



Functions represented as sums of two quasicontinuous functions with a closed graph

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ARTICLE INFO

Article history:

Received 31 March 2009

Available online 25 July 2009

Submitted by B.S. Thomson

Keywords:

Functions with closed graph

Quasicontinuity

ABSTRACT

Let $X = (X, d)$ be a metric space. We prove that if the set $D(f)$ of discontinuity points of a function $f : X \rightarrow \mathbf{R}$ is “sufficiently small,” then f can be decomposed into a sum of two quasicontinuous functions with a closed graph.

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1. Introduction

Let $X = (X, d)$ be an infinite metric space, and let $C(X)$, $U(X)$, $Q(X)$, $P(X)$, and $B_1^*(X)$ denote the sets of all functions $f : X \rightarrow \mathbf{R}$ that are continuous, with a closed graph, quasicontinuous, piecewise continuous, and Baire-one-star functions, respectively (the definitions of the latter three symbols are given below). All these sets consist of Baire class 1 functions [6,10].

If \mathcal{F} , \mathcal{G} are nonempty subsets of \mathbf{R}^X , then $\mathcal{F} + \mathcal{G} := \{f + g : f \in \mathcal{F}, g \in \mathcal{G}\}$, and $\mathcal{F} \cdot \mathcal{G} := \{f \cdot g : f \in \mathcal{F}, g \in \mathcal{G}\}$. We also write $\mathcal{F}\mathcal{G}$ instead of $\mathcal{F} \cap \mathcal{G}$; notice that $\mathcal{F}\mathcal{G} \neq \mathcal{F} \cdot \mathcal{G}$ in general. For a function $f : X \rightarrow \mathbf{R}$ the symbols $C(f)$ and $D(f)$ ($= X \setminus C(f)$) denote the sets of continuous and discontinuous points of f , respectively. The function f is *piecewise continuous* if there is a sequence (X_n) of closed subsets of X such that $X = \bigcup_{n=1}^{\infty} X_n$ and all the restrictions $f|_{X_n}$ are continuous. The function f is said to be *quasicontinuous* [9, p. 526] (and *Baire-one-star*, resp.) if for every $x \in X$ there is a sequence (x_n) in $C(f)$ such that $x_n \rightarrow x$ and $f(x_n) \rightarrow f(x)$ as $n \rightarrow \infty$ (and for every nonempty closed subset F of X the interior of $C(f|_F)$ is not empty, resp.).

It is known [6, Lemma 2.3] that $U(X) \subset P(X)$, and that $B_1^*(X) \subset P(X)$ with the equality $B_1^*(X) = P(X)$ for X complete [10, Theorem 2.3].

In this paper we consider the problem of decomposing a given “bad” function $f : X \rightarrow \mathbf{R}$ into a sum of two “nice” functions. For example, a classical result of Lindenbaum [12] asserts that every function $f : \mathbf{R} \rightarrow \mathbf{R}$ is a sum of two Darboux functions (i.e., mapping intervals onto intervals):

$$\mathbf{R}^{\mathbf{R}} = \mathcal{D} + \mathcal{D}$$

(here \mathcal{D} denotes the set of all Darboux functions on \mathbf{R}).

Within the last fifteen years a series of similar results were obtained (see [2–5,8] for decompositions into sums of quasicontinuous or closed graph functions). We list below only a few of them. In 2002 Borsík [5, Theorem 2] showed that

$$P(X) = U(X) + U(X). \quad (1)$$

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(It is worth noticing that the class $P(X)$ appears also in a dichotomy result by Solecki [14, Theorem 3.1] on the structure of Baire class 1 functions on X .)

Earlier, in 1999, Borsík, Doboš and Repický proved that, for X separable, every Baire-one-star function on X is a sum of three quasicontinuous functions with a closed graph:

$$B_1^*(X) \subset QU(X) + QU(X) + QU(X) \quad (2)$$

[6, Theorem 4.1], and that $B_1^*(X)$ is a linear subspace of \mathbf{R}^X with the equality

$$B_1^*(X) = P(X) = \text{lin}(QU(X)) = QU(X) + QU(X) + QU(X) \quad (3)$$

for X a Polish space [6, p. 680 and Theorem 2.5]. Hence, by (1), we have additionally

$$B_1^*(X) = U(X) + U(X) \quad \text{for } X \text{ a Polish space.} \quad (4)$$

In [6, Remark 2.8] the authors show that $B_1^*(X) \neq QU(X) + QU(X)$ for $X = \mathbf{R}$; hence, by (3), the set $QU(X) + QU(X)$ need not be a linear subspace of $B_1^*(X)$, in general (however, if the metric d on X is discrete then we evidently have $QU(X) (= \mathbf{R}^X) = B_1^*(X)$).

Moreover, in [6, Problem 2.9] the authors ask us to characterize the set $QU(X) + QU(X)$. Using some ideas from the paper by Borsík, Doboš and Repický [6] we shall show in Theorem 1 below that if the set $D(f)$ of $f \in \mathbf{R}^X$ is “relatively small” (at least closed, separable and nowhere dense), then f is a sum of two quasicontinuous functions with a closed graph. For this purpose we define a class $B_1^\#(X)$ consisting of the functions $f \in \mathbf{R}^X$ that are continuous or fulfil the following three conditions:

- (t_1) the set $D(f)$ is separable,
- (t_2) the restriction $f|_{D(f)}$ is continuous,
- (t_3) $\overline{\lim}_{u \rightarrow x} |f(u)| = \infty$ for every $x \in D(f)$ (i.e., by condition (t_2), for every $x \in D(f)$ there is a sequence (u_n) of continuity points of f such that $|f(u_n)| \uparrow \infty$).

