



Linearization of a free boundary problem in corrosion detection

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ABSTRACT

We consider a boundary identification problem arising in nondestructive testing of materials. The problem is to recover a part $\Gamma_I \subset \partial\Omega$ of the boundary of a bounded, planar domain Ω from one Cauchy data pair $(u, \partial u/\partial \nu)$ of a harmonic potential u in Ω collected on an accessible boundary subset $\Gamma_A \subset \partial\Omega$. We prove Fréchet differentiability of a suitably defined forward map, and discuss local uniqueness and Lipschitz stability results for the linearized problem.

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1. Introduction

In this paper we discuss an inverse problem arising in the nondestructive testing of materials, see e.g., [3,7,11,15,18]. Such materials are typically metallic specimens, as for instance pipes transporting water, gas, chemically aggressive fluids or bodywork of aircraft, cars, etc., whose surfaces have been damaged by a corrosion attack. In practice, it often happens that such surfaces are not accessible to direct inspection, hence in order to detect the possible presence of corrosion one has to rely on measurements only performed on the accessible part of the specimen surface. In what follows, we assume that a stationary (thermic or electric) potential u is available from direct measurements on the accessible boundary; for definiteness, we will refer generally to u as an electric potential. From these considerations one obtains an inverse problem for the following elliptic boundary value problem:

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = \phi & \text{on } \Gamma_A, \\ \frac{\partial u}{\partial \nu} + \gamma u = 0 & \text{on } \Gamma_I, \\ u = 0 & \text{on } \Gamma_D. \end{cases} \quad (1.1)$$

Boundary and parameter identification problems related to this elliptic equation have been considered by many authors even recently, see e.g., [3,5,7,8,11,15,18]. According to this model, Ω represents a conductor which contains no sources and

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no sinks, so that the potential u is harmonic. We assume that the boundary $\partial\Omega$ is decomposed in three open and disjoint subsets Γ_A , Γ_I , Γ_D . On the portion Γ_A , which is the one accessible to direct inspection, we prescribe a current density ϕ and we measure the corresponding voltage potential $u|_{\Gamma_A}$. The portion Γ_I , where corrosion took place, is out of reach. On such a portion, the potential u satisfies a Robin type condition $u_\nu + \gamma u = 0$, which models a resistive coupling with the exterior environment by means of the impedance coefficient $\gamma > 0$. The remaining portion of the boundary Γ_D is assumed to be grounded.

In this paper we are interested in the inverse problem of determining the location of the unknown and damaged boundary Γ_I from the data collected on the accessible part of the boundary Γ_A , that is, the Cauchy data pair $(u|_{\Gamma_A}, \phi)$. In particular, we improve and generalize the main results in [11,18] to much more general domains.

Many authors have treated analogous boundary identification problems where the unknown boundary is endowed by Neumann or Dirichlet boundary conditions, that correspond to a perfectly insulating or perfectly conducting boundary, see for instance [2,6,15–17,19]. In these cases, identifiability and stability issues are well known. However, the situation for the impedance case is quite different, since standard arguments fail when the Robin coefficient γ is finite and nonzero [20]. In fact, inverse problems concerning the identification of γ in (1.1) (or variants of it) have been addressed e.g., in [3,5,7,8,10,11,15,21]. For what concerns the determination of a portion of the boundary, where a Robin type condition is prescribed, in [11,18] the authors prove that, in a rather particular setting where both γ and ϕ are constant and Ω is a rectangular domain, one suitable data set collected in the accessible boundary identifies θ uniquely. On the other hand, in [7] it is proved, by counterexamples, that a single measurement is not sufficient to determine simultaneously the shape of Γ_I and the impedance coefficient γ , and the same holds if the only aim is to determine Γ_I when γ is a fixed constant.

However, we observe that the negative results in [7] concern domains whose unknown boundary Γ_I contain corners. Actually, a convergent numerical scheme for the reconstruction of Γ_I (with a known constant γ) is also shown in [7], under the assumption that Γ_I can be parametrized by a smooth function.

Furthermore, in [5] it has been achieved a global uniqueness result for the simultaneous determination of Γ_I and γ by means of two measurements, one of which is given for a positive current ϕ , and a logarithmic stability result for the same setting can be found in [22]. Moreover, [20] gives a local uniqueness result and reconstruction algorithms by two suitably chosen measurements for the related problem of recovering shape and impedance of an object buried into a conductive and homogeneous medium.

In the present paper we assume that the Robin coefficient γ is known and constant. Moreover, in order to have a solution u to (1.1) with constant sign, we will consider only positive fluxes ϕ , which is also in accordance with the hypotheses required in [5,10,11,18,15,20]. In order to investigate the location of the supposed damage, we adopt a model in which the undamaged domain $\Omega \subset \mathbb{R}^2$ is modified by a corrosion process localized on Γ_I . Since we are assuming that external physical conditions do not change significantly, we will consider small perturbations of Γ_I and we analyze the problem by a local approach. We describe such a situation by introducing a small vector field $\theta \in C_0^1(\Gamma_I)$ so that the damaged domain Ω_θ is such that

$$\partial\Omega_\theta = \overline{\Gamma_A} \cup \overline{\Gamma_D} \cup \overline{\Gamma_{I,\theta}}, \quad \Gamma_{I,\theta} = \{z \in \mathbb{R}^2: z = w + \theta(w), w \in \Gamma_I\}.$$

Hence our problem may be reformulated as follows. Find $\theta \in C_0^1(\Gamma_I)$ given a single measurement $(\phi, u|_{\Gamma_A})$ with $\phi \geq 0$.

