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A compactness result for perturbed semigroups and application to a transport model

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ABSTRACT

In this paper we are concerned with the compactness properties of remainder terms of the Dyson–Phillips expansion of perturbed semigroups on general Banach spaces. More specifically, we derive conditions which ensure the compactness of the remainder term $R_n(t)$ for some integer n. Our result applies directly to discuss the time asymptotic behaviour (for large times) of the solution of a one-dimensional transport equation with reentry boundary conditions on L^1-spaces without regularity conditions on the initial data.

1. Introduction

Let T be the generator of a strongly continuous semigroup $(U(t))_{t\geqslant 0}$ on a Banach space X and let $\mathcal{L}(X)$ denote the set of all bounded linear operators in X. If $K \in \mathcal{L}(X)$, by the classical perturbation theory, $T+K$ generates a strongly continuous semigroup $(V(t))_{t\geqslant 0}$ given by the Dyson–Phillips expansion

$$V(t) = \sum_{j=0}^{\infty} U_j(t) \tag{1}$$

where $U_0(t) = U(t)$, $U_j(t) = \int_0^t U(s)KU_{j-1}(t-s)\,ds$ $(j \geqslant 1)$ and the series (1) converges in $\mathcal{L}(X)$ uniformly in bounded times. The remainder term of order n, $R_n(t) = \sum_{j=n}^{\infty} U_j(t)$, is given by

$$R_n(t) = \sum_{j=n}^{\infty} U_j(t) = \int\limits_{s_1+\cdots+s_n\leqslant t,\, s_i\geqslant 0} U(s_1)K\cdots U(s_n)KV\left(t-\sum_{i=1}^{n} s_i\right) ds_1\cdots ds_n. \tag{2}$$

So the Cauchy problem

$$\frac{d\psi}{dt} = (T+K)\psi(t), \qquad \psi(0) = \psi_0 \tag{3}$$

has a unique classical solution given by $\psi(t) = V(t)\psi_0$ provided that ψ_0 belongs to $D(T)$. In general this result follows from the Hille–Yosida theorem. This procedure is not constructive, so in order to get more information on the solution, in particular, its behaviour for large times, the knowledge of the spectrum of $T+K$ or $(V(t))_{t\geqslant 0}$ plays a central role.

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Let $W \in \mathcal{L}(X)$. The essential spectral radius of W is defined by

$$r_e(W) := \sup\{|\lambda|;\ \lambda \in \sigma(W) \text{ but } \lambda \text{ is not an eigenvalue of finite algebraic multiplicity}\} \quad (\text{see [17]}).$$

Let $\omega(U)$ denote the type of the semigroup $(U(t))_{t\geqslant 0}$. It follows from [17, Lemma 2.1] that there exists $\omega_e \in [-\infty, \omega(U)]$ such that

$$r_e\big(U(t)\big) = e^{t\omega_e(U)} \quad (t \geqslant 0).$$

The number $\omega_e(U)$ is called the essential type of $(U(t))_{t\geqslant 0}$.

It is well known that, if some remainder term of the Dyson–Phillips expansion $R_n(t)$ is compact, then the operators $U(t)$ and $V(t)$ have the same essential type [8,20]. Therefore there are only isolated points in the spectrum of the perturbed semigroup $V(t)$ outside the circle $|\mu| = e^{t\omega(U)}$, all these points being eigenvalues with finite algebraic multiplicity. Assuming the existence of such eigenvalues, the semigroup $(V(t))_{t\geqslant 0}$ can be decomposed into two parts: the first containing the time development of finitely many eigenmodes, the second being of faster decay. Using the spectral mapping theorem for the point spectrum, we infer that, for any $\omega > \omega(U)$ we have $\sigma(T+K) \cap \{\lambda \in \mathbb{C} \text{ such that } \operatorname{Re}\lambda \geqslant \omega\}$ consists of finitely many isolated eigenvalues $\{\lambda_1, \ldots, \lambda_q\}$. Let $\beta_1 = \sup\{\operatorname{Re}\lambda,\ \lambda \in \sigma(T+K),\ \operatorname{Re}\lambda < \omega\}$, and $\beta_2 = \min\{\operatorname{Re}\lambda_j,\ 1 \leqslant j \leqslant q\}$. The solution of the problem (3) satisfies