Remark 1. From the conditions (t_2) and (t_3) it immediately follows that for every $f \in B_1^\#(X)$ the set $D(f)$ has empty interior (i.e., $C(f)$ is dense in X), with the convention that $f|_\emptyset$ is continuous. This simple observation allows us to indicate two nontrivial examples of elements of $B_1^\#(X)$:

- (a) every discontinuous $f : X \rightarrow \mathbf{R}$ with a closed graph and $D(f)$ finite;
- (b) every function f_F of the form $f_F(x) := 1/\text{dist}(x, F)$ if $x \notin F$, and $f_F \equiv 0$ on F , where F is an infinite, closed, separable and nowhere dense subset of X .

This example applies to the case X a nonseparable Banach space and F a (linear) separable and closed subspace of X : here $\text{dist}(x, F)$ equals the norm $\|[x]\|$ of the element $[x] = x + F$ of the quotient space X/F ; a classical example is $X = \ell_\infty$ and $F = c_0$. See also Remark 3 and Lemma 2 in Section 3, which provide more examples showing the class $B_1^\#(X)$ is sufficiently rich.

Remark 2. In the proof of our Theorem 1 we shall use essentially the fact that for every function $f \in B_1^\#(X)$ the set $D(f)$ is closed. This is a particular case (i.e., a consequence of conditions (t_2) and (t_3)) of a more general situation presented in Lemma 1 of Section 3. The lemma applies also to functions $g : X \rightarrow \mathbf{R}$ with a closed graph (giving $D(g)$ closed), which was discovered in 1964 by Kostyrko and Šalát [11], and generalized in 1985 by Doboš [7, Theorem 3].

2. The main result

Our main result reads as follows.

Theorem 1. Let X be an infinite metric space. Every element $f \in B_1^\#(X)$ is a sum of two quasicontinuous functions on X with a closed graph, i.e.,

$$B_1^\#(X) \subset QU(X) + QU(X). \quad (5)$$

Moreover, the above inclusion is proper for $X = \mathbf{R}$ (here $QU(\mathbf{R}) \subsetneq B_1^\#(\mathbf{R})$); in this case, by (3), $B_1^\#(X)$ is a proper subset of $B_1^*(X)$.

We shall show now that, in a sense, inclusion (5) is optimal: if $B_1^\#(\mathbf{R})$ in (5) is replaced by a natural superset, the inclusion becomes false. For this purpose consider the set $B_1^{**}(X)$ of all the functions $f : X \rightarrow \mathbf{R}$ that are continuous on $D(f)$. This set was defined and studied in 2000 by Pawlak [13] who proved, among other things, that for every such a function f the set $D(f)$ is nowhere dense, that f is continuous on the closure $\text{cl}(D(f))$, and that $B_1^{**}(\mathbf{R})$ is a proper subset of $B_1^*(\mathbf{R})$.

(see [13, Lemmas 1 and 2, Proposition 1]). Let $t_3(X)$ denote the set of all real-valued functions on X that fulfil axiom (t_3) . Then, by definition, we have the equality

$$B_1^\#(X) = B_1^{**}(X) \cap t_3(X) \quad \text{for } X \text{ a Polish space.}$$

It is easy to see that $B_1^\#(\mathbf{R})$ is a *proper* subset of $B_1^{**}(\mathbf{R})$, as the latter set contains *bounded* discontinuous elements. Let $f_0 \in B_1^{**}(\mathbf{R})$ denote the characteristic function of the singleton $\{0\}$. It is known [6, Remark 2.8] that $f_0 \notin QU(\mathbf{R}) + QU(\mathbf{R})$; this proves the inclusion (5) cannot be extended to the superset $B_1^{**}(X)$ of $B_1^\#(X)$, in general: $B_1^{**}(\mathbf{R}) \not\subset QU(\mathbf{R}) + QU(\mathbf{R})$.

An application of Theorem 1 to the function $g = \ln f$ gives immediately

Theorem 2. *Let X be a metric space, and let f be a strictly positive function on X that fulfils the above conditions (t_1) and (t_2) . If $\overline{\lim}_{u \rightarrow x} f(u) = \infty$ or $\underline{\lim}_{u \rightarrow x} f(u) = 0$ for every $x \in D(f)$, then f is a product of two strictly positive quasicontinuous functions on X with a closed graph.*

From Theorems 1 and 2 we obtain the following corollary.