In Section 2 we collect technical details required in the rest of the paper. In Section 3 we consider the forward map

$$F: \theta \mapsto u|_{\Gamma_A}$$

and, by adapting the techniques developed in [13,14], we show that F is Fréchet differentiable at Γ_I . In Section 4 we study the linearized problem and discuss some stability properties, provided that

$$2H(x) + \gamma > 0 \tag{1.2}$$

where $H(x)$ is the mean curvature of the undamaged boundary Γ_I . Let us observe that the hypothesis (1.2) is well justified when, for instance, Ω models a 2D transverse section of a metallic plate, the curvature being equal to zero in that case. In particular, we prove a local uniqueness result for θ and a local Lipschitz stability result, based on an argument developed by Bellout and Friedman [4], by establishing that the Gâteaux derivative does not vanish. Furthermore, under additional assumptions that look natural in the applicative context, we give an upper bound of the L^1 -norm of θ in an inner portion of Γ_I in terms of the solution u' to the linearized problem on Γ_A , providing a new conditional Lipschitz stability result for our free boundary problem. Finally, we observe that the Fréchet differential operator is compact over suitable spaces, hence the local identification issue of θ may be reformulated as the regularized inversion of a compact operator.

2. Definitions and assumptions

Throughout this paper, let Ω be a bounded domain in \mathbb{R}^2 . We will refer to Ω as the *undamaged* or *reference* domain. As already stated in the Introduction, we consider a boundary identification problem for the elliptic equation (1.1). Perturbations in the shape of Ω are assumed to be somewhat “small” and localized in the boundary Γ_I . Such perturbations will be described with the help of the vector field θ , see later. Accordingly, a perturbed domain is denoted by Ω_θ . Recall that γ is assumed to be a known positive constant.

We denote by B_r the ball in \mathbb{R}^2 centered in zero with radius r . We borrow from [12] the two following definitions:

Definition 2.1. We shall say that the boundary $\partial\Omega$ of Ω is of Lipschitz class with constants $r_0, M > 0$ if for every $P \in \partial\Omega$ there exists a rigid transformation of coordinates under which we have $P = 0$ and

$$\Omega \cap B_{r_0} = \{(x, y): y > g(x)\},$$

where

$$g: (-r_0, r_0) \subset \mathbb{R} \rightarrow \mathbb{R}$$

is a Lipschitz function that fulfills $g(0) = 0$ and

$$\|g\|_{C^{0,1}(-r_0, r_0)} \leq Mr_0,$$

with the notation

$$\|g\|_{C^{0,1}(-r_0, r_0)} = \|g\|_{L^\infty(-r_0, r_0)} + r_0 \sup_{\substack{x_1, x_2 \in (-r_0, r_0) \\ x_1 \neq x_2}} \frac{|g(x_1) - g(x_2)|}{|x_1 - x_2|}.$$

Definition 2.2. Given an integer $k \geq 1$ and a scalar $\alpha, 0 < \alpha < 1$, we shall say that a portion S of $\partial\Omega$ is of class $C^{k,\alpha}$ with constants $r_0, M > 0$ if for any $P \in S$ there exists a rigid transformation of coordinates under which we have $P = 0$ and

$$\Omega \cap B_{r_0} = \{(x, y): y > \varphi(x)\}$$

where

$$\varphi: (-r_0, r_0) \subset \mathbb{R} \rightarrow \mathbb{R}$$

is a $C^{k,\alpha}$ function satisfying $|D^\ell \varphi(0)| = 0$ for $0 \leq \ell \leq k$ and

$$\|\varphi\|_{C^{k,\alpha}(-r_0, r_0)} \leq Mr_0,$$

where we denote

$$\|\varphi\|_{C^{k,\alpha}(-r_0, r_0)} = \sum_{j=0}^k \|D^j \varphi\|_{L^\infty(-r_0, r_0)} + r_0^{k+\alpha} \sup_{\substack{x_1, x_2 \in (-r_0, r_0) \\ x_1 \neq x_2}} \frac{|D^k \varphi(x_1) - D^k \varphi(x_2)|}{|x_1 - x_2|^\alpha}.$$

Hereafter, we list the a priori informations that we will assume throughout this paper.

- *Prior information on the reference domain:* Recall that Ω is a bounded domain in \mathbb{R}^2 . We suppose that there exist constants $r_0, M > 0$, and $0 < \alpha < 1$, such that $\partial\Omega$ is of Lipschitz class with constants r_0, M , see Definition 2.1, and that the portion of the boundary Γ_I is of class $C^{2,\alpha}$ with constants r_0, M , see Definition 2.2.
- *Prior information on the prescribed current density:* We assume that the flux ϕ is such that

$$\|\phi\|_{H^{-\frac{1}{2}}(\Gamma_A)} \leq G$$

for some positive constant G .