$$\left\| \psi(t) - \sum_{j=1}^{q} e^{\lambda_j t} e^{D_j t} P_j \psi_0 \right\| = o\big(e^{\beta^* t}\big) \quad \text{for every } \beta^* \text{ with } \beta_1 < \beta^* < \beta_2, \tag{4}$$

where $\psi_0 \in D(T+K)$, P_j and D_j denote, respectively, the spectral projection and the nilpotent operator associated with λ_j, $j = 1, 2, \ldots, q$.

There are much works in this direction motivated by various problems arising in mathematical physics, bio-mathematics and, in particular, the time dependent neutron transport equation (see, for example, [1,4,5,8,9,11,10,12,14–16,18,20] and the references therein). For transport equations, the compactness of some order remainder term of the Dyson Phillips expansion in L^p-spaces, $1 \leqslant p < +\infty$, was established only for no-reentry boundary conditions (i.e. with zero incoming flux in the spacial domain) [7,9,14,18,20] and recently for bounce-back boundary conditions in bounded geometry for $1 < p < +\infty$ [5]. However, when dealing with reentry boundary conditions, except the one-dimensional case with reflective or periodic boundary conditions, the problem is open because it is difficult to compute $R_n(t)$ and its expression involves the boundary conditions.

Since the compactness of $R_n(t)$ is equivalent to that of $U_n(t)$ [8, p. 16], Brendle [1] characterizes the compactness of $R_n(t)$ in terms of the norm continuity of $U_n(t)$ and the compactness of $(\lambda - T)^{-1}(K(\lambda - T)^{-1})^n$, $n \in \mathbb{N}$. Extending Brendle's analysis, Sbihi [10] established that, if X is a Hilbert space, T is dissipative and there exists $\alpha > w(U)$ such that

$$(\alpha + i\beta - T)^{-1} K (\alpha + i\beta - T)^{-1} \text{ is compact for all } \beta \in \mathbb{R},$$

and

$$\lim_{\beta\to\infty} \big(\big\|K^*(\alpha + i\beta - T)^{-1}K\big\| + \big\|K(\alpha + i\beta - T)^{-1}K^*\big\|\big) = 0,$$

then $R_1(t) = V(t) - U(t)$ is compact on X for all $t \geqslant 0$. This implies that, for each $t > 0$, $U(t)$ and $V(t)$ have the same essential spectrum.

Even though Sbihi's result is a Hilbertian one, using approximation arguments and an interpolation result, it was applied successfully to transport equations for $1 < p < \infty$ [6,10,5].

Another approach based on the resolvent was used in [7]. It yields a similar description of the solution as Eq. (4) provided that the initial data ψ_0 belongs to $D((T+K)^2)$. In [11], Song showed that, for B-convex spaces, the condition $\psi_0 \in D((T+K)^2)$ may be weakened, and (4) holds true for all ψ_0 belonging to $D(T+K)$. Since L_p-spaces for $1 < p < \infty$ are B-convex spaces, this result was applied to a transport equation in one-dimensional geometry [4] (we refer also to this paper for the definition of B-convex spaces and more information on this approach). Despite all these efforts, the case of transport equations with reentry boundary conditions in L^1-spaces lies outside the scope of these works.

