{4\sigma}{3n}}, \tag{2.13}$$

where $|\Omega|$ denotes the volume of Ω, on the right side of (2.10). Thus, we have from (2.10), (2.11), and (2.13)

$$\int_\Omega v^{n+1}\,dx \leqslant C^{\frac{3}{4}}\lambda_1^{-\frac{1}{8}}|\Omega|^{\frac{3}{4}-\frac{\sigma}{n}}\left(\int_\Omega v^n\,dx\right)^{\frac{\sigma}{n}}\left(\int_\Omega \left|\nabla v^\alpha\right|^2 dx\right)^{\frac{1}{2}}$$

and from (2.4)

$$\varphi'(t) \leqslant K_1\left(\int_\Omega v^n\,dx\right)^{\frac{\sigma}{n}}\left(\int_\Omega \left|\nabla v^\alpha\right|^2 dx\right)^{\frac{1}{2}} - \frac{4nK\beta}{(n+\gamma)^2}\int_\Omega \left|\nabla v^\alpha\right|^2 dx,$$

where

$$K_1 = nC^{\frac{3}{4}}\lambda_1^{-\frac{1}{8}}|\Omega|^{\frac{n-4+2\gamma}{4n}}. \tag{2.14}$$

We now use the fundamental inequality

$$a^r b^s \leqslant ra + sb, \quad r + s = 1,\ a, b \geqslant 0, \tag{2.15}$$

with a positive undetermined weight factor θ, to write

$$\varphi'(t) \leqslant \frac{K_1}{2}\theta\int_\Omega \left|\nabla v^\alpha\right|^2 dx + \frac{K_1}{2}\theta^{-1}\left(\int_\Omega v^n\,dx\right)^{\frac{2\sigma}{n}} - \frac{4nK\beta}{(n+\gamma)^2}\int_\Omega \left|\nabla v^\alpha\right|^2 dx$$

and choose θ to be

$$\theta = \frac{8nK\beta}{K_1(n+\gamma)^2}. \tag{2.16}$$

Thus, we obtain the differential inequality

$$\varphi'(t) \leqslant K_2\left[\varphi(t)\right]^{\frac{2\sigma}{n}}, \tag{2.17}$$

where

$$K_2 = \frac{1}{2}K_1\theta^{-1}, \qquad \frac{2\sigma}{n} = \frac{n+2-\gamma}{n} > 1. \tag{2.18}$$

We now integrate (2.17) from 0 to t and obtain

$$-\left[\varphi(t)\right]^{-(\frac{2\sigma}{n}-1)} + \left[\varphi(0)\right]^{-(\frac{2\sigma}{n}-1)} \leqslant \left(\frac{2\sigma}{n} - 1\right)K_2 t.$$

Consequently, as $t \to t^*$, we have the lower bound for t^*

$$t^* \geqslant \frac{n}{K_2(2-\gamma)}\left[\int_\Omega \left(\int_{g(x)}^\infty \frac{d\eta}{f(\eta)}\right)^{-n} dx\right]^{\frac{\gamma-2}{n}} \tag{2.19}$$

in terms of the initial data and parameters. We summarize the result in this case in the following theorem.

Theorem 1. *If $\gamma \in (0,1)$ and u is a nonnegative classical solution of* (2.1), (2.2) *which becomes unbounded in the measure φ given by* (2.3) *at time t^*, then t^* is bounded below by* (2.19)*, where K_2 is given by* (2.18).

An example for which Theorem 1 holds is given by

$$\rho(s) = 1 + c_1 s^q, \qquad f(s) = c_2 s^p, \tag{2.20}$$

where $p > 1$, $0 < q < p-1$, $\gamma = \frac{q}{p-1}$, $n \geqslant (p-1)^{-1} + \beta$, c_1 and c_2 are positive constants and $K = c_1[c_2(p-1)]^\gamma$. We also note that the result in Theorem 1 is valid when the equality in the differential equation in (2.1) is replaced by the less than relation.