- *Prior information on u :* We assume that there exists a constant $U > 0$ such that

$$\|u\|_{C^2(\Gamma_I)} \leq U. \quad (2.1)$$

We observe that, based on the aforementioned assumptions on Ω and ϕ , the assumption (2.1) can be fulfilled by limiting ourselves to particular geometries, as for instance a cylinder, or by supposing that Γ_I is a connected component of the boundary $\partial\Omega$, see [12].

- *Prior information on θ :* We suppose that θ is a vector field in $C_0^1(\Gamma_I)$ having a nontrivial normal component on Γ_I , that is, if

$$\theta_\nu(x) \equiv \theta(x) \cdot \nu(x) = 0 \quad \forall x \in \Gamma_I$$

where $\nu(x)$ is the unit outward normal in $x \in \Gamma_I$, then $\theta \equiv 0$. Moreover, denoting with φ the acute angle such that

$$|\theta_\nu| = |\theta| |\cos(\varphi)|$$

we assume that

$$|\cos(\varphi)| \geq A > 0. \quad (2.2)$$

In what follows, the constants $r_0, M, \alpha, G, \gamma, U, A$ will be referred to as the *a priori data*.

Any sufficiently small vector field $\theta: \Gamma_I \mapsto \mathbb{R}^2$, $\theta \in C_0^1(\Gamma_I)$ induces a perturbation of $\partial\Omega$ which is still the boundary of a domain that we denote with Ω_θ , with

$$\partial\Omega_\theta = \overline{\Gamma_A} \cup \overline{\Gamma_D} \cup \overline{\Gamma_{I,\theta}}$$

where

$$\Gamma_{I,\theta} = \{z \in \mathbb{R}^2: z = w + \theta(w), w \in \Gamma_I\}.$$

For notational convenience, we will generally identify $\Gamma_{I,\theta}$ with the vector field θ defining it; in particular, the reference boundary Γ_I corresponds to $\theta = 0$. Furthermore, we denote by ν the outward normal to the boundary Γ_I , and by θ_ν and θ_t the normal and tangential components of the field θ , respectively. Finally, $\nabla_\nu f$ and $\nabla_t f$ denote the normal and tangential components of ∇f , respectively.

Definition 2.3. We shall denote with F the *forward map*

$$\begin{aligned} F: C_0^1(\Gamma_I) &\rightarrow H^{\frac{1}{2}}(\Gamma_A), \\ \theta &\mapsto u_\theta|_{\Gamma_A} \end{aligned} \quad (2.3)$$

where $u_\theta \in H_0^1(\Omega, \Gamma_D)$ is the solution to the elliptic problem

$$\begin{cases} \Delta u_\theta = 0 & \text{in } \Omega_\theta, \\ \frac{\partial u_\theta}{\partial \nu} = \phi & \text{on } \Gamma_A, \\ \frac{\partial u_\theta}{\partial \nu} + \gamma u_\theta = 0 & \text{on } \Gamma_{I,\theta}, \\ u = 0 & \text{on } \Gamma_D. \end{cases} \quad (2.4)$$

With the help of the foregoing definition, we can state our boundary identification problem as the solution on the nonlinear equation $F(\theta) = \eta$ for a given $\eta = u|_{\Gamma_A}$, the trace on the accessible boundary Γ_A of the potential u that solves (1.1) with a prescribed flux ϕ .

3. Fréchet differentiability of the forward map

This section contains the main results of this paper. In the forthcoming theorem, we prove that the forward map introduced in Definition 2.3 is Fréchet differentiable at $\theta = 0$, and provide the explicit form of the derivative. In the subsequent corollary, we specialize this result to the case where Ω is a rectangle, as considered in [11].

Theorem 3.1. *The operator F in (2.3) is Fréchet differentiable at Γ_I . Indeed, consider the linear operator $F': C_0^1(\Gamma_I) \rightarrow H^{\frac{1}{2}}(\Gamma_A)$ defined as $F'\theta = u'|_{\Gamma_A}$, where u' is the solution to the boundary value problem*

$$\begin{cases} \Delta u' = 0 & \text{in } \Omega, \\ \frac{\partial u'}{\partial \nu} = 0 & \text{on } \Gamma_A, \\ \frac{\partial u'}{\partial \nu} + \gamma u' = \frac{d}{ds} \left(\theta_\nu \frac{d}{ds} u \right) + \gamma \theta_\nu (\gamma + 2H)u & \text{on } \Gamma_I, \\ u' = 0 & \text{on } \Gamma_D, \end{cases} \quad (3.1)$$

the function u is the solution of (1.1) and H denotes the mean curvature of the boundary Γ_I . Then,

$$\frac{1}{\|\theta\|_{C_0^1(\Gamma_I)}} \|F(\theta) - F(0) - F'\theta\|_{H^{\frac{1}{2}}(\Gamma_A)} \rightarrow 0 \quad \text{as } \theta \rightarrow 0 \text{ in } C_0^1(\Gamma_I).$$

