



# Vanishing solutions of anisotropic parabolic equations with variable nonlinearity

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## ABSTRACT

We study the property of finite time vanishing of solutions of the homogeneous Dirichlet problem for the anisotropic parabolic equations

$$u_t - \sum_{i=1}^n D_i(a_i(x, t)|D_i u|^{p_i(x,t)-2} D_i u) + c(x, t)|u|^{\sigma(x,t)-2} u = f(x, t)$$

with variable exponents of nonlinearity  $p_i(x, t), \sigma(x, t) \in (1, \infty)$ . We show that the solutions of this problem may vanish in a finite time even if the equation combines the directions of slow and fast diffusion and estimate the extinction moment in terms of the data. If the solution does not identically vanish in a finite time, we estimate the rate of vanishing of the solution as  $t \rightarrow \infty$ . We establish conditions on the nonlinearity exponents which guarantee vanishing of the solution at a finite instant even if the equation eventually transforms into the linear one.

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## 1. Introduction

We study the behavior of solutions to the Dirichlet problem for the anisotropic parabolic equations with variable nonlinearity

$$\begin{cases} u_t - \sum_{i=1}^n D_i(a_i|D_i u|^{p_i(z)-2} D_i u) + c|u|^{\sigma(z)-2} u = f(z) & \text{in } Q_T, \\ u = 0 & \text{on } \Gamma_T, \quad u(x, 0) = u_0(x) & \text{in } \Omega, \end{cases} \quad (1.1)$$

where  $\Omega \subset \mathbb{R}^n$  is a domain with Lipschitz-continuous boundary  $\Gamma$ ,  $\Gamma_T = \Gamma \times (0, T)$ ,  $Q_T = \Omega \times (0, T)$ ,  $z = (x, t)$ .

Equations of the type (1.1) appear in various applications such as the mathematical modelling of heat and mass transfer in nonhomogeneous media, in description of the filtration processes, in the processes of recovery of digital images (see [4,9,10] and the references therein for an account of such models in the stationary case). For the sake of presentation, we will regard problem (1.1) as the mathematical model of a diffusion process.

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The questions we address in this paper are already studied for the evolutionary  $p$ -Laplacian equation

$$u_t = \Delta_p u \equiv \operatorname{div}(|\nabla u|^{p-2} \nabla u), \quad p \in (1, \infty). \quad (1.2)$$

If  $p \in (1, 2)$ , this equation describes processes of fast diffusion, the case  $p > 2$  corresponds to slow diffusion. It is well known that the solutions of the nonlinear equation (1.2) with  $p \neq 2$  possess some properties not displayed by the solutions of the linear equation. The solutions of the linear equation obey the strong maximum principle which prevents them from attaining the maximum and minimum values in the interior of the problem domain. Unlike the linear case, for  $p \neq 2$  the solutions of the Dirichlet problem for Eq. (1.2) are localized either in space, or in time. More precisely, the following alternative holds: if  $u$  is a solution of the Dirichlet problem for Eq. (1.2) with  $p \neq 2$ , then either

$$2 > p > 1 \quad (\text{fast diffusion}) \implies \exists T_1: u \equiv 0 \quad \text{for all } t \geq T_1,$$

or

$$\left. \begin{array}{l} p > 2 \quad (\text{slow diffusion}) \\ u_0 \equiv 0 \quad \text{in } B_r(x_0) \subset \Omega \end{array} \right\} \implies \exists t_*(x_0): u(x_0, t) \equiv 0 \quad \text{for } t \in [0, t_*(x_0)],$$

where  $B_s(x_0) = \{x \in \mathbb{R}^n: |x - x_0| < s\}$ . These properties complement each other: the former is called *extinction in a finite time*, the latter is usually referred to as *finite speed of propagation* of disturbances from the data. If  $p > 2$  and if the support of the initial function  $u_0$  is compact in  $\Omega$ , then the support of the solution is expanding with time and eventually covers the whole of  $\Omega$ .

It is proved in [11,12] that the same behavior is intrinsic for the solutions of Eq. (1.1) (anisotropic and variable diffusion) in the cases when the diffusion is either fast, or slow in every space direction:  $p_i(z) \geq p^- > 2$  or  $1 < p^- \leq p_i(z) \leq p^+ < 2$  for every  $i = 1, \dots, n$  and all  $z \in Q_T$ . At the same time, it is known that solutions of anisotropic equations may display specific localization properties caused by the anisotropy and impossible in the isotropic cases: it is shown in [8–10] that in the stationary diffusion processes the anisotropy of the diffusion operator acts like the strong absorption and may cause localization of the solution. This fact makes feasible the hypothesis that the solutions to problem (1.1) possess some specific properties of time localization.

In the present paper we study the influence of the anisotropy and the variable nonlinearity on the possibility of vanishing in a finite time of solutions to problem (1.1). We show that the solutions may vanish in a finite time even in the case when the diffusion operator in (1.1) combines the directions of locally slow diffusion ( $p_i(z) > 2$ ) with the directions of locally fast diffusion ( $p_i(z) \in (1, 2)$ ) or linear diffusion ( $p_i(z) = 2$ ). Our results apply to *energy solutions* of problem (1.1) (the rigorous definition is given in Section 2).

Let us illustrate the further results by the example of the model equation with two independent space variables:

$$\left\{ \begin{array}{l} u_t = (|u_x|^{p(z)-2} u_x)_x + (|u_y|^{q(z)-2} u_y)_y + c_0 |u|^{\sigma(z)-2} u + f(z) \quad \text{in } Q_T, \\ u = 0 \quad \text{on } \Gamma_T, \\ u(x, 0) = u_0(x) \quad \text{in } \Omega = (0, a) \times (0, a). \end{array} \right. \quad (1.3)$$

For the sake of definiteness we assume that  $c_0 > 0$ . By agreement, here and throughout the paper we use the notations

$$\phi^+(t) = \sup_{\Omega} \phi(\cdot, t), \quad \phi^-(t) = \inf_{\Omega} \phi(\cdot, t) \quad \text{for } \phi \in C^0(Q_T).$$

- *Vanishing in a finite time.* Let  $f(z) \equiv 0$  for all  $t \geq t_f$ . Then every (energy) solution of problem (1.3) vanishes in a finite time  $t_* \geq t_f$  if

$$\frac{1}{v^+(t)} \equiv \frac{1}{\sigma^+(t)} + \frac{1}{2} \left( \frac{1}{p^+(t)} + \frac{1}{q^+(t)} \right) > 1 \quad \text{in } Q_T$$

and the oscillation of the variable exponents of nonlinearity in  $\Omega$  is appropriately small: for every  $t \in (0, T)$ ,

$$\frac{1}{v^-(t)} \equiv \frac{1}{\sigma^-(t)} + \frac{1}{2} \left( \frac{1}{p^-(t)} + \frac{1}{q^-(t)} \right) \leq \frac{3}{2}.$$

- *Vanishing at a prescribed moment.* Let us additionally assume that in the above conditions

$$\int_{\Omega} |f(x, y, t)|^{\sigma(x, y, t)} dx dy \begin{cases} \equiv 0 & \text{as } t \geq t_f, \\ \leq C(t_f - t)^{\mu} & \text{for } t \in (0, t_f), \end{cases} \quad C, \mu = \text{const} > 0,$$

with a suitably big exponent  $\mu$ . Then every (energy) solution of problem (1.1) vanishes at the instant  $t = t_f$ , provided that  $C$  and  $\|u_0\|_{2, \Omega}$  sufficiently small.

- *Vanishing of solutions of eventually linear equations.* Although the effect of finite time vanishing is never displayed by the solutions of the linear parabolic equations, it may happen that Eq. (1.1) with variable nonlinearity transforms into the linear one as  $t \rightarrow \infty$  and nonetheless possesses localized in time solutions. A condition sufficient for such an effect can be formulated as a restriction on the rate of vanishing of  $v^+(t) - 1$  as  $t \rightarrow \infty$ : if  $f \equiv 0$  and  $\|u_0\|_{2,\Omega} \leq 1$ , then every solution of problem (1.3) vanishes at a finite moment, provided that

$$\int_0^\infty \|u_0\|_{2,\Omega}^{2(v^+(t)-1)} dt = \infty \quad \text{and} \quad \int_0^\infty \frac{dt}{e^{t(1-v^+(t))}} < \infty.$$

Notice that these conditions are surely fulfilled if  $v^+(t) \leq v_0 < 1$ .

- *Large time behavior.* In case that the sufficient conditions of time localization are not fulfilled, we study the behavior of the norm  $\|u(\cdot, t)\|_{2,\Omega}$  as  $t \rightarrow \infty$ . In dependence on the properties of the data, we establish the conditions of power or exponential decreasing of the  $L^2(\Omega)$ -norm of the solution.

Eq. (1.1) with constant and/or isotropic nonlinearity fall into the scope of our analysis as partial cases. In these cases the conditions of time localization coincide with the already known in the literature – see, e.g., [12] and [5, Ch. 2] for a survey of relevant results.

Existence, uniqueness and boundedness of solutions to anisotropic parabolic equations of the type (1.1) have been studied by many authors under various conditions on the data and with different methods – see, for example, [1–3, 6, 13–16, 21, 22, 26, 27] and the further references therein. In the present paper we are interested in the energy solutions of problem (1.1) – see Definition 2.1 below.

The localization properties of solutions of problem (1.1) with anisotropic diffusion operator and constant nonlinearity are studied in [13]. Now we extend the analysis of the phenomenon of time localization to equations with variable nonlinearity which may include the absorption terms. The study is based on the analysis of the energy functions associated with the solution. This method allows us to carry out the analysis in the situation when the equation under study need not admit explicit sub/super solutions.

The paper is organized as follows. In Section 2 we collect some facts from the theory of variable exponent Orlicz–Sobolev spaces used in the rest of the paper, define the weak (energy) solution of problem (1.1) and recall the results about its solvability. Section 3 contains results about the embedding of anisotropic Orlicz–Sobolev spaces the energy solutions of problem (1.1) belong to. Sections 4 and 5 are devoted to derivation of the ordinary nonlinear differential inequalities for the energy functions associated with the solution of problem (1.1). The proofs of the main results are based on the study of the differential inequalities and is given in Sections 6–9.

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## 2. The function spaces. Preliminaries

The definitions of the function spaces used throughout the paper and a brief description of their properties follow [17, 18, 20, 24]. The further references can be found in the survey papers [19, 25].