Corollary 1. *Let X be a metric space, and let f be a strictly positive function on X that fulfils the above conditions (t_1) and (t_2) . If $\overline{\lim}_{u \rightarrow x} f(u) = \infty$ for every $x \in D(f)$, then $f = g_1 + g_2 = g_3 \cdot g_4$, where g_1, g_2, g_3, g_4 are quasicontinuous functions on X with a closed graph.*

Remark 3. In 2006 Strońska [15, Theorems 1, 3] proved that if $f : \mathbf{R}^m \rightarrow \mathbf{R}$ is almost continuous (i.e., $D(f)$ is of Lebesgue measure 0 on \mathbf{R}^m) then f has two decompositions: $f = f_1 + f_2 = c + f_3 \cdot f_4$, where $c \in \mathbf{R}$ and f_1, f_2, f_3, f_4 are the so-called *strongly continuous* functions on \mathbf{R}^m . However, if the set $D(f)$ is of positive Lebesgue measure on $X = \mathbf{R}^m$, Strońska's result becomes useless. Nevertheless our Theorem 1 still admits a decomposition of f into “nice” functions as long as f fulfils conditions $(t_1), (t_2), (t_3)$. This is so, e.g., for the function $f_F \in B_1^\#(\mathbf{R})$ defined in Remark 1(b) with $F = C_p \subset [0, 1]$, where C_p is a Cantor-like set of Lebesgue measure $0 < p < 1$.

3. The set $B_1^\#(X)$

In Remark 1 we have noticed that for every function $f \in B_1^\#(X)$ the set $D(f)$ has empty interior. In this section we give a few additional informations about the elements of $B_1^\#(X)$.

The lemma below generalizes similar results for $f \in U(X)$ obtained earlier by Kostyrko and Šalát [11], Baggs [1] (for X a metric space), and by Doboš [7, Theorem 3] (for X a topological space), and it will be applied in the proof of Theorem 1.

Lemma 1. *Let X be a metric space, and let $f : X \rightarrow \mathbf{R}$ be a discontinuous function such that*

$$\overline{\lim}_{u \rightarrow x} |f(u)| = \infty \quad \text{for every } x \in D(f). \quad (6)$$

Then the set $D(f)$ is closed. In particular, this property possesses every discontinuous function $f \in B_1^\#(X) \cup U(X)$.

Proof. Assume the conclusion is false. Fix an element x in the set

$$\text{cl}(D(f)) \setminus D(f) = \text{cl}(D(f)) \cap C(f), \quad (7)$$

and let (x_j) be a sequence in $D(f)$ such that

$$\lim_{j \rightarrow \infty} \text{dist}(x_j, x) = 0. \quad (8)$$

By (6), there is a sequence (u_j) in X such that

$$\lim_{j \rightarrow \infty} \text{dist}(u_j, x_j) = 0 \quad \text{and} \quad |f(u_j)| \geq 1 + |f(x)| \quad \text{for all } j\text{'s}. \quad (9)$$

From (8) and the first condition in (9) we obtain $u_j \rightarrow x$ as $j \rightarrow \infty$, but since, by (7), x is a continuity point of f , from the second condition in (9) we obtain a contradiction: $|f(x)| \geq 1 + |f(x)|$. The contradiction implies the set $D(f)$ is closed, as claimed.

The particular case for $f \in B_1^\#(X)$ follows immediately from condition (t_3) ; and if the graph of f is closed, f is necessarily unbounded on every neighbourhood of any discontinuity point of f . \square

Let $B_{01}^\#(X)$ denote the subset of $B_1^\#(X)$ consisting of the elements f such that $f|_{D(f)} \equiv 0$. From Lemma 1, and from the Tietze extension theorem and the axiom (t_2) it now follows that every $f \in B_1^\#(X)$ has the decomposition $f = f_0 + g$, where $f_0 \in B_{01}^\#(X)$ and $g \in C(X)$; hence

$$B_1^\#(X) = B_{01}^\#(X) + C(X). \quad (10)$$

(The above decomposition will allow us to reduce the proof of Theorem 1 to a simpler case, i.e., to consider only elements of $B_{01}^\#(X)$.)

Let $U_c(X)$ be the set of the functions $f \in U(X)$ such that the restriction $f|_{D(f)}$ is continuous (recall that in Remark 1 we have agreed that $f|_\emptyset$ is continuous), and let $U_0(X) := \{f \in U_c(X) : f|_{D(f)} \equiv 0\}$. If X is a Baire space (e.g., if it is complete), then for every $f \in U(X)$ the set $D(f)$ is nowhere dense [11, Theorems 4 and 5] (cf. [7, p. 62]). The latter assumption on X is essential, and the example below, illustrating this, is surely known: let $X = \{x_n : n = 1, 2, \dots\}$ be the set of all rational numbers endowed with the natural topology; then the function f of the form $f(x_n) = n$ has a closed graph with $D(f) = X$. Hence, $U(X) \neq U_c(X)$, in general, and if X has the Baire property then $D(f)$ is a “small” subset of X . Despite of these “bad” and “nice” properties of $U(X)$, by the Tietze theorem (and by Lemma 1), for every metric space X the set $U_c(X)$ has a decomposition similar to $B_1^\#(X)$ in (10):

$$U_c(X) = U_0(X) + C(X). \quad (11)$$

Let $QU_c(X) = Q(X) \cap U_c(X)$. The next lemma complements partially Theorem 1.

Lemma 2. *Let X be a separable metric space. Then*

$$QU_c(X) \subset U_c(X) \subset B_1^\#(X), \quad (12)$$

yet $QU(\mathbf{R}) \not\subset B_1^\#(\mathbf{R})$. Hence, $U(\mathbf{R}) \not\subset B_1^\#(\mathbf{R})$ and $B_1^\#(\mathbf{R}) \neq QU(\mathbf{R}) + QU(\mathbf{R})$.

