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Approximation of the finite dimensional distributions of multiple fractional integrals

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ABSTRACT

We construct a family $I_{n_\varepsilon}(f)_t$ of continuous stochastic processes that converges in the sense of finite dimensional distributions to a multiple Wiener–Itô integral $I_n^H(f1_{[0,t]}^{\otimes n})$ with respect to the fractional Brownian motion. We assume that $H > \frac{1}{2}$ and we prove our approximation result for the integrands f in a rather general class.

1. Introduction

A first result concerning the approximation of iterated stochastic integrals has been given in [1]. Consider $(X^\varepsilon)_{\varepsilon>0}$ a family of semimartingales with paths in the Skorohod space $\mathcal{D}([0,1])$ that converges weakly in this space to another semimartingale X, as ε tends to zero. It has been proven in [1] that the couple $(X^\varepsilon, [X^\varepsilon, X^\varepsilon])$ converges weakly in $\mathcal{D}([0,1])$ as $\varepsilon \to 0$ to the couple $(X, [X,X])$ ($[X,X]$ denotes the usual semimartingale bracket) if and only if for every $m \geqslant 1$ the vector $(J_1(X^\varepsilon), \dots, J_m(X^\varepsilon))$ converges weakly in $\mathcal{D}([0,1])$ as $\varepsilon \to 0$ to the vector $(J_1(X), \dots, J_m(X))$. Here $J_1(X)_t = X_t$ and for $k \geqslant 2$, $J_k(X)_t = \int_0^t J_{k-1}(X)_{s-}\, dX_s$ (and similarly for $J_k(X^\varepsilon)$). This result shows that in order to obtain (joint) weak convergence of iterated Itô integrals we need the convergence of X^ε to X but also the convergence of the second order variations. When our semimartingale is the Wiener process, there are many examples of families of processes with absolutely continuous paths converging weakly to it in the topology of $\mathcal{C}([0,1])$. In this case it is obvious that we do not have convergence of the quadratic variations of such families to the quadratic variations of the Brownian motion. This led to the problem of approximating iterated stochastic integrals with respect to the Browniam motion and later, with respect to the fractional Brownian motion.

Let us recall some relatively recent results concerning the approximation of iterated integrals with respect to a standard Brownian motion by a family of processes with continuous paths. Consider a family of stochastic processes $(\rho_\varepsilon)_{\varepsilon>0}$ of the form

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$$\rho_\varepsilon(t) = \int_0^t \theta_\varepsilon(s)\,ds$$

such that $(\rho_\varepsilon)_{\varepsilon>0}$ converges weakly in $C_0([0,1])$ (the space of continuous function on $[0,1]$ which are null at zero) to the Wiener process. We will discuss two main examples: the case when $\theta_\varepsilon(s) := \frac{1}{\varepsilon}\sum_{k=1}^\infty \xi_k I_{[k-1,k)}(\frac{s}{\varepsilon^2})$, where $\{\xi_k\}$ is a sequence of independent, identically distributed random variables satisfying $E(\xi_1)=0$ and $\mathrm{Var}(\xi_1)=1$ (these kernels are traditionally called *Donsker kernels*) and the case when

$$\theta_\varepsilon(x) := \frac{1}{\varepsilon}(-1)^{N(\frac{x}{\varepsilon^2})},$$

where $N = \{N(s);\ s \geqslant 0\}$ is a standard Poisson process (these kernels are usually called *Stroock kernels* or *Kac–Stroock kernels* because they were introduced by Kac in [6] and used by Stroock, [10], in order to obtain weak approximations of the Brownian motion). In [2] the authors proved that, for a suitable function f defined on $[0,1]^{\otimes n}$, the family of multiple integrals $(I^1_{n_\varepsilon}(f))_{\varepsilon>0}$ with respect to ρ_ε given by

$$\begin{aligned} I^1_{n_\varepsilon}(f)_t &= \int_{[0,t]^n} f(t_1,\dots,t_n)\,d\rho_\varepsilon(t_1)\cdots d\rho_\varepsilon(t_n) \\ &= \int_{[0,t]^n} f(t_1,\dots,t_n)\theta_\varepsilon(t_1)\cdots\theta_\varepsilon(t_n)\,dt_1\cdots dt_n \end{aligned} \tag{1}$$

converges weakly in $C_0([0,1])$ to the nth multiple Stratonovich integral of $f1^{\otimes n}_{[0,t]}$ with respect to the standard Brownian motion. This is somehow expected because the Stratonovich integral usually satisfies the differential rules of the deterministic calculus. In order to obtain as a limit a multiple Itô integral (which has zero expectation) one needs to subtract the "trace" of $I^1_{n_\varepsilon}(f)$, that means, to suppress the values on the diagonals. This following result has been obtained in [4]: for any $f \in L^2([0,1]^n)$, the family $(I^2_{n_\varepsilon}(f))_{\varepsilon>0}$ given by

$$\begin{aligned} I^2_{n_\varepsilon}(f)_t &= \int_{[0,t]^n} f(x_1,x_2,\dots,x_n)\prod_{\substack{i,j=1\\ i\neq j}}^n I_{\{|x_i-x_j|>\varepsilon\}}\,d\rho_\varepsilon(x_1)\cdots d\rho_\varepsilon(x_n) \\ &= \int_{[0,t]^n} f(x_1,x_2,\dots,x_n)\prod_{i=1}^n\theta_\varepsilon(x_i)\prod_{\substack{i,j=1\\ i\neq j}}^n I_{\{|x_i-x_j|>\varepsilon\}}\,dx_1\cdots dx_n, \end{aligned} \tag{2}$$

converges weakly, in the sense of finite dimensional distributions (and in $C_0([0,1])$ for $n=2$), to the nth multiple Itô integral $I_n(f1^{\otimes n}_{[0,t]})$. Let us consider now the problem of approximating the fractional Brownian motion $(B^H_t)_{t\in[0,1]}$ and the multiple integrals with respect to it. Recall that the fractional Brownian motion is a centered Gaussian process with covariance $R(t,s) = \frac{1}{2}(t^{2H}+s^{2H}-|t-s|^{2H})$ with $H \in (0,1)$. It can be also expressed as a Wiener integral with respect to a Wiener process W by $B^H = \int_0^t K_H(t,s)\,dW_s$ where K_H is a deterministic kernel defined on the set $\{0<s<t\}$ and given by

$$K_H(t,s) = c_H(t-s)^{H-\frac{1}{2}} + c_H\left(\frac{1}{2}-H\right)\int_s^t (u-s)^{H-\frac{3}{2}}\left(1-\left(\frac{s}{u}\right)^{\frac{1}{2}-H}\right)du, \tag{3}$$

where c_H is the normalizing constant $c_H = (\frac{2H\Gamma(\frac{3}{2}-H)}{\Gamma(H+\frac{1}{2})\Gamma(2-2H)})^{\frac{1}{2}}$. From this representation and the weak convergence of ρ_ε to W it follows that (see [3]) for any $H\in(0,1)$ the family of processes $(\eta_\varepsilon)_{\varepsilon>0}$ with

$$\eta_\varepsilon(t) = \int_0^t K_H(t,s)\theta_\varepsilon(s)\,ds, \quad t\in[0,1]$$

converges weakly as $\varepsilon\to 0$ in $C_0([0,1])$ to B^H. When $H > \frac{1}{2}$ the paths of η_ε are even absolutely continuous. Moreover, if $H>\frac{1}{2}$, the multiple integral with respect to η_ε

$$I^3_{n_\varepsilon}(f)_t = \int_{[0,t]^{\otimes n}} f(t_1,\dots,t_n)\,d\eta_\varepsilon(t_1)\cdots d\eta_\varepsilon(t_n) \tag{4}$$

converges as $\varepsilon \to 0$ in $C_0([0,1])$ to the multiple Stratonovich integral of order n of the function $f 1_{[0,t]}^{\otimes n}$ with respect to B^H. The purpose of this work is to give an approximation result for the multiple Wiener–Itô integrals $I_n^H(f 1_{[0,t]}^{\otimes n})$ with respect to the fractional Brownian motion, for the integrand f in a rather general class of functions. Note that, as we recall in Section 2, the multiple fractional integral I_n^H can be expressed as a multiple Wiener–Itô integral with respect to the Brownian motion. In fact, we have $I_n^H(f 1_{[0,t]}^{\otimes n}) = I_n(\Gamma_H^{(n)} f 1_{[0,t]}^{\otimes n})$ where $\Gamma_H^{(n)}$ is a transfer operator. Concretely, we show here that the family $(I_{n_\varepsilon}(f))_{\varepsilon>0}$ defined by

$$I_{n_\varepsilon}(f)_t = \int_0^1 \cdots \int_0^1 \left(\Gamma_H^{(n)} f 1_{[0,t]}^{\otimes n}\right)(x_1,\dots,x_n)\left(\prod_{i=1}^n \theta_\varepsilon(x_i)\right) \prod_{i,j=1;\ i\neq j}^n 1_{\{|x_i-x_j|>\varepsilon\}}\, dx_1\cdots dx_n$$

converges, in the sense of finite dimensional distributions, to $(I_n^H(f 1_{[0,t]}^{\otimes n}))_{t\in[0,1]}$. Due to the rather complicate expression of the operator $\Gamma_H^{(n)}$ this result cannot be deduced from the result in [4] since the transfer principle for multiple fractional integrals actually implies that $I_n^H(f 1_{[0,t]}^{\otimes n})$ is equal to $I_n(g(t,\cdot) 1_{[0,t]}^{\otimes n})$ with some function g depending on f. Because of the appearance of the variable t in the argument of g, the main result in [4] cannot be directly applied. Another particularity of the multiple fractional integrals is that the expectation $E I_1^H(1_A) I_1^H(1_B)$ is not zero when A and B are disjoint subsets of $[0,1]$ and this fact makes the proofs considerably more complex than in the standard Brownian motion case.

We structured our paper in the following way. Section 2 contains some preliminaries on multiple Wiener–Itô integrals and multiple integrals with respect to the fractional Brownian motion. In Section 3 we prove our approximation result. We first regard the case when the integrand is a step function. We separated the cases $n=2$ and $n \geqslant 3$ because in the first case the proof is less complex and more intuitive and it helps to understand the general case. Finally we extend our result from simple functions to a bigger class of functions.

2. Preliminaries

2.1. Multiple Wiener–Itô integrals

In this paragraph we describe the basic elements of calculus on Wiener chaos. Let $(W_t)_{t\in[0,1]}$ be a classical Wiener process on a standard Wiener space $(\Omega, \mathcal{F}, \mathbf{P})$. If $f \in L^2([0,1]^n)$ with $n \geqslant 1$ integer, we introduce the multiple Wiener–Itô integral of f with respect to W. We refer to [7] for a detailed exposition of the construction and the properties of multiple Wiener–Itô integrals.

Let $f \in \mathcal{S}_m$ be an elementary functions with m variables that can be written as

$$f = \sum_{i_1,\dots,i_m} c_{i_1,\dots,i_m} 1_{A_{i_1}\times\cdots\times A_{i_m}}$$

where the coefficients satisfy $c_{i_1,\dots,i_m} = 0$ if two indices i_k and i_l are equal and the sets $A_i \in \mathcal{B}([0,1])$ are pairwise-disjoints. For a such step function f we define

$$I_m(f) = \sum_{i_1,\dots,i_m} c_{i_1,\dots,i_m} W(A_{i_1})\cdots W(A_{i_m})$$

where we put $W([a,b]) = W_b - W_a$. It can be seen that for every $m \geqslant 1$ the application I_m constructed above from $\mathcal{S}_m$ to $L^2(\Omega)$ is an isometry on $\mathcal{S}_m$, i.e.

$$E\big[I_n(f) I_m(g)\big] = n! \langle f, g\rangle_{L^2([0,1]^n)} \quad \text{if } m=n \tag{5}$$

and

$$E\big[I_n(f) I_m(g)\big] = 0 \quad \text{if } m \neq n.$$

It also holds that

$$I_n(f) = I_n(\tilde f)$$

where $\tilde f$ denotes the symmetrization of f defined by $\tilde f(x_1,\dots,x_n) = \frac{1}{n!}\sum_{\sigma\in S_n} f(x_{\sigma(1)},\dots,x_{\sigma(n)})$.