### 2.1. Spaces $L^{p(\cdot)}(\Omega)$ and $W_0^{1,p(\cdot)}(\Omega)$

Let  $\Omega \subset \mathbb{R}^n$  be a bounded domain with the Lipschitz-continuous boundary  $\partial\Omega$ , and let the function  $p(x): \Omega \mapsto (1, \infty)$  be continuous with the logarithmic module of continuity:

$$\begin{cases} \forall z, \zeta \in Q_T, |z - \zeta| < 1, \quad \sum_i |p(z) - p(\zeta)| \leq \omega(|z - \zeta|), \\ \lim_{\tau \rightarrow 0^+} \omega(\tau) \ln \frac{1}{\tau} = C < +\infty. \end{cases} \quad (2.1)$$

By  $L^{p(\cdot)}(\Omega)$  we denote the space of measurable functions  $f(x)$  on  $\Omega$  such that

$$A_{p(\cdot)}(f) = \int_\Omega |f(x)|^{p(x)} dx < \infty.$$

The space  $L^{p(\cdot)}(\Omega)$  equipped with the norm

$$\|f\|_{p(\cdot),\Omega} \equiv \|f\|_{L^{p(\cdot)}(\Omega)} = \inf\{\lambda > 0: A_{p(\cdot)}(f/\lambda) \leq 1\}$$

becomes a Banach space. The Banach space  $W_0^{1,p(\cdot)}(\Omega)$  with  $p(x) \in [p^-, p^+] \subset (1, \infty)$  is defined by

$$\begin{cases} W_0^{1,p(\cdot)}(\Omega) = \{f \in L^{p(\cdot)}(\Omega): |\nabla f|^{p(x)} \in L^1(\Omega), u = 0 \text{ on } \partial\Omega\}, \\ \|u\|_{W_0^{1,p(\cdot)}(\Omega)} = \sum_i \|D_i u\|_{p(\cdot),\Omega}. \end{cases} \quad (2.2)$$

It is known that condition (2.1) is sufficient for the density of smooth functions in  $W_0^{1,p(\cdot)}(\Omega)$  (see [29] and the references therein). It follows directly from the definition that

$$\min(\|f\|_{p(\cdot)}^-, \|f\|_{p(\cdot)}^+) \leq A_{p(\cdot)}(f) \leq \max(\|f\|_{p(\cdot)}^-, \|f\|_{p(\cdot)}^+). \quad (2.3)$$

The functions from the space  $L^{p(\cdot)}(\Omega)$  satisfy the Hölder inequality: for all  $f \in L^{p(\cdot)}(\Omega)$ ,  $g \in L^{p'(\cdot)}(\Omega)$  with  $p(x) \in (1, \infty)$ ,  $p'(x) = \frac{p(x)}{p(x)-1}$  (Hölder's conjugate),

$$\int_{\Omega} |fg| dx \leq 2 \|f\|_{p(\cdot)} \|g\|_{p'(\cdot)}. \quad (2.4)$$

## 2.2. Anisotropic spaces $\mathbf{V}(\Omega)$ and $\mathbf{W}(Q_T)$

For every fixed  $t \in [0, T]$  we introduce the Banach space

$$\begin{cases} \mathbf{V}(\Omega) = \{u(x): u(x) \in L^2(\Omega), |D_i u|^{p_i(x,t)} \in L^1(\Omega), u = 0 \text{ on } \Gamma\}, \\ \|u\|_{\mathbf{V}(\Omega)} = \|u\|_{2,\Omega} + \sum_{i=1}^n \|D_i u\|_{p_i(\cdot,t),\Omega} \end{cases}$$

and denote by  $\mathbf{W}(Q_T)$  the Banach space

$$\mathbf{W}(Q_T) = \{u: [0, T] \mapsto \mathbf{V}(\Omega): u \in L^2(Q_T), |D_i u|^{p_i(z)} \in L^1(Q_T), u = 0 \text{ on } \Gamma_T\}$$

with the norm

$$\|u\|_{\mathbf{W}(Q_T)} = \|u\|_{2,Q_T} + \sum_{i=1}^n \|D_i u\|_{p_i(\cdot),Q_T}.$$

If  $p_i(z) \in [p^-, p^+] \subset (1, \infty)$ , and if  $p_i(z)$  satisfy the log-continuity condition (2.1) in  $Q_T$ , then the space  $\mathbf{W}(Q_T)$  is separable and for every  $v \in \mathbf{W}(Q_T)$  there is a sequence  $v^{(m)} = \sum_{i=1}^m c_i^{(m)}(t) \psi_i(x)$ , where  $c_i^{(m)}(t) \in C^1(0, T)$  and  $\{\psi_i\}$  is an orthonormal basis of the space

$$\mathbf{V}_+(\Omega) = \{u(x) \mid u \in L^2(\Omega) \cap W_0^{1,1}(\Omega), |\nabla u| \in L^{p^+}(\Omega)\},$$

such that  $v^{(m)} \rightarrow v$  in the norm of  $\mathbf{W}(Q_T)$  (see [6, Sec. 2]). By  $\mathbf{W}'(Q_T)$  we denote the dual of  $\mathbf{W}(Q_T)$  (the space of linear functionals over  $\mathbf{W}(Q_T)$ ):

$$w \in \mathbf{W}'(Q_T) \iff \begin{cases} w = w_0 + \sum_{i=1}^n D_i w_i, & w_0 \in L^2(Q_T), w_i \in L^{p'_i(\cdot)}(Q_T), \\ \forall \phi \in \mathbf{W}(Q_T), & \langle w, \phi \rangle = \int_{Q_T} (w_0 \phi + \sum_i w_i D_i \phi) dz. \end{cases}$$

The norm in  $\mathbf{W}'(Q_T)$  is defined by

$$\|v\|_{\mathbf{W}'(Q_T)} = \sup\{\langle v, \phi \rangle \mid \phi \in \mathbf{W}(Q_T), \|\phi\|_{\mathbf{W}(Q_T)} \leq 1\}.$$

## 2.3. Solvability of problem (1.1)

**Definition 2.1.** A function  $u \in \mathbf{W}(Q_T)$  is called weak (energy) solution of problem (1.1) if  $u_t \in \mathbf{W}'(Q_T)$ , and if for every  $t_1, t_2 \in [0, T]$  and every test-function  $\phi \in \mathbf{W}(Q_T)$  such that  $\phi_t \in \mathbf{W}'(Q_T)$

$$\int_{\Omega} u \phi dx \Big|_{t=t_1}^{t=t_2} - \int_{Q_{t_2} \setminus Q_{t_1}} \left( u \phi_t - \sum_{i=1}^n a_i |D_i u|^{p_i-2} D_i u D_i \phi + c |u|^{\sigma-2} u \phi \right) dz = \int_{Q_{t_2} \setminus Q_{t_1}} f \phi dz. \quad (2.5)$$

It is assumed that the coefficients  $a_i(z, u)$ ,  $c(z, u)$  depend on  $z = (x, t)$ ,  $u(z)$  and are Carathéodory functions (measurable in  $z$  for every  $r \in \mathbb{R}$ , continuous in  $r$  for a.e.  $z \in Q_T$ ), such that

$$\forall (z, r) \in \overline{Q}_T \times \mathbb{R}, \quad \begin{aligned} 0 &< a_0 \leq a_i(z, r) \leq A_0 < \infty, & a_0, c_0, A_0 &= \text{const.} \\ 0 &\leq c_0 \leq c(z, r) \leq A_0, \end{aligned} \quad (2.6)$$

It is assumed throughout the paper that the exponents  $p_i(\cdot)$ ,  $\sigma(\cdot)$  are given functions defined on  $Q_T$  and subject to the following conditions:

$$\begin{cases} 1 < \underline{p} \leq p_i^-(t) \leq p_i(z) \leq p_i^+(t) \leq \bar{p} < \infty, \\ 1 < \underline{\sigma} \leq \sigma^-(t) \leq \sigma(z) \leq \sigma^+(t) \leq \bar{\sigma} < \infty, \end{cases} \quad (2.7)$$

with

$$\begin{aligned} p_i^+(t) &= \sup_{\Omega} p_i(x, t), & p_i^-(t) &= \inf_{\Omega} p_i(x, t), \\ \sigma^+(t) &= \sup_{\Omega} \sigma(x, t), & \sigma^-(t) &= \inf_{\Omega} \sigma(x, t). \end{aligned}$$

The constants  $\underline{p}$ ,  $\bar{p}$  are chosen so that  $1 < \underline{p} \leq p_i^-(t) \leq p_i^+(t) \leq \bar{p} < \infty$  for every  $i = 1, \dots, n$ .

**Theorem 2.1.** (See [6].) Let conditions (2.6), (2.7) be fulfilled and let the exponents  $p_i(z)$  and  $\sigma(z)$  satisfy condition (2.1) of log-continuity. For every  $u_0 \in L^2(\Omega)$  and  $f \in L^2(Q_T)$  problem (1.1) has at least one energy solution  $u \in \mathbf{W}(Q_T)$  such that  $u_t \in \mathbf{W}'(Q_T)$ . Moreover, if  $c_0 > 0$  and  $f \in L^{\sigma(\cdot)}(Q_T)$ , then  $u \in \mathbf{W}(Q_T) \cap L^{\sigma(\cdot)}(Q_T)$ .

A weak solution of problem (1.1) is constructed in [6] as the limit of a sequence of Galerkin's approximations. Such a procedure is possible if the variable exponents of nonlinearity satisfy the log-continuity condition (2.1). The proof relies on the monotonicity of the elliptic part of Eq. (1.1) (see [23, Ch. 2, Sec. 1.2]) and the formula of integration by parts for the products  $v w_t$  with  $v, w \in \mathbf{W}(Q_T)$ ,  $v_t, w_t \in \mathbf{W}'(Q_T) \cap L^1(Q_T)$  (see Lemma 4.3 below).

**Remark 2.1.** In this paper, we are not concerned with the question of uniqueness because all our further results apply to every energy solution. For the sake of completeness, let us notice here that under the conditions of Theorem 2.1 the energy solution is unique if either the coefficients  $a_i$  and  $c$  are independent of the solution, or are continuous with respect to the solution with moduli of continuity depending on  $p_i(z)$  and  $\sigma(z)$ . The rigorous formulations and proofs can be found in [6].

**Remark 2.2.** Under the conditions of Theorem 2.1 the energy solutions need not be bounded. However, it is shown in [6] that boundedness takes place if the data of problem (1.1) are subject to some additional regularity assumptions. For such solutions the property of vanishing in a finite time can be established via slightly different methods and under weaker restrictions on the exponents of nonlinearity. We leave this analysis for further publications.

### 3. Auxiliary propositions

**Theorem 3.1.** Let  $\Omega = \{x \in \mathbb{R}^n: x_i \in (0, a)\}$  and let  $p_i = \text{const} > 1$ . Then

$$\|v\|_{r, \Omega} \leq C(a, n) \left( \prod_{i=1}^n \|D_i v\|_{p_i, \Omega} \right)^{\frac{1}{n}},$$

with

$$r = \begin{cases} \frac{np}{n-p} & \text{if } p < n, \\ \text{any number from } [1, \infty) & \text{if } p \geq n, \end{cases} \quad \frac{1}{p} = \frac{1}{n} \sum_{i=1}^n \frac{1}{p_i}.$$

The assertion of Theorem 3.1 is a byproduct of results of [22,28]. Let us introduce the functions

$$\begin{aligned} \Theta(t) &= \int_{\Omega} u^2(z) dx, & z &= (x, t) \in Q_T, \\ \Lambda(t) &= \int_{\Omega} \sum_{i=1}^n |D_i u(z)|^{p_i(z)} dx, \\ \Lambda_{\sigma}(t) &= \int_{\Omega} \left( \sum_{i=1}^n |D_i u(z)|^{p_i(z)} + |u(z)|^{\sigma(z)} \right) dx. \end{aligned} \quad (3.1)$$

**Lemma 3.1.** Let the exponents  $p_i$ ,  $\sigma$  satisfy conditions (2.7). For  $u \in \mathbf{W}(Q_T)$  the functions  $\Theta(t)$ ,  $\Lambda(t)$  exist for a.e.  $t \in (0, T)$ . If  $u \in \mathbf{W}(Q_T) \cap L^{\sigma(\cdot)}(Q_T)$ , then for a.e.  $t \in (0, T)$  there exist  $\Theta(t)$  and  $\Lambda_{\sigma}(t)$ .