In this paper we are concerned with compactness properties of the remainder term $R_n(t)$ of the Dyson–Phillips expansion for some integer n on Banach spaces. Our result apply directly to discuss the time asymptotic behaviour (for large times) of the solution of one-dimensional transport equations on L_1-spaces. Our main result is the following:

Theorem 1.1. *Assume that there exist $m \in \mathbb{N}$, $\omega > \omega(U)$ and $C > 0$ satisfying*

($\imath$) $(\lambda - T)^{-1}(K(\lambda - T)^{-1})^m$ *is compact for all λ such that* $\operatorname{Re}\lambda \geqslant \omega$;
($\imath\imath$) $|\operatorname{Im}\lambda| \|(\lambda - T)^{-1}(K(\lambda - T)^{-1})^m\| \leqslant C$ *for every* $\lambda \in \mathcal{R}_\omega$,

where $\mathcal{R}_\omega := \{\lambda \in \mathbb{C}$ such that $\operatorname{Re}\lambda \geqslant \omega\}$. Then $R_{2m+1}(t)$ is compact on X for each $t > 0$ and therefore, $U(t)$ and $V(t)$ have the same essential type.

The next corollary is useful in applications.

Corollary 1.1. *Assume that there exist $m \in \mathbb{N}$ and $\omega > \omega(U)$ satisfying*

($\imath$) $(\lambda - T)^{-1}(K(\lambda - T)^{-1})^m$ *is compact for all* λ *such that* $\operatorname{Re}\lambda \geqslant \omega$;
($\imath\imath$) $\lim_{|\operatorname{Im}\lambda| \to \infty} |\operatorname{Im}\lambda| \|(\lambda - T)^{-1}(K(\lambda - T)^{-1})^m\| = 0$ *for every* $\lambda \in \mathcal{R}_\omega$.

Then $R_{2m+1}(t)$ is compact on X for each $t > 0$ and therefore $U(t)$ and $V(t)$ have the same essential type.

The outline of this work is as follows. In Section 2 we present the proofs of Theorem 1.1 and Corollary 1.1. In Section 3 we apply our result to discuss the time asymptotic behaviour of the solution to the Cauchy problem governed by a one-dimensional transport equation with reentry boundary conditions on L^1-spaces.

In a forthcoming paper we will apply Theorem 1.1 and its corollary to discuss the time asymptotic behaviour for large times to the Rotenberg model (a transport equation arising in growing cell populations) on L^1-spaces. The analysis of this model on L^p-spaces with $1 < p < \infty$ was already done in [6].

2. Proofs

Before proceeding to the proofs we first recall some facts from functional analysis required below.

Definition 2.1. (See [3, Definition 6.4.1].) Let X be a Banach space and let $\vartheta(.)$ denote an X-valued function defined on the half-plane $\{\lambda \in \mathbb{C}\colon \operatorname{Re}\lambda > \alpha\}$ where $\alpha \in \mathbb{R}$. We say that $\vartheta(.)$ belongs to the class $H_p(\alpha, X)$ if the following conditions are satisfied:

(a) $\vartheta(.)$ is a function on complex numbers to X which is holomorphic for $\operatorname{Re}\lambda > \alpha$;
(b) $\sup_{\gamma > \alpha}\{\int_{\gamma - i\infty}^{\gamma + i\infty} \|\vartheta(\lambda)\|^p d\lambda\}^{\frac{1}{p}} < \infty$;
(c) $\lim_{\gamma \to \alpha} \vartheta(\gamma + i\ell) = \vartheta(\alpha + i\ell)$ exists for almost all values of ℓ and

$$\int_{\alpha - i\infty}^{\alpha + i\infty} \|\vartheta(\lambda)\|^p \, d\lambda < \infty. \tag{5}$$

The following result shows that every function in $H_p(\alpha, X)$ may be represented by a generalized Laplace integral.