Proof. Since every discontinuous $f \in U_c(X)$ fulfils axiom (t_2) by definition, and f fulfils (t_3) (see the end of the proof of Lemma 1), the separability of X implies that $f \in B_1^\#(X)$. This proves inclusions (12).

Set $x_n = -1/n$, where $n \in \mathbf{N}$, and consider the function f on \mathbf{R} of the form

$$f(x) = \begin{cases} 0 & \text{if } x \geq 0, \\ \frac{1}{-x-1} & \text{if } x < -1, \\ \frac{1}{x_{n+1}-x} & \text{if } x \in [x_n, x_{n+1}), n = 1, 2, \dots \end{cases}$$

Then $f(x_n) = n \cdot (n+1)$ for all n 's, $D(f) = \{x_n : n \in \mathbf{N}\} \cup \{0\}$, and f has a closed graph and is quasicontinuous (as it is continuous on $(-\infty, -1)$ and right-continuous on $[-1, \infty)$), i.e., $f \in QU(\mathbf{R})$. Moreover, f fulfils conditions (t_1) and (t_3) , yet $f \notin B_1^\#(\mathbf{R})$ because the restriction $f|_{D(f)}$ is discontinuous at 0. Hence $QU(\mathbf{R}) \not\subset B_1^\#(\mathbf{R})$, as claimed. This immediately implies the equality $B_1^\#(\mathbf{R}) = QU(\mathbf{R}) + QU(\mathbf{R})$ is impossible. \square

4. The proof of Theorem 1

By equality (10), we shall prove the equivalent (to (5)) inclusion

$$B_{01}^\#(X) \subset QU(X) + QU(X) \quad (5')$$

is true. For this purpose, let us fix a function $f \in B_1^\#(X)$, and let $f|_{D(f)} \equiv 0$, i.e., $f \in B_{01}^\#(X)$. It is obvious we have to consider only the case f discontinuous. We shall give below explicit formulas defining two functions $h_1, h_2 \in QU(X)$ such that $f = h_1 + h_2$. Both the functions are built by the use of a countable family \mathcal{L} of open subsets of $C(f) = X \setminus D(f)$, whose properties are listed in Lemma 3 below.

To simplify notations, let, for a family \mathcal{F} of nonempty subsets of X , the symbols $K_{\mathcal{F}}$ and $U_{\mathcal{F}}$ denote the sets $\bigcup_{K \in \mathcal{F}} \text{cl}(K)$ and $\bigcup_{K \in \mathcal{F}} K$, respectively.

The (key) lemma below is an extended version of [6, Lemma 3.3]; its proof follows the proof of Theorem 1.

Lemma 3. *For every discontinuous function $f \in B_1^\#(X)$ there exist infinite families \mathcal{A}, \mathcal{B} of open subsets of X such that, for the family $\mathcal{L} := \mathcal{A} \cup \mathcal{B}$ we have:*

- (i) $K_{\mathcal{L}} \subset C(f)$ (i.e., $\text{cl}(K) \cap D(f) = \emptyset$ for every $K \in \mathcal{L}$; equivalently, $K_{\mathcal{L}} \cap D(f) = \emptyset$); in particular, $U_{\mathcal{A}}$ and $U_{\mathcal{B}}$ are open subsets of $C(f)$;
- (ii) for every $x \in C(f)$ there is an open neighbourhood U of x such that U meets at most one element of the family $\{\text{cl}(K) : K \in \mathcal{L}\}$;

(iii) for every sequence (K_p) of pairwise distinct elements of $K_{\mathcal{L}}$, and every sequence (y_p) with $y_p \in \text{cl}(K_p)$, $p = 1, 2, \dots$,

$$\overline{\lim}_{p \rightarrow \infty} |f(y_p)| = \infty;$$

in particular, the above equality holds for every convergent sequence $(y_p) \subset K_{\mathcal{L}}$ such that the limit $\lim_{p \rightarrow \infty} y_p$ is in $D(f)$;

(iv) $\text{cl}(G) \cap \text{cl}(H) = \emptyset$ for every pair of distinct elements $G, H \in \mathcal{L}$; in particular, $K_{\mathcal{A}} \cap K_{\mathcal{B}} = \emptyset$ and $\mathcal{A} \cap \mathcal{B} = \emptyset$;

(v) $D(f) \subset \text{cl}(U_{\mathcal{A}}) \cap \text{cl}(U_{\mathcal{B}}) = \text{cl}(K_{\mathcal{A}}) \cap \text{cl}(K_{\mathcal{B}})$ (notice that $\text{cl}(U_{\mathcal{A}}) = \text{cl}(K_{\mathcal{A}})$ and $\text{cl}(U_{\mathcal{B}}) = \text{cl}(K_{\mathcal{B}})$).