Proof. Let us recall that a weak solution to the problem (3.1) is a function $u' \in H_0^1(\Omega, \Gamma_D)$ such that

$$\int_{\Omega} \nabla u' \nabla v + \int_{\Gamma_I} \gamma u' v = \gamma \int_{\Gamma_I} \theta_\nu (\gamma + 2H) u v - \int_{\Gamma_I} \theta_\nu \frac{du}{ds} \frac{dv}{ds} \quad (3.2)$$

for all $v \in H_0^1(\Omega, \Gamma_D)$. Moreover, recall that the Sobolev space $H_0^1(\Omega, \Gamma_D)$ is defined as follows:

$$H_0^1(\Omega, \Gamma_D) = \{v \in H^1(\Omega): v = 0 \text{ on } \Gamma_D \text{ in the trace sense}\}.$$

With a little abuse of notation, we denote by $\theta \in C^1(\Omega)$ a smooth prolongation of the original vector field θ to the whole Ω which satisfies $\theta(x) = 0$ on Γ_A and $\|\theta\|_{C^1(\Omega)} \leq c\|\theta\|_{C_0^1(\Gamma)}$ where $c > 0$ depends on the a priori data only. Note that, with this convention, the theorem can be proved by showing the limit

$$\frac{\|u_\theta - u - u'\|_{H^{\frac{1}{2}}(\Gamma_A)}}{\|\theta\|_{C^1(\Omega)}} \rightarrow 0$$

as $\|\theta\|_{C^1(\Omega)} \rightarrow 0$, where u_θ is the solution of (2.4). Therefore, we introduce a change of variables defined onto the reference domain Ω :

$$\varphi: \Omega \rightarrow \Omega_\theta, \quad \varphi(x) = x + \theta(x).$$

Let J_θ denote the Jacobian of θ . Acting as [13, Theorem 2.1] and [14], we consider the function $\tilde{u}_\theta = u_\theta \circ \varphi$ and the bilinear form

$$R_\theta(\tilde{u}_\theta, v) := \int_{\Omega} (\nabla \tilde{u}_\theta J_\psi J_\psi^T \nabla v) \det J_\varphi + \int_{\Gamma_I} \gamma \tilde{u}_\theta v \det \tilde{J}_\varphi$$

for any $v \in H^1(\Omega)$, where J_φ denotes the Jacobian of φ , ψ the inverse of φ with Jacobian J_ψ and \tilde{J}_φ the Jacobian of φ with respect to the surface integral. Since u and u_θ have the same Neumann data ϕ on Γ_A we conclude that

$$R(u, v) = R_\theta(\tilde{u}_\theta, v) \quad \forall v \in H_0^1(\Omega, \Gamma_D), \quad (3.3)$$

where

$$R(u, v) = \int_{\Omega} \nabla u \nabla v + \int_{\Gamma_I} \gamma u v$$

is the bilinear form associated to (3.2). The regularity assumption (2.1) on u implies that there exists a unique solution $u' \in H_0^1(\Omega, \Gamma_D)$ to the problem (3.1), and its norm can be bounded in terms of the a priori data only (we will clarify this point in Theorem 4.5). We define $w = u' + \theta \cdot \nabla u$ and we notice that $u'|_{\Gamma_A} = w|_{\Gamma_A}$. By the coercivity of R , to complete the proof it is sufficient to prove that

$$\forall v \in H_0^1(\Omega, \Gamma_D), \quad \frac{1}{\|\theta\|_{C^1(\Omega)}} R(\tilde{u}_\theta - u - w, v) \rightarrow 0$$

when θ tends to zero. By (3.3) we obtain that

$$\begin{aligned} R(u - \tilde{u}_\theta, v) &= R_\theta(\tilde{u}_\theta, v) - R(\tilde{u}_\theta, v) \\ &= \int_{\Omega} \nabla \tilde{u}_\theta (J_\psi J_\psi^T \det J_\varphi - I) \nabla v + \int_{\Gamma_I} \gamma (\det \tilde{J}_\varphi - 1) \tilde{u}_\theta v. \end{aligned} \quad (3.4)$$

Since $J_\varphi = I + J_\theta$ and $J_\psi = J_\varphi^{-1} \circ \psi = I - J_\theta + \mathcal{O}(\|\theta\|_{C^1(\Omega)}^2)$, we have that

$$\|J_\psi J_\psi^T \det J_\varphi - I + J_\theta + J_\theta^T - \operatorname{div} \theta I\|_{C^0(\Omega)} = \mathcal{O}(\|\theta\|_{C^1(\Omega)}^2), \quad (3.5)$$

$$\|\det \tilde{J}_\varphi - 1 - \operatorname{div} \theta_t + 2H\theta_v\|_{C^0(\Gamma_I)} = \mathcal{O}(\|\theta\|_{C^1(\Omega)}^2). \quad (3.6)$$

By the estimates (3.5), (3.6) and by (3.4) we deduce by coercivity that

$$\|\tilde{u}_\theta - u\|_{H^1(\Omega)} \rightarrow 0 \quad \text{as } \theta \rightarrow 0.$$

