



S-shaped bifurcation curves in ecosystems

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ARTICLE INFO

Article history:

Received 14 November 2010

Available online 2 April 2011

Submitted by J. Shi

Keywords:

Ecological systems

S-shaped bifurcation curves

Sub-supersolutions

ABSTRACT

We consider the existence of multiple positive solutions to the steady state reaction diffusion equation with Dirichlet boundary conditions of the form:

$$\begin{cases} -\Delta u = \lambda \left[u - \frac{u^2}{K} - c \frac{u^2}{1+u^2} \right], & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases}$$

Here $\Delta u = \operatorname{div}(\nabla u)$ is the Laplacian of u , $\frac{1}{\lambda}$ is the diffusion coefficient, K and c are positive constants and $\Omega \subset \mathbb{R}^N$ is a smooth bounded region with $\partial\Omega$ in C^2 . This model describes the steady states of a logistic growth model with grazing in a spatially homogeneous ecosystem. It also describes the dynamics of the fish population with natural predation. In this paper we discuss the existence of multiple positive solutions leading to the occurrence of an S-shaped bifurcation curve. We prove our results by the method of sub-supersolutions.

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1. Introduction

Consider the nonlinear boundary value problem

$$\begin{cases} -\Delta u = \lambda \left[u - \frac{u^2}{K} - c \frac{u^2}{1+u^2} \right] =: \lambda f(u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \quad (1)$$

where $\Delta u = \operatorname{div}(\nabla u)$ is the Laplacian of u , $\frac{1}{\lambda}$ is the diffusion coefficient, K and c are positive constants and $\Omega \subset \mathbb{R}^N$ is a smooth bounded region with $\partial\Omega$ in C^2 . Here u is the population density and $u - \frac{u^2}{K}$ represents logistics growth. This model describes grazing of a fixed number of grazers on a logistically growing species (see [8,9]). The assumptions are that the ecosystem is spatially homogeneous and the herbivore density is a constant which are valid assumptions for managed grazing systems. The rate of grazing is given by $\frac{cu^2}{1+u^2}$. This term saturates to c at high levels of vegetation density as the grazing population is a constant. This model tries to capture the phenomena of bistability and hysteresis and provides qualitative and quantitative information for ecosystem managements. This model has also been applied to describe the dynamics of fish populations. In such cases the term $\frac{cu^2}{1+u^2}$ corresponds to natural predation. For more details see [8,10,12,14].

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¹ This author was supported by the National Research Foundation of Korea Grant funded by the Korean Govt. [NRF-2009-353-C00042].

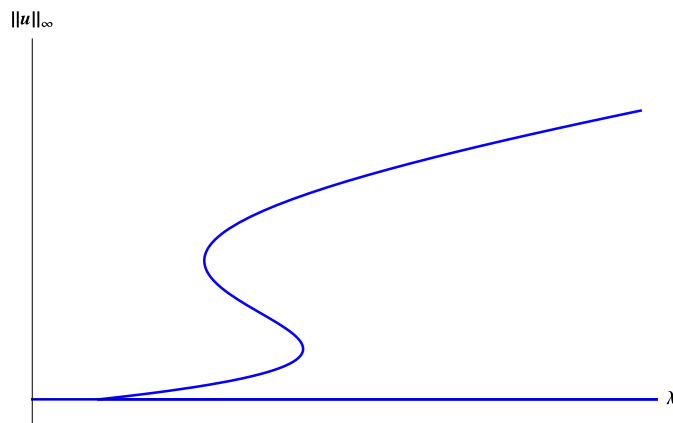


Fig. 1. S-shaped bifurcation curve.

Instead of working with the particular reaction term in (1) we will prove our results for a class of functions f which satisfy the following hypothesis:

(H1) $f \in C^2([0, \infty))$, $f(0) = 0$, $f'(0) = 1$, $f(u) > 0$ on $(0, r_0)$ and $f(u) < 0$ for $u > r_0$.

