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Inequalities and bounds for generalized complete elliptic integrals

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ABSTRACT

Computable bounds for the generalized complete elliptic integrals of the first and second kind are obtained. Also, bounds for some combinations and products for integrals under discussion are established. It has been proven that both families of integrals are logarithmically convex as functions of the first parameter. This property has been employed to obtain several inequalities involving integrals in question.

1. Introduction

In recent years progress in the study of inequalities and bounds for elliptic integrals was made. Emphasis was on Legendre elliptic integrals, symmetric elliptic integrals, and recently on generalized complete elliptic integrals of the first and second kind. The interested reader is referred to [1,3,4,6,9,13–15,17], and the references therein. A renewed interest in families of elliptic integrals mentioned above stems from the fact that they play a prominent role in theory of conformal mappings, astronomy, physics, and engineering, to mention the most important areas of applications.

The goal of this paper is to establish computable bounds for the generalized complete elliptic integrals of the first and second kind. Bounds for some combinations and products of these integrals are also obtained. Several inequalities involving integrals under discussion are also proven.

This paper is the fourth one in the series of papers devoted to the study of bounds and inequalities for elliptic integrals (see [13–15]) and is organized as follows. Notation and some definitions which are used in the sequel are introduced in Section 2. Computable lower and upper bounds for the Gauss hypergeometric function are included in Section 3. Most of the main results of this paper are established in Section 4 (see Theorems 4.1, 4.3, 4.6, 4.10, 4.12). In Section 5 we present another main result which provides a lower bound for the product of two zero-balanced generalized complete elliptic integrals of the second kind is given.

2. Notation and definitions

In this section we introduce notation and definitions which will be used throughout the sequel.

In what follows the letter k $(0 < k < 1)$ will stand for the modulus of the elliptic integral. The complementary modulus k' is given by $k' = \sqrt{1-k^2}$. The parameters a, b and c of the elliptic integrals discussed in this paper are assumed to satisfy

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$a > 0$ and

$$c > b > 0 \tag{2.1}$$

unless otherwise stated.

Definition 1. The generalized complete elliptic integrals of the first and second kind are defined, respectively, by

$$\mathcal{K} \equiv \mathcal{K}_{a,b,c} \equiv \mathcal{K}_{a,b,c}(k) = \frac{B(a,b)}{2} F\left(a, b; c; k^2\right) \tag{2.2}$$

and

$$\mathcal{E} \equiv \mathcal{E}_{a,b,c} \equiv \mathcal{E}_{a,b,c}(k) = \frac{B(a,b)}{2} F\left(a-1, b; c; k^2\right). \tag{2.3}$$

(See [2, (1.9), (1.10), (1.3), (1.5)] and [12, (1.6), (1.7)].) Here

$$B(a,b) = \int_0^1 u^{a-1}(1-u)^{b-1}\, du \tag{2.4}$$

($a > 0$, $b > 0$) is the beta function (see, e.g., [9]) and

$$F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a,n)(b,n)}{(c,n)n!} z^n \tag{2.5}$$

($a, b, c \in \mathbb{R}$, $c \neq 0, -1, \ldots,$ $|z| < 1$) is the celebrated Gauss hypergeometric function, which is also often denoted by ${}_2F_1(a, b; c; z)$. In (2.5) the symbol (a, n) stands for the shifted factorial also called the Appell symbol which is defined as $(a, 0) = 1$ for $a \neq 0$ and $(a, n) = a(a-1) \cdot \cdots \cdot (a-n+1)$ for $n = 1, 2, \ldots$ (see, e.g., [9]).

For later use, let us record Euler's integral formula

$$F(a, b; c; z) = \int_0^1 (1 - zt)^{-a} \mu(t)\, dt \tag{2.6}$$

($z < 1$), where

$$\mu(t) = \frac{1}{B(b, c-b)} t^{b-1}(1-t)^{c-b-1} \tag{2.7}$$

is the Dirichlet measure on $(0, 1)$ (see [9]).

Also, we will use the bivariate weighted power means of two positive numbers x_1 and x_2. The associated weights w_1 and w_2 are positive numbers which satisfy $w_1 + w_2 = 1$. With $X = (x_1, x_2)$ and $w = (w_1, w_2)$ the power mean of order p ($p \in \mathbb{R}$) is defined as follows

$$A_p(w; X) = \begin{cases} (w_1 x_1^p + w_2 x_2^p)^{1/p}, & p \neq 0, \\ x_1^{w_1} x_2^{w_2}, & p = 0. \end{cases} \tag{2.8}$$

It is well known that the function $p \mapsto A_p$ increases with increase in p.

3. Bounds for Gauss' hypergeometric function

The goal of this section is to present some known bounds for Gauss' hypergeometric function F. These results will be used in subsequent sections of this paper.