Therefore, due to Eq. (3.4) it remains to prove that

$$R(w, v) = \int_{\Omega} \nabla u (J_\theta + J_\theta^T - \operatorname{div} \theta I) \nabla v + \int_{\Gamma_I} \gamma u (\operatorname{div} \theta_t - 2H\theta_v) v$$

for all $v \in H_0^1(\Omega, \Gamma_D)$. Since u' is a solution to (3.1) we get from the boundary condition that

$$R(w, v) = \int_{\Omega} \nabla (\theta \cdot \nabla u) \nabla v - \int_{\Gamma_I} \gamma \left[\theta \cdot \nabla u - \theta_v \left(\frac{\partial u}{\partial v} - 2Hu \right) \right] v - \int_{\Gamma_I} \operatorname{div}(\theta_v \nabla_t u) v.$$

The formula

$$\operatorname{div}(v \times W) = -v \cdot \operatorname{curl} W$$

for a vector field $W \in H^1(\Omega)$ yields

$$R(w, v) = \int_{\Omega} \nabla(\theta \cdot \nabla u) \nabla v - \int_{\Gamma_1} \gamma \theta_v 2Hu v - \int_{\Gamma_1} \gamma \theta_t \cdot (\nabla_t u) v - \int_{\Gamma_1} v \cdot \text{curl}(\theta_v (\nabla u \times v)) v.$$

The Green's formula for test functions $v \in H^2(\Omega)$ leads to

$$R(w, v) = - \int_{\Omega} (\theta \cdot \nabla u) \Delta v - \int_{\Gamma_1} (\theta \cdot \nabla u) \frac{\partial v}{\partial \nu} - \int_{\Gamma_1} \gamma \theta_v 2Hu v - \int_{\Gamma_1} \gamma \theta_t \cdot (\nabla_t u) v - \int_{\Gamma_1} v \cdot \text{curl}(\theta_v (\nabla u \times v)) v.$$

Moreover, according to the Green's formula for a vector field $W \in H^2(\Omega)$ and a scalar function $v \in H^1(\Omega)$ we have that

$$\int_{\partial\Omega} v \cdot \text{curl} W v = \int_{\Omega} \text{curl} W \nabla v = \int_{\partial\Omega} v \times W \nabla v.$$

We let $W = \theta_v (\nabla u \times v)$ in the preceding formula. Hence, recalling that $\theta \in C^1(\Omega)$ and by Gauss theorem, we deduce that

$$\begin{aligned} R(w, v) = & \int_{\Omega} \text{div}[(\theta \cdot \nabla u) \nabla v + (\theta \cdot \nabla v) \nabla u - (\nabla u \cdot \nabla v) \theta] - (\theta \cdot \nabla u) \Delta v + \int_{\Gamma_1} [(\theta \cdot \nabla v) \nabla u - \theta (\nabla u \cdot \nabla v)] \cdot v \\ & - \int_{\Gamma_1} \gamma \theta_v 2Hu v - \int_{\Gamma_1} \gamma \theta_t \cdot (\nabla_t u) v + \int_{\Gamma_1} \theta_v \nabla_t u \cdot \nabla_t v. \end{aligned}$$

By the formula (see [13])

$$\nabla u (J_{\theta} + J_{\theta}^T - \text{div} \theta I) \nabla v = \text{div}[(\theta \cdot \nabla u) \nabla v + (\theta \cdot \nabla v) \nabla u - (\nabla u \cdot \nabla v) \theta] - (\theta \cdot \nabla u) \Delta v$$

we conclude that

$$R(w, v) = \int_{\Omega} \nabla u (J_{\theta} + J_{\theta}^T - \text{div} \theta I) \nabla v - \int_{\Gamma_1} \gamma \theta_v 2Hu v - \int_{\Gamma_1} \gamma \theta_t \cdot (\nabla_t u) v + \int_{\Gamma_1} \theta_t \cdot \nabla_t v \frac{\partial u}{\partial \nu}.$$

From the Robin boundary condition for u and the identity

$$\int_{\Gamma_1} \gamma \theta_t \cdot (\nabla_t u) v = - \int_{\Gamma_1} \gamma u v \text{div} \theta_t$$

for the surface gradient we obtain the thesis. \square

Corollary 3.2. Let $\Omega = (0, a) \times (0, b)$ be such that $\Gamma_A = (0, a) \times \{0\}$, $\Gamma_I = (0, a) \times \{b\}$, $\Gamma_D = \{0\} \times (0, b) \cup \{a\} \times (0, b)$ and let $\theta = (\theta_1, \theta_2) \in C_0^1((0, a))$. Hence $u' \in H^1(\Omega)$ is the solution to

$$\begin{cases} \Delta u' = 0 & \text{in } \Omega, \\ u'(0, y) = u'(a, y) = 0 & y \in (0, b), \\ u'_y(x, b) + \gamma u'(x, b) = \beta(x) & x \in (0, a), \\ u'_y(x, 0) = 0 & x \in (0, a) \end{cases}$$

where $\beta(x) = -\theta_2(x)(u_{yy}(x, b) - \gamma^2 u(x, b)) + \theta'_2(x)u_x(x, b)$.