Under this hypothesis it is well known that for $\lambda > \lambda_1(\Omega)$ there always exists a positive solution where $\lambda_1(\Omega)$ is the principle eigenvalue of the operator $-\Delta$ with Dirichlet boundary conditions. It is also trivial to prove that there is no solution for $\lambda \leq \lambda_1(\Omega)$. The main focus of this paper is to prove the existence of at least three positive solutions for a certain range of λ and hence produce an S-shaped bifurcation curve originating from the trivial branch at $(\lambda_1(\Omega), 0)$ (see Fig. 1). The study of S-shaped bifurcation curves for positone problems ($f(0) > 0$) has a rich history (see [2,3,6,11]). Even in the case of positone problems proving multiplicity results for nonlinearities with falling zeros is much harder and often not possible (see example (iv) in [3]). Here we deal with a situation when $f(0) = 0$ and when f is a rather complicated nonlinearity with a falling zero.

To precisely state our multiplicity result, for $0 < a < b$, let

$$Q(a, b, \Omega) := \frac{\max\{\lambda_1(B_R), \frac{b}{f(b)}(\frac{N+1}{N})^{N+1} \frac{N^2}{R^2}\}}{\min\{\frac{a}{\|e_\Omega\|_\infty f^*(a)}, \frac{2NM}{f(b)R^2}\}},$$

where $B_R = B(0, R)$ is the largest inscribed ball on Ω , e_Ω is the unique positive solution of $-\Delta e = 1$ in Ω , $e = 0$ on $\partial\Omega$ and $f^*(s) = \max_{t \in [0, s]} f(t)$. We establish:

Theorem 1.1. Let $m, M \in (0, r_0)$ be such that f is non-decreasing in (m, M) . Assume there exist $b \in [m, M]$ and $a \in (0, b)$ such that $Q(a, b, \Omega) < 1$. Then (1) has three positive solutions for $\lambda \in (\max\{\lambda_1(B_R), \frac{b}{f(b)}(\frac{N+1}{N})^{N+1} \frac{N^2}{R^2}\}, \min\{\frac{a}{\|e_\Omega\|_\infty f^*(a)}, \frac{2NM}{f(b)R^2}\})$.

We will use the method of sub-supersolutions to prove our results. By a subsolution (supersolution) of (1) we mean a function $\psi \in W^{1,2}(\Omega) \cap C(\bar{\Omega})$ that satisfies

$$\begin{aligned} \int_{\Omega} \nabla \psi \cdot \nabla q &\leq (\geq) \int_{\Omega} \lambda f(\psi) q, \\ \psi &\leq (\geq) 0, \quad x \in \partial\Omega \end{aligned}$$

for every $q \in \{\eta \in C_0^\infty(\Omega) : \eta \geq 0 \text{ in } \Omega\}$. Then the following lemma holds (see [1]).

Lemma 1.2. Let ψ be a subsolution of (1) and ϕ be a supersolution of (1) such that $\psi \leq \phi$. Then (1) has a solution u such that $\psi \leq u \leq \phi$.

To establish our main multiplicity result we use the following very useful result discussed in [1,13]. Note here that by $\psi_1 < \psi_2$ we mean that $\psi_1 \leq \psi_2$ and $\psi_1 \neq \psi_2$.

Lemma 1.3. Suppose there exist a subsolution ψ_1 , a strict supersolution Z_1 , a strict subsolution ψ_2 and a supersolution Z_2 for (1) such that $\psi_1 < Z_1 < Z_2$, $\psi_1 < \psi_2 < Z_2$ and $\psi_2 \not\leq Z_1$. Then (1) has at least three distinct solutions u_1, u_2 and u_3 such that $\psi_1 \leq u_1 < u_2 < u_3 \leq Z_2$.