Proposition 2.1. *(See [3, p. 230].) Let $\vartheta(.) \in H_p(\alpha, X)$ where $\alpha \geqslant 0$. Let $\gamma > \alpha$ and $\beta q > 1$ where $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\theta_\beta(t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{\lambda t} \lambda^{-\beta} \vartheta(\lambda) \, d\lambda$$

defines a continuous function on $(0, \infty)$ to X and

$$\vartheta(\lambda) = \lambda^\beta \int_0^\infty e^{-\lambda t} \theta_\beta(t) \, dt$$

the integral being absolutely convergent for $\operatorname{Re}\lambda > \alpha$.

For $p = 1$ (see [3, p. 230]), we may take $\beta = 0$, obtaining

$$\vartheta(\lambda) = \int_0^\infty e^{-\lambda t} \theta_0(t) \, dt.$$

Now we are in a position to prove Theorem 1.1.

Proof of Theorem 1.1. Set

$$\vartheta(\lambda) := (\lambda - T)^{-1}\left[K(\lambda - T)^{-1}\right]^{(2m+1)}$$

for $\lambda \in \mathcal{R}_\omega$. It follows from the hypothesis ($\iota\iota$) that there exists $\chi > 0$ such that

$$\|\vartheta(\lambda)\| \leqslant \frac{\chi}{|\mathrm{Im}\,\lambda|^2} \text{ uniformly on } \mathcal{R}_\omega.$$

Using the last estimate and the fact that $\vartheta(.)$ is holomorphic in the interior of $\mathcal{R}_\omega$, we infer from Definition 2.1 that $\vartheta(.)$ belongs to $H_1(\alpha, X)$ (with $\alpha > \max(\omega, 0)$). This implies, via Proposition 2.1, that there exists the continuous function

$$\theta(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t}\vartheta(\lambda)\,d\lambda, \quad \gamma > \alpha,\ t \geqslant 0, \tag{6}$$

satisfying

$$\int_0^\infty e^{-\lambda t}\theta(t)\,dt = \vartheta(\lambda),$$

where the integrals are considered in the strong sense. On the other hand, for any λ satisfying $\mathrm{Re}\,\lambda > \omega(U)$, one can write

$$\int_0^\infty e^{-\lambda t}U_{2m+1}(t)\,dt = (\lambda - T)^{-1}\big[K(\lambda - T)^{-1}\big]^{2m+1} = \vartheta(\lambda).$$

The uniqueness of the Laplace transform yields $U_{2m+1}(t) = \theta(t)$. Next, using the hypothesis (ι) we conclude that $\vartheta(\lambda)$ is compact for all λ such that $\mathrm{Re}\,\lambda \geqslant \alpha$. Applying the convex compactness property [19] to (6), we infer that $\theta(t)$ (and then $U_{2m+1}(t)$) is compact for all $t > 0$. Now the use of Theorem 2.6 in [8] concludes the proof. □

Proof of Corollary 1.1. For $\lambda \in \mathcal{R}_\omega$ set $\Delta_m(\lambda) = \|(\lambda - T)^{-1}(K(\lambda - T)^{-1})^m\|$. According to the assumption ($\iota\iota$) there exists $\nu > 0$ such that

$$|\mathrm{Im}\,\lambda|\Delta_m(\lambda) \leqslant 1 \text{ uniformly on } \big\{\lambda \in \mathbb{C}\colon\ \mathrm{Re}\,\lambda \geqslant \omega \text{ and } |\mathrm{Im}\,\lambda| > \nu\big\}. \tag{7}$$

On the other hand, for $\mathrm{Re}\,\lambda \geqslant \omega$, we have

$$|\mathrm{Im}\,\lambda|\Delta_m(\lambda) \leqslant |\mathrm{Im}\,\lambda|\|K\|^m\big\|(\lambda - T)^{-1}\big\|^{m+1}.$$

Since the operator T generates a strongly continuous semigroup on X, there exists $M > 0$ such that

$$\big\|(\lambda - T)^{-1}\big\| \leqslant \frac{M}{\mathrm{Re}\,\lambda - \omega} \quad \text{for } \mathrm{Re}\,\lambda > \omega.$$