Since the set $\mathcal{S}_n$ is dense in $L^2([0,1]^n)$ for every $n \geqslant 1$ the mapping I_n can be extended to an isometry from $L^2([0,1]^n)$ to $L^2(\Omega)$ and the above properties hold true for this extension. Note also that I_n can be viewed as an iterated stochastic integral

$$I_n(f) = n! \int_0^1 \int_0^{t_n} \cdots \int_0^{t_2} f(t_1,\dots,t_n)\, dW_{t_1}\cdots dW_{t_n};$$

here the integrals are of Itô type; this formula is easy to show for elementary functions f, and follows for general $f \in L^2([0,1]^n)$ by a density argument.

The product for two multiple integrals says that (see [7]): if $f \in L^2([0,1]^n)$ and $g \in L^2([0,1]^m)$ are symmetric functions, then

$$I_n(f)I_m(g) = \sum_{l=0}^{m\wedge n} l! \binom{m}{l}\binom{n}{l} I_{m+n-2l}(f \otimes_l g) \tag{6}$$

where the contraction $f \otimes_l g$ belongs to $L^2([0,1]^{m+n-2l})$ for $l = 0, 1, \dots, m \wedge n$ and it is given by

$$(f \otimes_l g)(s_1, \dots, s_{n-l}, t_1, \dots, t_{m-l}) = \int_{[0,1]^l} f(s_1, \dots, s_{n-l}, u_1, \dots, u_l) g(t_1, \dots, t_{m-l}, u_1, \dots, u_l)\, du_1 \cdots du_l. \tag{7}$$

When $l = 0$, we will denote, throughout this paper, by $f \otimes g := f \otimes_0 g$.

2.2. Multiple fractional integrals

Let us introduce here the multiple integrals with respect to the fractional Brownian motion. We follow the approach in [8] (see also [7,9]). Let $f \in L^1([0,1]^n)$ and for every $0 < \alpha < 1$ define the operator

$$\left(I_{t-}^{\alpha,n} f\right)(x_1, \dots, x_n) = \frac{1}{(\Gamma(\alpha))^n} \int_{x_1}^{t} \cdots \int_{x_n}^{t} \frac{f(t_1, \dots, t_n)}{\prod_{j=1}^n (t_j - x_j)^{1-\alpha}}\, dt_1 \cdots dt_n$$

for every $x_1, \dots, x_n \in [0,t]$ with $t \in [0,1]$.

We have the following properties:

- If $f = f_1 \otimes \cdots \otimes f_n$ with $f_i \in L^1([0,1])$ then
$$\left(I_{t-}^{\alpha,n} f\right)(x_1, \dots, x_n) = \left(I_{t-}^{\alpha,1} f_1\right)(x_1) \cdots \left(I_{t-}^{\alpha,1} f_n\right)(x_n)$$
for every $x_1, \dots, x_n \in [0,t]$.
- If $H > \frac{1}{2}$ then
$$c_H \Gamma\left(H + \frac{1}{2}\right) s^{\frac{1}{2}-H} \left(I_{1-}^{H-\frac{1}{2},1}\left(x^{H-\frac{1}{2}} 1_{[0,t]}\right)\right)(s) = K_H(t,s) \tag{8}$$
where K_H is the standard kernel of the fractional Brownian motion (3).

We introduce the space $|\mathcal{H}|^{\otimes n}$ of measurable functions $f : [0,1]^n \to \mathbb{R}$ such that

$$\int_{[0,1]^{2n}} \left|f(u_1, \dots, u_n) f(v_1, \dots, v_n)\right| \left(\prod_{j=1}^n \psi(u_j, v_j)\right) du_1 \cdots du_n\, dv_1 \cdots dv_n < \infty$$

where $\psi(s,t) = H(2H-1)|s-t|^{2H-2}$.

Remark 1. For any $H > \frac{1}{2}$ we have (see [7,8])

$$L^2\left([0,1]^n\right) \subset L^{\frac{1}{H}}\left([0,1]^n\right) \subset |\mathcal{H}|^{\otimes n}.$$

Define the operator $\Gamma_H^{(n)} : |\mathcal{H}|^{\otimes n} \to L^2([0,1]^n)$

$$\left(\Gamma_H^{(n)} f\right)(t_1, \dots, t_n) = \left[c_H \Gamma\left(H + \frac{1}{2}\right)\right]^n \prod_{j=1}^n t_j^{\frac{1}{2}-H} \left(I_{1-}^{H-\frac{1}{2},n}\right)\left(f(x_1, \dots, x_n) \prod_{j=1}^n x_j^{H-\frac{1}{2}}\right)(t_1, \dots, t_n). \tag{9}$$

Then the operator $\Gamma_H^{(n)}$ is an isometry between $|\mathcal{H}|^{\otimes n}$ and $L^2([0,1]^n)$ where we endow the space $|\mathcal{H}|^{\otimes n}$ with the following inner product

$$\langle f, g\rangle_{\mathcal{H}^{\otimes n}} = \int_{[0,1]^{2n}} f(u_1, \dots, u_n) g(v_1, \dots, v_n) \left(\prod_{j=1}^n \psi(u_j, v_j)\right) du_1 \cdots du_n\, dv_1 \cdots dv_n. \tag{10}$$

Note that:

- If $f = 1_{[0,b]}$ then by (8)

$$\left(\Gamma_H^{(1)} 1_{[0,b]}\right)(s) = c_H \Gamma\left(H + \frac{1}{2}\right) s^{\frac{1}{2}-H} I_{1-}^{H-\frac{1}{2},1}\left(x^{H-\frac{1}{2}} 1_{[0,b]}\right)(s) = K_H(b,s).$$

- If $f = 1_{(a,b]}$ then $(\Gamma_H^{(1)} 1_{(a,b]})(s) = K_H(b,s) - K_H(a,s)$.
- If $f_i \in |\mathcal{H}|$ $(i = 1, \dots, n)$ then

$$\Gamma_H^{(n)}(f_1 \otimes \cdots \otimes f_n) = \Gamma_H^{(1)} f_1 \otimes \cdots \otimes \Gamma_H^{(1)} f_n. \tag{11}$$

Let $f \in |\mathcal{H}|^{\otimes n}$. Then we define the multiple Wiener–Itô integral of f with respect to the fractional Brownian motion by

$$I_n^H(f) = I_n\left(\Gamma_H^{(n)} f\right) \tag{12}$$

where I_n denotes the standard Wiener–Itô integral with respect to the Wiener process as defined above. Note that $\Gamma_H^{(n)} f \in L^2([0,1]^n)$.

3. Approximation of multiple fractional Wiener integrals

Let us introduce some notation. We set

$$\eta_\varepsilon(t) = \int_0^t K_H(t,s)\theta_\varepsilon(s)\,ds, \quad t \in [0,1] \tag{13}$$

where θ_ε is such that $\int_0^t \theta_\varepsilon(s)\,ds$ converges weakly in the topology of the space $\mathcal{C}_0([0,1])$ to the standard Brownian motion.

Lemma 1. *Let θ_ε be either the Kac–Stroock kernels or the Donsker kernels. Then the family of processes η_ε converges weakly in $\mathcal{C}_0([0,1])$ as $\varepsilon \to 0$ to the fractional Brownian motion B^H for any $H \in (0,1)$.*

Proof. It has been proved in [3, Proposition 2.1]. $\square$

Denote, for every $\varepsilon > 0$

$$g_\varepsilon(x_1, \dots, x_n) = \prod_{i,j=1;\ i \neq j}^{n} 1_{\{|x_i - x_j| > \varepsilon\}} \tag{14}$$

and

$$I_{n_\varepsilon}(f)_t = \int_0^1 \cdots \int_0^1 \left(\Gamma_H^{(n)} f 1_{[0,t]}^{\otimes n}\right)(x_1, \dots, x_n) \left(\prod_{i=1}^n \theta_\varepsilon(x_i)\right) g_\varepsilon(x_1, \dots, x_n)\,dx_1 \cdots dx_n. \tag{15}$$

Remark 2. Note that it follows from a result in [9] that, if $f \in L^q([0,1])$ for some $q > \frac{1}{H}$ then the function $t \to I_{n_\varepsilon}(f)_t$ is continuous. Indeed, for every $s < t$

$$\begin{aligned}
\left|I_{n_\varepsilon}(f)_t - I_{n_\varepsilon}(f)_s\right| &\leqslant \sup_{0 \leqslant r \leqslant 1} \left|\theta_\varepsilon(r)\right|^n \int_{[0,1]^{\otimes n}} \left|\left(\Gamma_H^{(n)} f 1_{[0,t]}^{\otimes n}\right)(x_1, \dots, x_n) - \left(\Gamma_H^{(n)} f 1_{[0,s]}^{\otimes n}\right)(x_1, \dots, x_n)\right| dx_1 \cdots dx_n \\
&\leqslant \sup_{0 \leqslant r \leqslant 1} \left|\theta_\varepsilon(r)\right|^n \left(\int_{[0,1]^{\otimes n}} \left|\left(\Gamma_H^{(n)} f 1_{[0,t]}^{\otimes n}\right)(x_1, \dots, x_n) - \left(\Gamma_H^{(n)} f 1_{[0,s]}^{\otimes n}\right)(x_1, \dots, x_n)\right|^2 dx_1 \cdots dx_n\right)^{\frac{1}{2}} \\
&= \sup_{0 \leqslant r \leqslant 1} \left|\theta_\varepsilon(r)\right|^n \left(E\left|I_n^H\left(f 1_{[0,t]}^{\otimes n}\right) - I_n^H\left(f 1_{[0,s]}^{\otimes n}\right)\right|^2\right)^{\frac{1}{2}} \\
&\leqslant C_{H,n} \sup_{0 \leqslant r \leqslant 1} \left|\theta_\varepsilon(r)\right|^n |t-s|^{H-\frac{1}{q}},
\end{aligned}$$

where for the last inequality we used Theorem 3.2 in [9].

We first prove the following result.

Lemma 2. *Let f be a simple function of the form*

$$f(x_1,\dots,x_n)=\sum_{k=1}^m \alpha_k 1_{\Delta_k}(x_1,\dots,x_n) \tag{16}$$

where $m\in\mathbb{N}$, $\alpha_k\in\mathbb{R}$ for every $k=1,\dots,m$ and $\Delta_k=(a_k^1,b_k^1]\times\cdots\times(a_k^n,b_k^n]$ such that for every $k=1,\dots,m$, $(a_k^i,b_k^i]$ are disjoint intervals $(i=1,\dots,n)$. Then the finite dimensional distributions of the process $(Y^\varepsilon(f)_t)_{t\in[0,1]}$ given by

$$Y^\varepsilon(f)_t:=\int_0^1\cdots\int_0^1\big(\Gamma_H^{(n)}f1_{[0,t]}^{\otimes n}\big)(x_1,\dots,x_n)\left(\prod_{i=1}^n\theta_\varepsilon(x_i)\right)dx_1\cdots dx_n,$$

converge as $\varepsilon\to 0$ to the finite dimensional distributions of

$$\left(\sum_{k=1}^m\alpha_k I_1^H\big(1_{(a_k^1,b_k^1]}1_{[0,t]}\big)\cdots I_1^H\big(1_{(a_k^n,b_k^n]}1_{[0,t]}\big)\right)_{t\in[0,1]}=\left(\sum_{k=1}^m\alpha_k\big(B^H_{b_k^1\wedge t}-B^H_{a_k^1\wedge t}\big)\cdots\big(B^H_{b_k^n\wedge t}-B^H_{a_k^n\wedge t}\big)\right)_{t\in[0,1]}.$$