**Proof.** The functions  $\Theta(t)$ ,  $\Lambda(t)$  are nonnegative and, by the definition of  $\mathbf{W}(Q_T)$ ,  $\Theta(t)$ ,  $\Lambda(t) \in L^1(0, T)$ . It follows that the functions

$$\int_0^t \Theta(\tau) d\tau, \quad \int_0^t \Lambda(\tau) d\tau$$

are differentiable for a.e.  $t \in (0, T)$ . In the case when  $u \in L^{\sigma(\cdot)}(Q_T)$  the same is true for the function  $\int_0^t |u|^{\sigma(\cdot)} d\tau$ .  $\square$

**Lemma 3.2.** Let  $u \in \mathbf{W}(Q_T)$ , and let  $p_i, \sigma$  be constant. If  $p_i > 1$  and

$$\frac{1}{p} \equiv \frac{1}{n} \sum_{i=1}^n \frac{1}{p_i} \leq \frac{1}{2} + \frac{1}{n},$$

then for a.e.  $t \in (0, T)$ ,

$$\Theta(t) \leq C \Lambda^{\frac{1}{v}}(t) \quad \text{with } v = \frac{p}{2}. \quad (3.2)$$

Let  $u \in \mathbf{W}(Q_T) \cap L^\sigma(Q_T)$ . If  $p_i > 1$  and

$$\begin{cases} \frac{1}{\sigma} + \frac{1}{p} \geq 1 + \frac{1}{n} & \text{if } n > p, \\ \sigma > 1 & \text{if } n \leq p, \end{cases}$$

then for a.e.  $t \in (0, T)$ ,

$$\Theta(t) \leq C \Lambda_\sigma(t)^{\frac{1}{v}} \quad \text{with } \frac{1}{v} = \frac{1}{\sigma} + \frac{1}{p}. \quad (3.3)$$

**Proof.** We start by proving (3.2). By Hölder's inequality and due to Theorem 3.1 with

$$r = \frac{np}{n-p} \geq 2 \iff p \geq \frac{2n}{n+2}$$

we have

$$\Theta(t) \equiv \int_\Omega u^2 dx \leq C' \|u\|_{r,\Omega}^2 \leq C \prod_{i=1}^n \|D_i u\|_{p_i,\Omega}^{\frac{2}{n}} \leq C \Lambda(t)^{\frac{1}{v}}. \quad (3.4)$$

To prove (3.3) we use Hölder's inequality and Theorem 3.1 with

$$\sigma' = \frac{\sigma}{\sigma-1} \leq r = \frac{np}{n-p} \quad \text{if } n > p \iff \frac{1}{\sigma} + \frac{1}{p} \geq 1 + \frac{1}{n},$$

which gives:

$$\Theta(t) = \int_\Omega u^2 dx \leq \|u\|_{\sigma,\Omega} \|u\|_{\sigma',\Omega} \leq C \|u\|_{\sigma,\Omega} \left( \sum_{i=1}^n \int_\Omega |D_i u|^{p_i} dx \right)^{\frac{1}{p}} \leq C \Lambda_\sigma^{\frac{1}{\sigma} + \frac{1}{p}}(t) = C \Lambda_\sigma^{\frac{1}{v}}(t). \quad \square$$

We will use the notations

$$\frac{1}{p^\pm(t)} = \frac{1}{n} \sum_{i=1}^n \frac{1}{p_i^\pm(t)}.$$

**Lemma 3.3.** Let conditions (2.7) be fulfilled and

$$\frac{1}{v^-(t)} = \frac{1}{\sigma^-(t)} + \frac{1}{p^-(t)} \leq \frac{n+1}{n} \quad \text{for } \frac{1}{n} < \frac{1}{p^-(t)}. \quad (3.5)$$

Then for every  $u \in \mathbf{W}(Q_T) \cap L^{\sigma(\cdot)}(Q_T)$  and a.e.  $t \in (0, T)$ ,

$$\min\{\Theta^{v^+(t)}(t), \Theta^{v^-(t)}(t)\} \leq C \Lambda_\sigma(t) \quad \text{with } \frac{1}{v^+(t)} = \frac{1}{\sigma^+(t)} + \frac{1}{p^+(t)}. \quad (3.6)$$

**Proof.** Following the proof of Lemma 3.2 we find that for every fixed  $t \in (0, T)$ ,

$$\int_{\Omega} u^2 dx \leq C \|u\|_{\sigma^-(t), \Omega} \|u\|_{(\sigma^-(t))', \Omega} \leq C \|u\|_{\sigma^-(t), \Omega} \left( \prod_{i=1}^n \|D_i u\|_{p_i^-(t), \Omega} \right)^{\frac{1}{n}}, \quad (3.7)$$

provided that (3.5) is fulfilled. Let us denote  $\rho(t) = \|u(\cdot, t)\|_{\sigma(\cdot, t), \Omega}$ . Applying Young's inequality and then using the definition of the norm of  $L^{\sigma(\cdot, t)}(\Omega)$  we estimate

$$\begin{aligned} \int_{\Omega} \left( \frac{|u(z)|}{\rho(t)} \right)^{\sigma^-(t)} dx &\leq \int_{\Omega} \left( \frac{\sigma^-(t)}{\sigma(z)} \left( \frac{|u(z)|}{\rho(t)} \right)^{\sigma(z)} + 1 - \frac{\sigma^-(t)}{\sigma(z)} \right) dx \\ &\leq \int_{\Omega} \left( \frac{|u(z)|}{\rho(t)} \right)^{\sigma(z)} dx + \left( 1 - \frac{\sigma^-(t)}{\sigma^+(t)} \right) |\Omega| \\ &\leq 1 + \left( 1 - \frac{\sigma^-(t)}{\sigma^+(t)} \right) |\Omega|. \end{aligned}$$

It follows that

$$\|u(\cdot, t)\|_{\sigma^-(t), \Omega} \equiv \left( \int_{\Omega} |u(z)|^{\sigma^-(t)} dx \right)^{\frac{1}{\sigma^-(t)}} \leq \left( 1 + \left( 1 - \frac{\sigma^-(t)}{\sigma^+(t)} \right) |\Omega| \right)^{\frac{1}{\sigma^-(t)}} \rho(t) \equiv \tilde{C} \|u(\cdot, t)\|_{\sigma(\cdot, t), \Omega}$$

with  $\tilde{C} \equiv \tilde{C}(\sigma^{\pm}(t), |\Omega|)$ . Combining this inequality with (2.3), we have

$$\|u\|_{\sigma^-(t), \Omega} \leq \tilde{C} \|u(\cdot, t)\|_{\sigma(\cdot, t), \Omega} \leq C \max \left\{ \left( \int_{\Omega} |u|^{\sigma(z)} dx \right)^{\frac{1}{\sigma^+(t)}}, \left( \int_{\Omega} |u|^{\sigma(z)} dx \right)^{\frac{1}{\sigma^-(t)}} \right\}$$

or, equivalently,

$$C' \int_{\Omega} |u|^{\sigma(z)} dx \geq \begin{cases} \|u\|_{\sigma^-(t), \Omega}^{\sigma^+(t)} & \text{if } \|u\|_{\sigma^-(t), \Omega} \leq 1, \\ \|u\|_{\sigma^-(t), \Omega}^{\sigma^-(t)} & \text{if } \|u\|_{\sigma^-(t), \Omega} > 1. \end{cases}$$

For every  $i = 1, 2, \dots, n$ ,

$$C'' \int_{\Omega} |D_i u|^{p_i(z)} dx \geq \begin{cases} \|D_i u\|_{p_i^-(t), \Omega}^{p_i^+(t)} & \text{if } \|D_i u\|_{p_i^-(t), \Omega} \leq 1, \\ \|D_i u\|_{p_i^-(t), \Omega}^{p_i^-(t)} & \text{if } \|D_i u\|_{p_i^-(t), \Omega} > 1. \end{cases}$$

It follows that

$$\|u\|_{\sigma^-(t), \Omega} \left( \prod_{i=1}^n \|D_i u\|_{p_i^-(t), \Omega} \right)^{\frac{1}{n}} \leq C A_{\sigma}^{\frac{1}{\mu(t)}}(t),$$

with the exponents

$$\frac{1}{\mu(t)} = \frac{1}{\tilde{\sigma}(t)} + \frac{1}{n} \sum_{i=1}^n \frac{1}{\tilde{p}_i(t)} \in \left[ \frac{1}{v^+(t)}, \frac{1}{v^-(t)} \right],$$

and

$$\tilde{p}_i(t) = \begin{cases} p_i^-(t) & \text{if } \|D_i u\|_{p_i^-(t), \Omega} > 1, \\ p_i^+(t) & \text{if } \|D_i u\|_{p_i^-(t), \Omega} \leq 1, \end{cases} \quad \tilde{\sigma}(t) = \begin{cases} \sigma^-(t) & \text{if } \|u\|_{\sigma^-(t), \Omega} > 1, \\ \sigma^+(t) & \text{if } \|u\|_{\sigma^-(t), \Omega} \leq 1. \end{cases}$$

Plugging these formulas to (3.7) we find that

$$\Theta^{\mu(t)}(t) \leq C A_{\sigma}(t).$$

The assertion now follows because

$$\Theta^{\mu(t)}(t) \geq \min \{ \Theta^{v^+(t)}(t), \Theta^{v^-(t)}(t) \} = \begin{cases} \Theta^{v^+(t)} & \text{if } \Theta(t) \leq 1, \\ \Theta^{v^-(t)} & \text{if } \Theta(t) \geq 1. \end{cases} \quad \square$$

**Lemma 3.4.** Let  $u \in \mathbf{W}(Q_T)$ . If

$$p^-(t) \geq \frac{2n}{n+2}, \quad (3.8)$$

then for a.e.  $t \in (0, T)$ ,

$$\min\{\Theta^{v^+(t)}(t), \Theta^{v^-(t)}(t)\} \leq C\Lambda(t), \quad \frac{1}{v^\pm(t)} = \frac{2}{p^\pm(t)}, \quad (3.9)$$

with a constant  $C = C(\Omega, \underline{p}, \bar{p}, n)$ .