This together with the boundedness of K implies the existence of a real $\tau > 0$ such that

$$|\mathrm{Im}\,\lambda|\Delta(\lambda) \leqslant 1 \text{ uniformly on } \big\{\lambda \in \mathbb{C}\colon\ \mathrm{Re}\,\lambda \geqslant \tau \text{ and } |\mathrm{Im}\,\lambda| \leqslant \nu\big\}. \tag{8}$$

Next, using (7) and (8) together with the continuity of the map $\lambda \mapsto |\mathrm{Im}\,\lambda|\Delta(\lambda)$ on the compact set $\{\lambda \in \mathbb{C}\colon\ \omega \leqslant \mathrm{Re}\,\lambda \leqslant \tau$ and $|\mathrm{Im}\,\lambda| \leqslant \nu\}$ we infer that $|\mathrm{Im}\,\lambda|\|(\lambda - T)^{-1}(K(\lambda - T)^{-1})^m\|$ is uniformly bounded on $\mathcal{R}_\omega$. Now the result follows from Theorem 1.1. □

3. Application to a transport model

The goal of this section is to apply our result to the following neutron transport equation

$$\begin{cases} \dfrac{\partial\psi}{\partial t}(x,\nu,\mu,t) = -\nu\mu\dfrac{\partial\psi}{\partial x}(x,\nu,\mu,t) - \sigma(x,\nu,\mu)\psi(x,\nu,\mu,t) + \displaystyle\int_D\int_V \kappa(x,\nu,\nu',\mu,\mu')\psi(x,\nu',\mu',t)\,d\nu'\,d\mu', \\ \psi(-a,\nu,\mu,t) = \alpha(\nu,\mu)\psi(-a,\nu,-\mu,t), \quad 0 < \mu \leqslant 1, \\ \psi(a,\nu,-\mu,t) = \gamma(\nu,\mu)\psi(a,\nu,\mu,t), \quad 0 < \mu \leqslant 1, \\ \psi(x,\nu,\mu,0) = \psi_0(x,\nu,\mu), \\ x \in Q := [-a,a], \quad \nu \in V := (0,\nu_M], \quad \mu \in D := [-1,1], \quad t > 0, \end{cases} \tag{9}$$

where $a > 0$, $0 < \nu_M < +\infty$, $\alpha(\nu,\mu)$ and $\gamma(\nu,\mu)$ are reflection coefficients, $\psi(x,\nu,\mu,t)$ is the particle density within the slab of thickness $2a$, $\sigma(x,\nu,\mu)$ is the total cross section frequency, $\kappa(x,\nu,\nu',\mu,\mu')$ is the scattering fission kernel, and $\psi_0(x,\nu,\mu)$ is the initial distribution.

Let us now introduce the following hypotheses:

$$(\mathcal{H})\quad \begin{cases} \bullet\ \alpha(\nu,\mu) \text{ and } \gamma(\nu,\mu) \text{ are measurable functions satisfying } 0 \leqslant \alpha(\nu,\mu)=\alpha(\nu,-\mu)\leqslant 1, \\ \quad 0\leqslant \gamma(\nu,\mu)=\gamma(\nu,-\mu)\leqslant 1. \\ \bullet\ \sigma(x,\nu,\mu) \text{ is an essentially bounded real measurable function.} \\ \bullet\ \kappa(x,\nu,\nu',\mu,\mu') \text{ is a real measurable function satisfying } |\kappa(x,\nu,\nu',\mu,\mu')| \leqslant C\big(\nu|\mu|\big)^{-\delta}, \\ \quad \text{where } C \text{ and } \delta \text{ are nonnegative constants, } 0\leqslant\delta<1. \\ \bullet\ \alpha(\nu,\mu),\ \gamma(\nu,\mu),\ \sigma(r,\nu,\mu) \text{ and } \tilde\kappa(x,\nu,\nu',\mu,\mu') := \big(\nu|\mu|\big)^{\delta}\kappa(x,\nu,\nu',\mu,\mu') \\ \quad \text{are partially differentiable with respect to } \mu,\mu'\in D \text{ a.e.,} \\ \quad \text{and the corresponding partial derivatives } (\partial\alpha/\partial\mu),\ (\partial\gamma/\partial\mu),\ (\partial\sigma/\partial\mu), \\ \quad (\partial\tilde\kappa/\partial\mu) \text{ and } (\partial\tilde\kappa/\partial\mu') \text{ are essentially bounded.} \end{cases}$$