Proof. We have, by using the property (11) of the operator $\Gamma_H^{(n)}$,

$$\begin{aligned}
Y^\varepsilon(f)_t&=\int_0^1\cdots\int_0^1\big(\Gamma_H^{(n)}f1_{[0,t]}^{\otimes n}\big)(x_1,\dots,x_n)\left(\prod_{i=1}^n\theta_\varepsilon(x_i)\right)dx_1\cdots dx_n\\
&=\sum_{k=1}^m\alpha_k\int_0^1\cdots\int_0^1\big(\Gamma_H^{(n)}1_{(a_k^1,b_k^1]\times\cdots\times(a_k^n,b_k^n]}1_{[0,t]}^{\otimes n}\big)(x_1,\dots,x_n)\left(\prod_{i=1}^n\theta_\varepsilon(x_i)\right)dx_1\cdots dx_n\\
&=\sum_{k=1}^m\alpha_k\prod_{i=1}^n\int_0^1\big(\Gamma_H^{(1)}1_{(a_k^i,b_k^i]}1_{[0,t]}\big)(x_i)\theta_\varepsilon(x_i)\,dx_i\\
&=\sum_{k=1}^m\alpha_k\prod_{i=1}^n\int_0^1\big(K_H\big(b_k^i\wedge t,x_i\big)-K_H\big(a_k^i\wedge t,x_i\big)\big)\theta_\varepsilon(x_i)\,dx_i\\
&=\sum_{k=1}^m\alpha_k\prod_{i=1}^n\big(\eta_\varepsilon\big(b_k^i\wedge t\big)-\eta_\varepsilon\big(a_k^i\wedge t\big)\big),
\end{aligned}$$

and then for every fixed $t_1,\dots,t_r\in[0,1]$ the vector $(Y^\varepsilon(f)_{t_1},\dots,Y^\varepsilon(f)_{t_r})$ converges as in the statement because by Lemma 1 η_ε converges weakly to the fractional Brownian motion. □

Remark 3. Let f be a simple function. It can be seen that (here ∂_1K_H denotes the partial derivative of K_H with respect to the first variable)

$$\begin{aligned}
Y^\varepsilon(f)_t&=\int_{[0,t]^n}dx_1\cdots dx_n\left(\int_{x_1}^t\cdots\int_{x_n}^t\partial_1K_H(t_1,x_1)\cdots\partial_1K_H(t_n,x_n)f(t_1,\dots,t_n)\,dt_1\cdots dt_n\right)\left(\prod_{i=1}^n\theta_\varepsilon(x_i)\right)\\
&=\int_{[0,t]^n}f(t_1,\dots,t_n)\,d\eta_\varepsilon(t_1)\cdots d\eta_\varepsilon(t_n).
\end{aligned}$$

Therefore $Y^\varepsilon(f)$ coincides with $I^3_{\eta_\varepsilon}$ defined by (4).

Note also that in the case of multiple Wiener–Itô integrals ($H=\frac{1}{2}$) the random variable $Y^\varepsilon(f)_t$ coincides with $I_{\eta_\varepsilon}(f)_t$ for ε small enough if f is a simple function.

3.1. The case $n = 2$

Let us consider first the case of a multiple integral in the second Wiener chaos. Suppose that f is a simple function of two variables of the form

$$f(x, y) = \sum_{k=1}^{m} \alpha_k 1_{(a_k^1, b_k^1]}(x) 1_{(a_k^2, b_k^2]}(y)$$

where for every k, $(a_k^1, b_k^1]$ and $(a_k^2, b_k^2]$ are disjoint intervals. In this case, by using the product formula for multiple stochastic integrals (6), the multiple integral of f with respect to B^H can be expressed as

$$\begin{aligned} I_2^H\big(f 1_{[0,t]}^{\otimes 2}\big) &= \sum_{k=1}^{m} \alpha_k I_2^H\big(1_{(a_k^1,b_k^1]} 1_{(a_k^2,b_k^2]} 1_{[0,t]}^{\otimes 2}\big) \\ &= \sum_{k=1}^{m} \alpha_k \big(B^H_{b_k^1 \wedge t} - B^H_{a_k^1 \wedge t}\big)\big(B^H_{b_k^2 \wedge t} - B^H_{a_k^2 \wedge t}\big) - \sum_{k=1}^{m} \alpha_k \langle 1_{(a_k^1,b_k^1]} 1_{[0,t]}, 1_{(a_k^2,b_k^2]} 1_{[0,t]} \rangle_{\mathcal{H}}. \end{aligned} \tag{17}$$

The main difference with respect to the case of the standard Brownian motion is given by the fact that the scalar product in $\mathcal{H}$ of two indicator functions of disjoint intervals is not zero anymore (recall that this scalar product is given by the formula (10)).

Let us show that the sequence

$$I_{2_\varepsilon}(f)_t = \int_0^1 \int_0^1 \big(\Gamma_H^{(2)} f 1_{[0,t]}^{\otimes 2}\big)(x_1, x_2) \left(\prod_{i=1}^{2} \theta_\varepsilon(x_i) \right) 1_{\{|x_1 - x_2| > \varepsilon\}}\, dx_1\, dx_2 \tag{18}$$

converges in the sense of finite dimensional distributions to the process $I_2^H(f 1_{[0,t]}^{\otimes 2})$. We can write

$$\begin{aligned} I_{2_\varepsilon}(f)_t = {} & \int_0^1 \int_0^1 \big(\Gamma_H^{(2)} f 1_{[0,t]}^{\otimes 2}\big)(x_1, x_2) \left(\prod_{i=1}^{2} \theta_\varepsilon(x_i) \right) dx_1\, dx_2 \\ & - \int_0^1 \int_0^1 \big(\Gamma_H^{(2)} f 1_{[0,t]}^{\otimes 2}\big)(x_1, x_2) \left(\prod_{i=1}^{2} \theta_\varepsilon(x_i) \right) 1_{\{|x_1 - x_2| < \varepsilon\}}\, dx_1\, dx_2. \end{aligned} \tag{19}$$

Note that, using the properties of the transfer operator $\Gamma_H^{(2)}$, the first term can be written as

$$\begin{aligned} & \sum_k \alpha_k \int_0^1 \Gamma_H^{(1)}\big(1_{(a_k^1,b_k^1] \cap [0,t]}\big)(x_1) \theta_\varepsilon(x_1)\, dx_1 \int_0^1 \Gamma_H^{(1)}\big(1_{(a_k^2,b_k^2] \cap [0,t]}\big)(x_2) \theta_\varepsilon(x_2)\, dx_2 \\ & \quad = \sum_k \alpha_k \big(\eta_\varepsilon\big(b_k^1 \wedge t\big) - \eta_\varepsilon\big(a_k^1 \wedge t\big)\big)\big(\eta_\varepsilon\big(b_k^2 \wedge t\big) - \eta_\varepsilon\big(a_k^2 \wedge t\big)\big) \end{aligned}$$

and by Lemma 2, its finite dimensional distributions converge to those of the stochastic process

$$\sum_{k=1}^{m} \alpha_k \big(B^H_{b_k^1 \wedge t} - B^H_{a_k^1 \wedge t}\big)\big(B^H_{b_k^2 \wedge t} - B^H_{a_k^2 \wedge t}\big).$$

Next we will discuss the behavior as $\varepsilon \to 0$ of the second term. We need the following lemma, which will play an important role in the sequel.

Lemma 3. *Consider two functions $f, g \in L^2([0,1])$ and denote by*

$$Y_\varepsilon = \int_0^1 \int_0^1 dx_1\, dx_2\, f(x_1) g(x_2) \theta_\varepsilon(x_1) \theta_\varepsilon(x_2) 1_{\{|x_1 - x_2| < \varepsilon\}}$$

where θ_ε are the Kac–Stroock kernels or the Donsker kernels. Then

$$Y_\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} Y = \int_0^1 f(x)g(x)\,dx \quad \text{in } L^2(\Omega).$$

Proof. *The case when θ_ε are the Kac–Stroock kernels.* In this case, $\theta_\varepsilon(x) = \frac{1}{\varepsilon}(-1)^{N(\frac{x}{\varepsilon^2})}$, where $\{N(t);\ t \geqslant 0\}$ is a standard Poisson process. We have

$$E(Y_\varepsilon - Y)^2 = E(Y_\varepsilon)^2 - 2YE(Y_\varepsilon) + E(Y)^2.$$

We first calculate,

$$\begin{aligned} E(Y_\varepsilon) &= \int_0^1\int_0^1 dx_1\,dx_2\, f(x_1)g(x_2)\frac{1}{\varepsilon^2}e^{\frac{-2}{\varepsilon^2}|x_1-x_2|}1_{\{|x_1-x_2|<\varepsilon\}} \\ &= \int_0^1 dx_1\, f(x_1)\int_0^{x_1} dx_2\, g(x_2)\frac{1}{\varepsilon^2}e^{\frac{-2}{\varepsilon^2}(x_1-x_2)}1_{\{0<x_1-x_2<\varepsilon\}} \\ &\quad + \int_0^1 dx_2\, g(x_2)\int_0^{x_2} dx_1\, f(x_1)\frac{1}{\varepsilon^2}e^{\frac{-2}{\varepsilon^2}(x_2-x_1)}1_{\{0<x_2-x_1<\varepsilon\}}. \end{aligned}$$

Note that

$$\int_0^{x_1} dx_2\, g(x_2)\frac{2}{\varepsilon^2}e^{\frac{-2}{\varepsilon^2}(x_1-x_2)}1_{\{0<x_1-x_2<\varepsilon\}} = g*\varphi_\varepsilon(x_1)$$

where $\varphi_\varepsilon(z) = 1_{(0,\varepsilon)}(z)\frac{2}{\varepsilon^2}e^{-\frac{2z}{\varepsilon^2}}$ is an approximation of the identity. Therefore the convolution $g*\varphi_\varepsilon$ converges to g in $L^2([0,1])$ because $g \in L^2([0,1])$. We obtain

$$E(Y_\varepsilon) \underset{\varepsilon\to 0}{\longrightarrow} \int_0^1 f(x)g(x)\,dx = Y.$$

On the other hand,

$$\begin{aligned} E\big((Y_\varepsilon)^2\big) &= \int_{[0,1]^4} dx_1\,dx_2\,dx_3\,dx_4\, f(x_1)g(x_2)f(x_3)g(x_4)E\big(\theta_\varepsilon(x_1)\cdots\theta_\varepsilon(x_4)\big)1_{\{|x_1-x_2|<\varepsilon\}}1_{\{|x_3-x_4|<\varepsilon\}} \\ &:= I_1^\varepsilon + I_2^\varepsilon \end{aligned}$$

with

$$\begin{aligned} I_1^\varepsilon = \int_{[0,1]^4} & dx_1\,dx_2\,dx_3\,dx_4\, f(x_1)g(x_2)f(x_3)g(x_4)\frac{1}{\varepsilon^4}e^{\frac{-2}{\varepsilon^2}(|x_2-x_1|)}e^{\frac{-2}{\varepsilon^2}(|x_4-x_3|)} \\ &\times 1_{\{|x_2-x_1|<\varepsilon\}}1_{\{|x_4-x_3|<\varepsilon\}}\big(1_{\{x_1\vee x_2<x_3\wedge x_4\}} + 1_{\{x_3\vee x_4<x_1\wedge x_2\}}\big) \end{aligned}$$

and

$$\begin{aligned} I_2^\varepsilon = \int_{[0,1]^4} & dx_1\,dx_2\,dx_3\,dx_4\, f(x_1)g(x_2)f(x_3)g(x_4)\frac{1}{\varepsilon^4}e^{\frac{-2}{\varepsilon^2}(x_{(2)}-x_{(1)})}e^{\frac{-2}{\varepsilon^2}(x_{(4)}-x_{(3)})} \\ &\times 1_{\{|x_2-x_1|<\varepsilon\}}1_{\{|x_4-x_3|<\varepsilon\}}1_A(x_1,x_2,x_3,x_4) \end{aligned}$$

where we denoted by $A = \{\{x_1\vee x_2 < x_3\wedge x_4\}\cup\{x_3\vee x_4 < x_1\wedge x_2\}\}^C$. Here we used the fact that $E(\theta_\varepsilon(x_1)\cdots\theta_\varepsilon(x_4)) = \frac{1}{\varepsilon^4}e^{-\frac{2}{\varepsilon^2}(x_{(2)}-x_{(1)})}e^{-\frac{2}{\varepsilon^2}(x_{(4)}-x_{(3)})}$ where $x_{(1)}, x_{(2)}, x_{(3)}, x_{(4)}$ are the variables x_1, x_2, x_3, x_4 in increasing order.