We continue the proof of Theorem 1. Put $f^+ = \max\{f, 0\}$, $f^- = \max\{-f, 0\}$, $F = D(f)$, and $W := F \cup K_{\mathcal{L}}$. To prove the theorem we shall use the above Lemma 3 and two properties below of the sets $K_{\mathcal{A}}$, $K_{\mathcal{B}}$, and W , which will be verified in the second part of this proof:

$$W = \text{cl}(W), \quad (13)$$

$$K_{\mathcal{A}} \cap \text{cl}(K_{\mathcal{B}}) = K_{\mathcal{B}} \cap \text{cl}(K_{\mathcal{A}}) = \emptyset. \quad (14)$$

Let ℓ_W denote a continuous function on $X \setminus W$ of the form $\ell_W(x) = 1/\text{dist}(x, W)$. Now we define two functions h_1, h_2 by the formulas:

$$h_1(x) = \begin{cases} f(x) & \text{for } x \in K_{\mathcal{A}}, \\ 0 & \text{for } x \in K_{\mathcal{B}} \cup F, \\ f^+(x) + \ell_W(x) & \text{for } x \in X \setminus W, \end{cases}$$

and

$$h_2(x) = \begin{cases} 0 & \text{for } x \in K_{\mathcal{A}}, \\ f(x) & \text{for } x \in K_{\mathcal{B}} \cup F, \\ -f^-(x) - \ell_W(x) & \text{for } x \in X \setminus W. \end{cases}$$

We obviously have $f = h_1 + h_2$. We prove that both the functions h_1, h_2 are quasicontinuous with a closed graph. They are continuous on the following open sets: on $X \setminus W$ (by (13)), and on $U_{\mathcal{A}} \cup U_{\mathcal{B}}$ (by Lemma 3(i)). Since $U_{\mathcal{A}}$ and $U_{\mathcal{B}}$ are dense in $K_{\mathcal{A}}$ and $K_{\mathcal{B}} \cup W$ (by Lemma 3(v)), respectively, from the forms of h_1 and h_2 it follows both the functions are quasicontinuous on the set $W = K_{\mathcal{A}} \cup K_{\mathcal{B}} \cup F = K_{\mathcal{L}} \cup F$. Finally, h_1, h_2 are quasicontinuous on $(X \setminus W) \cup W = X$, as claimed.

To prove the graphs of h_1, h_2 are closed, let us fix $x_0 \in X$, and let $x_n \rightarrow x_0$ and $h_j(x_n) \rightarrow \alpha_j \in \mathbf{R}$ as $n \rightarrow \infty$, $j = 1, 2$. We have to show $\alpha_j = h_j(x_0)$, $j = 1, 2$. Since the set $X \setminus W$ is open, from the forms of h_j , $j = 1, 2$, it is enough to consider only the following subcases of the case $x_0 \in W = K_{\mathcal{A}} \cup (K_{\mathcal{B}} \cup F)$:

- (a) $x_0 \in K_{\mathcal{A}}$,
- (b) $x_0 \in K_{\mathcal{B}}$,
- (c) $x_0 \in F$.

Notice first that

$$x_n \in W \quad \text{for almost all } n\text{'s} \quad (15)$$

(otherwise $|h_j(x_n)| \geq \ell_W(x_n)$ for infinitely many n 's, whence

$$\overline{\lim}_{n \rightarrow \infty} |h_j(x_n)| = \infty$$

for $j = 1$ or $j = 2$, a contradiction).

In subcase (a), $\alpha_1 = f(x_0) = h_1(x_0)$ because $K_{\mathcal{A}} \subset C(f)$ (see Lemma 3(i)). To prove $\alpha_2 = h_2(x_0) = 0$, it is enough to show $x_n \in K_{\mathcal{A}}$ for almost all n 's. If this were not the case, by (15) we would have $x_n \in K_{\mathcal{B}} \cup F$ for infinitely many n 's, whence $x_0 \in \text{cl}(K_{\mathcal{B}} \cup F) = \text{cl}(K_{\mathcal{B}}) \cup F$ (because $\text{cl}(F) = F = D(f)$: see Lemma 1). Hence, by (14) and Lemma 3(i), we would obtain a contradiction:

$$x_0 \in K_{\mathcal{A}} \cap (\text{cl}(K_{\mathcal{B}}) \cup F) = (K_{\mathcal{A}} \cap \text{cl}(K_{\mathcal{B}})) \cup (K_{\mathcal{A}} \cap F) = \emptyset.$$

In subcase (b), $x_n \in K_{\mathcal{B}}$ for almost all n 's because, by (14) and Lemma 3(i) again, $F \cap K_{\mathcal{B}} = \emptyset = \text{cl}(K_{\mathcal{A}}) \cap K_{\mathcal{B}}$. It follows that $h_1(x_0) = 0 = h_1(x_n) = 0 \rightarrow 0 = \alpha_1$, and $h_2(x_n) = f(x_n) \rightarrow f(x_0) = \alpha_2$ (since $x_0 \in C(f)$: see Lemma 3(i)), as $n \rightarrow \infty$.

In subcase (c), notice that, by the forms of h_1, h_2 and the hypothesis that $f|_F = 0$, we have $h_1(x_0) = 0 = f(x_0) = h_2(x_0)$. Further, h_1 and h_2 agree with f on $K_{\mathcal{A}}$ and $K_{\mathcal{B}}$, respectively, and hence, by Lemma 3(iii), $x_n \in K_{\mathcal{A}}$ for a finite number of n 's when we consider h_1 , and $x_n \in K_{\mathcal{B}}$ for a finite number of n 's when we consider h_2 . Finally, $x_n \in K_{\mathcal{B}} \cup F$ for almost all n 's for h_1 , and $x_n \in K_{\mathcal{A}} \cup F$ for almost all n 's for h_2 . But then, since h_1 and h_2 vanish on $K_{\mathcal{B}} \cup F$ and $K_{\mathcal{A}} \cup F$, respectively, $h_j(x_n) = 0 \rightarrow 0 = h_j(x_0) = \alpha_j$, $j = 1, 2$, as $n \rightarrow \infty$.