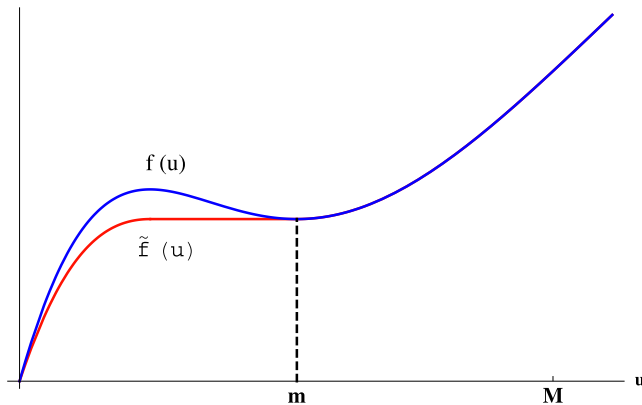


Fig. 2. Graph of $\tilde{f}(u)$.

We prove Theorem 1.1 in Section 2. In Section 3 we apply our results in the case when $f(u) = u - \frac{u^2}{K} - c \frac{u^2}{1+u^2}$. This study is motivated by the results in [10,14] where such a multiplicity result for the case $N = 1$ was discussed. Here we extend this study for the higher dimension case. In Appendix A we provide more detailed results for the case $N = 1$ using the quadrature method discussed in [3,7].

2. Proof of Theorem 1.1

To establish the multiplicity result we have to construct a subsolution ψ_1 , a strict supersolution Z_1 , a strict subsolution ψ_2 and a supersolution Z_2 for (1) such that $\psi_1 < Z_1 < Z_2$, $\psi_1 < \psi_2 < Z_2$ and $\psi_2 \not\leq Z_1$.

Let B_R be the largest inscribed ball in Ω . Define

$$\psi_1(x) = \begin{cases} \tilde{\epsilon} \phi_1(x), & x \in B_R, \\ 0, & x \in \Omega - B_R, \end{cases}$$

where $\phi_1 > 0$ is an eigenfunction corresponding to $\lambda_1(B_R)$ and $\tilde{\epsilon} > 0$. Then $\psi_1 \in W^{1,2}(\Omega) \cap C(\bar{\Omega})$ and $\psi_1 = 0$ on $\partial\Omega$. Now

$$-\Delta \psi_1 = -\tilde{\epsilon} \Delta \phi_1 = \tilde{\epsilon} \lambda_1(B_R) \phi_1, \quad x \in B_R. \quad (2)$$

Let $H(s) = \lambda f(s) - \lambda_1(B_R)s$. Then $H'(s) = \lambda f'(s) - \lambda_1(B_R)$, $H(0) = 0$ and $H'(0) = \lambda - \lambda_1(B_R)$. Since we are interested in the λ range $\lambda > \lambda_1(B_R)$, clearly $H'(0) > 0$. So for $\tilde{\epsilon} \approx 0$ we have $H(\tilde{\epsilon} \phi_1) = \lambda f(\tilde{\epsilon} \phi_1) - \lambda_1(B_R)(\tilde{\epsilon} \phi_1) \geq 0$. Hence from (2), for $\tilde{\epsilon} \approx 0$ we have

$$-\Delta \psi_1 = \tilde{\epsilon} \lambda_1(B_R) \phi_1 \leq \lambda f(\tilde{\epsilon} \phi_1) = \lambda f(\psi_1), \quad x \in B_R,$$

while outside B_R we have $-\Delta \psi_1 = 0 = \lambda f(0) = \lambda f(\psi_1)$. Thus ψ_1 is a positive subsolution.

For the large supersolution choose $Z_2 = r_0$. Then $-\Delta Z_2 = 0 \geq f(r_0) = f(Z_2)$ making Z_2 a positive supersolution. For $\tilde{\epsilon} \approx 0$ clearly $\psi_1 \leq Z_2$.