We begin studying the convergence of the term I_2^ε. In the set A, there are 16 possible orders for the variables x_1, x_2, x_3, x_4. We will make the calculation for the case $x_1 < x_3 < x_2 < x_4$ but for the other 15 possible orders we can proceed in a similar way. In this case we have

$$\int_{[0,1]^4} dx_1\,dx_2\,dx_3\,dx_4\, f(x_1)g(x_2)f(x_3)g(x_4)\frac{1}{\varepsilon^4}e^{\frac{-2}{\varepsilon^2}(x_3-x_1)}e^{\frac{-2}{\varepsilon^2}(x_4-x_2)}1_{\{x_1<x_3<x_2<x_4\}}1_{\{|x_2-x_1|<\varepsilon\}}1_{\{|x_4-x_3|<\varepsilon\}}$$

$$\leqslant \frac{1}{2}\Bigg(\int_{[0,1]^2} dx_1\,dx_2\big[f(x_1)g(x_2)\big]^2 1_{\{|x_2-x_1|<\varepsilon\}}\Bigg(\int_{[0,1]^2} dx_3\,dx_4\,\frac{1}{\varepsilon^4}e^{\frac{-2}{\varepsilon^2}(x_3-x_1)}e^{\frac{-2}{\varepsilon^2}(x_4-x_2)}1_{\{x_1<x_3\}}1_{\{x_2<x_4\}}1_{\{|x_4-x_3|<\varepsilon\}}\Bigg)$$

$$+\int_{[0,1]^2} dx_3\,dx_4\big[f(x_3)g(x_4)\big]^2 1_{\{|x_4-x_3|<\varepsilon\}}\Bigg(\int_{[0,1]^2} dx_1\,dx_2\,\frac{1}{\varepsilon^4}e^{\frac{-2}{\varepsilon^2}(x_3-x_1)}e^{\frac{-2}{\varepsilon^2}(x_4-x_2)}1_{\{x_1<x_3\}}1_{\{x_2<x_4\}}1_{\{|x_2-x_1|<\varepsilon\}}\Bigg)\Bigg).$$

When we integrate the first integral with respect to $dx_3\,dx_4$ and the second integral with respect to $dx_1\,dx_2$ we obtain that the last expression can be bounded by

$$C\int_{[0,1]^2} dx_1\,dx_2\big[f(x_1)g(x_2)\big]^2 1_{\{|x_2-x_1|<\varepsilon\}}.$$

Proceeding in a similar way for the other 15 possible orders we obtain that

$$I_2^\varepsilon \leqslant C\int_{[0,1]^2} dx_1\,dx_2\big[f(x_1)g(x_2)\big]^2 1_{\{|x_2-x_1|<\varepsilon\}}.$$

This implies that I_2^ε converges to 0, by using the dominated convergence theorem.

Let us regard the behavior of the term I_1^ε. This term will give the convergence of $E(Y^\varepsilon)^2$. We have

$$I_1^\varepsilon = 8\int_{[0,1]^4} dx_1\,dx_2\,dx_3\,dx_4\, f(x_1)g(x_2)f(x_3)g(x_4)\frac{1}{\varepsilon^4}e^{\frac{-2}{\varepsilon^2}(|x_2-x_1|)}e^{\frac{-2}{\varepsilon^2}(|x_4-x_3|)}1_{\{|x_2-x_1|<\varepsilon\}}1_{\{|x_4-x_3|<\varepsilon\}}1_{\{x_1<x_2<x_3<x_4\}}$$

$$=2\int_0^1 dx_2\, g(x_2)1_{\{x_2<x_4\}}\int_0^{x_2} dx_1\, f(x_1)\frac{2}{\varepsilon^2}e^{\frac{-2}{\varepsilon^2}(x_2-x_1)}1_{\{0<x_2-x_1<\varepsilon\}}$$

$$\times\int_0^1 dx_4\, g(x_4)\int_{x_2}^{x_4} dx_3\, f(x_3)\frac{2}{\varepsilon^2}e^{\frac{-2}{\varepsilon^2}(x_4-x_3)}1_{\{0<x_4-x_3<\varepsilon\}}.$$

We obtain that I_1^ε converges to $2[\int_0^1\int_0^1 f(x)g(x)f(y)g(y)1_{\{x<y\}}\,dx\,dy]$. Thus I_1^ε converges to $[\int_0^1 f(x)g(x)\,dx]^2 = Y^2$.

The case when θ_ε are the Donsker kernels. In this case, $\theta_\varepsilon(x) = \frac{1}{\varepsilon}\sum_{k=1}^\infty \xi_k 1_{[k-1,k)}(\frac{x}{\varepsilon^2})$ where (ξ_k) is a sequence of independent, identically distributed random variables satisfying $E(\xi_1)=0$ and $E(\xi_1^2)=1$ with $E(\xi_1^{2n})<+\infty$. In this case we have

$$E(Y_\varepsilon) = \int_0^1\int_0^1 dx_1\,dx_2\, f(x_1)g(x_2)E\big(\theta_\varepsilon(x_1)\theta_\varepsilon(x_2)\big)1_{\{|x_1-x_2|<\varepsilon\}}$$

$$=\frac{1}{\varepsilon^2}\sum_{k=1}^\infty\int_0^1\int_0^1 dx_1\,dx_2\, f(x_1)g(x_2)1_{[k-1,k)^2}\left(\frac{x_1}{\varepsilon^2},\frac{x_2}{\varepsilon^2}\right)$$

$$=\int_0^1 dx_1\, f(x_1)\sum_{k=1}^\infty 1_{[(k-1)\varepsilon^2,k\varepsilon^2)}(x_1)\frac{1}{\varepsilon^2}\int_{(k-1)\varepsilon^2}^{k\varepsilon^2} dx_2\, g(x_2)$$

$$=\int_0^1 dx_1\, f(x_1)\sum_{k=1}^{[\frac{1}{\varepsilon^2}]+1} 1_{[(k-1)\varepsilon^2,k\varepsilon^2)}(x_1)\frac{1}{\varepsilon^2}\int_{(k-1)\varepsilon^2}^{k\varepsilon^2} dx_2\, g(x_2)$$

$$=\int_0^1 dx_1\, f(x_1)G_\varepsilon(x_1),$$

where

$$G_\varepsilon(x) := \sum_{k=1}^{[\frac{1}{\varepsilon^2}]+1} 1_{[(k-1)\varepsilon^2, k\varepsilon^2)}(x) \frac{1}{\varepsilon^2} \int_{(k-1)\varepsilon^2}^{k\varepsilon^2} dy\, g(y).$$

Fix $x_1 \in (0,1)$. Then for every $\varepsilon > 0$ close to zero, there exists a $k(x_1,\varepsilon) \in \{1,\dots,[\frac{1}{\varepsilon^2}]+1\}$ such that $(k(x_1,\varepsilon)-1)\varepsilon^2 \leqslant x_1 < k(x_1,\varepsilon)\varepsilon^2$. Then $0 \leqslant x_1 - (k(x_1,\varepsilon)-1)\varepsilon^2 < \varepsilon^2$ and this implies that $(k(x_1,\varepsilon)-1)\varepsilon^2 \to x_1$ as $\varepsilon \to 0$. Thus

$$G_\varepsilon(x_1) = \frac{1}{\varepsilon^2} \int_{(k(x_1,\varepsilon)-1)\varepsilon^2}^{k(x_1,\varepsilon)\varepsilon^2} dx_2\, g(x_2)$$

converges to $g(x_1)$ as $\varepsilon \to 0$. Consequently

$$E(Y_\varepsilon) \underset{\varepsilon\to 0}{\longrightarrow} \int_0^1 f(x_1) g(x_1)\, dx_1.$$

Now, we calculate $E(Y_\varepsilon^2)$. We have that,

$$\begin{aligned}
E\big(\theta_\varepsilon(x_1)\cdots\theta_\varepsilon(x_4)\big) &= \frac{1}{\varepsilon^4} \sum_{k\neq j=1}^{\infty} 1_{[(k-1)\varepsilon^2,k\varepsilon^2)^2}(x_1,x_2) 1_{[(j-1)\varepsilon^2,j\varepsilon^2)^2}(x_3,x_4) \\
&\quad + \frac{1}{\varepsilon^4} \sum_{k\neq j=1}^{\infty} 1_{[(k-1)\varepsilon^2,k\varepsilon^2)^2}(x_1,x_3) 1_{[(j-1)\varepsilon^2,j\varepsilon^2)^2}(x_2,x_4) \\
&\quad + \frac{1}{\varepsilon^4} \sum_{k\neq j=1}^{\infty} 1_{[(k-1)\varepsilon^2,k\varepsilon^2)^2}(x_1,x_4) 1_{[(j-1)\varepsilon^2,j\varepsilon^2)^2}(x_2,x_3) \\
&\quad + \frac{E(\xi_1^4)}{\varepsilon^4} \sum_{k=1}^{\infty} 1_{[(k-1)\varepsilon^2,k\varepsilon^2)^4}(x_1,x_2,x_3,x_4) \\
&:= G_\varepsilon^1(x_1,x_2,x_3,x_4) + G_\varepsilon^2(x_1,x_2,x_3,x_4) + G_\varepsilon^3(x_1,x_2,x_3,x_4) + G_\varepsilon^4(x_1,x_2,x_3,x_4).
\end{aligned}$$

Thus

$$\begin{aligned}
E\big((Y_\varepsilon)^2\big) &= \int_{[0,1]^4} dx_1\, dx_2\, dx_3\, dx_4\, f(x_1)g(x_2)f(x_3)g(x_4) G_\varepsilon^1(x_1,x_2,x_3,x_4) 1_{\{|x_1-x_2|<\varepsilon\}} 1_{\{|x_3-x_4|<\varepsilon\}} \\
&\quad + \int_{[0,1]^4} dx_1\, dx_2\, dx_3\, dx_4\, f(x_1)g(x_2)f(x_3)g(x_4) G_\varepsilon^2(x_1,x_2,x_3,x_4) 1_{\{|x_1-x_2|<\varepsilon\}} 1_{\{|x_3-x_4|<\varepsilon\}} \\
&\quad + \int_{[0,1]^4} dx_1\, dx_2\, dx_3\, dx_4\, f(x_1)g(x_2)f(x_3)g(x_4) G_\varepsilon^3(x_1,x_2,x_3,x_4) 1_{\{|x_1-x_2|<\varepsilon\}} 1_{\{|x_3-x_4|<\varepsilon\}} \\
&\quad + \int_{[0,1]^4} dx_1\, dx_2\, dx_3\, dx_4\, f(x_1)g(x_2)f(x_3)g(x_4) G_\varepsilon^4(x_1,x_2,x_3,x_4) 1_{\{|x_1-x_2|<\varepsilon\}} 1_{\{|x_3-x_4|<\varepsilon\}} \\
&:= J_\varepsilon^1 + J_\varepsilon^2 + J_\varepsilon^3 + J_\varepsilon^4.
\end{aligned}$$

The convergence of J_ε^1: Fix x_1 and x_3 in $[0,1]$. Then for every $\varepsilon > 0$ close to zero, there exists $k(x_1,\varepsilon) \in \{1,\dots,[\frac{1}{\varepsilon^2}]+1\}$ and $j(x_3,\varepsilon) \in \{1,\dots,[\frac{1}{\varepsilon^2}]+1\}$ such that $k(x_1,\varepsilon) \neq j(x_3,\varepsilon)$, $(k(x_1,\varepsilon)-1)\varepsilon^2 \leqslant x_1 < k(x_1,\varepsilon)\varepsilon^2$ and $(j(x_3,\varepsilon)-1)\varepsilon^2 \leqslant x_3 < j(x_3,\varepsilon)\varepsilon^2$, this implies that $(k(x_1,\varepsilon)-1)\varepsilon^2 \to x_1$ and $(j(x_3,\varepsilon)-1)\varepsilon^2 \to x_3$ as $\varepsilon \to 0$. Then we can write

$$\int_{[0,1]^2} dx_2\, dx_4\, g(x_2)g(x_4) G_\varepsilon^1(x_1,x_2,x_3,x_4) = \frac{1}{\varepsilon^2} \int_{(k(x_1,\varepsilon)-1)\varepsilon^2}^{k(x_1,\varepsilon)\varepsilon^2} dx_2\, g(x_2) \times \frac{1}{\varepsilon^2} \int_{(j(x_3,\varepsilon)-1)\varepsilon^2}^{j(x_3,\varepsilon)\varepsilon^2} dx_4\, g(x_4).$$