Proof. The claim follows from Theorem 3.1, noticing that in this special geometry we have $\theta_v = \theta_2$, $H = 0$, and the solution u of problem (1.1) is harmonic up to the boundary Γ_I . \square

4. Applications

In this section we prove some consequences of our main results in the previous section, which are relevant for the analysis and numerical solution of our boundary identification problem. Indeed, Theorem 4.2 proves that the “domain derivative” operator F' is injective, under some reasonable hypotheses. This fact is relevant to conclude that the solution of our problem is identifiable (i.e., unique whenever it exists), at least for sufficiently small perturbations. Theorem 4.3 gives a “local stability” result analogous to the ones found in e.g., [4,8] for related problems, by establishing that the Gâteaux derivative does not vanish. Moreover, in Theorem 4.4 we prove a conditional Lipschitz type upper bound for θ on a suitable

portion of Γ_I in terms of $u'|_{\Gamma_A} = F'\theta$, thus showing that the inversion of F' is not too much ill-behaved, at least in the conditions stated therein.

From a computational point of view, the availability of the expression of the operator F' allows to tackle the solution of the boundary identification problem by a regularized Newton-type iteration, see e.g., [9, Chap. 11]. In this case, the main computational task consists of the solution of a sequence of linear operator equations, associated to the operator F' . The ill-posed character of these linearized problems is clarified by Theorem 4.5.

Hereafter, we denote by Γ_I^ρ a portion of the boundary Γ_I sufficiently distant from its endpoints; more precisely, given $\rho > 0$, we set

$$\Gamma_I^\rho = \{x \in \Gamma_I: \text{dist}(x, \partial\Gamma_I) > \rho\}.$$

Lemma 4.1 (Positivity). *Let $\phi \in H^{-\frac{1}{2}}(\Gamma_A)$ be nonnegative in the sense of distributions, and let u be the solution to the problem (1.1). Then, $u(x) \geq 0$ for all $x \in \overline{\Omega}$. Moreover, for any sufficiently small $\rho > 0$, there exists a positive constant c_ρ depending on the a priori data and on ρ only such that*

$$u(x) \geq c_\rho \quad \forall x \in \Gamma_I^\rho. \quad (4.1)$$

Proof. Let $\phi_n \in C_0^\infty(\Gamma_A)$ be a sequence of nonnegative functions such that $\phi_n \rightarrow \phi$ in $H^{-\frac{1}{2}}(\Gamma_A)$. Let u_n be the solution to

$$\begin{cases} \Delta u_n = 0 & \text{in } \Omega, \\ \frac{\partial u_n}{\partial \nu} = \phi_n & \text{on } \overline{\Gamma_A}, \\ \frac{\partial u_n}{\partial \nu} + \gamma u_n = 0 & \text{on } \overline{\Gamma_I}, \\ \frac{\partial u_n}{\partial \nu} = 0 & \text{on } \overline{\Gamma_D}. \end{cases} \quad (4.2)$$

Let $x_n \in \partial\Omega$ and let $m_n < 0$ be such that

$$m_n = \min_{\overline{\Omega}} u_n = \min_{\partial\Omega} u_n = u(x_n).$$

By the Hopf principle and since $\phi_n \geq 0$ we have that $x_n \in \Gamma_I$. By the Robin condition we have that

$$\frac{\partial u}{\partial \nu}(x_n) = -\gamma m_n > 0$$

which is in contradiction with the Hopf principle. Hence we deduce that $m_n \geq 0$ and thus $u_n \geq 0$ in $\overline{\Omega}$. Moreover, the existence of the lower estimate $u(x) > c_\rho$ is shown in [3, Proposition 2.3] also in the more difficult case when ϕ has a variable sign. By the classical estimate for solutions to elliptic boundary value problems [12]

$$\|u_n - u\|_{H^1(\Omega)} \leq C \|\phi_n - \phi\|_{H^{-1/2}(\Gamma_A)},$$

where $C > 0$ is a constant depending on the a priori data only, we have that $u_n \rightarrow u$ in $H^1(\Omega)$ and therefore $u \geq 0$ in $\overline{\Omega}$ and $u \geq c_\rho$ in Γ_I^ρ which ends the proof. \square

Theorem 4.2 (Injectivity of F'). *Let the hypothesis of Lemma 4.1 be satisfied. Let us assume that $2H(x) + \gamma > 0$ and $\theta_\nu(x) \leq 0$ for any $x \in \Gamma_I$. Then F' is injective.*

Proof. Let us suppose that $F'\theta = 0$. By Holmgren unique continuation theorem, we have that $u' \equiv 0$ in $\overline{\Omega}$. Hence by (3.1) and by the a priori regularity assumption (2.1) we have that

$$\frac{d}{ds} \left(\theta_\nu \frac{d}{ds} u \right) = -\gamma \theta_\nu (\gamma + 2H) u \quad \text{on } \Gamma_I.$$