We now consider the second case when $\gamma > 1$. We recall that φ given by (2.3) satisfies the inequality (2.4) and by the Rayleigh principle

$$\int_\Omega \left|\nabla v^{\frac{n+\gamma}{2}}\right|^2 dx \geqslant \lambda_1 \int_\Omega v^{n+\gamma}\, dx$$

from (2.12). It follows by Hölder's inequality that

$$\int_\Omega \left|\nabla v^{\frac{n+\gamma}{2}}\right|^2 dx \geqslant \lambda_1 \left(\int_\Omega v^{n+1}\, dx\right)^{\frac{n+\gamma}{n+1}} |\Omega|^{\frac{1-\gamma}{n+1}}$$

and by (2.4) that

$$\varphi'(t) \leqslant n \int_\Omega v^{n+1}\, dx \left\{1 - \frac{4K\beta\lambda_1}{(n+\gamma)^2} |\Omega|^{\frac{1-\gamma}{n+1}} \left(\int_\Omega v^{n+1}\, dx\right)^{\frac{\gamma-1}{n+1}}\right\}. \tag{2.21}$$

Since by Hölder's inequality we have

$$\int_\Omega v^{n+1} dx \geqslant \left(\int_\Omega v^n\, dx\right)^{\frac{n+1}{n}} |\Omega|^{-\frac{1}{n}},$$

the inequality (2.21) becomes

$$\varphi'(t) \leqslant n \int_\Omega v^{n+1}\, dx \left\{1 - \frac{4K\beta\lambda_1}{(n+\gamma)^2} |\Omega|^{\frac{1-\gamma}{n}} \left[\varphi(t)\right]^{\frac{\gamma-1}{n}}\right\}. \tag{2.22}$$

It now follows from (2.22) that the function u cannot blow up. For suppose contrariwise that $u \to \infty$ at some finite time t^*. Then $v(u) \to \infty$ would imply that $\varphi(t) \to \infty$ as $t \to t^*$. But from (2.22), we would conclude that $\varphi'(t)$ is negative for t sufficiently large which leads to a contradiction.

We formulate this result in the following theorem.

Theorem 2. *If $\gamma > 1$ and u is a nonnegative classical solution of the problem* (2.1), (2.2)*, then u cannot blow up in the measure φ in finite time.*

The previous example (2.20) exemplifies Theorem 2 where now $q > p-1$ and $\gamma = \frac{q}{p-1}$. In the threshold case $\gamma = 1$, it follows from (2.22) that if

$$\frac{4K\beta\lambda_1}{(n+\gamma)^2} \leqslant 1, \tag{2.23}$$

then the solution of (2.1), (2.2) will not blow up. However, if (2.23) is violated, then there may or may not be blow-up.

3. Special case

In the previous section we required that the functions ρ and f be related by the condition (2.2)(v) which may be difficult to verify because we are unable to evaluate v in (2.3). In this section, we consider another set of conditions on the functions ρ and f which are easier to verify and which lead to analogous results. We again consider problem (2.1) and ask that

$$0 < f(s) \leqslant a_1 + a_2 s^p, \qquad \rho(s) \geqslant b_1 + b_2 s^q > 0, \quad \text{for } s > 0, \tag{3.1}$$

where $p > 1$ and $q > 0$ and a_1, a_2, b_1, and b_2 are positive constants. We shall consider the two cases, $q > p-1$ and $q < p-1$. We note that if the equality signs hold in (3.1) and $a_1 = 0$, then the conditions on ρ and f satisfy (2.2) and if $a_1 = 0 = b_2$, the conditions on ρ and f are basically those covered in [10].