We thus have shown that in each of the above (essential) subcases (a), (b), and (c) the conditions $x_n \rightarrow x_0$ and $h_j(x_n) \rightarrow \alpha_j \in \mathbf{R}$ (as $n \rightarrow \infty$), $j = 1, 2$, imply $\alpha_j = h_j(x_0)$, i.e., both the functions h_1, h_2 have a closed graph, as claimed.

The second part of Theorem 1 follows from the second part of Lemma 2.

The proof of inclusion (5') is complete.

Now we come to proofs of conditions (13), (14), and Lemma 3.

Proof of condition (13). By Lemma 1, the set $F = D(f)$ is closed, whence $\text{cl}(W) = F \cup \text{cl}(K_{\mathcal{L}})$. Fix an element $x \in \text{cl}(W)$. Without loss of generality we assume

$$x \in \text{cl}(K_{\mathcal{L}}) \setminus F = (\text{cl}(K_{\mathcal{L}}) \cap C(f)). \quad (16)$$

We claim that (16) implies $x \in K_{\mathcal{L}}$, i.e., $x \in \text{cl}(K')$ for some $K' \in \mathcal{L}$ (as this further follows that $x \in W$, and hence $\text{cl}(W) = W$). Assume this is not the case: $x \notin \text{cl}(K)$ for every $K \in \mathcal{L}$. Since, by (16), $x \in C(f)$, Lemma 3(ii) implies there is an open neighbourhood U of x such that, either $U \cap \text{cl}(K_0) \neq \emptyset$ for exactly one set $K_0 \in \mathcal{L}$, or $U \cap \text{cl}(K) = \emptyset$ for every $K \in \mathcal{L}$. Now let us put $U_x := U \setminus \text{cl}(K_0)$ in the first case, and $U_x := U$ in the second case. Then U_x is an open neighbourhood of x with $U_x \cap K_{\mathcal{L}} = \emptyset$. Hence we obtain $x \notin \text{cl}(K_{\mathcal{L}})$, and this contradicts (16). Thus, our claim is true, and the proof of condition (13) is complete. \square

Proof of condition (14). Assume condition (14) is false, e.g., there is $w \in K_{\mathcal{A}} \cap \text{cl}(K_{\mathcal{B}})$. Hence there exist: $K \in \mathcal{A}$ such that $w \in \text{cl}(K)$, and a sequence (w_r) in $K_{\mathcal{B}}$ converging to x . By the pigeonhole principle, there are infinitely many r 's such that: (a) $w_r \in \text{cl}(H_0)$ for some $H_0 \in \mathcal{B}$, or (b) $w_r \in \text{cl}(H_r)$, where $H_r \in \mathcal{B}$ and $H_{r'} \neq H_{r''}$ for $r' \neq r''$. In case (a) we obtain $x \in \text{cl}(K) \cap \text{cl}(H_0)$, which contradicts condition (iv) in Lemma 3. In case (b), by part (iii) of Lemma 3, we obtain $\overline{\lim}_{r \rightarrow \infty} |f(w_r)| = \infty$, a contradiction with the continuity of f in $x \in K_{\mathcal{A}} \subset C(f)$. Hence, condition (14) cannot be false. \square

Proof of Lemma 3. Let U be a nonempty subset of X , and let an element $x \in X$ be fixed. Then $\text{diam}(U)$ denotes the diameter of X , and $\text{dist}(x, U)$ denotes the distance of x to U .

For our purposes we shall use the following notation: for a sequence (U_n) of nonempty subsets of X we say that (U_n) converges to x , and we write

$$\lim_{n \rightarrow \infty} U_n = x,$$

whenever both the sequences, $(\text{diam}(U_n))$ and $(\text{dist}(x, U_n))$, converge to 0. A sample sequence (U_n) in \mathbf{R} is $U_n = (1/(n+1), 1/n)$, converging to 0.

Put $F = D(f)$. By Lemma 1 and property (t_3) , the set F is closed and its points are nonisolated. Let G be a countable and dense subset of F . If F is countable, we set $G = F$. Write $G = \{x_k\}_{k=1}^M$, where $x_k \neq x_n$ for $k \neq n$, and $M = \text{card}(F)$ for F finite, and $M = \infty$ for F infinite.

The main idea in constructing the class \mathcal{L} consists largely in careful choosing sequences $(U_j(k))_{j=1}^{\infty}$ of open subsets of $C(f)$ such that $U_j^{(k)} \rightarrow x_k$ as $j \rightarrow \infty$, and that $|f(x)| \geq k + j$ on $U_j^{(k)}$ for all j, k . The construction will depend on mathematical induction, and will be done in three steps.