Moreover, this term converges to $g(x_1)g(x_3)$. We conclude that

$$J_\varepsilon^1 \underset{\varepsilon\to 0}{\longrightarrow} \int_0^1\int_0^1 dx_1\,dx_3\,f(x_1)g(x_1)f(x_3)g(x_3) = Y^2. \tag{20}$$

The convergence of J_ε^2 and J_ε^3: Fix x_1 and x_2 in $[0,1]$. Then for every $\varepsilon > 0$ close to zero, there exists $k(x_1,\varepsilon) \in \{1,\dots,[\frac{1}{\varepsilon^2}]+1\}$ and $j(x_2,\varepsilon) \in \{1,\dots,[\frac{1}{\varepsilon^2}]+1\}$ such that $k(x_1,\varepsilon) \neq j(x_2,\varepsilon)$, $(k(x_1,\varepsilon)-1)\varepsilon^2 \leqslant x_1 < k(x_1,\varepsilon)\varepsilon^2$ and $(j(x_2,\varepsilon)-1)\varepsilon^2 \leqslant x_2 < j(x_2,\varepsilon)\varepsilon^2$, this implies that $(k(x_1,\varepsilon)-1)\varepsilon^2 \to x_1$ and $(j(x_2,\varepsilon)-1)\varepsilon^2 \to x_2$ as $\varepsilon \to 0$. Hence

$$\int_{[0,1]^2} dx_3\,dx_4\,f(x_3)g(x_4)G_\varepsilon^2(x_1,x_2,x_3,x_4)1_{\{|x_1-x_2|<\varepsilon\}}1_{\{|x_3-x_4|<\varepsilon\}}$$

$$\leqslant 1_{\{|x_1-x_2|<\varepsilon\}}\frac{1}{\varepsilon^2}\int_{(k(x_1,\varepsilon)-1)\varepsilon^2}^{k(x_1,\varepsilon)\varepsilon^2} dx_3\,f(x_3) \times \frac{1}{\varepsilon^2}\int_{(j(x_2,\varepsilon)-1)\varepsilon^2}^{j(x_2,\varepsilon)\varepsilon^2} dx_4\,g(x_4).$$

This last term converges to $f(x_1)g(x_2)1_{\{x_1=x_2\}}$, this implies that

$$J_\varepsilon^2 \underset{\varepsilon\to 0}{\longrightarrow} 0. \tag{21}$$

In the same way, we obtain that

$$J_\varepsilon^3 \underset{\varepsilon\to 0}{\longrightarrow} 0. \tag{22}$$

The convergence of J_ε^4: Fix x_1 in $[0,1]$. Then for every $\varepsilon > 0$ close to zero, there exists $k(x_1,\varepsilon) \in \{1,\dots,[\frac{1}{\varepsilon^2}]+1\}$ such that $(k(x_1,\varepsilon)-1)\varepsilon^2 \leqslant x_1 < k(x_1,\varepsilon)\varepsilon^2$, this implies that $(k(x_1,\varepsilon)-1)\varepsilon^2 \to x_1$ as $\varepsilon \to 0$. Then we can write

$$\int_{[0,1]^3} dx_2\,dx_3\,dx_4\,g(x_2)f(x_3)g(x_4)G_\varepsilon^4(x_1,x_2,x_3,x_4) = \frac{E(\xi_1^4)}{\varepsilon^2}\int_{(k(x_1,\varepsilon)-1)\varepsilon^2}^{k(x_1,\varepsilon)\varepsilon^2} dx_2\,g(x_2)$$
$$\times \frac{1}{\varepsilon^2}\int_{(k(x_1,\varepsilon)-1)\varepsilon^2}^{k(x_1,\varepsilon)\varepsilon^2} dx_3\,f(x_3) \times \int_{(k(x_1,\varepsilon)-1)\varepsilon^2}^{k(x_1,\varepsilon)\varepsilon^2} dx_4\,g(x_4).$$

The last term converges to zero, thus

$$J_\varepsilon^4 \underset{\varepsilon\to 0}{\longrightarrow} 0. \tag{23}$$

Consequently, by combining the above convergences we obtain that

$$E\big((Y_\varepsilon)^2\big) \underset{\varepsilon\to 0}{\longrightarrow} Y^2. \qquad \square \tag{24}$$

We will also need the following lemma.

Lemma 4. *Let us consider a family of stochastic processes $(X^\varepsilon)_{t\in[0,1]}$ converging as $\varepsilon \to 0$ to $(X_t)_{t\in[0,1]}$ in the sense of finite dimensional distributions and a family of stochastic processes $(Y^\varepsilon)_{t\in[0,1]}$ such that for every $t \in [0,1]$ the sequence of random variables Y_t^ε converges, as $\varepsilon \to 0$ to Y_t in $L^2(\Omega)$ where Y_t is a constant for every t. Then $X^\varepsilon + Y^\varepsilon$ converges to $X + Y$ in the sense of finite dimensional distributions.*

Proof. Fix $t_1,\dots,t_r \in [0,1]$ and let us show that the vector

$$\big(X_{t_1}^\varepsilon + Y_{t_1}^\varepsilon,\dots,X_{t_r}^\varepsilon + Y_{t_r}^\varepsilon\big)$$

converges in law to the vector

$$(X_{t_1} + Y_{t_1},\dots,X_{t_r} + Y_{t_r}).$$

Take $g \in \mathcal{C}_b^1(\mathbb{R}^r)$, then

$$\begin{aligned}
&\big|E\big(g\big(X^{\varepsilon}_{t_1}+Y^{\varepsilon}_{t_1},\dots,X^{\varepsilon}_{t_r}+Y^{\varepsilon}_{t_r}\big)\big)-E\big(g(X_{t_1}+Y_{t_1},\dots,X_{t_r}+Y_{t_r})\big)\big|\\
&\quad\leqslant\big|E\big(g\big(X^{\varepsilon}_{t_1}+Y^{\varepsilon}_{t_1},\dots,X^{\varepsilon}_{t_r}+Y^{\varepsilon}_{t_r}\big)\big)-E\big(g\big(X^{\varepsilon}_{t_1}+Y_{t_1},\dots,X^{\varepsilon}_{t_r}+Y_{t_r}\big)\big)\big|\\
&\qquad+\big|E\big(g\big(X^{\varepsilon}_{t_1}+Y_{t_1},\dots,X^{\varepsilon}_{t_r}+Y_{t_r}\big)\big)-E\big(g(X_{t_1}+Y_{t_1},\dots,X_{t_r}+Y_{t_r})\big)\big|.
\end{aligned}$$

The first term converges to zero due to the L^2 convergence of Y^{ε} to Y since

$$\big|E\big(g\big(X^{\varepsilon}_{t_1}+Y^{\varepsilon}_{t_1},\dots,X^{\varepsilon}_{t_r}+Y^{\varepsilon}_{t_r}\big)\big)-E\big(g\big(X^{\varepsilon}_{t_1}+Y_{t_1},\dots,X^{\varepsilon}_{t_r}+Y_{t_r}\big)\big)\big|\leqslant KE\big[\big(Y^{\varepsilon}_{t_1}-Y_{t_1}\big)^2+\dots+\big(Y^{\varepsilon}_{t_r}-Y_{t_r}\big)^2\big]^{\frac{1}{2}}$$

and the second one converges to zero as $\varepsilon\to 0$ because, by Slutsky's theorem, $X^{\varepsilon}+Y$ converges to $X+Y$ in the sense of finite dimensional distributions. □

We obtain the following result:

Proposition 1. *Let $f\in\mathcal{S}_2$ and let $I_{2_\varepsilon}(f)_t$ be given by (18). Then $(I_{2_\varepsilon}(f)_t)_{t\in[0,1]}$ converges as $\varepsilon\to 0$ in the sense of finite dimensional distributions to the process $(I^H_2(f1^{\otimes 2}_{[0,t]}))_{t\in[0,1]}$.*

Proof. Recall the expressions (17) and (19) of $I^H_2(f1^{\otimes 2}_{[0,t]})$ and $I_{2_\varepsilon}(f)_t$. By Lemma 2 the first term in (17) converges in the sense of finite dimensional distributions to the first term in (19) and applying Lemma 3 for $f=\Gamma^{(1)}_H 1_{(a^1_k,b^1_k]}1_{[0,t]}$ and $g=\Gamma^{(1)}_H 1_{(a^2_k,b^2_k]}1_{[0,t]}$ we obtain that the term

$$\int_0^1\int_0^1\big(\Gamma^{(2)}_H f1^{\otimes 2}_{[0,t]}\big)(x_1,x_2)\left(\prod_{i=1}^2\theta_\varepsilon(x_i)\right)1_{\{|x_1-x_2|<\varepsilon\}}\,dx_1\,dx_2$$

converges in $L^2(\Omega)$ for every $t\in[0,1]$ to

$$\sum_k\alpha_k\int_0^1\Gamma^{(1)}_H\big(1_{(a^1_k,b^1_k]}1_{[0,t]}\big)(x)\Gamma^{(1)}_H\big(1_{(a^2_k,b^2_k]}1_{[0,t]}\big)(x)\,dx=\sum_k\alpha_k\langle 1_{(a^1_k,b^1_k]}1_{[0,t]},1_{(a^2_k,b^2_k]}1_{[0,t]}\rangle_{\mathcal{H}}.$$

The above Lemma 4 gives the conclusion. □

3.2. The case $n\geqslant 3$

In the case of multiple integrals of order $n\geqslant 3$, the structure of $I_{n_\varepsilon}(f)_t$ is more complex because of the appearance of all diagonals. The first step is to express the multiple integral of a tensor product of one-variable functions.

Lemma 5. *Let $f_1,\dots,f_n\in|\mathcal{H}|$. Then*

$$I^H_n(f_1\otimes f_2\otimes\dots\otimes f_n)=\prod_{i=1}^n I^H_1(f_i)+\sum_{l=1}^{[n/2]}(-1)^l\sum_{\substack{k_1,\dots,k_{2l}=1;\\ k_j\ \text{distinct}}}^{n}\left(\prod_{u\in\{1,\dots,n\}\setminus\{k_1,\dots,k_{2l}\}}I^H_1(f_u)\right)\langle f_{k_1},f_{k_2}\rangle_{\mathcal{H}}\cdots\langle f_{k_{2l-1}},f_{k_{2l}}\rangle_{\mathcal{H}}.\tag{25}$$

Proof. We will prove the result by induction. For $n=1,2$ it is trivial. Let us show how it works for $n=3$ because it is useful to understand the general case. We have, using (11), (12) and the product formula for multiple integrals (6)

$$\begin{aligned}
I^H_3(f_1\otimes f_2\otimes f_3)&=I_3\big(\Gamma^{(3)}_H(f_1\otimes f_2\otimes f_3)\big)\\
&=I_3\big(\Gamma^{(1)}_H(f_1)\otimes\Gamma^{(1)}_H(f_2)\otimes\Gamma^{(1)}_H(f_3)\big)\\
&=I_3\big(\big(\Gamma^{(1)}_H(f_1)\otimes\Gamma^{(1)}_H(f_2)\otimes\Gamma^{(1)}_H(f_3)\big)^{\sim}\big)\\
&=I_2\big(\Gamma^{(1)}_H(f_1)\tilde{\otimes}\Gamma^{(1)}_H(f_2)\big)I_1\big(\Gamma^{(1)}_H(f_3)\big)-2I_1\big(\big(\Gamma^{(1)}_H(f_1)\tilde{\otimes}\Gamma^{(1)}_H(f_2)\big)\otimes_1\Gamma^{(1)}_H(f_3)\big).
\end{aligned}$$

Note that

$$\big(\Gamma^{(1)}_H(f_1)\tilde{\otimes}\Gamma^{(1)}_H(f_2)\big)(t_1,t_2)=\frac{1}{2}\big(\Gamma^{(1)}_H(f_1)(t_1)\Gamma^{(1)}_H(f_2)(t_2)+\Gamma^{(1)}_H(f_1)(t_2)\Gamma^{(1)}_H(f_2)(t_1)\big)$$

and thus

$$\big(\Gamma_H^{(1)}(f_1)\tilde{\otimes}\Gamma_H^{(1)}(f_2)\big)\otimes_1 \Gamma_H^{(1)}(f_3)=\frac{1}{2}\big(\langle f_1,f_3\rangle_{\mathcal H}\Gamma_H^{(1)}(f_2)+\langle f_2,f_3\rangle_{\mathcal H}\Gamma_H^{(1)}(f_1)\big).$$