In the case $q > p-1$, we define the auxiliary function

$$\Phi(t) = \int_\Omega u^2\,dx \tag{3.2}$$

and compute

$$\begin{aligned}
\Phi'(t) &= 2\int_\Omega u\big[\big(\rho(u)u_{,i}\big)_{,i} + f(u)\big]\,dx = -2\int_\Omega \rho(u)|\nabla u|^2\,dx + 2\int_\Omega uf(u)\,dx \\
&\leqslant -2\int_\Omega \big(b_1 + b_2 u^q\big)|\nabla u|^2\,dx + 2\int_\Omega u\big[a_1 + a_2 u^p\big]\,dx \\
&= 2\bigg[-b_1\int_\Omega |\nabla u|^2\,dx + a_1\int_\Omega u\,dx\bigg] + 2\bigg[-b_2\int_\Omega u^q|\nabla u|^2\,dx + a_2\int_\Omega u^{p+1}\,dx\bigg] \\
&= I_1 + I_2,
\end{aligned} \tag{3.3}$$

where I_1 and I_2 have the obvious meaning.

By the membrane inequality and Schwarz's inequality, we bound I_1 by

$$I_1 \leqslant 2\big[\Phi(t)\big]^{\frac{1}{2}}\big\{a_1|\Omega|^{\frac{1}{2}} - \lambda_1 b_1\big[\Phi(t)\big]^{\frac{1}{2}}\big\}. \tag{3.4}$$

To bound I_2, we use

$$\big|\nabla u^{\frac{q+2}{2}}\big|^2 = \bigg(\frac{q+2}{2}\bigg)^2 u^q|\nabla u|^2,$$

the Rayleigh principle, and Hölder's inequality to obtain

$$\begin{aligned}
I_2 &= 2\bigg[-\frac{4b_2}{(q+2)^2}\int_\Omega \big|\nabla u^{\frac{q+2}{2}}\big|^2\,dx + a_2\int_\Omega u^{p+1}\,dx\bigg] \\
&\leqslant 2\bigg[-\frac{4b_2\lambda_1}{(q+2)^2}\int_\Omega u^{q+2}\,dx + a_2\bigg(\int_\Omega u^{q+2}\,dx\bigg)^{\frac{p+1}{q+2}}|\Omega|^{\frac{q+1-p}{q+2}}\bigg] \\
&= 2\bigg(\int_\Omega u^{q+2}\,dx\bigg)^{\frac{p+1}{q+2}}\bigg\{a_2|\Omega|^{\frac{q+1-p}{q+2}} - \frac{4b_2\lambda_1}{(q+2)^2}\bigg(\int_\Omega u^{q+2}\,dx\bigg)^{\frac{q+1-p}{q+2}}\bigg\} \\
&\leqslant 2\bigg(\int_\Omega u^{q+2}\,dx\bigg)^{\frac{p+1}{q+2}}\bigg\{a_2|\Omega|^{\frac{q+1-p}{q+2}} - \frac{4b_2\lambda_1}{(q+2)^2}\big(\big[\Phi(t)\big]^{\frac{q+2}{2}}|\Omega|^{-\frac{q}{2}}\big)^{\frac{q+1-p}{q+2}}\bigg\}.
\end{aligned} \tag{3.5}$$

It is clear that if u blows up in the Φ measure, then by (3.3), (3.4), and (3.5) it follows that $\Phi'(t) \leqslant 0$ which leads to a contradiction. Thus u cannot blow up in this case and we have the following theorem. We note, in fact, that the theorem is not restricted to three dimensions.

Theorem 3. *If $q > p-1$ and u is a nonnegative classical solution of* (2.1), (3.1)*, then u cannot blow up in the measure Φ in finite time.*

We now consider the problem (2.1), (3.1), in the case $0 < q < p-1$. We define the function

$$\psi(t) = \int_\Omega u^{n(p-1)}\,dx, \tag{3.6}$$

where $p > 1$ and $n > \frac{2}{p-1}$. As in the previous computations for (2.4) and (3.3), we obtain

$$\psi'(t) \leqslant -n(p-1)\big[n(p-1)-1\big]\int_\Omega u^{n(p-1)-2}|\nabla u|^2\big(b_1+b_2u^q\big)\,dx$$
$$+\,n(p-1)\int_\Omega \big[a_1u^{n(p-1)-1}+a_2u^{(n+1)(p-1)}\big]\,dx. \tag{3.7}$$