**Proof.** By Theorem 3.1

$$\Theta(t) \leq C \left( \prod_{i=1}^n \|D_i u\|_{p_i^-(t), \Omega} \right)^{\frac{2}{n}}$$

and the conclusion follows like in Lemma 3.3.  $\square$

**Corollary 3.1.** Let  $\Theta(t) \leq 1$ . Since  $v^+(t) \geq v^-(t)$ , in this case the left-hand sides of inequalities (3.9) and (3.6) can be changed to  $\Theta^{v^+(t)}(t)$ .

#### 4. The energy relations

Let us introduce the notations

$$\Omega(s) = \Omega \cap \{x_1 > s\}, \quad \omega(s) = \Omega \cap \{x_1 = s\}.$$

##### 4.1. Formulas of integration by parts

Let  $\rho$  be the Friedrichs mollifying kernel

$$\rho(y) = \begin{cases} \kappa \exp(-\frac{1}{1-|y|^2}) & \text{if } |y| < 1, \\ 0 & \text{if } |y| > 1, \end{cases} \quad \kappa = \text{const}: \int_{\mathbb{R}^{n+1}} \rho(z) dz = 1.$$

Given a function  $v \in \mathbf{W}(Q_T)$ , we extend it by zero to the whole  $\mathbb{R}^{n+1}$  (keeping the same notation for the continued function) and then define

$$v_h(z) = \int_{\mathbb{R}^{n+1}} v(y) \rho_h(z-y) dy \quad \text{with } \rho_h(y) = \frac{1}{h^{n+1}} \rho\left(\frac{y}{h}\right), \quad h > 0.$$

**Lemma 4.1.** If  $u \in \mathbf{W}(Q_T)$  with the exponents  $p_i(z)$  satisfying (2.1), then

$$\|u_h\|_{\mathbf{W}(Q_T)} \leq C(\|u\|_{W_0^{1,1}(Q_T)} + \|u\|_{\mathbf{W}(Q_T)}) \quad \text{and} \quad \|u_h - u\|_{\mathbf{W}(Q_T)} \rightarrow 0 \quad \text{as } h \rightarrow 0.$$

Lemma 4.1 is an immediate byproduct of [29, Theorem 2.1].

**Lemma 4.2.** Let in the conditions of Lemma 4.1  $u_t \in \mathbf{W}'(Q_T) \cap L^1(Q_T)$ . Then  $(u_h)_t \in \mathbf{W}'(Q_T)$ , and for every  $\psi \in \mathbf{W}(Q_T)$ ,

$$\langle (u_h)_t, \psi \rangle \rightarrow \langle u_t, \psi \rangle \quad \text{as } h \rightarrow 0.$$

**Proof.** By the definition of  $\mathbf{W}'(Q_T)$  there exist  $\phi_0 \in L^2(Q_T)$ ,  $\phi_i \in L^{p_i'(\cdot)}(Q_T)$  such that

$$\langle u_t, \psi \rangle = (\phi_0, \psi)_{2, Q_T} + \sum_i (\phi_i, D_i \psi)_{2, Q_T} \quad \forall \psi \in \mathbf{W}(Q_T).$$



It follows that

$$\begin{aligned} \langle (u_h)_t, \psi \rangle &= \int_{Q_T} (u_t)_h \psi \, dz = \int_{Q_T} u_t \psi_h \, dz = \int_{Q_T} \left( \phi_0 \psi_h + \sum_i \phi_i D_i \psi_h \right) dz \\ &= \int_{Q_T} \left( (\phi_0)_h \psi + \sum_i (\phi_i)_h D_i \psi \right) dz \rightarrow \langle u_t, \psi \rangle \quad \text{as } h \rightarrow 0 \end{aligned}$$

by virtue of Lemma 4.1.  $\square$

**Lemma 4.3.** Let  $u \in \mathbf{W}(Q_T)$  and  $u_t \in \mathbf{W}'(Q_T) \cap L^1(Q_T)$  with the exponents  $p_i(z)$  satisfying (2.1). Then

$$\forall \text{ a.e. } t_1, t_2 \in (0, T) \quad \int_{t_1}^{t_2} \int_{\Omega} u u_t \, dz = \frac{1}{2} \|u(\cdot, t_2)\|_{2, \Omega}^2 - \frac{1}{2} \|u(\cdot, t_1)\|_{2, \Omega}^2.$$

**Proof.** Let  $t_1 < t_2$ . Take

$$\chi_k(t) = \begin{cases} 0 & \text{for } t \leq t_1, \\ k(t - t_1) & \text{for } t_1 \leq t \leq t_1 + \frac{1}{k}, \\ 1 & \text{for } t_1 + \frac{1}{k} \leq t \leq t_2 - \frac{1}{k}, \\ k(t_2 - t) & \text{for } t_2 - \frac{1}{k} \leq t \leq t_2, \\ 0 & \text{for } t \geq t_2. \end{cases}$$

Integrating by parts we have that for every sufficiently large  $k \in \mathbb{N}$ ,

$$2 \int_{Q_T} u_h (u_h)_t \chi_k(t) \, dz = k \int_{t_2 - \frac{1}{k}}^{t_2} \int_{\Omega} u_h^2 \, dz - k \int_{t_1}^{t_1 + \frac{1}{k}} \int_{\Omega} u_h^2 \, dz.$$

The integrals on the right-hand side exist because  $u_h \in L^2(Q_T)$ . Letting  $h \rightarrow 0$ , we obtain the equality

$$2 \lim_{h \rightarrow 0} \int_{Q_T} u_h (u_h)_t \chi_k(t) \, dz = k \int_{t_2 - \frac{1}{k}}^{t_2} \int_{\Omega} u^2 \, dz - k \int_{t_1}^{t_1 + \frac{1}{k}} \int_{\Omega} u^2 \, dz.$$

According to Lemmas 4.1, 4.2  $u_h \rightarrow u$  weakly in  $\mathbf{W}(Q_T)$ ,  $(u_h)_t = (u_t)_h \rightarrow u_t$  weakly in  $\mathbf{W}'(Q_T)$  as  $h \rightarrow 0$ , and  $\|u_h\|_{\mathbf{W}}$ ,  $\|(u_h)_t\|_{\mathbf{W}'}$  are uniformly bounded. It follows that

$$\begin{aligned} \lim_{h \rightarrow 0} \int_{Q_T} u_h (u_h)_t \chi_k(t) \, dz &= \lim_{h \rightarrow 0} \int_{Q_T} (u_h - u)(u_h)_t \chi_k(t) \, dz + \lim_{h \rightarrow 0} \int_{Q_T} u((u_h)_t - u_t) \chi_k(t) \, dz + \int_{Q_T} u u_t \chi_k(t) \, dz \\ &= \int_{Q_T} u u_t \chi_k(t) \, dz. \end{aligned}$$

By the Lebesgue differentiation theorem

$$\forall \text{ a.e. } \theta > 0, \quad \lim_{k \rightarrow 0} k \int_{\theta - \frac{1}{k}}^{\theta} \left( \int_{\Omega} u^2(x, t) \, dx \right) dt = \|u(\cdot, \theta)\|_{2, \Omega}^2,$$

whence for almost every  $t_1, t_2 \in [0, T]$ ,

$$\int_{t_1}^{t_2} \int_{\Omega} u u_t \, dz = \lim_{k \rightarrow \infty} \int_{Q_T} u u_t \chi_k(t) \, dz$$

$$\begin{aligned}
&= \lim_{k \rightarrow \infty} k \int_{t_2 - \frac{1}{k}}^{t_2} \int_{\Omega} u^2 dz - \lim_{k \rightarrow \infty} k \int_{t_1}^{t_1 + \frac{1}{k}} \int_{\Omega} u^2 dz \\
&= \frac{1}{2} \|u(\cdot, t_2)\|_{2, \Omega}^2 - \frac{1}{2} \|u(\cdot, t_1)\|_{2, \Omega}^2. \quad \square
\end{aligned}$$

Let us introduce the notation  $Q(s, t) = \Omega(s) \times (0, t)$ .

**Lemma 4.4.** *Let conditions (2.1) and (2.6) be fulfilled. For every solution  $u \in \mathbf{W}(Q_T)$  of problem (1.1) and a.e.  $s$  such that  $\Omega \cap \{x_1 = s\} \neq \emptyset$*

$$\begin{aligned}
&\frac{1}{2} \int_{\Omega(s)} u^2(x, \tau) dx \Big|_{\tau=0}^{\tau=t} + \sum_{i=1}^n \int_{Q(s, t)} a_i(z, u) |D_i u|^{p_i(z)} dz + \int_{Q(s, t)} c(z, u) |u|^{\sigma(z)} dz \\
&\quad + \int_0^t dt \int_{\omega(s)} a_1(z, u) u |D_1 u|^{p_1-2} D_1 u dx' \\
&= \int_{Q(s, t)} u f dz.
\end{aligned} \tag{4.1}$$

**Proof.** The energy solution can be taken for the test-function in identity (2.5). Let us set

$$\phi_k(x_1, x', s) = \begin{cases} 1 & \text{for } x_1 > s + \frac{1}{k}, \\ k(x_1 - s) & \text{for } x_1 \in [s, s + \frac{1}{k}], \\ 0 & \text{for } x_1 < s, \quad k \in \mathbb{N}, \end{cases}$$

and choose  $u(x, t) \phi_k(x_1, x', s)$ . The resulting identity has the form

$$\begin{aligned}
\sum_{j=1}^5 I_j(k, s) &\equiv \sum_{i=1}^n \int_{Q(s, t)} a_i(z, u) |D_i u|^{p_i(z)} \phi_k dz + \int_{Q(s, t)} c(z, u) |u|^{\sigma(z)} \phi_k dz \\
&\quad + k \int_0^t dt \int_{\Omega(s+1/k) \setminus \Omega(s)} a_1(z, u) u |D_1 u|^{p_1(z)-2} D_1 u dx \\
&\quad + \frac{1}{2} \int_{\Omega(s)} \phi_k u^2 dx \Big|_{\tau=0}^{\tau=t} - \int_{Q(s, t)} u f \phi_k dz = 0.
\end{aligned} \tag{4.2}$$

By the definition of  $\mathbf{W}(Q_T)$

$$|u|^{\sigma} \phi_k, |D_i u|^{p_i} \phi_k, u \phi_k \in L^1(Q_T), \quad u^2 \phi_k \in L^1(\Omega) \quad \text{for a.e. } t \in (0, T),$$

which allows us to pass to the limit as  $k \rightarrow \infty$  in  $I_1, I_2, I_4$  and  $I_5$ :

$$\begin{aligned}
\lim_{k \rightarrow \infty} I_1 &= \sum_{i=1}^n \int_{Q(s, t)} a_i(z, u) |D_i u|^{p_i} dz, \\
\lim_{k \rightarrow \infty} I_2 &= \int_{Q(s, t)} c(z, u) |u|^{\sigma(z)} dz, \\
\lim_{k \rightarrow \infty} I_4 &= \frac{1}{2} \int_{\Omega(s)} u^2 dx - \frac{1}{2} \int_{\Omega(s)} u_0^2 dx, \\
\lim_{k \rightarrow \infty} I_5 &= \int_{Q(s, t)} u f dz.
\end{aligned}$$