Step 1. Since the element $x_1 \in G$ is nonisolated in X , the restriction $f|_F$ is continuous, and $\overline{\lim}_{t \rightarrow x_1} |f(t)| = \infty$, there exists an infinite sequence $(t_j^{(1)})$ in $C(f)$ such that

$$t_j^{(1)} \neq t_m^{(1)} \quad \text{for } j \neq m, \quad (17)$$

$$\lim_{j \rightarrow \infty} t_j^{(1)} = x_1, \quad (18)$$

and

$$|f(t_{j+1}^{(1)})| > |f(t_j^{(1)})| > 1 + j \quad (19)$$

for all j 's. Let $\bar{K}(t, \varepsilon)$ denote the closed ball centered at $t \in X$ with radius $\varepsilon > 0$. Since the set $C(f)$ is open, there is a sequence of strictly positive numbers $(\varepsilon_j^{(1)})$ with $\varepsilon_j^{(1)} \downarrow 0$ as $j \rightarrow \infty$, and such that

$$\bar{K}(t_j^{(1)}, \varepsilon_j^{(1)}) \subset C(f) \quad \text{for all } j\text{'s}, \quad (20)$$

and, by (17),

$$\bar{K}(t_j^{(1)}, \varepsilon_j^{(1)}) \cap \bar{K}(t_m^{(1)}, \varepsilon_m^{(1)}) = \emptyset \quad \text{for } j \neq m. \quad (21)$$

Moreover, by (19) and (20), we may assume all the numbers $\varepsilon_j^{(1)}$, $j = 1, 2, \dots$, are such that

$$|f(x)| > 1 + j \quad \text{for every } x \in \bar{K}(t_j^{(1)}, \varepsilon_j^{(1)}) \text{ and for all } j\text{'s.} \quad (22)$$

Let us set now $\mathcal{K}_1 := \{U_j^{(1)} : j = 1, 2, \dots\}$, where $U_j^{(1)} := K(t_j^{(1)}, \varepsilon_j^{(1)})$, $j = 1, 2, \dots$. The family \mathcal{K}_1 possesses the following properties:

- (a₁) $\text{cl}(U_j^{(1)}) \subset C(f)$ for every $j = 1, 2, \dots$,
- (b₁) $|f(x)| > 1 + j$ for every $x \in \text{cl}(U_j^{(1)})$ and every $j = 1, 2, \dots$,
- (c₁) $\text{cl}(U_j^{(1)}) \cap \text{cl}(U_m^{(1)}) = \emptyset$ for $j \neq m$,
- (d₁) $\lim_{j \rightarrow \infty} \text{cl}(U_j^{(1)}) = x_1$.

If $M = 1$, the construction of the class \mathcal{L} is complete. Indeed, set $\mathcal{L} := \mathcal{K}_1$, $\mathcal{A} := \{U_{2j-1}^{(1)} : j \geq 1\}$, and $\mathcal{B} := \{U_{2j}^{(1)} : j \geq 1\}$. It is now obvious that the above properties (a₁), (b₁), (c₁), and (d₁) imply conditions (i), (iii), (iv), and (v), respectively, of Lemma 3. We have to show condition (ii) of the lemma is satisfied too. Assume this is not the case: there is $x_0 \in C(f)$ such that, for every positive integer p , there exist *distinct* positive integers j_p, j'_p with

$$K\left(x_0, \frac{1}{p}\right) \cap \text{cl}(U_{j_p}^{(1)}) \neq \emptyset \neq K\left(x_0, \frac{1}{p}\right) \cap \text{cl}(U_{j'_p}^{(1)}). \quad (23)$$

Notice that the sequences $(j_p), (j'_p)$ are bounded (otherwise, we may assume there exists a sequence (u_p) such that, e.g., $u_p \in K(x_0, \frac{1}{p}) \cap \text{cl}(U_{j_p}^{(1)})$ for all p 's, and $\lim_{p \rightarrow \infty} u_p = x_0$; hence, since x_0 is a continuity point of f , $\lim_{p \rightarrow \infty} f(u_p) = f(x_0)$, but this contradicts the inequality in condition (b₁)). Passing to subsequences, we may further assume both the sequences are constant, whence $U_{j_p}^{(1)} = K$ and $U_{j'_p}^{(1)} = L$ for all p 's, and, by (c₁),

$$\text{cl}(K) \cap \text{cl}(L) = \emptyset. \quad (24)$$

On the other hand, from (23) we obtain that

$$K\left(x_0, \frac{1}{p}\right) \cap \text{cl}(K) \neq \emptyset \neq K\left(x_0, \frac{1}{p}\right) \cap \text{cl}(L)$$

for infinitely many p 's. Hence $x_0 \in \text{cl}(K) \cap \text{cl}(L)$, which contradicts condition (24). The contradiction implies the condition (ii) of Lemma 3 cannot be false. This finishes the proof of Step 1; and, at the same time, for $M = 1$, the proof of the lemma is complete.

Step 2. Let $M \geq 2$, and let us assume that, for a positive integer $r < M$, we have already constructed a countable family \mathcal{K}_r of open subsets of the space X of the form $\mathcal{K}_r = \{U_j^{(i)} : i = 1, \dots, r; j \geq 1\}$, that fulfils the four conditions below, corresponding to the conditions for $r = 1$ obtained in Step 1:

- (a_r) $\text{cl}(U_j^{(i)}) \subset C(f)$ for every $j = 1, 2, \dots$, and every $i = 1, \dots, r$,
- (b_r) $|f(x)| > i + j$ for every $x \in \text{cl}(U_j^{(i)})$, every $j = 1, 2, \dots$, and every $i = 1, \dots, r$,
- (c_r) $\text{cl}(U_{j_1}^{(i_1)}) \cap \text{cl}(U_{j_2}^{(i_2)}) = \emptyset$ for $(i_1, j_1) \neq (i_2, j_2)$, with $i_1, i_2 \leq r$ and $j_1, j_2 \geq 1$,
- (d_r) $\lim_{j \rightarrow \infty} \text{cl}(U_j^{(i)}) = x_i$ for $i = 1, \dots, r$.