We obtain

$$\begin{aligned}I_3^H(f_1\otimes f_2\otimes f_3)&=\big(I_1^H(f_1)I_1^H(f_2)-\langle f_1,f_2\rangle_{\mathcal H}\big)I_1^H(f_3)+\big(\langle f_1,f_3\rangle_{\mathcal H}I_1^H(f_2)+\langle f_2,f_3\rangle_{\mathcal H}I_1^H(f_1)\big)\\&=I_1^H(f_1)I_1^H(f_2)I_1^H(f_3)-\big(\langle f_1,f_3\rangle_{\mathcal H}I_1^H(f_2)+\langle f_2,f_3\rangle_{\mathcal H}I_1^H(f_1)+\langle f_1,f_2\rangle_{\mathcal H}I_1^H(f_3)\big).\end{aligned}$$

Concerning the general case, assume that (25) holds for $1,2,\dots,n-1$. Again by the multiplication formula (6),

$$\begin{aligned}I_n^H(f_1\otimes\cdots\otimes f_n)&=I_{n-1}\big(\Gamma_H^{(1)}(f_1)\otimes\cdots\otimes\Gamma_H^{(1)}(f_{n-1})\big)I_1\big(\Gamma_H^{(1)}(f_n)\big)\\&\quad-(n-1)I_{n-2}\big(\big(\Gamma_H^{(1)}(f_1)\otimes\cdots\otimes\Gamma_H^{(1)}(f_{n-1})\big)^{\sim}\otimes_1\Gamma_H^{(1)}(f_n)\big)\\&=I_1^H(f_n)\cdot\prod_{i=1}^{n-1}I_1^H(f_i)+\sum_{l=1}^{[(n-1)/2]}(-1)^l\sum_{\substack{k_1,\dots,k_{2l}=1;\\ k_j\text{ distinct}}}^{n-1}\Big(\prod_{u\in\{1,\dots,n-1\}\setminus\{k_1,\dots,k_{2l}\}}I_1^H(f_u)I_1^H(f_n)\Big)\\&\quad\times\langle f_{k_1},f_{k_2}\rangle_{\mathcal H}\cdots\langle f_{k_{2l-1}},f_{k_{2l}}\rangle_{\mathcal H}-\sum_{i=1}^{n-1}I_{n-2}\big(\big(\Gamma_H^{(1)}(f_1)\otimes\cdots\hat{i}\cdots\otimes\Gamma_H^{(1)}(f_{n-1})\big)^{\sim}\big)\langle f_i,f_n\rangle_{\mathcal H}\end{aligned}$$

and this equal to

$$\begin{aligned}&I_1^H(f_n)\cdot\prod_{i=1}^{n-1}I_1^H(f_i)+\sum_{l=1}^{[(n-1)/2]}(-1)^l\sum_{\substack{k_1,\dots,k_{2l}=1;\\ k_j\text{ distinct}}}^{n-1}\Big(\prod_{u\in\{1,\dots,n-1\}\setminus\{k_1,\dots,k_{2l}\}}I_1^H(f_u)I_1^H(f_n)\Big)\langle f_{k_1},f_{k_2}\rangle_{\mathcal H}\cdots\langle f_{k_{2l-1}},f_{k_{2l}}\rangle_{\mathcal H}\\&-\sum_{i=1}^{n-1}\Big(\prod_{j=1;\ j\neq i}^{n-1}I_1^H(f_j)+\sum_{l=1}^{[(n-2)/2]}(-1)^l\sum_{\substack{k_1,\dots,k_{2l}=1;\ k_j\neq i;\\ k_j\text{ distinct}}}^{n-1}\Big(\prod_{u\in\{1,\dots,\hat{i},\dots,n-1\}\setminus\{k_1,\dots,k_{2l}\}}I_1^H(f_u)\Big)\\&\times\langle f_{k_1},f_{k_2}\rangle_{\mathcal H}\cdots\langle f_{k_{2l-1}},f_{k_{2l}}\rangle_{\mathcal H}\Big)\langle f_i,f_n\rangle_{\mathcal H}\end{aligned}$$

and it is not difficult to see that the last quantity is equal to the right side of (25). □

The next auxiliary two lemmas will be used in the proof of the main result.

Lemma 6. *Suppose that $(X_t^\varepsilon)_{t\in[0,1]}$ is a family of stochastic processes whose finite dimensional distributions converges to the finite dimensional distributions of a stochastic processes $(X_t)_{t\geqslant 0}$. Suppose also that $(Y_{s,t}^\varepsilon)_{s,t\in[0,1]}$ is a two-parameter stochastic process such that for every $s,t\in[0,1]$ we have that $Y_{s,t}^\varepsilon$ converge in $L^2(\Omega)$ to $a_{s,t}$, when ε tends to 0, where $a_{s,t}$ is a real constant. Then for every $t_1,\dots,t_{r_1}\in[0,1]$ and $s_1,\dots,s_{r_2},u_1,\dots,u_{r_2}\in[0,1]$ the vector*

$$\big(X_{t_1}^\varepsilon,\dots,X_{t_{r_1}}^\varepsilon,Y_{s_1,u_1}^\varepsilon,\dots,Y_{s_{r_2},u_{r_2}}^\varepsilon\big)$$

converges weakly to the vector

$$(X_{t_1},\dots,X_{t_{r_1}},a_{s_1,u_1},\dots,a_{s_{r_2},u_{r_2}}).$$

Proof. Consider $f\in\mathcal C_b^1(\mathbb R^{r_1+r_2})$. We have

$$\begin{aligned}&\big|E\big(f\big(X_{t_1}^\varepsilon,\dots,X_{t_{r_1}}^\varepsilon,Y_{s_1,u_1}^\varepsilon,\dots,Y_{s_{r_2},u_{r_2}}^\varepsilon\big)\big)-E\big(f(X_{t_1},\dots,X_{t_{r_1}},a_{s_1,u_1},\dots,a_{s_{r_2},u_{r_2}})\big)\big|\\&\quad\leqslant\big|E\big(f\big(X_{t_1}^\varepsilon,\dots,X_{t_{r_1}}^\varepsilon,Y_{s_1,u_1}^\varepsilon,\dots,Y_{s_{r_2},u_{r_2}}^\varepsilon\big)\big)-E\big(f\big(X_{t_1}^\varepsilon,\dots,X_{t_{r_1}}^\varepsilon,a_{s_1,u_1},\dots,a_{s_{r_2},u_{r_2}}\big)\big)\big|\\&\qquad+\big|E\big(f\big(X_{t_1}^\varepsilon,\dots,X_{t_{r_1}}^\varepsilon,a_{s_1,u_1},\dots,a_{s_{r_2},u_{r_2}}\big)\big)-E\big(f(X_{t_1},\dots,X_{t_{r_1}},a_{s_1,u_1},\dots,a_{s_{r_2},u_{r_2}})\big)\big|\\&\quad\leqslant KE\big[\big(Y_{s_1,u_1}^\varepsilon-a_{s_1,u_1}\big)^2+\cdots+\big(Y_{s_{r_2},u_{r_2}}^\varepsilon-a_{s_{r_2},u_{r_2}}\big)^2\big]^{\frac12}+\big|E\big(f\big(X_{t_1}^\varepsilon,\dots,X_{t_{r_1}}^\varepsilon,a_{s_1,u_1},\dots,a_{s_{r_2},u_{r_2}}\big)\big)\\&\qquad-E\big(f(X_{t_1},\dots,X_{t_{r_1}},a_{s_1,u_1},\dots,a_{s_{r_2},u_{r_2}})\big)\big|.\end{aligned}$$

The first term converges to zero because $Y^{\varepsilon}_{s,t}$ converge in $L^2(\Omega)$ to $a_{s,t}$ and the second one converges to 0 because the finite dimensional distributions of $(X^{\varepsilon}_t)_{t\in[0,1]}$ converge weakly to those of $(X_t)_{t\geqslant 0}$. □

Lemma 7. *Let $f_i \in L^2([0,1])$ for $i = 1, \dots, n$ and define*

$$F_\varepsilon = \int\limits_{[0,1]^n} f_1(x_1)\cdots f_n(x_n) 1_{\{|x_1-x_2|<\varepsilon\}} 1_{\{|x_1-x_3|<\varepsilon\}} \left(\prod_{j=1}^n \theta_\varepsilon(x_j)\right) h_\varepsilon(x_1,\dots,x_n)\,dx_1\cdots dx_n$$

where θ_ε are the Kac–Stroock's or the Donsker's kernels and we assume that for every $\varepsilon > 0$, $|h_\varepsilon(x_1,\dots,x_n)| \leqslant 1$ for every $x_1,\dots,x_n \in [0,1]$. Then

$$E\left(F_\varepsilon^2\right) \underset{\varepsilon\to 0}{\longrightarrow} 0. \tag{26}$$

Proof. (1) Assume that θ_ε are Kac–Stroock kernels. For $\varepsilon > 0$ and $x \geqslant 0$, we set $Q_\varepsilon(x) = \frac{1}{\varepsilon^2} e^{\frac{-2x}{\varepsilon^2}}$ and $f_j := f_{j-n}$ for every $j = n+1, \dots, 2n$. We introduce some operators on the set of permutations. $\mathfrak{S}_k$ denote the set of permutations on $1, \dots, k$. When $\tau \in \mathfrak{S}_{2m}$ and $\sigma \in \mathfrak{S}_m$, we note $\sigma \star \tau$ the element of $\mathfrak{S}_{2m}$ defined by

$$\sigma \star \tau(2j-1) = \tau\bigl(2\sigma(j)-1\bigr) \quad\text{and}\quad \sigma \star \tau(2j) = \tau\bigl(2\sigma(j)\bigr).$$

We have $id \star \tau = \tau$ and $\sigma' \star (\sigma \star \tau) = (\sigma'\sigma) \star \tau$, hence $\star : \mathfrak{S}_m \times \mathfrak{S}_{2m} \to \mathfrak{S}_{2m}$ defines a (right) group action of $\mathfrak{S}_m$ on $\mathfrak{S}_{2m}$. For any $\tau \in \mathfrak{S}_{2m}$, the orbit of τ has exactly $m!$ elements. Consequently, the set $\mathcal{O}$ of the orbits under the group action $\star$ has $\frac{(2m)!}{m!}$ elements and we have, by denoting τ_i one particular element of the orbit $o_i = o(\tau_i) \in \mathcal{O}$: for $r_1, \dots, r_{2m} \in [0,1]$,

$$\begin{aligned}
1_{\{\forall i\neq j,\ r_i\neq r_j\}} &= \sum_{\tau\in\mathfrak{S}_{2m}} 1_{\{r_{\tau(1)}>\cdots>r_{\tau(2m)}\}} = \sum_{o_i\in\mathcal{O}} \sum_{\tau\in o_i} 1_{\{r_{\tau(1)}>\cdots>r_{\tau(2m)}\}} \\
&\leqslant \prod_{i=1}^{\frac{(2m)!}{m!}} \prod_{j=1}^{m} 1_{\{r_{2\tau_i(j)-1}>r_{2\tau_i(j)}\}}.
\end{aligned} \tag{27}$$

Then for any $\varepsilon > 0$, we have

$$\begin{aligned}
E\left(F_\varepsilon^2\right) &= \Biggl| \int\limits_{[0,1]^{2n}} f_1(x_1)\cdots f_{2n}(x_{2n}) 1_{\{|x_1-x_2|<\varepsilon\}} 1_{\{|x_1-x_3|<\varepsilon\}} 1_{\{|x_{n+1}-x_{n+2}|<\varepsilon\}} 1_{\{|x_{n+1}-x_{n+3}|<\varepsilon\}} \\
&\quad \times E\left(\prod_{j=1}^n \theta_\varepsilon(x_j)\theta_\varepsilon(x_{j+n}) h_\varepsilon(x_1,\dots,x_n) h_\varepsilon(x_{j+n},\dots,x_{2n})\right) dx_1\cdots dx_{2n} \Biggr| \\
&\leqslant \int\limits_{[0,1]^{2n}} \bigl|f_1(x_1)\bigr|\cdots\bigl|f_{2n}(x_{2n})\bigr| 1_{\{|x_1-x_2|<\varepsilon\}} 1_{\{|x_1-x_3|<\varepsilon\}} \left| E\left(\prod_{j=1}^{2n} \theta_\varepsilon(x_j)\right)\right| dx_1\cdots dx_{2n} \\
&= \sum_{o_i\in\mathcal{O}} \sum_{\tau\in o_i} \int\limits_{[0,1]^{2n}} \bigl|f_1(x_1)\bigr|\cdots\bigl|f_{2n}(x_{2n})\bigr| 1_{\{|x_1-x_2|<\varepsilon\}} 1_{\{|x_1-x_3|<\varepsilon\}} \\
&\quad \times 1_{\{x_{\tau(1)}>\cdots>x_{\tau(2n)}\}} Q_\varepsilon\left(\sum_{j=1}^n (x_{\tau(2j-1)} - x_{\tau(2j)})\right) dx_1\cdots dx_{2n}.
\end{aligned}$$

Among the addends of the last term there are two possible situations.