By virtue of (4.2)  $I_3$  is bounded uniformly with respect to  $k$ , provided that so are the integrals  $I_1$ ,  $I_2$  and  $I_4$ . Writing  $I_3$  in the form

$$I_3 = k \int_s^{s+1/k} \left( \int_0^t \left( \int_{\omega(y)} a_i(z, u) u |D_1 u|^{p_1-2} D_1 u dx' \right) dt \right) dy, \quad \omega(y) = \Omega \cap \{x_1 = y\},$$

and applying the Lebesgue differentiation theorem we conclude that there exists

$$\lim_{k \rightarrow \infty} I_3(k, s) = \int_0^t dt \int_{\omega(s)} a_1(z, u) u |D_1 u|^{p_1-2} D_1 u dx'.$$

Equality (4.2) transforms into (4.1) as  $k \rightarrow \infty$ .  $\square$

**Remark 4.1.** Let

$$\Omega \subset \Omega_a \equiv \{x \in \mathbb{R}^n : x_i \in (0, a), i = 1, \dots, n\}$$

and let  $u \in \mathbf{W}(Q_T)$  be a solution of problem (1.1). Set

$$u^*(x, t) = \begin{cases} u(x, t) & \text{in } Q_T, \\ 0 & \text{in } Q_T^{(a)} \equiv (\Omega_a \setminus \Omega) \times (0, T). \end{cases}$$

The function  $u^*(x, t)$  belongs to  $\mathbf{W}(Q_T^{(a)})$  and formally satisfies identity (4.1) in the cylinder  $Q_T^{(a)}$ . If  $u^*(x, t)$  possesses the property of time localization (i.e. vanishes at a finite instant), so does the function  $u(x, t)$ , which is why in what follows we study the localization properties of the energy solutions of problem (1.1) formulated in the domain  $\Omega \equiv \{x \in \mathbb{R}^n : x_i \in (0, a)\}$ .

#### 4.2. Estimates on the total energy

In the further study of the behavior of the energy functions  $\Theta(t)$ ,  $\Lambda(t)$ ,  $\Lambda_\sigma(t)$  we will need the following uniform in  $t$  estimates.

**Lemma 4.5.** Let conditions (2.1), (2.6), (2.7) be fulfilled,  $c_0 \geq 0$  and  $u_0 \in L^2(\Omega)$ .

(a) If  $f \in L^2(Q_T)$ , then every solution  $u \in \mathbf{W}(Q_T)$  of problem (1.1) satisfies the estimate

$$\frac{1}{4} \operatorname{ess\,sup}_{(0,T)} \Theta(t) + a_0 \int_0^T \Lambda(t) dt \leq \frac{1}{2} \Theta(0) + 4 \left( \int_0^T \|f(\cdot, t)\|_{2,\Omega} dt \right)^2. \quad (4.3)$$

(b) If

$$p^-(t) \geq \frac{2n}{n+2} \quad \text{and} \quad f \in L^{(p^-)'(t)}(0, T; L^2(\Omega)),$$

then

$$\frac{1}{4} \operatorname{ess\,sup}_{(0,T)} \Theta(t) + \frac{a_0}{2} \int_0^T \Lambda(t) dt \leq \frac{1}{2} \Theta(0) + C \int_0^T (\|f(\cdot, t)\|_{2,\Omega}^{(p^+)'(t)} + \|f(\cdot, t)\|_{2,\Omega}^{(p^-)'(t)}) dt \quad (4.4)$$

with a constant  $C \equiv C(a_0, n, |\Omega|)$ .

**Proof.** (a) It follows from the energy relation (4.1) with  $s = 0$  that

$$\frac{1}{2} \operatorname{ess\,sup}_{t \in (0,T)} \int_{\Omega(0)} u^2(x, t) dx + a_0 \sum_{i=1}^n \int_{Q_T} |D_i u|^{p_i(z)} dz + c_0 \int_{Q_T} |u|^{\sigma(z)} dz \leq \int_{Q_T} |u| |f| dz + \frac{1}{2} \int_{\Omega} u_0^2 dx. \quad (4.5)$$

Using Hölder's and Young's inequalities, we estimate

$$\begin{aligned}
\int_{Q_T} u f dz &\leq \int_0^T \|u(\cdot, t)\|_{2, \Omega} \|f(\cdot, t)\|_{2, \Omega} dt \\
&\leq \operatorname{ess\,sup}_{t \in (0, T)} \|u\|_{2, \Omega} \int_0^T \|f(\cdot, t)\|_{2, \Omega} dt \\
&\leq \frac{1}{4} \operatorname{ess\,sup}_{t \in (0, T)} \|u\|_{2, \Omega}^2 + 4 \left( \int_0^T \|f(\cdot, t)\|_{2, \Omega} dt \right)^2,
\end{aligned}$$

and (4.3) follows.

(b) We apply Lemma 3.4 and Young's inequality

$$\begin{aligned}
\int_{Q_T} u f dz &\leq \int_0^T \|u(\cdot, t)\|_{2, \Omega} \|f(\cdot, t)\|_{2, \Omega} dt \\
&\leq C \int_0^T \max \{ \Lambda^{\frac{1}{p^+(t)}}(t), \Lambda^{\frac{1}{p^-(t)}}(t) \} \|f(\cdot, t)\|_{2, \Omega} dt \\
&\leq \frac{a_0}{2} \int_0^T \Lambda(t) dt + C' \int_0^T \max \{ \|f(\cdot, t)\|_{2, \Omega}^{(p^+)'(t)}, \|f(\cdot, t)\|_{2, \Omega}^{(p^-)'(t)} \} dt. \quad \square
\end{aligned}$$

**Lemma 4.6.** Let conditions (2.1), (2.6), (2.7) be fulfilled. If  $c_0 > 0$  and  $f \in L^{\sigma'(\cdot)}(Q_T)$ , then the solution of problem (1.1),  $u \in \mathbf{W}(Q_T) \cap L^{\sigma(\cdot)}(Q_T)$  satisfies the estimate

$$\frac{1}{2} \operatorname{ess\,sup}_{t \in (0, T)} \Theta(t) + \frac{1}{2} \min\{a_0, c_0\} \int_0^T \Lambda_\sigma(t) dt \leq \frac{1}{2} \Theta(0) + \left( \frac{2}{\min\{2; c_0\}} \right)^{\frac{1}{\sigma^-(z)-1}} \int_{Q_T} |f|^{\sigma'(z)} dz. \quad (4.6)$$

**Proof.** The assertion follows from (4.5) and Young's inequality:

$$|u| |f| \leq \frac{c_0}{2} |u|^{\sigma(z)} + \left( \frac{2}{c_0} \right)^{\frac{\sigma'(z)}{\sigma(z)}} |f|^{\sigma'(z)} \leq \frac{c_0}{2} |u|^{\sigma(z)} + \left( \frac{2}{\min\{2; c_0\}} \right)^{\frac{1}{\sigma(z)-1}} |f|^{\sigma'(z)},$$

whence

$$\int_{Q_T} |u f| dz \leq \frac{c_0}{2} \int_{Q_T} |u|^{\sigma(z)} dz + \left( \frac{2}{\min\{2; c_0\}} \right)^{\frac{1}{\sigma^-(z)-1}} \int_{Q_T} |f|^{\sigma'(z)} dz. \quad \square$$

## 5. Differential inequality for the energy function

Let  $u(z) \in \mathbf{W}(Q_T)$  be a solution of problem (1.1). Letting in (4.1)  $s = 0$ , we find that for every  $t, t + \Delta t \in [0, T]$  the solution satisfies the identity

$$\frac{1}{2} \int_{\Omega} u^2(z) \Big|_{\tau=t}^{\tau=t+\Delta t} dx + \int_t^{t+\Delta t} \int_{\Omega} \left( \sum_i a_i |D_i u|^{p_i(z)} + c |u|^{\sigma(z)} \right) dz = \int_t^{t+\Delta t} \int_{\Omega} f u dz. \quad (5.1)$$

**Lemma 5.1.** Let in the conditions of Lemma 4.4  $c_0 > 0$ . Then the functions  $\Theta(t)$ ,  $\Lambda(t)$  satisfy the differential inequality

$$\frac{1}{2} \Theta'(t) + \min\{a_0, c_0\} \Lambda_\sigma(t) \leq \int_{\Omega} |f u| dx \quad \forall \text{ a.e. } t \in (0, T). \quad (5.2)$$

**Proof.** For every  $t, t + \Delta t \in [0, T]$  equality (5.1) gives the inequality

$$\frac{1}{2|\Delta t|} \Theta(\tau) \Big|_{\tau=t}^{\tau=t+\Delta t} \leq -\frac{1}{|\Delta t|} \min\{a_0, c_0\} \int_t^{t+\Delta t} \Lambda_\sigma(\tau) d\tau + \frac{1}{|\Delta t|} \int_t^{t+\Delta t} \int_\Omega |fu| dz. \quad (5.3)$$

For  $u \in \mathbf{W}(Q_T)$

$$\int_\Omega |D_i u|^{p_i(z)} dx, \int_\Omega |u|^{\sigma(z)} dx, \int_\Omega |fu| dx \in L^1(0, T),$$

whence for a.e.  $t \in (0, T)$  every term on the right-hand side of (5.3) has a limit as  $\Delta t \rightarrow 0$ . It follows that there exists a limit of the left-hand side as  $\Delta t \rightarrow 0$ , whence

$$\frac{1}{2} \Theta'(t) + \min\{a_0, c_0\} \int_\Omega \left( \sum_{i=1}^n |D_i u|^{p_i(z)} + |u|^{\sigma(z)} \right) dx \leq \int_\Omega |fu| dx. \quad \square$$

**Corollary 5.1.** *Let in the conditions of Lemma 5.1  $c_0 \geq 0$ . Then*

$$\frac{1}{2} \Theta'(t) + a_0 \Lambda(t) \leq \int_\Omega |fu| dx \quad \forall \text{ a.e. } t \in (0, T).$$

Let us denote

$$\frac{1}{v^+(t)} = \frac{1}{\sigma^+(t)} + \frac{1}{p^+(t)}, \quad \frac{1}{p^+(t)} \equiv \frac{1}{n} \sum_{i=1}^n \frac{1}{p_i^+(t)}.$$

**Lemma 5.2.** *Let in the conditions of Lemma 4.6*

$$\frac{1}{\sigma^-(t)} + \frac{1}{p^-(t)} = \frac{1}{v^-(t)} \leq \frac{n+1}{n} \quad \text{for } \frac{1}{n} < \frac{1}{p^-(t)} \equiv \frac{1}{n} \sum_{i=1}^n \frac{1}{p_i^-(t)}. \quad (5.4)$$