We shall show now there is a family $\mathcal{K}^{(r+1)}$ of open subsets $U_j^{(r+1)}$ of X , where $j = 1, 2, \dots$, such that, the family $\mathcal{K}_{r+1} := \mathcal{K}_r \cup \mathcal{K}^{(r+1)}$ fulfils the above four conditions when r is replaced by $r + 1$.

We argue similarly as in Step 1. Since the element $x_{r+1} \in G \setminus \{x_1, \dots, x_r\}$ is nonisolated in X , there are two sequences: $(t_j^{(r+1)})$ in $C(f)$, and $(\varepsilon_j^{(r+1)})$ in \mathbf{R} , such that

$$t_j^{(r+1)} \rightarrow x_{r+1} \quad \text{and} \quad \varepsilon_j^{(r+1)} \downarrow 0 \quad \text{as } j \rightarrow \infty, \quad (18')$$

and, for all indices $j, m \geq 1$,

$$\bar{K}(t_j^{(r+1)}, \varepsilon_j^{(r+1)}) \subset C(f) \quad \text{for all } j\text{'s}, \quad (20')$$

$$\bar{K}(t_j^{(r+1)}, \varepsilon_j^{(r+1)}) \cap \bar{K}(t_m^{(r+1)}, \varepsilon_m^{(r+1)}) = \emptyset \quad \text{for } j \neq m, \quad (21')$$

and

$$|f(x)| > (r+1) + j \quad \text{for every } x \in \bar{K}(t_j^{(r+1)}, \varepsilon_j^{(r+1)}) \text{ and for all } j\text{'s.} \quad (22')$$

Since $x_{r+1} \notin \{x_1, \dots, x_r\}$, from condition (d_r) and from the pigeonhole principle it follows that the intersection $\bar{K}(t_j^{(r+1)}, \varepsilon_j^{(r+1)}) \cap \bigcup \mathcal{K}_r$ is not empty only for a finite number of j 's. Hence, there is j_0 such that

$$\bar{K}(t_{j_0+j}^{(r+1)}, \varepsilon_{j_0+j}^{(r+1)}) \cap \bigcup_{i=1}^r \bigcup_{j=1}^{\infty} \text{cl}(U_j^{(i)}) = \emptyset \quad \text{for } j = 1, 2, \dots \quad (25)$$

Put $U_j^{(r+1)} := K(t_{j_0+j}^{(r+1)}, \varepsilon_{j_0+j}^{(r+1)})$, $j = 1, 2, \dots$, and $\mathcal{K}^{(r+1)} := \{U_j^{(r+1)} : j \geq 1\}$. From the above conditions $(18')$, $(20')$, $(21')$, $(22')$, and (25) it now follows that $\mathcal{K}^{(r+1)}$ is the searched family of open subsets of X , i.e., such that the family $\mathcal{K}_{r+1} := \mathcal{K}_r \cup \mathcal{K}^{(r+1)}$ fulfils the required conditions (a_{r+1}) , (b_{r+1}) , (c_{r+1}) , and (d_{r+1}) .

We thus have proved that, for $M \geq 2$, there is an increasing sequence $\mathcal{K}_1 \subset \mathcal{K}_2 \subset \dots \subset \mathcal{K}_r \subset \dots$ of countable classes of open subsets of X , of the form as above, that fulfil, for each r , conditions (a_r) – (d_r) . For M finite, the sequence is finite and ends with $r = M$; and for $M = \infty$, by the mathematical induction, the sequence is infinite countable.

Step 3. Now we set $\mathcal{L} := \bigcup_{r=1}^M \mathcal{K}_r = \{U_j^{(i)} : i, j \geq 1\}$, $\mathcal{A} := \{U_{2j-1}^{(i)} : i, j \geq 1\}$, and $\mathcal{B} := \{U_{2j}^{(i)} : i, j \geq 1\}$. Similarly as in Step 1, the properties (a_r) , (b_r) , (c_r) , and (d_r) , respectively, imply conditions (i), (iii), (iv), and (v), respectively, of Lemma 3.

Moreover, arguments similar to those of the proof of condition (ii) in Step 1 (i.e., by the use of conditions (b_r) and (c_r)) can be used to show that the condition

$$K\left(x_0, \frac{1}{p}\right) \cap \text{cl}(U_{j_p}^{s_p}) \neq \emptyset \neq K\left(x_0, \frac{1}{p}\right) \cap \text{cl}(U_{j'_p}^{s'_p}), \quad p = 1, 2, \dots, (s_p, j_p) \neq (s'_p, j'_p), \quad (23')$$

for some $x_0 \in C(f)$, is impossible. This proves condition (ii) of Lemma 3 is true too.

The proof of Lemma 3 is complete. \square

Acknowledgment

The author wishes to express his gratitude to Professor Marek Wójciewicz for his helpful attention in preparation of this manuscript.

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