Now for the smaller strict supersolution define $Z_1 = \frac{ae_\Omega}{\|e_\Omega\|_\infty}$, where e_Ω is the unique positive solution of $-\Delta e = 1$ in Ω , $e = 0$ on $\partial\Omega$. Since $\lambda < \frac{a}{\|e_\Omega\|_\infty f^*(a)}$, $-\Delta Z_1 = \frac{a}{\|e_\Omega\|_\infty} > \lambda f^*(a) \geq \lambda f^*\left(\frac{ae_\Omega}{\|e_\Omega\|_\infty}\right) \geq \lambda f\left(\frac{ae_\Omega}{\|e_\Omega\|_\infty}\right) = \lambda f(Z_1)$ in Ω . Here $f^*(s) = \max_{t \in [0,s]} f(t)$. Hence Z_1 is a strict supersolution.

We will now construct the strict subsolution ψ_2 . Let

$$\tilde{f}(u) = \begin{cases} \hat{f}(u), & u < m, \\ f(u), & u \geq m \end{cases}$$

where $\hat{f}(u)$ is defined so that the function $\tilde{f}(u)$ is strictly increasing on $(0, M)$ and $\tilde{f}(u) \leq f(u)$ (see Fig. 2). Let

$$\rho(r) = \begin{cases} 1, & r \in [0, \epsilon], \\ 1 - [1 - (\frac{R-r}{R-\epsilon})^\beta]^\alpha, & r \in (\epsilon, R], \quad \alpha, \beta > 1. \end{cases}$$

Note that

$$\rho'(r) = \begin{cases} 0, & r \in [0, \epsilon], \\ -\alpha \beta [1 - (\frac{R-r}{R-\epsilon})^\beta]^{\alpha-1} (\frac{R-r}{R-\epsilon})^{\beta-1}, & r \in (\epsilon, R], \quad \alpha, \beta > 1 \end{cases}$$

and $|\rho'(r)| < \frac{\alpha\beta}{R-\epsilon}$. Now define $w(r) := b\rho(r)$ and

$$\psi_2(x) = \begin{cases} \tilde{\psi}_2, & x \in B_R, \\ 0, & x \in \Omega - B_R, \end{cases}$$

where $\tilde{\psi}_2$ is the solution of

$$\begin{cases} -\tilde{\psi}_2''(r) - \frac{N-1}{r}\tilde{\psi}_2'(r) = \lambda \tilde{f}(w(r)), & r \in (0, R), \\ \tilde{\psi}_2'(0) = 0 = \tilde{\psi}_2(R). \end{cases}$$

Then $\psi_2 \in W^{1,2}(\Omega) \cap C(\bar{\Omega})$ and $\psi_2 = 0$ on $\partial\Omega$. We will now establish that $\tilde{\psi}_2(r) \in (w(r), M]$ on $[0, R)$. Then $-\Delta\psi_2 = \lambda \tilde{f}(w(r)) < \lambda \tilde{f}(\psi_2(r)) \leq \lambda f(\psi_2(r))$ on $[0, R)$ while outside B_R we have $-\Delta\psi_2 = 0 = \lambda f(0) = \lambda f(\psi_2)$ and hence ψ_2 will be a strict subsolution.

First we will show that $\tilde{\psi}_2(r) \leq M$. Now

$$\begin{aligned} (r^{N-1}\tilde{\psi}_2'(r))' &= -\lambda r^{N-1}\tilde{f}(w(r)), \\ \tilde{\psi}_2'(r) &= \frac{-\lambda}{r^{N-1}} \int_0^r s^{N-1}\tilde{f}(w(s)) ds, \\ \tilde{\psi}_2(t) - \tilde{\psi}_2(0) &= -\int_0^t \frac{\lambda}{r^{N-1}} \left\{ \int_0^r s^{N-1}\tilde{f}(w(s)) ds \right\} dr. \end{aligned}$$