Then the energy solution  $u \in \mathbf{W}(Q_T)$  of problem (1.1) for a.e.  $t \in (0, T)$  satisfies the differential inequality

$$\frac{1}{2} \Theta'(t) + K_1 L(t) \Theta^{v^+(t)}(t) \leq K_2 \int_\Omega |f|^{\sigma'(z)} dx \quad (5.5)$$

with the constants  $M = \text{ess sup } \Theta(t)$ ,  $K_1 = \frac{1}{2C} \min\{a_0, c_0\}$ ,  $K_2 = \left(\frac{2}{\min\{2; c_0\}}\right)^{\frac{1}{\sigma^- - 1}}$ ,  $C$  from (3.6), and the coefficient

$$L(t) = \begin{cases} 1 & \text{if } M \leq 1, \\ M^{v^-(t) - v^+(t)} & \text{otherwise.} \end{cases} \quad (5.6)$$

**Proof.** By Young's inequality

$$\int_\Omega |fu| dx \leq \frac{c_0}{2} \int_\Omega |u|^{\sigma(z)} dx + \left(\frac{2}{\min\{2; c_0\}}\right)^{\frac{1}{\sigma^- - 1}} \int_\Omega |f|^{\sigma'(z)} dx.$$

Combining this inequality with (5.2) we have

$$\frac{1}{2} \Theta'(t) + \frac{1}{2} \min\{a_0, c_0\} \Lambda_\sigma(t) \leq \left(\frac{2}{\min\{2; c_0\}}\right)^{\frac{1}{\sigma^- - 1}} \int_\Omega |f|^{\sigma'(z)} dx.$$

By Lemma 3.3

$$\begin{aligned} C \Lambda_\sigma(t) &\geq \min\{\Theta^{v^+(t)}(t), \Theta^{v^-(t)}(t)\} \\ &= \min\left\{M^{v^+(t)} \left(\frac{\Theta(t)}{M}\right)^{v^+(t)}, M^{v^-(t)} \left(\frac{\Theta(t)}{M}\right)^{v^-(t)}\right\} \end{aligned}$$

$$\begin{aligned}
&\geq \left(\frac{\Theta(t)}{M}\right)^{v^+(t)} \min\{M^{v^+(t)}, M^{v^-(t)}\} \\
&= \Theta^{v^+(t)}(t) \min\{1, M^{v^-(t)-v^+(t)}\} \\
&\geq L(t)\Theta^{v^+(t)}(t). \quad \square
\end{aligned}$$

**Remark 5.1.** Estimates on the “total energy”  $M$  are given in Lemmas 4.5, 4.6.

**Lemma 5.3.** Let  $c_0 \geq 0$  and let  $\text{ess sup } \Theta(t) = M$ . If

$$\frac{2}{p^-(t)} \leq 1 + \frac{2}{n},$$

then the solution of problem (1.1) satisfies the differential inequality

$$\frac{1}{2}\Theta'(t) + \frac{a_0}{C}L(t)\Theta^{v^+(t)}(t) \leq F(t) \quad (5.7)$$

with the exponent and the right-hand side

$$v^+(t) = \frac{p^+(t)}{2} \geq \frac{p^-(t)}{2} \geq \max\left\{\frac{n}{n+2}, \frac{1}{2}\right\}, \quad F(t) = K_2 \|f(\cdot, t)\|_{2, \Omega}^{\frac{2v^+(t)}{2v^+(t)-1}},$$

the coefficients  $K_1$ , the constant  $C$  from (3.6),  $L(t)$  from Lemma 5.2 and a finite constant  $K_2 \equiv K_2(K_1, M, C, a_0, p^\pm)$ .

**Proof.** By virtue of Corollary 5.1 and (3.9)

$$\Theta'(t) + \frac{2a_0}{C}L(t)\Theta^{v^+(t)} \leq 2 \int_{\Omega} |fu| dx.$$

Applying Hölder's and Young's inequalities we estimate the right-hand side

$$\int_{\Omega} |u||f| dx \leq \sqrt{\Theta(t)} \|f(\cdot, t)\|_{2, \Omega} \leq \frac{a_0}{2C}L(t)\Theta^{v^+(t)}(t) + K_2 \|f(\cdot, t)\|_{2, \Omega}^{\frac{2v^+(t)}{2v^+(t)-1}},$$

whence the assertion.  $\square$

## 6. Vanishing in a finite time

### 6.1. Sufficient conditions for the finite time extinction

**Theorem 6.1.** Let  $u_0 \in L^2(\Omega)$ ,  $f \in L^2(Q_T) \cap L^{\sigma'(\cdot)}(Q_T)$ ,  $c_0 > 0$ , and let the exponents  $\sigma(z)$ ,  $p_i(z)$  satisfy the conditions of Lemma 5.2. If  $f \equiv 0$  for  $t \geq t_f$ , and if

$$\frac{1}{v^+(t)} = \frac{1}{\sigma^+(t)} + \frac{1}{n} \sum_{i=1}^n \frac{1}{p_i^+(t)} \geq \frac{1}{v_0} > 1, \quad v_0 = \text{const}, \quad (6.1)$$

then there exists  $t_* \geq t_f$  such that

$$\|u(\cdot, t)\|_{2, \Omega} \equiv 0 \quad \text{in } \Omega \text{ for all } t \geq t_*.$$

The extinction moment  $t_*$  depends on  $\|u_0\|$ ,  $\|f\|_{\sigma'(\cdot), Q_T}$ ,  $n$ ,  $\bar{p}$ ,  $\underline{p}$ ,  $\sup v(t)$  and  $\inf v(t)$ .

**Proof.** By Lemma 5.2 the energy function  $\Theta(t)$  satisfies the ordinary differential inequality

$$\frac{1}{2}\Theta'(t) + K_1 L(t)\Theta^{v^+(t)}(t) \leq 0 \quad \text{in } (t_f, T). \quad (6.2)$$

By virtue of (4.6),  $\Theta(t) \leq M$  with a constant  $M$  depending only on the data. Let us introduce the new energy function  $Z(t) \equiv \Theta(t)/M$  and write (6.2) in the form

$$Z'(t) + 2K_1 L(t)M^{v^+(t)-1}Z^{v^+(t)}(t) \leq 0 \quad \text{in } (t_f, T), \quad Z(t_f) \leq 1. \quad (6.3)$$

The function  $Z(t)$  is nonnegative and monotone decreasing. Let us denote

$$T^* = \sup\{t > 0: Z(t) > 0\}.$$

If  $T^* = t_f$ , the assertion is proven. Let  $T^* > t_f$ . Since  $Z^{v^+(t)} \geq Z^{v_0}(t)$ , inequality (6.3) leads to the differential inequality

$$Z'(t) + 2K_1 L(t) M^{v^+(t)-1} Z^{v_0}(t) \leq 0 \quad \text{in } (t_f, T^*), \quad Z(t_f) \leq 1.$$

Integration of this inequality gives: for  $t \in [t_f, T^*)$ ,

$$\begin{aligned} Z^{1-v_0}(t) &\leq 1 - 2K_1(1-v_0) \int_{t_f}^t \min\{1, M^{v^-(\tau)-v^+(\tau)}\} M^{v^+(\tau)-1} d\tau \\ &\leq 1 - 2K_1(1-v_0) \max\{1, 1/M\}(t-t_f). \end{aligned}$$

The conclusion follows now because  $Z(t) \geq 0$ .  $\square$

The assertion of Theorem 6.1 remains true in the case  $c_0 = 0$ , but under different conditions on the nonlinearity exponents.

**Theorem 6.2.** Let  $c_0 \geq 0$ ,  $u_0 \in L^2(\Omega)$ , and let  $f$  satisfy the conditions of Lemma 4.5. Assume that the exponents  $p_i(z)$  satisfy the conditions of Lemma 5.3. If  $f \equiv 0$  for  $t \geq t_f$ , and

$$1 + \frac{2}{n} \geq \frac{2}{p^-(t)} \geq \frac{1}{v^+(t)} = \frac{2}{p^+(t)} \geq \frac{1}{v_0} > 1, \quad (6.4)$$

then there exists  $t_* \geq t_f$  such that

$$\|u(\cdot, t)\|_{2,\Omega} \equiv 0 \quad \text{in } \Omega \text{ for all } t \geq t_*.$$

The proof is an imitation of the proof Theorem 6.1: the energy function  $\Theta(t)$  satisfies the differential inequality (6.2) with the exponent  $v^+(t)$  given in (6.4) and is globally bounded by virtue of (4.3).

## 6.2. On the balance between slow and fast diffusion

Let us briefly discuss the conditions on the nonlinearity exponents  $p_i(z)$  which guarantee finite time vanishing of the solutions in the isotropic and anisotropic cases. For the diffusion equation

$$u_t = \sum_i D_i (|D_i u|^{p_i(z)-2} D_i u)$$

these conditions are given in Theorem 6.2. In the special case of the isotropic diffusion and constant exponents of nonlinearity,  $p_i \equiv p = \text{const}$ , they reduce to the well-known condition

$$2 > p \geq \frac{2n}{n+2}. \quad (6.5)$$

The former inequality of (6.5) guarantees extinction in a finite time (see [5, Ch. 2, Sec. 2]), while the latter provides the inclusion  $W_0^{1,p}(\Omega) \subset L^2(\Omega)$ . Moreover, it is known that in the case of constant isotropic diffusion and better regularity of the data this condition can be relaxed and transformed to the inequalities  $2 > p > 1$  – see, e.g., [5, pp. 79–83]. We also refer to [7,12] for the case of isotropic and variable nonlinearity.

Let us turn to the case of anisotropic and variable diffusion.

**$n = 2$ .** Condition (6.4) becomes

$$1 < p_1^+(t) < \frac{p_2^+(t)}{p_2^+(t) - 1}, \quad 1 < p_2^+(t). \quad (6.6)$$

It follows that unlike the isotropic case, the solutions always extinct in a finite time, provided that the diffusion in the direction  $x_1$  is fast in comparison with the diffusion in the direction  $x_2$ . In particular, even in the case of very slow diffusion in the direction  $x_2$ , i.e. when  $p_2^+(t) \rightarrow \infty$ , the localization effect still takes place if the diffusion in the direction  $x_1$  is very fast.

Fig. 1 illustrates the difference between the equations of anisotropic and isotropic diffusion in the case  $n = 2$ : the solutions of the equation of anisotropic diffusion vanish in a finite time if the point  $(p_1^+(t), p_2^+(t))$  belongs to the shadowed

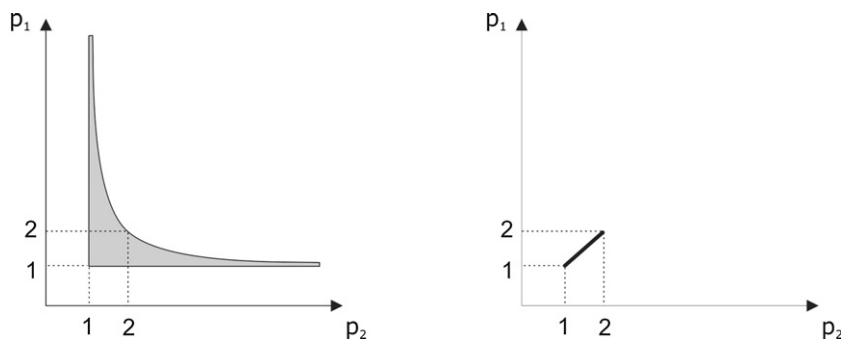


Fig. 1. The admissible ranges of the exponents in the case  $n = 2$ .

zone for all  $t$ , while in the equation of isotropic diffusion the same effect takes place only if the point  $(p_1(t), p_2(t))$  belongs to the open interval with the endpoints  $(1, 1)$  and  $(2, 2)$ .