We may drop the b_1 term on the right side of (3.7) and use Hölder's inequality to bound the a_1 term by

$$n(p-1)a_1\int_\Omega u^{n(p-1)-1}\,dx \leqslant C_1\Big(\int_\Omega u^{n(p-1)}\,dx\Big)^{1-\frac{1}{n(p-1)}}, \tag{3.8}$$

where

$$C_1 = n(p-1)a_1|\Omega|^{\frac{1}{n(p-1)}}. \tag{3.9}$$

We now need to consider the remaining two terms in (3.7), i.e.,

$$n(p-1)\Big\{-\big[n(p-1)-1\big]b_2\int_\Omega u^{n(p-1)-2+q}|\nabla u|^2\,dx + a_2\int_\Omega u^{(n+1)(p-1)}\,dx\Big\}. \tag{3.10}$$

For simplicity, we let

$$v = u^{p-1}, \qquad \alpha = \frac{1}{2}\Big(n+\frac{q}{p-1}\Big), \tag{3.11}$$

and rewrite (3.10) as

$$n(p-1)\Big\{-\frac{[n(p-1)-1]}{\alpha^2(p-1)^2}b_2\int_\Omega \big|\nabla v^\alpha\big|^2\,dx + a_2\int_\Omega v^{n+1}\,dx\Big\}. \tag{3.12}$$

Now by Hölder's inequality, we have

$$\int_\Omega v^{n+1}\,dx \leqslant \Big(\int_\Omega v^{4\alpha}\,dx\Big)^{\frac14}\Big(\int_\Omega v^{\frac23(n+2-\frac{q}{p-1})}\,dx\Big)^{\frac34}$$

and

$$\int_\Omega v^{\frac23(n+2-\frac{q}{p-1})}\,dx \leqslant \Big(\int_\Omega v^n\,dx\Big)^{\frac23(1+\frac2n-\frac{q}{n(p-1)})}|\Omega|^{\frac23(\frac12-\frac2n+\frac{q}{n(p-1)})},$$

and by the Sobolev inequality (see (2.7), (2.9), and (2.11))

$$\int_\Omega v^{4\alpha}\,dx \leqslant C^3\lambda_1^{-\frac12}\Big(\int_\Omega \big|\nabla v^\alpha\big|^2\,dx\Big)^2.$$

It follows that

$$a_2\int_\Omega v^{n+1}\,dx \leqslant C_2\Big(\int_\Omega \big|\nabla v^\alpha\big|^2\,dx\Big)^{\frac12}\Big(\int_\Omega v^n\,dx\Big)^{\tau} \tag{3.13}$$

where

$$C_2 = a_2 C^{\frac{3}{4}} \lambda_1^{-\frac{1}{8}} |\Omega|^{\frac{1}{4}-\frac{1}{n}+\frac{q}{2n(p-1)}}, \tag{3.14}$$

$$\tau = \frac{1}{2} + \frac{1}{n} - \frac{q}{2n(p-1)}. \tag{3.15}$$

We make use of (2.15) with an, as yet, undetermined positive weight factor δ to obtain

$$a_2 \int_\Omega v^{n+1}\,dx \leqslant \frac{1}{2} C_2 \delta \int_\Omega \left|\nabla v^\alpha\right|^2 dx + \frac{1}{2} C_2 \delta^{-1} \left(\int_\Omega v^n\,dx \right)^{2\tau} \tag{3.16}$$