Assume that parameters b and c satisfy (2.1). We let

$$w = \left(\frac{b}{c}, 1 - \frac{b}{c}\right) \quad \text{and} \quad X = \left(k', 1\right). \tag{3.1}$$

Assumption (2.1) guarantees positivity of weights w_1 and w_2 of the power mean $A_p(w; X)$. The lower bounds in

$$A_p(w; X)^{-2a} < F\left(a, b; c; k^2\right) < A_0(w; X)^{-2a} \tag{3.2}$$

($0 < a < c \leqslant \min(1 + 2a, 2b)$, $p \geqslant 2\frac{c-a}{c+1}$) and

$$A_p(w;X)^{-2a} < F\big(a,b;c;k^2\big) < A_2(w;X)^{-2a} \tag{3.3}$$

($-1 < a < 0$, $c \geqslant \max(1+2a, 2b)$, $p \leqslant 2\frac{c-a}{c+1}$) follow from (6) and (5), respectively, in [22] while the respective upper bounds are special cases of (2.14), (2.15), and (3.4) in [8].

Also, we will need the following.

Proposition 3.1. *Assume that the parameters b and c satisfy* (2.1) *and let w and X be the same as in* (3.1)*. Then the inequality*

$$A_p(w;X)^{2(1-2a)} < F\big(2a-1,b;c;k^2\big) \tag{3.4}$$

holds true if either

(i) $0 < a < \frac{1}{2}$, $c \geqslant \max(4a-1, 2b)$ *and* $p \leqslant 2\frac{c-2a+1}{c+1}$

or if

(ii) $\frac{1}{2} < a < 1$, $2a-1 < c < \min(4a-1, 2b)$ *and* $p \geqslant 2\frac{c-2a+1}{c+1}$.

Proof. Suppose that the conditions (i) are satisfied. Making use of the first inequality in (3.3) with $a := 2a-1$, we obtain the assertion (3.4). Similarly, if conditions (ii) are satisfied, then (3.4) follows from the left inequality in (3.2) when a is replaced by $2a-1$. □

4. Bounds and inequalities for elliptic integrals $\mathcal{K}$ and $\mathcal{E}$

Most of the main results of this paper are established in this section. We begin with the following.

Theorem 4.1. *Assume that the parameters b and c satisfy* (2.1) *and let w and X be the same as in* (3.1)*. If $0 < a < c \leqslant \min(1+2a, 2b)$ and $p \geqslant 2\frac{c-a}{c+1}$, then*

$$A_p(w;X)^{-2a} < \frac{2}{B(a,b)}\mathcal{K} < A_0(w;X)^{-2a}. \tag{4.1}$$

If $0 < a < 1$, $c \geqslant \max(2a-1, 2b)$ and $p \leqslant 2\frac{c-a+1}{c+1}$, then

$$A_p(w;X)^{2(1-a)} < \frac{2}{B(a,b)}\mathcal{E} < A_2(w;X)^{2(1-a)}. \tag{4.2}$$

Proof. Inequalities (4.1) follow from (2.2) and (3.2). For the proof of (4.2) we use (3.3) with $a := a-1$ followed by application of (2.3). □

For Legendre complete elliptic integrals $K = \mathcal{K}_{\frac{1}{2},\frac{1}{2},1}$ and $E = \mathcal{E}_{\frac{1}{2},\frac{1}{2},1}$ inequalities (4.1) and (4.2) simplify to

$$A_p(w;X)^{-1} < \frac{2}{\pi}K < A_0(w;X)^{-1}$$

($p \geqslant \frac{1}{2}$) and

$$A_p(w;X) < \frac{2}{\pi}E < A_2(w;X) \tag{4.3}$$

($p \leqslant \frac{3}{2}$) where now $w = (\frac{1}{2}, \frac{1}{2})$. Alzer and Qiu [1] have obtained a tighter upper bound in (4.3)

$$\frac{2}{\pi}E < A_\gamma(w;X),$$

where $\gamma = \ln(2)/\ln(\pi/2) = 1.53\ldots$.

To this end we will assume that X is the same as defined in (3.1) and also that parameters b and c satisfy inequalities (2.1).

Bounds for certain combinations of integrals under discussion are obtained in the following.

Theorem 4.2. *Let $0 < b+1 < c+1$ and let $w = (\frac{b+1}{c+1}, 1-\frac{b+1}{c+1})$. If $0 < a < c+1 \leqslant \min(1+2a, 2+2b)$, then*

$$A_p(w;X)^{-2a} < \frac{2c}{bB(a,b)}\frac{\mathcal{K}-\mathcal{E}}{k^2} < A_0(w;X)^{-2a} \tag{4.4}$$

holds for all $p \geqslant 2\frac{c+1-a}{c+1}$. *If* $0 < b < c+1$ *and if*

$$0 < a < c+1 < \min(1+2a, 2b), \tag{4.5}$$

then for $p \geqslant 2\frac{c+1-a}{c+2}$

$$A_p(w;X)^{-2a} < \frac{2c}{(c-b)B(a,b)}\frac{\mathcal{E}-k'^2\mathcal{K}}{k^2} < A_0(w;X)^{-2a}, \tag{4.6}$$

where now $w = (\frac{b}{c+1}, 1-\frac{b}{c+1})$. *If* $c+1 > \min(1+2a, 2b)$, *then*

$$A_2(w;X)^{-2a} < \frac{2c}{(c-b)B(a,b)}\frac{\mathcal{E}-k'^2\mathcal{K}}{k^2} < A_0(w;X)^{-2a}, \tag{4.7}$$

where the vector w *is the same as in* (4.6).