**$n = 3$ .** In this case the solutions of the equation of isotropic diffusion vanish in a finite time if

$$\frac{6}{5} < p^+(t) < 2.$$

Let us consider the situation when the diffusion is fast in the directions  $x_1, x_2$ . For the sake of simplicity we assume that  $p_1^+(t) = p_2^+(t)$ . The solutions extinct in a finite if

$$\frac{3}{2} < \frac{1}{p_1^+(t)} + \frac{1}{p_2^+(t)} + \frac{1}{p_3^+(t)},$$

$$1 < p_1^+(t) = p_2^+(t) < \frac{4p_3^+(t)}{3p_3^+(t) - 2} \searrow \frac{4}{3} \text{ as } p_3^+(t) \nearrow \infty.$$

It follows that if the diffusion in the direction  $x_3$  is very slow, the solutions vanish in a finite time, provided that the diffusion in the directions  $x_1, x_2$  is suitably fast.

In case of fast diffusion in the direction  $x_1$  ( $p_1(z) \in (1, 2)$ ) and slow diffusion of the same rate in the directions  $x_2, x_3$  ( $p_2(z) = p_3(z) > 2$ ) we find that extinction takes place if

$$1 < p_1^+(t) < \frac{2p_2^+(t)}{3p_2^+(t) - 4} \quad \text{and} \quad p_2^+(t) = p_3^+(t) < 4.$$

In the limit case of fast diffusion in the direction  $x_1$ ,  $p_1^+(t) \searrow 1$ , we arrive at the upper limit of the admissible rate of slow diffusion in the directions  $x_2, x_3$ :  $p_2^+(t) = p_3^+(t) \nearrow 4$ .

**$n \geq 3$ .** A similar analysis shows that in the case of higher space dimension the effect of finite time extinction in solutions of anisotropic equation takes place under stronger restrictions on the admissible rate of slow and fast diffusion in different directions.

## 7. Vanishing at a prescribed moment

**Theorem 7.1.** Let the conditions of Theorem 6.1 be fulfilled. Set

$$S^{1-v_0}(t) = 2K_1(1 - v_0) \int_t^{t_f} \min\{M^{v^+(\theta)-1}; M^{v^-(\theta)-1}\} d\theta, \quad F(t) \equiv 2K_2 \int_{\Omega} |f|^{\sigma'(z)} dx$$

with the constants  $K_1, K_2$  from the conditions of Lemma 5.2 and  $M = \text{ess sup } \Theta(t)$ . If the integral

$$K(t_f) = \int_0^{t_f} \frac{F(\tau)}{S(\tau)} d\tau$$

is convergent and

$$G(t_f) \equiv \|u_0\|_{2,\Omega}^2 - S(0)(M - K(t_f)) \leq 0,$$



then

$$\|u(\cdot, t)\|_{2, \Omega} \equiv 0 \quad \text{in } \Omega \text{ for all } t \geq t_f.$$

**Proof.** By Lemma 5.2 the energy function  $\Theta(t)$  satisfies the differential inequality

$$\Theta'(t) + 2K_1 L(t) \Theta^{\nu^+(t)}(t) \leq F(t) \quad \text{in } (0, T), \quad \Theta(0) = \|u_0\|_{2, \Omega}^2.$$

Following the proof of Theorem 6.1 we introduce the function  $Z(t) \equiv \Theta(t)/M \leq 1$ . Since  $Z \leq 1$  and  $\nu^+(t) \leq \nu_0 < 1$ , we have  $Z^{\nu^+(t)} \geq Z^{\nu_0}$ . The differential inequality for  $Z$  transforms to the form

$$Z'(t) + 2K_1 \mu(t) Z^{\nu_0}(t) \leq \frac{1}{M} F(t) \quad \text{in } (0, T), \quad Z(0) = \frac{1}{M} \|u_0\|_{2, \Omega}^2 \leq 1,$$

$$\mu(t) = L(t) M^{\nu^+(t)-1} \equiv \min\{M^{\nu^+(t)-1}; M^{\nu^-(t)-1}\}.$$

By  $S(t)$  we denote the nonnegative solution of the problem

$$S'(t) + 2K_1 \mu(t) S^{\nu_0}(t) = 0, \quad S(0) = S_0 > 0,$$

which is given by the explicit formula

$$S(t) = \max \left\{ 0, \left( S(0)^{1-\nu_0} - 2K_1(1-\nu_0) \int_0^t \mu(\tau) d\tau \right)^{\frac{1}{1-\nu_0}} \right\}.$$

Let us fix the initial value  $S(0)$  by the condition

$$\frac{S^{1-\nu_0}(0)}{2K_1(1-\nu_0)} = \int_0^{t_f} \mu(\tau) d\tau,$$

i.e.,  $S(t_f) = 0$ . Notice that this choice of  $S(0)$  transforms the function  $S(t)$  to the form

$$S^{1-\nu_0}(t) = 2K_1(1-\nu_0) \int_t^{t_f} \mu(\tau) d\tau \quad \text{if } t \leq t_f.$$

Let us consider the function  $W(t) \equiv Z(t) - S(t)$  which satisfies on the interval  $(0, t_f)$  the inequality

$$W'(t) + 2K_1 \nu_0 \mu(t) \int_0^1 \frac{d\lambda}{(\lambda Z(t) + (1-\lambda)S(t))^{1-\nu_0}} W(t) \leq \frac{1}{M} F(t).$$

Multiplying this inequality by

$$\exp \left( 2K_1 \nu_0 \int_0^t \mu(\tau) \int_0^1 \frac{d\lambda d\tau}{(\lambda Z(\tau) + (1-\lambda)S(\tau))^{1-\nu_0}} \right)$$

and then integrating over the interval  $(0, t_f)$ , we transform it to the form

$$Z(t_f) \leq Z(0) - S(0) + \frac{1}{M} \int_0^{t_f} F(\tau) \exp \left( 2K_1 \nu_0 \int_0^\tau \mu(z) \int_0^1 \frac{d\lambda dz}{(\lambda Z(z) + (1-\lambda)S(z))^{1-\nu_0}} \right) d\tau.$$

Let us notice that by virtue of the equation for  $S(t)$ ,

$$2K_1 \nu_0 \int_0^\tau \mu(z) \int_0^1 \frac{d\lambda dz}{(\lambda Z(z) + (1-\lambda)S(z))^{1-\nu_0}} \leq 2K_1 \int_0^\tau \frac{\mu(z) dz}{S^{1-\nu_0}(z)} \int_0^1 \frac{\nu_0 d\lambda}{(1-\lambda)^{1-\nu_0}} = - \int_0^\tau \frac{S'(z)}{S(z)} dz = - \ln \frac{S(\tau)}{S(0)}.$$

It follows that

$$Z(t_f) \leq Z(0) - S(0) \left( 1 - \frac{1}{M} \int_0^{t_f} \frac{F(\tau)}{S(\tau)} d\tau \right) \equiv \frac{1}{M} G(t_f) \leq 0.$$

Since  $Z(t) \geq 0$  for all  $t > 0$ , it is necessary that  $Z(t_f) = 0$ . Considering now the differential inequality for  $Z(t)$  on the interval  $(t_f, T)$ ,

$$\begin{cases} Z'(t) + 2K_1\mu(t)Z^{v_0}(t) \leq 0 & \text{for } t \geq t_f, \\ Z(t_f) = 0, & Z(t) \geq 0, \end{cases}$$

we conclude that these conditions are satisfied only if  $Z(t) \equiv 0$  for  $t \geq t_f$ , that is, if  $\|u\|_{2,\Omega}(t) \equiv 0$ .  $\square$

**Remark 7.1.** It is easy to see that the class of data  $\{u_0, f\}$  for which the equation  $G(t_f) = 0$  has a solution is nonempty. Let us assume, for example, that  $u_0 \equiv 0$ ,  $c_0 \geq 2$  and

$$\int_{\Omega} |f|^{\sigma'(z)} dx \leq \epsilon(t_f - t)^{\mu} \quad \text{for } t \leq t_f \text{ with an exponent } \mu > \frac{v_0}{1 - v_0}.$$

In this special case  $K_2 = (\frac{2}{\min\{2, c_0\}})^{\frac{1}{\sigma^- - 1}} = 1$  and, by virtue of Lemma 4.6,

$$M = \operatorname{ess\,sup}_{t \in (0, T)} \Theta(t) \leq 2K_2 \int_{Q_T} |f|^{\sigma'(z)} dz \leq \frac{2\epsilon t_f^{\mu+1}}{1 + \mu} \leq 1$$

for all sufficiently small  $\epsilon$ . The condition  $G(t_f) \leq 0$  reads then  $K(t_f) \leq M$ , that is,

$$K(t_f) \leq \frac{2\epsilon}{(2K_1(1 - v_0))^{\frac{1}{1-v_0}}} \frac{t_f^{\mu+1 - \frac{1}{1-v_0}}}{\mu + 1 - \frac{1}{1-v_0}} \leq M$$

with the constants  $K_1 = \frac{1}{2C} \min\{a_0, c_0\}$  independent of  $\epsilon$ ,  $t_f$  and  $M$ . The last inequality is surely fulfilled if we claim that  $\min\{a_0, c_0\}$  is suitably large.

The next theorem refers to the case  $c_0 \geq 0$  and is a byproduct of Theorem 7.1.

**Theorem 7.2.** Let the conditions of Theorem 6.2 be fulfilled. If the integral  $K(t_f)$  with

$$F(t) \equiv 2K_2 \|f(\cdot, t)\|_{2,\Omega}^{\frac{2v^+(t)}{2v^+(t)-1}}$$

is convergent and if  $G(t_f) \leq 0$ , then

$$\|u(\cdot, t)\|_{2,\Omega} \equiv 0 \quad \text{in } \Omega \text{ for all } t \geq t_f.$$

## 8. Large time behavior

According to Theorems 6.1, 6.2, every solution of problem (1.1) vanishes at a finite moment  $t_*$ , provided that  $v^+(t) < 1$  and  $f(x, t) \equiv 0$  from some  $t_f$  on. Let us now study the behavior of  $\|u\|_{2,\Omega}(t)$  in the cases when at least one of these conditions is violated.