and then choose δ so that

$$\frac{1}{2} C_2 \delta = \frac{n(p-1)-1}{\alpha^2 (p-1)^2} b_2. \tag{3.17}$$

It follows from (3.7), (3.8), (3.12), and (3.16) that

$$\psi'(t) \leqslant C_1 \left[\psi(t)\right]^{1-\frac{1}{n(p-1)}} + C_3 \left[\psi(t)\right]^{1+\frac{2}{n}-\frac{q}{n(p-1)}}, \tag{3.18}$$

where

$$C_3 = \frac{1}{2} n(p-1) C_2 \delta^{-1}. \tag{3.19}$$

An integration of (3.18) leads to

$$\int_{\psi(0)}^{\psi(t)} \frac{d\eta}{C_1 \eta^{1-\frac{1}{n(p-1)}} + C_3 \eta^{1+\frac{2}{n}-\frac{q}{n(p-1)}}} \leqslant t$$

and if u blows up in the measure ψ, then we obtain a lower bound for t^* given by

$$t^* \geqslant \int_{\psi(0)}^{\infty} \frac{d\eta}{C_1 \eta^{1-\frac{1}{n(p-1)}} + C_3 \eta^{1+\frac{2}{n}-\frac{q}{n(p-1)}}}, \tag{3.20}$$

where

$$\psi(0) = \int_\Omega \left[g(x)\right]^{n(p-1)} dx.$$

We summarize this result in the follows theorem.

Theorem 4. *If* $0 < q < p-1$ *and* u *is a nonnegative classical solution of the problem* (2.1), (3.1) *which becomes unbounded in the measure* ψ *given by* (3.6) *at time* t^**, then* t^* *is bounded below by* (3.20)*, where* C_1 *and* C_3 *are given by* (3.9) *and* (3.19)*, respectively.*

We remark again that the results in this section are valid in the case that equality is replaced by the less than relation in the differential equation in (2.1).

4. Criterion for blow-up

We are now interested in determining a criterion that ensures blow-up of the solution in the problem (2.1) when

$$\rho(s) = b_1 + b_2 s^q, \qquad f(s) = a s^p, \quad s \geqslant 0, \tag{4.1}$$

for $0 < q < p-1$ and a, b_1 and b_2 positive constants.

We define the function

$$\chi(t)=\int_{\Omega} uw^2\,dx, \tag{4.2}$$

where (w,λ) is the first eigenpair in the fixed membrane problem (2.12) and w is normalized by the condition

$$\int_{\Omega} w^2\,dx=1. \tag{4.3}$$

We compute

$$\begin{aligned}\chi'(t)&=\int_{\Omega} w^2\big[b_1\Delta u+b_2\big(u^q u_{,i}\big)_{,i}+au^p\big]\,dx\\&=-2\lambda b_1\int_{\Omega} w^2u\,dx+2b_1\int_{\Omega} w_{,i}w_{,i}u\,dx-\frac{2\lambda b_2}{q+1}\int_{\Omega} w^2u^{q+1}\,dx\\&\quad+\frac{2b_2}{q+1}\int_{\Omega} w_{,i}w_{,i}u^{q+1}\,dx+a\int_{\Omega} w^2u^p\,dx\end{aligned}$$

on integrating by parts. We drop the second and fourth terms on the right side and use Hölder's inequality and (4.3) on the third term to obtain

$$\chi'(t)\geqslant -2\lambda b_1\chi(t)-\frac{2\lambda b_2}{q+1}\bigg(\int_{\Omega} w^2u^p\,dx\bigg)^{\frac{q+1}{p}}+a\int_{\Omega} w^2u^p\,dx,$$

which we rewrite as

$$\chi'(t)\geqslant -2\lambda b_1\chi(t)+\bigg(\int_{\Omega} w^2u^p\,dx\bigg)^{\frac{q+1}{p}}\bigg\{a\bigg(\int_{\Omega} w^2u^p\,dx\bigg)^{\frac{\mu}{p}}-\frac{2\lambda b_2}{q+1}\bigg\}, \tag{4.4}$$