Proof. We shall establish (4.4) using known formula

$$\frac{2c}{bB(a,b)}\frac{\mathcal{K}-\mathcal{E}}{k^2} = F\bigl(a, b+1; c+1; k^2\bigr)$$

(see, e.g., [12]). Application of (3.2), with $b := b+1$ and $c := c+1$, to the right side of the last formula gives the assertion (4.4). Let the middle term in (4.6) and (4.7) be denoted by Λ. It is known that

$$\Lambda = F\bigl(a, b; c+1; k^2\bigr) \tag{4.8}$$

(see, e.g., [12]). Assume that the conditions (4.5) are satisfied. Using (3.2) with c replaced by $c+1$ we obtain the two-sided inequality (4.6). If, however, conditions (4.5) are not satisfied, then we apply (3.4) and (2.15) in [8] to the right side of (4.8) to obtain (4.7). The proof is complete. □

Before we state and prove our next result, let us recall an important result which is commonly called Chebyshev inequality for integrals.

Let μ be a probability measure on an interval Ω and let f and g be integrable functions with respect to μ. If both f and g are monotonically increasing or decreasing on Ω, then

$$\int_{\Omega} f(t)\mu(t)\,dt \int_{\Omega} g(t)\mu(t)\,dt \leqslant \int_{\Omega} f(t)g(t)\mu(t)\,dt.$$

The last inequality is reversed if either f is increasing and g is decreasing or vice versa and it becomes an equality if and only if either f or g is constant almost everywhere (see [21, p. 197]).

We are in a position to prove the following.

Theorem 4.3. *Let* w *be the same as in* (3.1). *If* $0 < a < \frac{1}{2}$, $c \geqslant \max(4a-1, 2b)$ *and* $p \leqslant 2\frac{c-2a+1}{c+1}$ *or if* $\frac{1}{2} < a < 1$, $2a-1 < c \leqslant \min(4a-1, 2b)$ *and* $p \geqslant 2\frac{c-2a+1}{c+1}$, *then*

$$\left[\frac{2}{B(a,b)}\right]^2 \mathcal{K}\mathcal{E} > A_p(w;X)^{2(1-2a)}. \tag{4.9}$$

Proof. It follows from (2.2), (2.3), and (2.6) that

$$\left[\frac{2}{B(a,b)}\right]^2 \mathcal{K}\mathcal{E} = \int_0^1 \bigl(1-k^2t\bigr)^{-a}\mu(t)\,dt \int_0^1 \bigl(1-k^2t\bigr)^{1-a}\mu(t)\,dt, \tag{4.10}$$

where the Dirichlet measure μ is defined in (2.7). Taking into account that the function $t \mapsto (1-k^2t)^{-a}$ is an increasing function on $(0,1)$ while $t \mapsto (1-k^2t)^{1-a}$ is a decreasing function on the same domain we obtain, using Chebyshev's inequality for integrals applied to (4.10), that

$$\left[\frac{2}{B(a,b)}\right]^2 \mathcal{K}\mathcal{E} > \int_0^1 \bigl(1-k^2t\bigr)^{1-2a}\mu(t)\,dt = F\bigl(2a-1, b; c; k^2\bigr). \tag{4.11}$$

The equality in (4.11) follows from (2.6). A lower bound for the third member of (4.11) is obtained by use of Proposition 3.1. The proof is complete. □

Corollary 4.4. *Under the assumptions of Theorem* 4.3 *the following inequality*

$$\mathcal{K}+\mathcal{E} > B(a,b)A_p(w;X)^{1-2a} \tag{4.12}$$

holds true.

Proof. It follows from (4.9) that

$$(\mathcal{K}\mathcal{E})^{1/2} > \frac{B(a,b)}{2}A_p(w;X)^{1-2a}.$$

Application of the inequality of the arithmetic and geometric means to the left side of the last inequality gives the assertion. □

We shall now prove the following.

Theorem 4.5. *Let* $0<k,l<1$ *and let* $p=\sqrt{1-(u/v)^2}$, *where* $u=\min(k',l')$ *and* $v=\max(k',l')$. *Then*

$$\mathcal{K}_{a,b,c}(k)\mathcal{K}_{a,c-b,c}(l) < \frac{B(a,c-b)}{2v^{2a}}\mathcal{K}_{a,b,c}(p) \tag{4.13}$$

and

$$\frac{B(a,c-b)}{2}v^{2(1-a)}\mathcal{E}_{a,b,c}(p) < \mathcal{E}_{a,b,c}(k)\mathcal{E}_{a,c-b,c}(l) \tag{4.14}$$

provided $0<a<1$.