But $\tilde{\psi}_2(R) = 0$. Hence we get

$$\begin{aligned} \tilde{\psi}_2(0) &= \int_0^R \frac{\lambda}{r^{N-1}} \left\{ \int_0^r s^{N-1}\tilde{f}(w(s)) ds \right\} dr \\ &\leq \frac{\lambda \tilde{f}(b)}{N} \int_0^R r ds \\ &= \frac{\lambda \tilde{f}(b)}{2N} R^2 \quad (\text{since } b \geq m \text{ and } \tilde{f}(s) = f(s) \text{ for } s \geq m). \end{aligned}$$

But $\lambda < \frac{2NM}{f(b)R^2}$. Hence $\|\tilde{\psi}_2\|_\infty = \tilde{\psi}_2(0) < M$.

Next to establish $\tilde{\psi}_2 > w$ on $[0, R]$ we will show that $\tilde{\psi}_2' < w' \leq 0$ on $[0, R]$. This will be sufficient since $\tilde{\psi}_2(R) = w(R) = 0$. Now $w' = 0$ and $\tilde{\psi}_2' < 0$ in the interval $[0, \epsilon)$ and hence $\tilde{\psi}_2' < w' \leq 0$ in that interval. For $r > \epsilon$ we have

$$\begin{aligned} -\tilde{\psi}_2'(r) &= \frac{\lambda}{r^{N-1}} \int_0^r s^{N-1}\tilde{f}(w(s)) ds \\ &\geq \frac{\lambda}{r^{N-1}} \int_0^\epsilon s^{N-1}\tilde{f}(w(s)) ds \\ &= \frac{\lambda}{r^{N-1}} \int_0^\epsilon s^{N-1}\tilde{f}(b) ds \quad (\text{since } \rho(s) = 1, s < \epsilon) \\ &\geq \frac{\lambda \tilde{f}(b)}{R^{N-1}} \int_0^\epsilon s^{N-1} ds \\ &= \frac{\lambda \tilde{f}(b)}{R^{N-1}} \frac{\epsilon^N}{N} \quad (\text{since } b \geq m \text{ and } \tilde{f}(s) = f(s) \text{ for } s \geq m). \end{aligned}$$

We also know that $|w'(r)| \leq \frac{b\alpha\beta}{R-\epsilon}$. Hence $|\tilde{\psi}_2'(r)| > |w'(r)|$ if $\lambda > \alpha\beta \frac{b}{f(b)} \frac{NR^{N-1}}{(R-\epsilon)\epsilon^N}$. But $\min_{0 < \epsilon < R} \frac{1}{(R-\epsilon)\epsilon^N} = \frac{(N+1)^{N+1}}{N^N R^{N+1}}$ and this minimum is achieved at $\epsilon_0 = \frac{NR}{N+1}$. Since $\lambda > \frac{b}{f(b)} \frac{N^2}{R^2} \left(\frac{N+1}{N}\right)^{N+1} = \frac{b}{f(b)} \frac{NR^{N-1}}{(R-\epsilon_0)\epsilon_0^N}$ we can choose $\epsilon = \epsilon_0$ and $\alpha, \beta > 1$ such that $\lambda > \alpha\beta \frac{b}{f(b)} \frac{NR^{N-1}}{(R-\epsilon_0)\epsilon_0^N}$. Hence $|\tilde{\psi}_2'(r)| > |w'(r)|$ on $(0, R)$. This implies $w < \tilde{\psi}_2$. Thus ψ_2 is a strict subsolution of (1) if