where $\mu=p-q-1>0$. Since

$$\int_{\Omega} w^2u\,dx\leqslant\bigg(\int_{\Omega} w^2u^p\,dx\bigg)^{\frac{1}{p}},$$

from (4.4), we have

$$\begin{aligned}\chi'(t)&\geqslant -2\lambda b_1\chi(t)+\bigg(\int_{\Omega} w^2u^p\,dx\bigg)^{\frac{q+1}{p}}\bigg\{a\bigg(\int_{\Omega} w^2u\,dx\bigg)^{\mu}-\frac{2\lambda b_2}{q+1}\bigg\}\\&=\bigg(\int_{\Omega} w^2u^p\,dx\bigg)^{\frac{q+1}{p}}\bigg\{a\big[\chi(t)\big]^{\mu}-\frac{2\lambda b_2}{q+1}-\frac{2\lambda b_1\chi(t)}{(\int_{\Omega} w^2u^p\,dx)^{\frac{q+1}{p}}}\bigg\}\\&\geqslant\bigg(\int_{\Omega} w^2u^p\,dx\bigg)^{\frac{q+1}{p}}\bigg\{a\big[\chi(t)\big]^{\mu}-\frac{2\lambda b_2}{q+1}-\frac{2\lambda b_1}{[\chi(t)]^q}\bigg\}.\end{aligned} \tag{4.5}$$

As the expression in the brace in (4.5) is increasing in χ, we deduce that a sufficient condition for blow up is

$$a\bigg(\int_{\Omega} gw^2\,dx\bigg)^{\mu}-\frac{2\lambda b_2}{q+1}-\frac{2\lambda b_1}{(\int_{\Omega} gw^2\,dx)^q}>0 \tag{4.6}$$

and that

$$\chi'(t) \geqslant [\chi(t)]^{q+1}\left\{a[\chi(t)]^{\mu} - \frac{2\lambda b_2}{q+1} - \frac{2\lambda b_1}{[\chi(t)]^q}\right\} = a[\chi(t)]^p - \frac{2\lambda b_2}{q+1}[\chi(t)]^{q+1} - 2\lambda b_1[\chi(t)]. \tag{4.7}$$

On integrating (4.7) from 0 to t, we have

$$t \leqslant \int\limits_{\chi(0)}^{\chi(t)} \frac{d\eta}{a\eta^p - \frac{2\lambda b_2}{q+1}\eta^{q+1} - 2\lambda b_1\eta} \leqslant \int\limits_{\chi(0)}^{\infty} \frac{d\eta}{a\eta^p - \frac{2\lambda b_2}{q+1}\eta^{q+1} - 2\lambda b_1\eta} = M. \tag{4.8}$$

It follows that the solution u blows up in the measure χ since (4.8) cannot hold for all time t, i.e.,

$$t^* \leqslant M. \tag{4.9}$$

We formulate this result in the following theorem.

Theorem 5. *If u is a nonnegative classical solution of the problem* (2.1), (4.1)*, where $0 < q < p - 1$ and the initial data $g(x)$ satisfies condition* (4.6)*, then u blows up in the measure χ at some finite time $t^* \leqslant M$, where M is given by* (4.8).

We remark that the advantage of using $\chi(t)$ rather than the first Fourier coefficient is two fold. First, the inequalities are "cleaner" unless one uses the unconventional normalization $\int_\Omega w\,dx = 1$ and second, if (w, λ) is unknown for the domain Ω in (2.12), then one may choose an interior domain $\widetilde{\Omega}$ for which $(\widetilde{w}, \widetilde{\lambda})$ are known and follow the some procedure with

$$\widetilde{\chi}(t) = \int\limits_{\widetilde{\Omega}} \widetilde{w}^2 u\,dx.$$

Finally, we remark that Theorem 5, as was Theorem 3, is not restricted to a domain Ω in $\mathbb{R}^3$ but is valid for Ω in $\mathbb{R}^n$, $\mathbb{N} \geqslant 2$.

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