- On one hand we have terms of the type:

$$\begin{aligned}
&\int\limits_{[0,1]^4} 1_{\{x_3>x_4\}} \bigl|f_{2\tau_i(k)-1}(x_1)\bigr| \bigl|f_{2\tau_i(k)}(x_2)\bigr| Q_\varepsilon(x_1-x_2) \\
&\quad \times 1_{\{0<x_1-x_2<\varepsilon\}} 1_{\{0<x_1-x_3<\varepsilon\}} \bigl|f_{2\tau_i(k')-1}(x_3)\bigr| \bigl|f_{2\tau_i(k')}(x_4)\bigr| Q_\varepsilon(x_3-x_4)\,dx_1\,dx_2\,dx_3\,dx_4 \\
&\quad \times \prod_{j=1;\ j\neq k,k'}^{n} \int\limits_{[0,1]^2} 1_{\{x_1>x_2\}} \bigl|f_{2\tau_i(j)-1}(x_1)\bigr| \bigl|f_{2\tau_i(j)}(x_2)\bigr| Q_\varepsilon(x_1-x_2)\,dx_1\,dx_2,
\end{aligned}$$

  where $\tau_i(k) > \tau_i(k') + 1$.

Notice that, using (27) (as in [5]), we obtain

$$\int_{[0,1]^2} 1_{\{x_1>x_2\}}\big|f_{2\tau_i(j)-1}(x_1)\big|\big|f_{2\tau_i(j)}(x_2)\big|Q_\varepsilon(x_1-x_2)\,dx_1\,dx_2 \leqslant \frac{1}{2}\|f_{2\tau_i(j)-1}\|_{L^2}\|f_{2\tau_i(j)}\|_{L^2}.$$

Moreover, given $h_i \in L^2([0,1])$, $i=1,\dots,4$, we have that

$$\begin{aligned}
&\int_{[0,1]^4} \big|h_1(x_1)\big|\big|h_2(x_2)\big|\big|h_3(x_3)\big|\big|h_4(x_4)\big|1_{\{0<x_1-x_2<\varepsilon\}}1_{\{0<x_1-x_3<\varepsilon\}}Q_\varepsilon(x_1-x_2)Q_\varepsilon(x_3-x_4)\,dx_1\,dx_2\,dx_3\,dx_4\\
&\quad\leqslant \int_{[0,1]^4} \big|h_1(x_1)\big|\big|h_2(x_2)\big|\big|h_3(x_3)\big|\big|h_4(x_4)\big|1_{\{0<x_1-x_2<\varepsilon\}}1_{\{0<x_1-x_3<\varepsilon\}}1_{\{0<x_3-x_4<\varepsilon\}}\\
&\qquad\times Q_\varepsilon(x_1-x_2)Q_\varepsilon(x_3-x_4)\,dx_1\,dx_2\,dx_3\,dx_4\\
&\qquad+\int_{[0,1]^4} \big|h_1(x_1)\big|\big|h_2(x_2)\big|\big|h_3(x_3)\big|\big|h_4(x_4)\big|1_{\{0<x_1-x_2<\varepsilon\}}1_{\{0<x_1-x_3<\varepsilon\}}1_{\{\varepsilon<x_3-x_4\}}\\
&\qquad\times Q_\varepsilon(x_1-x_2)Q_\varepsilon(x_3-x_4)\,dx_1\,dx_2\,dx_3\,dx_4\\
&\quad:= A_\varepsilon^1 + A_\varepsilon^2.
\end{aligned}$$

The term A_ε^1 converges to zero by using the same manner of the convergence of I_2^ε in the proof of Lemma 3. For the term A_ε^2 we have that

$$\begin{aligned}
A_\varepsilon^2 \leqslant \frac{1}{2}\bigg(&\int_{[0,1]^4} h_1^2(x_1)h_3^2(x_3)1_{\{0<x_1-x_3<\varepsilon\}}Q_\varepsilon(x_1-x_2)Q_\varepsilon(x_3-x_4)\,dx_1\,dx_2\,dx_3\,dx_4\\
&+\int_{[0,1]^4} h_2^2(x_2)h_4^2(x_4)1_{\{0<x_1-x_3<\varepsilon\}}1_{\{\varepsilon<x_3-x_4\}}Q_\varepsilon(x_1-x_2)Q_\varepsilon(x_3-x_4)\,dx_1\,dx_2\,dx_3\,dx_4\bigg).
\end{aligned}$$

Integrating with respect to x_2 and x_4 in the first addend we obtain the convergence to zero by using the dominated convergence theorem. Moreover, using the fact that for $y>\varepsilon$, $Q_\varepsilon(y) \leqslant e^{-2}$, and integrating after with respect to x_3 and x_1 we can bound the second addend by

$$\varepsilon\int_{[0,1]^2} h_2^2(x_2)h_4^2(x_4)\,dx_2\,dx_4$$

that clearly converge also to zero.

- We have also terms of the type:

$$\begin{aligned}
&\int_{[0,1]^6} 1_{\{x_1>x_4\}}\big|f_{2\tau_i(k)-1}(x_1)\big|\big|f_{2\tau_i(k)}(x_4)\big|Q_\varepsilon(x_1-x_4)1_{\{x_2>x_5\}}\big|f_{2\tau_i(k')-1}(x_2)\big|\big|f_{2\tau_i(k')}(x_5)\big|Q_\varepsilon(x_2-x_5)\\
&\quad\times 1_{\{x_3>x_6\}}\big|f_{2\tau_i(k'')-1}(x_3)\big|\big|f_{2\tau_i(k'')}(x_6)\big|Q_\varepsilon(x_3-x_6)1_{\{0<x_1-x_2<\varepsilon\}}1_{\{0<x_1-x_3<\varepsilon\}}\,dx_1\,dx_2\,dx_3\,dx_4\,dx_5\,dx_6\\
&\quad\times \prod_{j=1;\ j\neq k,k',k''}^{n}\int_{[0,1]^2} 1_{\{x_1>x_2\}}\big|f_{2\tau_i(j)-1}(x_1)\big|\big|f_{2\tau_i(j)}(x_2)\big|Q_\varepsilon(x_1-x_2)\,dx_1\,dx_2,
\end{aligned}$$

  where $\tau_i(k) > \tau_i(k')+1 > \tau_i(k'')+2$.
  But, using arguments similar to those presented in the previous situation it is not difficult to see that also this type of terms converges to zero.

Combining the above convergences we conclude that $E(F_\varepsilon^2)$ converges to zero and thus Lemma 7 satisfied.
(2) Assume now that θ_ε are Donsker kernels. For any $m \geqslant 3$

$$G_{\varepsilon,m}(x_1,\dots,x_m) := \frac{E(\xi_1^m)}{\varepsilon^m}\sum_{k=1}^{\infty} 1_{[(k-1)\varepsilon^2,k\varepsilon^2)^m}(x_1,\dots,x_m).$$

Fix x_1 in $[0,1]$. Then for every $\varepsilon>0$ close to zero, there exists $k(x_1,\varepsilon) \in \{1,\dots,[\frac{1}{\varepsilon^2}]+1\}$ such that $(k(x_1,\varepsilon)-1)\varepsilon^2 \leqslant x_1 < k(x_1,\varepsilon)\varepsilon^2$, this implies that $(k(x_1,\varepsilon)-1)\varepsilon^2 \to x_1$ as $\varepsilon \to 0$. Then we can write

$$J_{\varepsilon,m} := \int\limits_{[0,1]^m} f_1(x_1) f_2(x_2) \cdots f_m(x_m) G_{\varepsilon,m}(x_1,\ldots,x_m)\,dx_1\cdots dx_m$$
$$= E\big(\xi_1^m\big)\varepsilon^{m-2}\int\limits_0^1 f_1(x_1)\prod_{j=2}^m\left[\frac{1}{\varepsilon^2}\int\limits_{(k(x_1,\varepsilon)-1)\varepsilon^2}^{k(x_1,\varepsilon)\varepsilon^2} f_j(x_j)\,dx_j\right]dx_1.$$

Moreover for each $j=2,\ldots,m$, the term $\frac{1}{\varepsilon^2}\int_{(k(x_1,\varepsilon)-1)\varepsilon^2}^{k(x_1,\varepsilon)\varepsilon^2} f_j(x_j)\,dx_j$ converges to $f_j(x_1)$. Combining this with $m \geqslant 3$, we obtain that

$$J_{\varepsilon,m} \underset{\varepsilon\to 0}{\longrightarrow} 0. \tag{28}$$

On the other hand, if we denote by

$$\bar G_{\sigma,\varepsilon}(x_1,\ldots,x_{2n}) = \frac{1}{\varepsilon^{2n}}\prod_{j=0}^{n-1}\left(\sum_{k=1}^\infty 1_{[(k-1)\varepsilon^2,k\varepsilon^2)^2}(x_{\sigma(2j+1)},x_{\sigma(2j+2)})\right).$$

Fix $x_{\sigma(2j+1)}\in[0,1]$ for any $j=0,\ldots,n-1$. Then for every $\varepsilon>0$ close to zero, there exists $k(x_{\sigma(2j+1)},\varepsilon)\in\{1,\ldots,[\frac{1}{\varepsilon^2}]+1\}$ such that

$$k(x_{\sigma(2j+1)},\varepsilon)\neq k(x_{\sigma(2j'+1)},\varepsilon),\quad \forall j'\neq j \quad\text{and}\quad \big(k(x_{\sigma(2j+1)},\varepsilon)-1\big)\varepsilon^2 \leqslant x_{\sigma(2j+1)} < k(x_{\sigma(2j+1)},\varepsilon)\varepsilon^2.$$

This implies that $(k(x_{\sigma(2j+1)},\varepsilon)-1)\varepsilon^2 \to x_{\sigma(2j+1)}$ as $\varepsilon\to 0$. Then we can write

$$\bar J_{\sigma,\varepsilon} := \int\limits_{[0,1]^{2n}} f_1(x_1)\cdots f_{2n}(x_{2n})\bar G_{\sigma,\varepsilon}(x_1,\ldots,x_{2n}) 1_{\{|x_1-x_2|<\varepsilon\}}1_{\{|x_1-x_3|<\varepsilon\}}\,dx_1\cdots dx_{2n}$$
$$\leqslant \int\limits_{[0,1]^n}\prod_{l=0}^{n-1}\Bigg[f_{\sigma(2l+1)}(x_{\sigma(2l+1)})\frac{1}{\varepsilon^2}\int\limits_{(k(x_{\sigma(2l+1)},\varepsilon)-1)\varepsilon^2}^{k(x_{\sigma(2l+1)},\varepsilon)\varepsilon^2} 1_{\{|x_1-x_2|<\varepsilon\}}1_{\{|x_1-x_3|<\varepsilon\}}$$
$$\times f_{\sigma(2l+2)}(x_{\sigma(2l+2)})\,dx_{\sigma(2l+2)}\Bigg]dx_{\sigma(1)}\,dx_{\sigma(3)}\cdots dx_{\sigma(2n-1)}.$$

Moreover, this last term converges to

$$\int\limits_{[0,1]^n} 1_{\{x_{\sigma(2k+1)}=x_{\sigma(2k'+1)};\ \text{for some } k\neq k'\}}\prod_{l=0}^{n-1}\big[f_{\sigma(2l+1)}(x_{\sigma(2l+1)})f_{\sigma(2l+2)}(x_{\sigma(2l+1)})\big]\,dx_{\sigma(1)}\,dx_{\sigma(3)}\cdots dx_{\sigma(2n-1)}=0. \tag{29}$$

From (28), (29) and the fact that the term $E(\prod_{j=1}^n\theta_\varepsilon(x_j))$ is written as a sum of terms of type $G_{\varepsilon,m}$ or $\bar G_{\sigma,\varepsilon}$ we conclude that

$$E\big(F_\varepsilon^2\big)\underset{\varepsilon\to 0}{\longrightarrow} 0. \qquad \square$$

We can state now our approximation result for multiple fractional integrals when the integrand is a simple function.