By Lemma 4.1 we have that

$$\frac{d}{ds} \left(\theta_\nu \frac{d}{ds} u \right) < 0 \quad \text{on } \Gamma_I$$

and thus $\theta_\nu \frac{d}{ds} u$ is decreasing in Γ_I . Since $\theta \in C_0^1(\Gamma_I)$ we have that $\theta_\nu \frac{d}{ds} u = 0$ on $\partial\Gamma_I$ and thus by the monotonicity we infer that $\theta_\nu \frac{d}{ds} u \equiv 0$ in Γ_I . This implies that

$$0 \equiv \frac{d}{ds} \left(\theta_\nu \frac{d}{ds} u \right) = -\gamma \theta_\nu (\gamma + 2H) u \quad \text{on } \Gamma_I.$$

Hence by Lemma 4.1 and by the assumptions on θ we deduce that $\theta_\nu \equiv 0$ and thus $\theta \equiv 0$. \square

Theorem 4.3 (Local Lipschitz stability). *Let the hypothesis of Lemma 4.1 be satisfied and let $2H(x) + \gamma > 0$ for any $x \in \Gamma_I$. Given $\bar{\theta} \in C_0^1(\Gamma_I)$ such that $\bar{\theta}_v = \bar{\theta} \cdot v \leq 0$ and given $h \in (-h_0, h_0)$ for some $h_0 > 0$, we set $\theta_h = h \cdot \bar{\theta}$. Denoting by u_h the solution to (2.4) with u_θ and $\theta = \theta_h$ we have that*

$$\lim_{h \rightarrow 0} \frac{\|u_h - u\|_{H^{\frac{1}{2}}(\Gamma_A)}}{|h|} > 0. \quad (4.3)$$

Proof. By Theorem 3.1 we can infer that there exists $\varepsilon(h) \in H_0^1(\Omega)$ such that

$$u_h = u + hv' + \varepsilon(h) \quad (4.4)$$

where $\|\varepsilon(h)\|_{H_0^1(\Omega)} \rightarrow 0$ as $h \rightarrow 0$ and where $v' \in H_0^1(\Omega)$ is the weak solution to (1.1) with $u' = v'$ and $\theta = \bar{\theta}$. According to (4.4), we have that (4.3) is equivalent to

$$\|v'\|_{H^{\frac{1}{2}}(\Gamma_A)} > 0.$$

Let us assume that $v' = 0$ on Γ_A . Then, arguing as in Theorem 4.2, we will obtain that

$$\gamma \bar{\theta}_v (\gamma + 2H)u \equiv 0 \quad \text{on } \Gamma_I.$$

This would imply that u vanishes in a set of positive measure of Γ_I , which is in contradiction with Lemma 4.1. \square

Theorem 4.4 (Conditional stability). *Let the hypothesis of Lemma 4.1 be satisfied. Moreover, let us assume that $2H(x) + \gamma > 0$ and $\theta_v \leq 0$ for any $x \in \Gamma_I$. Then, for any $\rho > 0$ there exists a positive constant C_ρ depending on the a priori data and on ρ only such that*

$$\|u'\|_{H^{\frac{1}{2}}(\Gamma_A)} \geq C_\rho \int_{\Gamma_I^\rho} |\theta|.$$

Proof. From the weak formulation of the problem (3.1), which is shown in (3.2), we have that

$$\int_{\Omega} \nabla u' \nabla u = - \int_{\Gamma_I} \gamma u' u + \gamma^2 \int_{\Gamma_I} \theta_v u^2 + \gamma \int_{\Gamma_I} \theta_v 2Hu^2 + \int_{\Gamma_I} \theta_v \left(\frac{du}{ds} \right)^2.$$

On the other hand, we have that

$$\int_{\Omega} \nabla u' \nabla u = - \int_{\Gamma_I} \gamma u' u + \int_{\Gamma_A} u' \frac{\partial u}{\partial v}.$$

Combining the last two equalities we have that

$$\int_{\Gamma_A} u' \frac{\partial u}{\partial v} = \gamma^2 \int_{\Gamma_I} \theta_v u^2 + \gamma \int_{\Gamma_I} \theta_v 2Hu^2 + \int_{\Gamma_I} \theta_v \left(\frac{du}{ds} \right)^2.$$

By the Schwartz inequality and the hypotheses, we infer that

$$\begin{aligned} \|u'\|_{H^{\frac{1}{2}}(\Gamma_A)} \|\phi\|_{H^{-\frac{1}{2}}(\Gamma_A)} &\geq \left| \int_{\Gamma_I} \gamma \theta_v (\gamma + 2H) u^2 + \int_{\Gamma_I} \theta_v \left(\frac{du}{ds} \right)^2 \right| \\ &\geq \int_{\Gamma_I} |\gamma \theta_v (\gamma + 2H)| u^2 \\ &\geq \int_{\Gamma_I^\rho} |\gamma \theta_v (\gamma + 2H)| u^2. \end{aligned}$$