Proof. Let the numbers x and y be such that $0<x,y<1$. If $a>0$, then the following inequality

$$F(a,b;c;1-x)F(a,c-b;c;1-y) < y^{-a}F\left(a,b;c;1-\frac{x}{y}\right) \tag{4.15}$$

follows from Theorem 4.1 in [10]. Inequality (4.15) is reversed if $a<0$. Let $0<l<k<1$. With $x=k'^2$ and $y=l'^2$ we have $0<x<y<1$ and also that $u=k'$, $v=l'$ and $p^2=1-(u/v)^2$. Multiplying both sides of (4.15) by $\frac{B(a,b)}{2}\frac{B(a,c-b)}{2}$ and next using (2.2) we obtain the desired result when $l<k$. The case when $k<l$ can be treated in an analogous manner. We omit further details. This completes the proof of (4.13). In order to establish inequality (4.14) we let in (4.15) $a:=a-1$. Then the reversed inequality to (4.15)

$$y^{1-a}F\left(a-1,b;c;1-\frac{x}{y}\right) < F(a-1,b;c;1-x)F(a-1,c-b;c;1-y) \tag{4.16}$$

is valid because $a-1<0$. Again, assume that $l<k$. With $x=k'^2$ and $y=l'^2$ inequality (4.16) implies (4.14) with $u=k'$, $v=l'$ and $p^2=1-(u/v)^2$. The case when $k<l$ can be established in a similar way. The proof is complete. □

Inequalities (4.13) and (4.14), when $a=b=\frac{1}{2}$ and $c=1$, are established in [14].

For later use, let α and β be positive numbers such that $\alpha+\beta=1$. In the subsequent results the vectors w and X are defined as

$$w=(\alpha,\beta), \qquad X=(k,l), \tag{4.17}$$

where $0<k,l<1$.

Our next result reads as follows.

Theorem 4.6. *Let* w *and* X *be the same as in* (4.17) *and assume that k and l are not equal. Then for* $p\leqslant 2$

$$\mathcal{K}\big(A_p(w;X)\big) < \big[\mathcal{K}(k)\big]^{\alpha}\big[\mathcal{K}(l)\big]^{\beta} \tag{4.18}$$

and

$$\alpha\mathcal{E}(k)+\beta\mathcal{E}(l) < \mathcal{E}\big(A_p(w;X)\big). \tag{4.19}$$

Proof. In the proof of (4.18) we shall utilize a fact that the function $x \to F(a,b;c;x)$ $(|x|<1)$ is strictly logarithmically-convex (log-convex), i.e.,

$$F(a,b;c;\alpha x+\beta y) < F(a,b;c;x)^{\alpha} F(a,b;c;y)^{\beta},$$

$|y|<1$. This follows immediately by applying Theorem B.6 in [9] to (2.6) or by using Proposition 2.1 in [19] and formula (5.9-11) in [9]. In the last inequality we let $x=k^2$, $y=l^2$. Next, we multiply both sides by $B(a,b)/2$ and apply (2.2) to obtain (4.18) when $p=2$. Taking into account that $A_p \leqslant A_2$ for $p \leqslant 2$ and also the fact that the function $k \mapsto \mathcal{K}(k)$ is nondecreasing we obtain the assertion (4.18).

In the proof of (4.19) we shall employ the inequality

$$\alpha f\big(g(x)\big)+\beta f\big(g(y)\big) \leqslant f\big(g(\alpha x+\beta y)\big)$$

which is valid if g is a convex function and f is a decreasing and concave function defined on the range of g. Letting in the last inequality $f(t)=F(a-1,b;c;t)$, $g(t)=t^2$, $x=k$, and $y=l$ we obtain the inequality

$$\alpha F\big(a-1,b;c;k^2\big)+\beta F\big(a-1,b;c;l^2\big) < F\big(a-1,b;c;\alpha k^2+\beta l^2\big)$$

which holds true because the function $z \mapsto F(a-1,b;c;z)$ is decreasing and concave on its domain. The last statement follows from (2.6)–(2.7). Multiplying both sides of the last inequality by $B(a,b)/2$ and next using (2.3) we obtain

$$\alpha\mathcal{E}(k)+\beta\mathcal{E}(l) < \mathcal{E}\big(A_2(w;X)\big). \tag{4.20}$$

Since the function $k \mapsto \mathcal{E}(k)$ is decreasing and $A_2 \geqslant A_p$ for $p \leqslant 2$, $\mathcal{E}(A_2(w;X)) \leqslant \mathcal{E}(A_p(w;X))$. This in conjunction with (4.20) gives the assertion. The proof is complete. □

A. Baricz [5] has established a special case of the inequality (4.18) for the Legendre complete elliptic integrals of the first kind.

Before we state and prove the next theorem, let us recall the following result.

Proposition A. *(See [18].) Let $f:[0,1] \mapsto \mathbb{R}$ be a convex (concave) function on its domain. Then the function*

$$g(x)=f(x)+f(1-x)$$

is decreasing (increasing) on $[0,\frac{1}{2}]$ and increasing (decreasing) on $[\frac{1}{2},1]$.