Set $G = Q\times V\times D$, and define the streaming operator T on the complex Banach space $L^1(G)$ as follows:

$$(T\psi)(x,\nu,\mu) = -\nu\mu\frac{\partial\psi}{\partial x}(x,\nu,\mu) - \sigma(x,\nu,\mu)\psi(x,\nu,\mu),$$

with

$$D(T) = \big\{\psi\in L^1(G)\colon\ T\psi\in L^1(G),\ \psi(-a,\nu,\mu)=\alpha(\nu,\mu)\psi(-a,\nu,-\mu) \\ \text{and } \psi(a,\nu,-\mu)=\gamma(\nu,\mu)\psi(a,\nu,\mu) \text{ for every } \mu\in(0,1]\big\}.$$

The integral part of Eq. (9) defines the collision operator. It is a bounded operator on $L^1(G)$ given by

$$(K\psi)(x,\nu,\mu) = \int_D\int_V \kappa(x,\nu,\nu',\mu,\mu')\psi(x,\nu',\mu')\,d\nu'\,d\mu',$$

while the transport operator is the bounded perturbation of T by K, i.e. $A := T+K$ with $D(A)=D(T)$. Now Eq. (9) can be written as

$$\frac{d\psi}{dt}(t) = A\psi(t), \qquad \psi(0)=\psi_0.$$

Let λ^* be the real defined by

$$\lambda^* = \operatorname*{ess\text{-}inf}_{(x,\nu,\mu)\in G}\sigma(x,\nu,\mu).$$

Recall that the half-plane $\{\lambda\in\mathbb{C}\colon\ \operatorname{Re}\lambda > -\lambda^*\}$ is contained in the resolvent set of T [13] and, for any λ such that $\operatorname{Re}\lambda>-\lambda^*$ and $\psi\in L^1(G)$, we have

$$\big((\lambda-T)^{-1}\psi\big)(x,\nu,\mu) = \frac{P_1\psi + \alpha(\nu,\mu)P_2\psi + \alpha(\nu,\mu)\gamma(\nu,\mu)P_3\psi}{1-\alpha(\nu,\mu)\gamma(\nu,\mu)\exp[-(2/\nu\mu)\int_{-a}^{a}\Delta(\lambda)\,ds]}$$

for $\mu>0$, and

$$\big((\lambda-T)^{-1}\psi\big)(x,\nu,\mu) = \frac{P_4\psi + \gamma(\nu,\mu)P_5\psi + \alpha(\nu,\mu)\gamma(\nu,\mu)P_6\psi}{1-\alpha(\nu,\mu)\gamma(\nu,\mu)\exp[-(2/\nu\mu)\int_{-a}^{a}\Delta(\lambda)\,ds]}$$

for $\mu<0$, where $\Delta(\lambda)=\lambda+\sigma(s,\nu,\mu)$, the symmetry of α, γ has been employed, and $P_1, P_2, \ldots, P_6$ are bounded operators on $L^1(G)$ given by [13, p. 105]

$$(P_1\psi)(x,\nu,\mu) = \frac{1}{\nu\mu}\int_{-a}^{x}\psi(x',\nu,\mu)\exp\left[-\frac{1}{\nu\mu}\int_{x'}^{x}\Delta(\lambda)\,ds\right]dx',$$