Proposition 2. *Let f be a simple function of the form* (16)*. Then the finite dimensional distribution of the process* (15) *converge as $\varepsilon\to 0$ to the finite dimensional distributions of $(I_n^H(f1_{[0,t]}^{\otimes n}))_{t\in[0,1]}$.*

Proof. If f is a simple function of the form (16) then for every $t\in[0,1]$

$$I_n^H\big(f1_{[0,t]}^{\otimes n}\big) = \sum_{k=1}^m \alpha_k I_n^H\big(1_{(a_k^1,b_k^1]}\times\cdots\times 1_{(a_k^n,b_k^n]}1_{[0,t]}^{\otimes n}\big)$$
$$=\sum_{k=1}^m\alpha_k\Bigg(\prod_{i=1}^n I_1^H(1_{(a_k^i,b_k^i]}1_{[0,t]}) + \sum_{l=1}^{[n/2]}(-1)^l\sum_{\substack{j_1,\ldots,j_{2l}=1;\\ j_i \text{ distinct}}}^n\Bigg(\prod_{u\in\{1,\ldots,n\}\setminus\{j_1,\ldots,j_{2l}\}} I_1^H(1_{(a_k^u,b_k^u]}1_{[0,t]})\Bigg)$$
$$\times\langle 1_{(a_k^{j_1},b_k^{j_1}]}1_{[0,t]},1_{(a_k^{j_2},b_k^{j_2}]}1_{[0,t]}\rangle_{\mathcal H}\cdots\langle 1_{(a_k^{j_{2l-1}},b_k^{j_{2l-1}}]}1_{[0,t]},1_{(a_k^{j_{2l}},b_k^{j_{2l}}]}1_{[0,t]}\rangle_{\mathcal H}\Bigg).$$

The approximation $I_{n_\varepsilon}(f)_t$ can be expressed as

$$\begin{aligned} I_{n_\varepsilon}(f)_t &= \int_{[0,1]^n} \big(\Gamma_H^{(n)} f 1_{[0,t]}^{\otimes n}\big)(x_1,\dots,x_n) \prod_{j=1}^n \theta_\varepsilon(x_j) \Big(\prod_{i,j=1;\ i\neq j} (1 - 1_{\{|x_i-x_j|<\varepsilon\}})\Big)\, dx_1\cdots dx_n \\ &= \int_{[0,1]^n} \big(\Gamma_H^{(n)} f 1_{[0,t]}^{\otimes n}\big)(x_1,\dots,x_n) \prod_{j=1}^n \theta_\varepsilon(x_j) \\ &\quad \times \Bigg(1 + \sum_{l=1}^{[n/2]} (-1)^l \Bigg(\sum_{\substack{k_1,\dots,k_{2l}=1;\\ k_j \text{ distinct}}}^n 1_{\{|x_{k_1}-x_{k_2}|<\varepsilon\}} \cdots 1_{\{|x_{k_{2l-1}}-x_{k_{2l}}|<\varepsilon\}}\, dx_1\cdots dx_n\Bigg)\Bigg) + R. \end{aligned}$$

The term R above contains terms of the type

$$\int_{[0,1]^n} \Bigg(\prod_{j=1}^n \big(\Gamma_H^{(1)} 1_{(a_k^j,b_k^j]} 1_{[0,t]}\big)(x_j)\theta_\varepsilon(x_j)\Bigg) 1_{\{|x_1-x_2|<\varepsilon\}} 1_{\{|x_1-x_3|<\varepsilon\}} 1_A(x_1,\dots,x_n)\, dx_1\cdots dx_n$$

where A is a Borel subset of $[0,1]^{\otimes n}$. It will converge to zero by using Lemma 7 for $h_\varepsilon(x_1,\dots,x_n) = 1_A(x_1,\dots,x_n)$. The behavior of $I_{n_\varepsilon}(f)_t$ will be then given by the behavior of

$$\begin{aligned} &\int_{[0,1]^n} \big(\Gamma_H^{(n)} f 1_{[0,t]}^{\otimes n}\big)(x_1,\dots,x_n) \prod_{j=1}^n \theta_\varepsilon(x_j) \\ &\quad \times \Bigg(1 + \sum_{l=1}^{[n/2]} (-1)^l \Bigg(\sum_{\substack{k_1,\dots,k_{2l}=1;\\ k_j \text{ distinct}}}^n 1_{\{|x_{k_1}-x_{k_2}|<\varepsilon\}} \cdots 1_{\{|x_{k_{2l-1}}-x_{k_{2l}}|<\varepsilon\}}\, dx_1\cdots dx_n\Bigg)\Bigg). \end{aligned}$$

First we note that by Lemma 2 the first term in the above expression converges in the sense of finite dimensional distributions to

$$\sum_k \alpha_k I_1^H(1_{(a_k^1,b_k^1]} 1_{[0,t]}) \cdots I_1^H(1_{(a_k^n,b_k^n]} 1_{[0,t]}) = \sum_k \alpha_k \big(B^H_{b_k^1\wedge t} - B^H_{a_k^1\wedge t}\big)\cdots\big(B^H_{b_k^n\wedge t} - B^H_{a_k^n\wedge t}\big).$$

We will show that for every $l = 1,\dots,[\frac{n}{2}]$ and for every $j_1,\dots,j_{2l} = 1,\dots,n$ distinct the sequence

$$\int_{[0,1]^n} dx_1\cdots dx_n\, \Gamma_H^{(n)}\big(f 1_{[0,t]}^{\otimes n}\big)(x_1,\dots,x_n) \Bigg(\prod_{j=1}^n \theta_\varepsilon(x_j)\Bigg) 1_{\{|x_{j_1}-x_{j_2}|<\varepsilon\}} \cdots 1_{\{|x_{j_{2l-1}}-x_{j_{2l}}|<\varepsilon\}}$$

converges in the sense of finite dimensional distributions to the stochastic process

$$\begin{aligned} &\sum_k \alpha_k \Bigg(\prod_{u\in\{1,\dots,n\}\setminus\{j_1,\dots,j_{2l}\}} I_1^H(1_{(a_k^u,b_k^u]} 1_{[0,t]})\Bigg) \\ &\quad \times \langle 1_{(a_k^{j_1},b_k^{j_1}]} 1_{[0,t]}, 1_{(a_k^{j_2},b_k^{j_2}]} 1_{[0,t]}\rangle_{\mathcal{H}} \cdots \langle 1_{(a_k^{j_{2l-1}},b_k^{j_{2l-1}}]} 1_{[0,t]}, 1_{(a_k^{j_{2l}},b_k^{j_{2l}}]} 1_{[0,t]}\rangle_{\mathcal{H}}. \end{aligned}$$

Indeed, since

$$\Gamma_H^{(n)}\big(f 1_{[0,t]}^{\otimes n}\big)(x_1,\dots,x_n) = \sum_{k=1}^m \alpha_k \big(\Gamma_H^{(1)} 1_{(a_k^1,b_k^1]}\big)(x_1)\cdots\big(\Gamma_H^{(1)} 1_{(a_k^n,b_k^n]}\big)(x_n)$$

we can write, for every $j_1,\dots,j_{2l} = 1,\dots,n$ distinct

$$\begin{aligned} &\int_{[0,1]^n} dx_1\cdots dx_n\, \Gamma_H^{(n)}\big(f 1_{[0,t]}^{\otimes n}\big)(x_1,\dots,x_n) \Bigg(\prod_{j=1}^n \theta_\varepsilon(x_j)\Bigg) 1_{\{|x_{j_1}-x_{j_2}|<\varepsilon\}} \cdots 1_{\{|x_{j_{2l-1}}-x_{j_{2l}}|<\varepsilon\}} \\ &\quad = \sum_{k=1}^m \alpha_k \Bigg(\prod_{u\in\{1,\dots,n\}\setminus\{j_1,\dots,j_{2l}\}} \int_{[0,1]} dx_u\, \big(\Gamma_H^{(1)} 1_{(a_k^u,b_k^u]} 1_{[0,t]}\big)(x_u)\theta_\varepsilon(x_u)\Bigg) \end{aligned}$$

$$\times \int_{[0,1]^2} \big(\Gamma_H^{(1)} 1_{(a_k^{j_1}, b_k^{j_1}]} 1_{[0,t]}\big)(x_{j_1}) \big(\Gamma_H^{(1)} 1_{(a_k^{j_2}, b_k^{j_2}]} 1_{[0,t]}\big)(x_{j_2}) \theta_\varepsilon(x_{j_1}) \theta_\varepsilon(x_{j_2}) 1_{\{|x_{j_1}-x_{j_2}|<\varepsilon\}}\, dx_{j_1}\, dx_{j_2} \times \cdots$$

$$\times \int_{[0,1]^2} \big(\Gamma_H^{(1)} 1_{(a_k^{j_{2l-1}}, b_k^{j_{2l-1}}]} 1_{[0,t]}\big)(x_{j_{2l-1}}) \big(\Gamma_H^{(1)} 1_{(a_k^{j_{2l}}, b_k^{j_{2l}}]} 1_{[0,t]}\big)(x_{j_{2l}}) \theta_\varepsilon(x_{j_{2l-1}}) \theta_\varepsilon(x_{j_{2l}}) 1_{\{|x_{j_{2l-1}}-x_{j_{2l}}|<\varepsilon\}}\, dx_{j_{2l-1}}\, dx_{j_{2l}}$$

$$= \sum_{k=1}^m \alpha_k \Bigg(\prod_{u \in \{1,\dots,n\} \setminus \{j_1,\dots,j_{2l}\}} \big(\eta_\varepsilon\big(b_k^u \wedge t\big) - \eta_\varepsilon\big(a_k^u \wedge t\big)\big) \Bigg) \big(Y^\varepsilon_{b_k^{j_1}\wedge t, b_k^{j_2}\wedge t} - Y^\varepsilon_{a_k^{j_1}\wedge t, b_k^{j_2}\wedge t} - Y^\varepsilon_{b_k^{j_1}\wedge t, a_k^{j_2}\wedge t} + Y^\varepsilon_{a_k^{j_1}\wedge t, a_k^{j_2}\wedge t}\big)$$

where, for $v = 1, \dots, l$ we denoted by

$$Y^\varepsilon_{s,t} = \int_{[0,1]^2} \big(\Gamma_H^{(1)} 1_{[0,t]}\big)(x_1) \big(\Gamma_H^{(1)} 1_{[0,s]}\big)(x_2) \theta_\varepsilon(x_1) \theta_\varepsilon(x_2) 1_{\{|x_1-x_2|<\varepsilon\}}\, dx_1\, dx_2.$$

The conclusion follows by using Lemma 6 and the results obtained for the case $n = 2$. $\square$

We state now our main result.

Theorem 1. *Let f be a function in the space $|\mathcal{H}|^{\otimes n}$. Then the finite dimensional distribution of the process* (15) *converges as $\varepsilon \to 0$ to the finite dimensional distributions of $(I_n^H(f 1_{[0,t]}^{\otimes n}))_{t\in[0,1]}$.*

Proof. It is a consequence of Lemma 2.1 and Theorem 2.3 in [4], of the isometry of multiple integrals and of the fact that the simple functions are dense in $|\mathcal{H}|^{\otimes n}$ since for every $t \in [0,1]$ it holds (see Section 2.2 in [4])

$$E\big|I_{n_\varepsilon}(f)_t\big| \leqslant c \big\| \Gamma_H^{(n)} f 1_{[0,t]}^{\otimes n} \big\|_{L^2([0,1]^{\otimes n})} = c \big\| f 1_{[0,t]}^{\otimes n} \big\|_{\mathcal{H}^{\otimes n}}. \qquad \square$$

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