This implies the following.

Corollary 4.7. *If $f:[0,1] \mapsto \mathbb{R}_+$ is a log-convex (log-concave) on its domain, then the function*

$$h(x)=f(x)f(1-x)$$

is decreasing (increasing) on $[0,\frac{1}{2}]$ and increasing (decreasing) on $[\frac{1}{2},1]$.

Here $\mathbb{R}_+=\{x\colon x>0\}$.

Before we state and prove the next theorem, let us introduce four subsets of the unit square $[0,1]\times[0,1]$:

$$d_{11}=\left\{(k,l)\colon \frac{1}{\sqrt{2}} \leqslant k \leqslant l < 1\right\}, \qquad d_{12}=\left\{(k,l)\colon 0<l \leqslant k \leqslant \frac{1}{\sqrt{2}}\right\},$$
$$d_{21}=\left\{(k,l)\colon \frac{1}{\sqrt{2}} \leqslant l \leqslant k < 1\right\}, \qquad d_{22}=\left\{(k,l)\colon 0<k \leqslant l \leqslant \frac{1}{\sqrt{2}}\right\}.$$

Theorem 4.8. *If $(k,l) \in d_{11} \cup d_{12}$, then the following inequalities*

$$\begin{aligned}
&\mathcal{K}(k)+\mathcal{K}\big(k'\big) \leqslant \mathcal{K}(l)+\mathcal{K}\big(l'\big),\\
&\mathcal{K}(k)\mathcal{K}\big(k'\big) \leqslant \mathcal{K}(l)\mathcal{K}\big(l'\big),\\
&\mathcal{E}(k)+\mathcal{E}\big(k'\big) \geqslant \mathcal{E}(l)+\mathcal{E}\big(l'\big),\\
&\mathcal{E}(k)\mathcal{E}\big(k'\big) \geqslant \mathcal{E}(l)\mathcal{E}\big(l'\big)
\end{aligned} \tag{4.21}$$

hold true. If $(k,l) \in d_{21} \cup d_{22}$, then the reversed inequalities are valid.

Proof. We shall establish inequalities (4.21) only in the case when $(k,l) \in d_{11} \cup d_{12}$. This is due to the fact that when $(k,l) \in d_{21} \cup d_{22}$ then interchanging k with l we see that d_{21} becomes d_{11} and d_{22} becomes d_{12}. We shall prove now the first inequality in (4.21). To this aim recall that the function $f(x) = F(a,b;c;x)$ $(a > 0,\ c > b > 0)$ is log-convex on $(0,1)$. This in turn implies that $f(x)$ is also convex on $(0,1)$. Thus by virtue of Proposition A a function $f(x) + f(1-x)$ is increasing on $[\frac{1}{2}, 1)$. With $x = k^2$ $(\frac{1}{2} \leqslant k^2 < 1)$ we see that the function

$$k \mapsto F\big(a,b;c;k^2\big) + F\big(a,b;c;k'^2\big) \tag{4.22}$$

is increasing on $[\frac{1}{2}, 1)$. If $(k,l) \in d_{11}$, then $\frac{1}{2} \leqslant k^2 \leqslant l^2 < 1$. Thus the following inequality

$$F\big(a,b;c;k^2\big) + F\big(a,b;c;k'^2\big) \leqslant F\big(a,b;c;l^2\big) + F\big(a,b;c;l'^2\big) \tag{4.23}$$

is satisfied. Multiplying both sides of (4.23) by $B(a,b)/2$ and next using (2.2) we obtain the first inequality in (4.21) for $(k,l) \in d_{11}$. If $(k,l) \in d_{12}$, then $0 < l^2 \leqslant k^2 < \frac{1}{2}$. In this case inequality (4.23) is still satisfied because the function defined in (4.22) is decreasing on $(0, \frac{1}{2}]$. Thus we conclude that the first inequality in (4.21) is also valid when $(k,l) \in d_{12}$. The second inequality in (4.21) can be established in an analogous manner using Corollary 4.7. We omit further details. The proof of the remaining two inequalities in (4.21) goes along the lines introduced above and utilizes the fact that the function $f(x) = F(a-1,b;c;x)$ $(0 < a < 1)$ is concave on (0,1) and therefore is also log-concave on (0,1). This completes the proof. □

Inequalities (4.21) for Legendre integrals K and E have been established in [14, Corollary 3.3].

In the proof of the next theorem we shall utilize the following result.

Proposition B. *(See [16].) Let $f : \mathbb{R}_+ \mapsto \mathbb{R}_+$ be a differentiable log-convex function. If $0 < q < 1$, then the function*

$$g(x) = \frac{[f(x)]^q}{f(qx)}$$

is an increasing function on its domain and is decreasing if $q > 1$.

Our next result reads as follows.