**Theorem 8.1.** Let the exponents  $p_i(z)$  and  $\sigma(z)$  satisfy the conditions of Lemma 5.2 if  $c_0 > 0$ , or Lemma 5.3 if  $c_0 = 0$ . Let  $\Theta(t) \leq M$  for all  $t > 0$ . Denote

$$\frac{1}{v^+(t)} = \begin{cases} \frac{1}{\sigma^+(t)} + \frac{1}{n} \sum_{i=1}^n \frac{1}{p_i^+(t)} & \text{if } c_0 > 0, \\ \frac{2}{n} \sum_{i=1}^n \frac{1}{p_i^+(t)} & \text{if } c_0 = 0. \end{cases}$$

(a) If  $v^+(t) \leq 1$ , then

$$\|u\|_{2,\Omega}^2(t) \leq e^{-2K_1 \min\{1, 1/M\}t} \left( \|u_0\|_{2,\Omega}^2 + \int_0^t \Phi(\tau) e^{2K_1 \min\{1, 1/M\}\tau} d\tau \right)$$

with

$$\Phi(t) = \begin{cases} 2K_2 \int_{\Omega} |f|^{\sigma'(z)} dx & \text{if } c_0 > 0, \\ 2K_2 \|f\|_{2,\Omega}^{\frac{2\nu^+(t)}{2\nu^+(t)-1}}(t) & \text{if } c_0 = 0 \end{cases}$$

and the constant  $K_2$  from (5.5) if  $c_0 > 0$ , or from (5.7) if  $c_0 = 0$ .

(b) If  $\nu_0 \geq \nu^+(t) > 1$  and

$$\Phi(t) \leq f_0(1+t)^{-\frac{\nu_0}{\nu_0-1}}$$

from some  $t_0$  on with a positive constant  $f_0$ , then there exists  $C = \text{const}$  such that

$$\|u\|_{2,\Omega}^2(t) \leq C(1+t)^{-\frac{\nu_0}{\nu_0-1}} \quad \text{for } t \geq t_0.$$

**Remark 8.1.** Gathering assertions (a) and (b), we have that for  $f \equiv 0$  for  $t \geq t_*$  and  $\nu^\pm(t) = 1$ ,

$$\|u\|_{2,\Omega}^2 \leq e^{-Ct} \|u_0\|_{2,\Omega}^2 \quad \text{as } t \rightarrow \infty, \quad C = \text{const}.$$

**Proof.** (a) Since  $\Theta(t) \leq M$ , we may apply (5.6) and continue inequalities (5.5), (5.7) as follows:

$$\Theta'(t) + 2K_1 \min\{1, 1/M\} \Theta(t) \leq \Theta'(t) + 2K_1 L(t) M^{\nu^+(t)-1} \Theta(t) \leq \Theta'(t) + 2K_1 L(t) M^{\nu^+(t)} \left( \frac{\Theta(t)}{M} \right)^{\nu^+(t)} \leq \Phi(t).$$

The conclusion now follows by Gronwall's inequality.

(b) Without loss of generality we may assume that  $t_0 = 0$ . The function  $\Theta(t)$  satisfies the differential inequality

$$\begin{aligned} \Theta'(t) + 2K_1(1+M)^{1-\nu_0} \Theta^{\nu_0}(t) &\leq \Theta'(t) + 2K_1(1+M)^{\nu^+(t)-\nu_0} \Theta^{\nu_0}(t) \\ &\leq \Theta'(t) + 2K_1 L(t) (1+M)^{\nu^+(t)} \left( \frac{\Theta(t)}{1+M} \right)^{\nu^+(t)} \\ &= \Theta'(t) + 2K_1 L(t) \Theta^{\nu^+(t)}(t) \leq f_0(1+t)^{-\frac{\nu_0}{\nu_0-1}} \quad \text{for } t \geq 0. \end{aligned}$$

Let  $Y(t)$  be a solution of the equation

$$Y'(t) + 2K_1(1+M)^{1-\nu_0} Y^{\nu_0}(t) = f_*(1+t)^{-\frac{\nu_0}{\nu_0-1}}$$

with the constant  $f_* > 0$  to be defined. A solution of this equation is given by the explicit formula

$$Y(t) = A(1+t)^{-\frac{1}{\nu_0-1}}$$

with the parameter  $A$  chosen from the condition

$$G(A) \equiv -\frac{A}{\nu_0-1} + 2K_1(1+M)^{1-\nu_0} A^{\nu_0} - f_* = 0.$$

This algebraic equation always has a solution  $A^* > 0$  because  $G(0) = -f_* < 0$ ,  $G(\infty) = \infty$ . Moreover, the solution  $A^*$  is estimated from below

$$A^* = \left[ \frac{(1+M)^{\nu_0-1}}{2K_1} \left( \frac{A^*}{\nu_0-1} + f_* \right) \right]^{\frac{1}{\nu_0}} \geq \left[ \frac{(1+M)^{\nu_0-1}}{2K_1} f_* \right]^{\frac{1}{\nu_0}}.$$

If we claim that  $f_* \geq \max\{f_0, 2K_1(1+M)^{1-\nu_0} \Theta^{\nu_0}(0)\}$ , then

$$\Theta(0) \leq Y(0) = A^* \quad \text{and} \quad f_* \geq f_0.$$

Let us introduce the function  $W(t) = \Theta(t) - Y(t)$ . The function  $W(t)$  satisfies the linear differential inequality

$$W'(t) + DW(t) \leq (f_0 - f_*)(1+t)^{-\frac{\nu_0}{\nu_0-1}} \leq 0 \quad \text{for } t > 0, \quad W(0) \leq 0,$$

with the coefficient

$$D = \frac{2K_1}{(1+M)^{\nu_0-1}} \frac{\Theta^{\nu_0}(t) - Y^{\nu_0}}{\Theta(t) - Y} \equiv \nu \frac{2K_1}{(1+M)^{\nu_0-1}} \int_0^1 (\lambda \Theta(t) + (1-\lambda) Y(t))^{\nu_0-1} d\lambda \geq 0.$$

It follows that  $W(t) \leq 0$  for all  $t > 0$ , i.e.,

$$0 \leq \Theta(t) \leq Y(t) = A^*(1+t)^{-\frac{1}{\nu_0-1}}. \quad \square$$

## 9. Limit cases

Let us now turn to the situation when in inequalities (5.5), (5.7),

$$v^+(t) \nearrow 1 \quad \text{as } t \rightarrow \infty.$$

If this happens, the arguments based on comparison with solutions of equations with constant exponents of nonlinearity are no longer valid because the nonlinear differential inequality (5.5) eventually transforms into the linear one. We will rely on the following assertion.

**Lemma 9.1.** *Let a nonnegative function  $\Theta(t)$  satisfy the conditions*

$$\begin{cases} \Theta'(t) + C\Theta^{\mu(t)}(t) \leq 0 & \text{for a.e. } t \geq 0 \text{ with } \mu(t) \in (0, 1) \text{ and } C = \text{const} > 0, \\ \Theta(t) \leq \Theta(0) < \infty, & \Theta(0) > 0. \end{cases} \quad (9.1)$$

*If the exponent  $\mu(t)$  is monotone increasing, then  $\Theta(t) \equiv 0$  for all  $t \geq t_*$  with  $t_*$  defined from the equality*

$$C \int_0^{t_*} \Theta^{\mu(s)-1}(0) ds = \int_0^\infty \frac{dz}{e^{z(1-\mu(z))}}.$$

**Proof.** Let us consider the function  $J(t) = \Theta(t)/\Theta(0)$ , satisfying the conditions

$$\forall \text{ a.e. } t > 0, \quad J'(t) + C\Theta^{\mu(t)-1}(0)J^{\mu(t)} \leq 0, \quad J(0) = 1, \quad J'(t) \leq 0.$$

Introducing the new independent variable  $\tau = C \int_0^t \Theta^{\mu(t)-1}(0) dt$ , the new exponent  $a(\tau) \equiv \mu(t)$  and the new thought function  $I(\tau) \equiv J(t)$ , we find that the function  $I(\tau)$  satisfies the conditions

$$\forall \text{ a.e. } \tau > 0, \quad I'(\tau) + I^{a(\tau)} \leq 0, \quad I(0) = 1, \quad I(\tau) \geq 0, \quad I'(\tau) \leq 0. \quad (9.2)$$

By monotonicity of  $I(\tau)$  there is an interval  $[0, \epsilon)$  where  $I(\tau) > 0$ , otherwise  $I(\tau) \equiv 0$  for all  $\tau \geq 0$ . Since  $I(\tau) \leq 1$  and  $a(\tau) \in (0, 1)$ , then  $I(\tau) \leq I^{a(\tau)}(\tau)$ , which leads to the inequality

$$I'(\tau) + I(\tau) \leq I'(\tau) + I^{a(\tau)}(\tau) \leq 0. \quad (9.3)$$

This inequality yields  $I(\tau) \leq e^{-\tau}$ , whence

$$\tau \leq -\ln I(\tau) \quad \text{and} \quad a(\tau) \leq a(-\ln I(\tau)). \quad (9.4)$$

Combining this inequality with (9.2) we have that

$$I'(\tau) + I^{a(-\ln I(\tau))}(\tau) \leq 0. \quad (9.5)$$

The straightforward integration of this inequality over the interval  $(0, \tau)$  gives:

$$\int_1^{I(\tau)} \frac{dr}{r^{a(-\ln r)}} \leq -\tau.$$

Introducing the new variable  $z = -\ln r$ , we rewrite it in the form

$$\int_0^{-\ln I(\tau)} \frac{dz}{e^{z(1-a(z))}} \geq \tau \quad \text{for } \tau \geq 0.$$

According to the choice of  $t_*$

$$\int_0^{-\ln I(\tau_*)} \frac{dz}{e^{z(1-a(z))}} \geq \tau_* = \int_0^\infty \frac{dz}{e^{z(1-a(z))}},$$

which is impossible unless  $I(\tau_*) = \Theta(t_*)/\Theta(0) = 0$ .  $\square$

**Theorem 9.1.** *Let in the conditions of Theorem 8.1  $f \equiv 0$  and  $\Theta(t) \leq M$ . Assume that  $v^+(t)$  is monotone increasing and  $v^+(t) \nearrow 1$  as  $t \rightarrow \infty$ . If the equation*

$$2K_1 \min\{1, 1/M\} \int_0^R \|u_0\|_{2,\Omega}^{2(v^+(s)-1)} ds = \int_0^\infty \frac{dz}{e^{z(1-v^+(z))}}$$

*has a root  $R = t_*$ , then the solution of problem (1.1)  $u \in \mathbf{W}(Q)$  vanishes at the instant  $t = t_*$ :*

$$\|u(\cdot, t)\|_{2,\Omega} = 0 \quad \text{for } t \geq t_*.$$

**Proof.** By Lemmas 5.2, 5.3 for every finite  $T > 0$ , the energy function  $\Theta(t)$  satisfies the ordinary differential inequality (5.5) or (5.7), which can be written in the form

$$\Theta'(t) + 2K_1 \min\{1, 1/M\} \Theta^{v^+(t)}(t) \leq 0.$$

The assertion follows then from Lemma 9.1.  $\square$

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