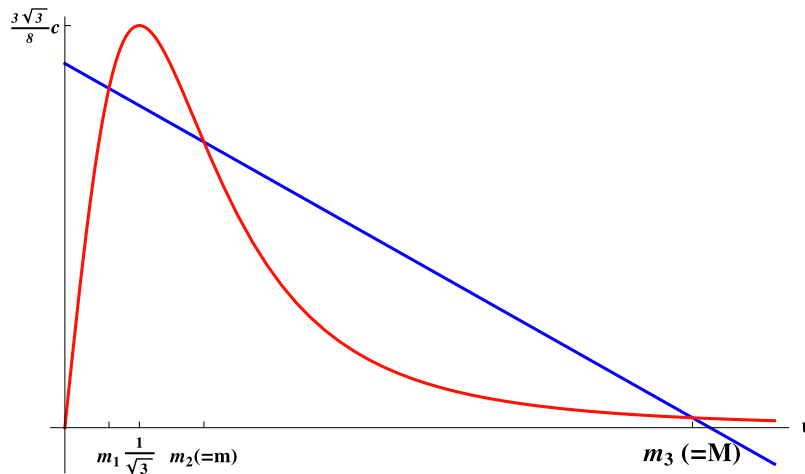


Fig. 3. Graph of the line $1 - \frac{2u}{K}$ and $g(u)$.

$\frac{b}{f(b)} \frac{N^2}{R^2} \left(\frac{N+1}{N}\right)^{N+1} < \lambda < \frac{2NM}{f(b)R^2}$. Moreover $\tilde{\psi}_2(0) > w(0) = b > a = \|Z_1\|_\infty$, i.e. $\psi_2 \notin Z_1$. Hence by Lemma 1.3 Theorem 1.1 holds.

3. Results for the example $f(u) = u - \frac{u^2}{K} - c \frac{u^2}{1+u^2}$

First we will analyze some properties of this nonlinearity. We will show that for large K we can find values of c for which the function $f(u) = u - \frac{u^2}{K} - c \frac{u^2}{1+u^2}$ satisfies (H1) and we will also identify m, M such that f is increasing in (m, M) . Clearly $f \in C^2([0, \infty))$, $f(0) = 0$ and $f'(0) = 1$. Now we will prove that the other assumptions in Theorem 1.1 hold in the given example.

Proposition 3.1. *If $c > \frac{8}{3\sqrt{3}}$ then for K large there exist three points m_1, m_2 and m_3 such that $0 < m_1 < m_2 < m_3$ and $f'(m_i) = 0$ for $i = 1, 2, 3$.*

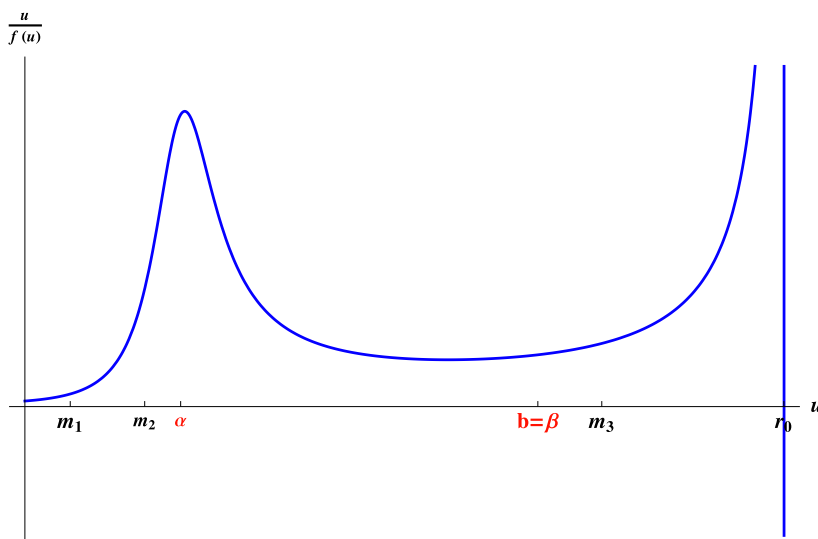
Proof. We have $f'(u) = 1 - \frac{2u}{K} - \frac{2cu}{(1+u^2)^2}$. So $f'(u) = 0$ when $1 - \frac{2u}{K} = \frac{2cu}{(1+u^2)^2}$. Let $g(u) := \frac{2cu}{(1+u^2)^2}$. Here $1 - \frac{2u}{K}$ is a line passing through $(0, 1)$ and with slope $-\frac{2}{K}$. We will prove that for $K \gg 1$, this line will cut $g(u)$ at three different points. We have $g(u) \geq 0$, $g(0) = 0$ and $\lim_{u \rightarrow \infty} g(u) = 0$. Since $g'(u) = 2c \frac{1-3u^2}{(1+u^2)^3}$, we can see that $g(u)$ achieves a maximum of $\frac{3\sqrt{3}c}{8}$ at $u = \frac{1}{\sqrt{3}}$, if $\max_{u \in (0, \infty)} g(u) = \frac{3\sqrt{3}c}{8} > 1$ then for K large the line $1 - \frac{2u}{K}$ will cut $g(u)$ at exactly three different points. Hence if $c > \frac{8}{3\sqrt{3}}$ and K is large, then there are exactly three positive points $m_1 < m_2 < m_3$ such that $f'(m_i) = 0$ for $i = 1, 2, 3$. \square