Theorem 4.9. *Let $0 < k < l < 1$ and let $0 < q < 1$. Then*

$$\frac{\mathcal{K}(\sqrt{q}l)}{\mathcal{K}(\sqrt{q}k)} < \left[\frac{\mathcal{K}(l)}{\mathcal{K}(k)}\right]^q. \tag{4.24}$$

Inequality (4.24) is reversed if $q > 1$ and $\sqrt{q}k < \sqrt{q}l$.

Proof. Let $0 < q < 1$. It follows from Proposition B that for a log-convex function $f : \mathbb{R}_+ \mapsto \mathbb{R}_+$

$$\frac{f(qy)}{f(qx)} < \left[\frac{f(y)}{f(x)}\right]^q$$

provided $0 < x < y$. Letting above $f(x) = F(a,b;c;x)$, $x = k^2$ and $y = l^2$ we obtain

$$\frac{F(a,b;c;ql^2)}{F(a,b;c;qk^2)} < \left[\frac{F(a,b;c;l^2)}{F(a,b;c;k^2)}\right]^q.$$

Application of (2.2) to the last inequality gives the assertion (4.24). The case $q > 1$ can be treated in an analogous manner. We omit further details. □

To this end we will deal with inequalities in parameter a of $\mathcal{K}_{a,b,c}$ and $\mathcal{E}_{a,b,c}$, where $0 < a < 1$.

Our next result reads as follows.

Theorem 4.10. *Elliptic integrals $\mathcal{K}_{a,b,c}$ and $\mathcal{E}_{a,b,c}$ are strictly log-convex functions of a, i.e., the inequalities*

$$\mathcal{K}_{\alpha a_1 + \beta a_2,b,c} < \mathcal{K}^{\alpha}_{a_1,b,c}\mathcal{K}^{\beta}_{a_2,b,c} \tag{4.25}$$

and

$$\mathcal{E}_{\alpha a_1 + \beta a_2,b,c} < \mathcal{E}^{\alpha}_{a_1,b,c}\mathcal{E}^{\beta}_{a_2,b,c} \tag{4.26}$$

are valid provided $0 < a_1, a_2 < 1$, $a_1 \neq a_2$.

Proof. Firstly we apply Theorem B.6 in [9] to (2.6) to conclude that the Gauss function is strictly log-convex in a, i.e.,

$$F(\alpha a_1 + \beta a_2, b; c; x) < \left[F(a_1, b; c; x)\right]^{\alpha}\left[F(a_2, b; c; x)\right]^{\beta}. \tag{4.27}$$

Secondly it follows from (2.7) and Theorem B.6 in [9] that the beta function is log-convex in each of its variables. Thus, in particular,

$$B(\alpha a_1 + \beta a_2, b) \leqslant \left[B(a_1, b)\right]^{\alpha}\left[B(a_2, b)\right]^{\beta} \tag{4.28}$$

$(a_1, a_2, b > 0)$. Making use of (2.2), (4.27) and (4.28) we obtain

$$\begin{aligned}
\mathcal{K}_{\alpha a_1+\beta a_2,b,c} &= \frac{B(\alpha a_1 + \beta a_2, b)}{2} F(\alpha a_1 + \beta a_2, b; c; \cdot) \\
&< \left[\frac{B(a_1, b)}{2} F(a_1, b; c; \cdot)\right]^{\alpha}\left[\frac{B(a_2, b)}{2} F(a_2, b; c; \cdot)\right]^{\beta} \\
&= \mathcal{K}^{\alpha}_{a_1,b,c}\mathcal{K}^{\beta}_{a_2,b,c}.
\end{aligned}$$

For the proof of (4.26) we use (2.3), (4.27), and (4.28) to obtain

$$\begin{aligned}
\mathcal{E}_{\alpha a_1+\beta a_2,b,c} &= \frac{B(\alpha a_1 + \beta a_2, b, c)}{2} F\big(\alpha(a_1 - 1) + \beta(a_2 - 1), b; c; \cdot\big) \\
&< \left[\frac{B(a_1, b)}{2} F(a_1 - 1, b; c; \cdot)\right]^{\alpha}\left[\frac{B(a_2, b)}{2} F(a_2 - 1, b; c; \cdot)\right]^{\beta} \\
&= \mathcal{E}^{\alpha}_{a_1,b,c}\mathcal{E}^{\beta}_{a_2,b,c}.
\end{aligned}$$

The proof is complete. □

A. Baricz has established logarithmic convexity of the zero-balanced integrals $\mathcal{K}_{a,1-a,1} \equiv K_a$ and $\mathcal{E}_{a,1-a,1} \equiv E_a$ (see [4, Theorem 2.6]). In the same paper logarithmic convexity, of the functions $a \mapsto \mathcal{K}_{a,c-a,c}$ and $a \mapsto \mathcal{E}_{a,c-a,c}$ is established (see Theorem 2.14).

In the proof of the next theorem we shall utilize the following result.