$$(P_2\psi)(x,\nu,\mu) = \frac{1}{\nu\mu}\int_Q\psi(x',\nu,-\mu)\exp\left[-\frac{1}{\nu\mu}\left(\int_{-a}^{x}\Delta(\lambda)\,ds + \int_{-a}^{x'}\Delta(\lambda)\,ds\right)\right]dx',$$

$$(P_3\psi)(x,\nu,\mu) = \frac{1}{\nu\mu}\exp\left[-\frac{2}{\nu\mu}\int_Q\Delta(\lambda)\,ds\right]\int_x^a\psi(x',\nu,\mu)\exp\left[-\frac{1}{\nu\mu}\int_{x'}^{x}\Delta(\lambda)\,ds\right]dx',$$

$$(P_4\psi)(x,\nu,\mu) = \frac{1}{\nu\mu}\int_x^a \psi(x',\nu,\mu)\exp\left[-\frac{1}{\nu\mu}\int_x^{x'}\Delta(\lambda)\,ds\right]dx',$$

$$(P_5\psi)(x,\nu,\mu) = -\frac{1}{\nu\mu}\int_Q \psi(x',\nu,-\mu)\exp\left[\frac{1}{\nu\mu}\left(\int_x^a \Delta(\lambda)ds + \int_{x'}^a \Delta(\lambda)\,ds\right)\right]dx',$$

$$(P_6\psi)(x,\nu,\mu) = -\frac{1}{\nu\mu}\exp\left[\frac{2}{\nu\mu}\int_Q \Delta(\lambda)\,ds\right]\int_{-a}^x \psi(x',\nu,\mu)\exp\left[\frac{1}{\nu\mu}\int_x^{x'}\Delta(\lambda)\,ds\right]dx'.$$

It is well known that the streaming operator T generates a strongly continuous positive semigroup $(S(t))_{t\geqslant 0}$ on $L^1(G)$ (see, for example, [2]). Since the collision operator K is bounded and positive, the transport operator A generates also a strongly continuous positive semigroup $(\tilde{S}(t))_{t\geqslant 0}$ on $L^1(G)$ given by the Dyson–Phillips expansion (see the Introduction).

Theorem 3.1. *Assume that $(\mathcal{H})$ holds true, then the semigroups $(S(t))_{t\geqslant 0}$ and $(\tilde{S}(t))_{t\geqslant 0}$ have the same essential type ω_e, that is,*

$$r_e\big(\tilde{S}(t)\big) = r_e\big(S(t)\big) = e^{\omega_e t} \leqslant e^{-t\lambda^*} \quad \textit{for all } t \geqslant 0.$$

To prove this result it suffices to establish that a remainder term of the Dyson–Phillips expansion $R_n(t)$, for some integer n, is compact. So, we have only to check that the conditions $(\imath)$ and $(\imath\imath)$ of Corollary 1.1 are satisfied. This will be done in the following lemmas.

Lemma 3.1. *If the hypotheses of Theorem* 3.1 *are satisfied, then, for every $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > -\lambda^*$ and $m \geqslant 3$, the operator $(\lambda I - T)^{-1}[K(\lambda - T)^{-1}]^m$ is compact on $L^1(G)$.*

Proof. Since $(\mathcal{H})$ is satisfied, Theorem 3.8 in [12] implies that, for $\operatorname{Re}\lambda > -\lambda^*$, the operator $[K(\lambda - T)^{-1}]^2 K$ is compact on $L^1(G)$. Now the result follows from the fact that, for $m \geqslant 3$, the operator $(\lambda - T)^{-1}[K(\lambda - T)^{-1}]^m$ can be written in the form

$$(\lambda - T)^{-1}\big[K(\lambda - T)^{-1}\big]^{m-3}\big(\big[K(\lambda - T)^{-1}\big]^2 K\big)(\lambda - T)^{-1}. \qquad \square$$