By the estimates (4.1) and (2.2), we infer that there exists a constant $C_\rho > 0$ depending on the a priori data and on ρ only such that

$$\|u'\|_{H^{\frac{1}{2}}(\Gamma_A)} \geq C_\rho \int_{\Gamma_I^\rho} |\theta|,$$

and the proof is complete. \square

In the next theorem, we consider $L^2(\Gamma_A)$ as codomain space of the operator F' introduced in Theorem 3.1. This choice is fully justified from the standpoint of the computational practice [7,10,11,15,18]. Indeed, facing the numerical solution of the nonlinear equation $F(\theta) = u|_{\Gamma_A}$, or its linearized form $F'\theta = u'|_{\Gamma_A}$, we are obliged to take into account the presence of measurement errors in the data $u|_{\Gamma_A}$. These errors are usually modeled as finite energy perturbations [9]. Additionally, computational and regularization errors that enter into the solution procedure, are controlled by means of the (possibly discretized) residual $\|F'\theta - v\|_{L^2(\Gamma_A)}$, where $v \in L^2(\Gamma_A)$ is an available, noisy approximation for $u'|_{\Gamma_A}$.

Theorem 4.5 (Compactness). *The linear operator*

$$F' : C_0^1(\Gamma_I) \rightarrow L^2(\Gamma_A), \\ \theta \mapsto u'|_{\Gamma_A}$$

where u' is the solution to the boundary value problem (3.1), is compact.

Proof. Let us first prove that F' as operator from $C_0^1(\Gamma_I)$ to $H^{\frac{1}{2}}(\Gamma_A)$ is bounded. In what follows, we will denote by C a generic positive constant depending on the a priori data only, whose value may change from one occurrence to another.

By the weak formulation (3.2) with $v = u'$ we have that

$$\int_{\Omega} |\nabla u'| + \gamma \int_{\Gamma_I} |u'|^2 = \gamma \int_{\Gamma_I} \theta_v (\gamma + 2H) u u' - \int_{\Gamma_I} \theta_v \frac{du'}{ds} \frac{du}{ds}.$$

By a Poincaré type inequality we have that there exists a constant C such that

$$\|u'\|_{H^1(\Omega)}^2 \leq C \left(\gamma \int_{\Gamma_I} \theta_v (\gamma + 2H) u' u - \int_{\Gamma_I} \theta_v \frac{du'}{ds} \frac{du}{ds} \right).$$

Moreover, by the a priori hypothesis (2.1) and the continuous embedding $H^2(\Gamma_I) \hookrightarrow C^1(\Gamma_I)$ (see for instance [1, Chap. 8]) we have that

$$\begin{aligned} \|u'\|_{H^1(\Omega)}^2 &\leq C \|u'\|_{C^1(\Gamma_I)} \left(\gamma \int_{\Gamma_I} |\theta_v (\gamma + 2H) u'| + \int_{\Gamma_I} \left| \theta_v \frac{du'}{ds} \right| \right) \\ &\leq C \left(\int_{\Gamma_I} |\theta_v (\gamma + 2H) u'| + \int_{\Gamma_I} \left| \theta_v \frac{du'}{ds} \right| \right). \end{aligned}$$

Furthermore, by the inequality $2|ab| \leq a^2/\varepsilon + \varepsilon b^2$ and standard trace inequalities [1] we have

$$\|u'\|_{H^1(\Omega)}^2 \leq C \left(\frac{\|\theta\|_{L^2(\Gamma_I)}^2}{\varepsilon} + \varepsilon \|u'\|_{H^1(\Omega)}^2 \right).$$

Hence, choosing $\varepsilon = \frac{1}{2C}$ (with the same C of the previous formula), we deduce that

$$\|u'\|_{H^1(\Omega)}^2 \leq C \|\theta\|_{C_0^1(\Gamma_I)}^2.$$

Finally, by a standard trace inequality, we deduce that

$$\|u'\|_{H^{\frac{1}{2}}(\Gamma_A)}^2 \leq C \|\theta\|_{C_0^1(\Gamma_I)}^2.$$

Hence $F' : C_0^1(\Gamma_I) \rightarrow H^{\frac{1}{2}}(\Gamma_A)$ is bounded. The thesis follows immediately by the compact embedding

$$H^{\frac{1}{2}}(\Gamma_A) \hookrightarrow L^2(\Gamma_A),$$

see for instance [1, Chap. 8]. \square

5. Conclusions

In this paper we considered a free boundary problem arising in corrosion detection for conductive materials having an impedance-type interface with the aggressive environment. We proved that a properly defined forward map is Fréchet differentiable and obtained the explicit expression of the derivative, which turns out to be an injective, compact operator. As a consequence, the unknown boundary is (locally) identifiable from the knowledge of a single measurement, although the linearized problem is severely ill-posed [22].

In view of the above results, the issue of the stable identification of the unknown boundary may be reformulated as the regularized inversion of a compact operator. However, classical computational methods, e.g., based on singular values decomposition or Tikhonov regularization [9], are oblivious to the one-signed character of the solution, that originates from the physical meaning of the starting problem.

On the other hand, we also proved in Theorem 4.4 a conditional stability result of Lipschitz type that addresses boundary deformations due to a corrosion attack. Hence, our problem could be not too much ill-behaved, when restricted to an admissible solution set endowed by a nonnegativity constraint. We argue that a computational method taking that result into proper account would be less sensitive to data errors and provide more accurate approximations, than conventional approximate inversion methods.

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