Proposition C. *(See [20].) Let D be an interval with nonempty interior and let $f : D \mapsto \mathbb{R}_+$ be a log-convex function. If $u, v \in D$ $(u \leqslant v)$ and if a number $\lambda > 0$ is such that $u + \lambda$, $v + \lambda \in D$, then*

$$\frac{f(u + \lambda)}{f(v + \lambda)} \leqslant \frac{f(u)}{f(v)}.$$

Theorem 4.11. *Let $0 < a_1 < a_2 < 1$ and let $\lambda > 0$ be such that $0 < a_1 + \lambda < a_2 + \lambda < 1$. Then*

$$\frac{\mathcal{K}_{a_1+\lambda,b,c}}{\mathcal{K}_{a_2+\lambda,b,c}} \leqslant \frac{\mathcal{K}_{a_1,b,c}}{\mathcal{K}_{a_2,b,c}} \tag{4.29}$$

and

$$\frac{\mathcal{E}_{a_1+\lambda,b,c}}{\mathcal{E}_{a_2+\lambda,b,c}} \leqslant \frac{\mathcal{E}_{a_1,b,c}}{\mathcal{E}_{a_2,b,c}}. \tag{4.30}$$

Proof. Inequalities (4.29) and (4.30) follow immediately from Proposition C and Theorem 4.10. □

Theorem 4.12. *Let $0 < a_2 < a_1 < 1$. Then*

$$\left(\frac{2}{B(a_2, b)}\mathcal{K}_{a_2,b,c}\right)^{\frac{1}{a_2}} < \left(\frac{2}{B(a_1, b)}\mathcal{K}_{a_1,b,c}\right)^{\frac{1}{a_1}} \tag{4.31}$$

and

$$\left(\frac{2}{B(a_1, b)}\mathcal{E}_{a_1,b,c}\right)^{\frac{1}{1-a_1}} < \left(\frac{2}{B(a_2, b)}\mathcal{E}_{a_2,b,c}\right)^{\frac{1}{1-a_2}}. \tag{4.32}$$

Proof. We shall utilize the following inequality

$$\left[F(a_2,b;c;x)\right]^{\frac{1}{a_2}} < \left[F(a_1,b;c;x)\right]^{\frac{1}{a_1}} \tag{4.33}$$

which follows from Theorem 3 in [7] and from formula (5.9-12) in [9]. Letting $x = k^2$ in (4.33) and next using (2.2) we obtain inequality (4.31). For the proof of (4.32) we let $a_2 := a_2 - 1$ and $a_1 := a_1 - 1$ in (4.33) and next apply formula (2.3). This completes the proof. □

Before we state and prove the last theorem of this section let us recall Lyapunov's inequality for integrals (see, e.g., [21]). Let $f \in C[0,1]$ and let μ be a probability measure on (0,1). If $0 < r < s < t$, then

$$\left[\int_0^1 f^s(y)\mu(y)\,dy\right]^{t-r} < \left[\int_0^1 f^r(y)\mu(y)\,dy\right]^{t-s}\left[\int_0^1 f^t(y)\mu(y)\,dy\right]^{s-r}. \tag{4.34}$$

Theorem 4.13. *Let* $0 < a_1 < a_2 < a_3 < 1$. *Then*

$$\left[\frac{2}{B(a_2,b)}\mathcal{E}_{a_2,b,c}\right]^{a_3-a_1} < \left[\frac{2}{B(a_1,b)}\mathcal{E}_{a_1,b,c}\right]^{a_3-a_2}\left[\frac{2}{B(a_3,b)}\mathcal{E}_{a_3,b,c}\right]^{a_2-a_1}. \tag{4.35}$$

Proof. It follows from (2.6) that

$$F\big(a-1,b;c;k^2\big) = \int_0^1 f^{1-a}(y)\mu(y)\,dy, \tag{4.36}$$

where $f(y) = 1 - k^2 y$ and μ is the Dirichlet measure defined in (2.7). Letting in (4.34) $r = 1 - a_3$, $s = 1 - a_2$ and $t = 1 - a_1$ we obtain using (4.36)

$$\left[F\big(a_2-1,b;c;k^2\big)\right]^{a_3-a_1} < \left[F\big(a_3-1,b;c;k^2\big)\right]^{a_2-a_1}\left[F\big(a_1-1,b;c;k^2\big)\right]^{a_3-a_2}.$$

This in conjunction with (2.3) gives the assertion (4.35). □

Corollary 4.14. *Under the assumptions of Theorem* 4.13 *the following inequality*

$$\mathcal{E}_{a_2,b,c} < \frac{a_3-a_2}{a_3-a_1}\frac{B(a_2,b)}{B(a_1,b)}\mathcal{E}_{a_1,b,c} + \frac{a_2-a_1}{a_3-a_1}\frac{B(a_2,b)}{B(a_3,b)}\mathcal{E}_{a_3,b,c} \tag{4.37}$$

is valid.