Lemma 3.2. *Let ω be a constant satisfying $\omega > -\lambda^*$ and assume that the hypotheses of Theorem* 3.1 *hold true. Then, for $m > (1-\delta)^{-1}$, we have*

$$\lim_{|\operatorname{Im}\lambda|\to\infty} |\operatorname{Im}\lambda| \big\|(\lambda - T)^{-1}\big(K(\lambda - T)^{-1}\big)^{2m}\big\| = 0$$

for every λ in $\mathcal{R}_\omega = \{\lambda \in \mathbb{C}$ such that $\operatorname{Re}\lambda \geqslant \omega\}$.

Proof. If $(\mathcal{H})$ is satisfied, then, according to Lemma 3.1 in [12], there exist constants $\Xi_0 > 0$ and $\tilde{\tau}$ such that

$$\big\|K(\lambda - T)^{-1}K\big\| \leqslant \Xi_0\big|\omega + \lambda^* + i\tau\big|^{\delta-1}\ln\big|\omega + \lambda^* + i\tau\big|$$

uniformly on $\{\lambda = \beta + i\tau\colon\ \beta \geqslant \omega,\ |\tau| \geqslant \tilde{\tau}\}$. Further, for every λ with $\operatorname{Re}\lambda > -\lambda^*$ we have

$$\big\|(\lambda - T)^{-1}\big\| \leqslant \big(\operatorname{Re}\lambda + \lambda^*\big)^{-1}.$$

Accordingly, putting $\zeta(\lambda) = \|(\lambda - T)^{-1}(K(\lambda - T)^{-1})^{2m}\|$, we get

$$\zeta(\lambda) \leqslant \big\|(\lambda - T)^{-1}\big\|^{m+1}\big\|K(\lambda - T)^{-1}K\big\|^m \leqslant \Xi_1\frac{[\ln|\omega + \lambda^* + i\tau|]^m}{|\omega + \lambda^* + i\tau|^{m(1-\delta)}}$$

uniformly on $\{\lambda = \beta + i\tau\colon\ \beta \geqslant \omega,\ |\tau| \geqslant \tilde{\tau}\}$, where $\Xi_1 = \frac{\Xi_0{}^m}{(\omega+\lambda^*)^{m+1}}$.

Next, since $m(1-\delta) > 1$, it follows that there exists $\delta' > 0$ such that

$$|\operatorname{Im}\lambda|\zeta(\lambda) \leqslant \Xi_1\frac{[\ln|\omega + \lambda^* + i\tau|]^m}{|\omega + \lambda^* + i\tau|^{\delta'}}$$

uniformly on $\{\lambda = \beta + i\tau\colon\ \beta \geqslant \omega,\ |\tau| \geqslant \tilde{\tau}\}$. This implies

$$\lim_{|\operatorname{Im}\lambda|\to\infty} |\operatorname{Im}\lambda| \big\|(\lambda - T)^{-1}\big(K(\lambda - T)^{-1}\big)^{2m}\big\| = 0$$

uniformly on $\mathcal{R}_\omega$ which ends the proof. $\square$

Proof of Theorem 3.1. Since the semigroup $(S(t))_{t\geqslant 0}$ is positive and the spectral bound of T is equal to $-\lambda^*$ (see [12] or [13]), it follows from [21] that $\omega(S) = -\lambda^*$. Hence the essential type $\omega_e(S)$ of $(S(t))_{t\geqslant 0}$ satisfies $\omega_e(S) \leqslant -\lambda^*$ and therefore

$$r_e\big(S(t)\big) = e^{\omega_e(S)t} \leqslant e^{-\lambda^* t} \quad \text{for all } t \geqslant 0.$$

Now, using Lemmas 3.1 and 3.2 together with Corollary 1.1 we conclude the result for $m > \max\{3, 2(1-\delta)^{-1}\}$ and $\omega > -\lambda^*$. □

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