Proof. Raising both sides of (4.35) to the power of $1/(a_3 - a_1)$ and next using inequality of arithmetic and geometric means with weights we obtain desired inequality (4.37). □

5. A lower bound for $E_a E'_a$

Recall that the zero-balanced generalized complete elliptic integrals of the first and second kind, denoted by K_a and E_a, respectively, are defined by

$$K_a \equiv K_a(k) = \mathcal{K}_{a,1-a,1}(k) = \frac{\pi}{2\sin(\pi a)}F\big(a,1-a;1;k^2\big) \tag{5.1}$$

and

$$E_a \equiv E_a(k) = \mathcal{E}_{a,1-a,1}(k) = \frac{\pi}{2\sin(\pi a)}F\big(a-1,1-a;1;k^2\big), \tag{5.2}$$

where $0 < a < 1$. See, e.g., [6, Section 5.5] where the factor $\sin(\pi a)$ is replaced by 1.

The main result of this section reads as follows.

Theorem 5.1. *The following inequality*

$$E_a E'_a \geqslant \frac{\pi}{4(1-a)\sin(\pi a)} + \left[2kk'\left(K_a\left(\frac{1}{\sqrt{2}}\right) - E_a\left(\frac{1}{\sqrt{2}}\right)\right)\right]^2 \tag{5.3}$$

is valid, where

$$K_a\left(\frac{1}{\sqrt{2}}\right)=\frac{1}{4\sqrt{\pi}}\Gamma\left(\frac{1-a}{2}\right)\Gamma\left(\frac{a}{2}\right) \tag{5.4}$$

(*see* [6, p. 191]) *and*

$$E_a\left(\frac{1}{\sqrt{2}}\right)=\left[\frac{\pi}{4(1-a)\sin(\pi a)}+K_a^2\left(\frac{1}{\sqrt{2}}\right)\right]\Big/\left(2K_a\left(\frac{1}{\sqrt{2}}\right)\right). \tag{5.5}$$

Inequality (5.3) *becomes an equality if* $k=k'=\frac{1}{\sqrt{2}}$.

Proof. The following result

$$K_a-E_a=k^2\frac{\pi(1-a)}{2\sin(\pi a)}F\big(a,2-a;2;k^2\big) \tag{5.6}$$

is known (see, e.g., [12]). Replacing k by k' and next multiplying corresponding sides of the resulting equation and (5.6) we obtain

$$(K_a-E_a)\big(K_a'-E_a'\big)=\left[kk'\frac{\pi(1-a)}{2\sin(\pi a)}\right]^2F\big(a,2-a;2;k^2\big)F\big(a,2-a;2;k'^2\big). \tag{5.7}$$

Logarithmic convexity of Gauss' function F in its variable implies the inequality

$$F\big(a,2-a;2;k^2\big)F\big(a,2-a;2;k'^2\big)\geqslant\left[F\left(a,2-a;2;\frac{1}{2}\right)\right]^2 \tag{5.8}$$

which becomes an equality if and only if $k=k'=\frac{1}{\sqrt{2}}$. This in conjunction with (5.7) gives

$$E_aE_a'-\big(K_aE_a'+K_a'E_a-K_aK_a'\big)\geqslant\left[kk'\frac{\pi(1-a)}{2\sin(\pi a)}F\left(a,2-a;2;\frac{1}{2}\right)\right]^2. \tag{5.9}$$

Application of generalized Legendre's identity

$$K_aE_a'+K_a'E_a-K_aK_a'=\frac{\pi}{4(1-a)\sin(\pi a)}$$

(see [6, (5.5.6)], [3, (5.8)]) to (5.9) gives

$$E_aE_a'\geqslant\frac{\pi}{4(1-a)\sin(\pi a)}+\left[kk'\frac{\pi(1-a)}{2\sin(\pi a)}F\left(a,2-a;2;\frac{1}{2}\right)\right]^2. \tag{5.10}$$

To complete the proof of (5.3) we let $k=1/\sqrt{2}$ in (5.6) to obtain

$$F\left(a,2-a;2;\frac{1}{2}\right)=\frac{4\sin(\pi a)}{\pi(1-a)}\left[K_a\left(\frac{1}{\sqrt{2}}\right)-E_a\left(\frac{1}{\sqrt{2}}\right)\right].$$

This in conjunction with (5.10) gives the assertion. To prove formula (5.5) we let in (5.3) $k=k'=\frac{1}{\sqrt{2}}$ to obtain

$$E_a^2\left(\frac{1}{\sqrt{2}}\right)=\frac{\pi}{4(1-a)\sin(\pi a)}\left[K_a\left(\frac{1}{\sqrt{2}}\right)-E_a\left(\frac{1}{\sqrt{2}}\right)\right]^2.$$

Solving for $E_a(\frac{1}{\sqrt{2}})$ we obtain the desired result (5.5). The proof is complete. □

One can prove that

$$E_a\left(\frac{1}{\sqrt{2}}\right)=\frac{\Gamma(1-a)}{2^{2-a}\pi}\left[\frac{1}{2}\sin\left(\frac{\pi a}{2}\right)\Gamma^2\left(\frac{a}{2}\right)+\frac{1}{1-a}\cos\left(\frac{\pi a}{2}\right)\Gamma^2\left(\frac{1+a}{2}\right)\right].$$

We omit further details.

For more results about the integrals K_a and E_a, the interested reader is referred to [23] and [11].

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