Proposition 3.2. *If $c < 2$ then for $K \gg 1$ there exists a unique $r_0 > 0$ such that $f(r_0) = 0$.*

Proof. If $c < \frac{8}{3\sqrt{3}}$, then from the geometry discussed above it is clear that for K large $f(u)$ has a unique positive zero. Now consider the case $\frac{8}{3\sqrt{3}} < c < 2$. Since $c > \frac{8}{3\sqrt{3}}$ by Proposition 3.1 we have for K large, there exist three positive numbers m_1, m_2 and m_3 such that $f'(m_i) = 0$ for $i = 1, 2, 3$. Clearly the function $f(u)$ has a relative minimum at $u = m_2$. We will prove that $f(m_2) > 0$ for $K \gg 1$. This implies that $f(u)$ has a unique positive zero. It is clear from Fig. 3 that there exists a constant M_2 such that $m_2 < M_2$ for all K . In fact $m_2 = m_2(K)$ is a continuous decreasing function of K such that $m_2(K) \in (\frac{1}{\sqrt{3}}, M_2)$. Also $\lim_{K \rightarrow \infty} f(m_2) = z - c \frac{z^2}{1+z^2}$ for some $z \in (\frac{1}{\sqrt{3}}, M_2)$. But $h(z) = z - c \frac{z^2}{1+z^2} > 0$ for $z > 0$ if $c < 2$. Hence $\lim_{K \rightarrow \infty} f(m_2) > 0$. Thus for K large there exists a unique $r_0 > m_3$ such that $f(r_0) = 0$. \square

Thus given $c \in (\frac{8}{3\sqrt{3}}, 2)$ we can find K large so that $f(u)$ is increasing on $(m_2(=m), m_3(=M))$ and there exists a unique $r_0 > 0$ such that $f(r_0) = 0$, i.e. $f(u)$ satisfies (H1).

Next we will select candidates for $b \in [m_2, m_3]$ and $a \in (0, b)$ using which later we will show that $Q(a, b, \Omega) < 1$. The point at which the function $\frac{u}{f(u)}$ has a minimum would be an ideal value for b (see Fig. 4). We have $(\frac{u}{f(u)})' = \frac{f(u) - uf'(u)}{(f(u))^2}$. Hence the critical points of $\frac{u}{f(u)}$ are given by $f(u) - uf'(u) = 0$ and in particular the non-zero critical points are given by

Fig. 4. Graph of $\frac{u}{f(u)}$.

$\frac{1}{K} - \frac{c}{1+u^2} + \frac{2c}{(1+u^2)^2} = 0$. Solving for u we get the positive critical points as $\alpha = \sqrt{\frac{cK-2-\sqrt{cK(cK-8)}}{2}}$ and $\beta = \sqrt{\frac{cK-2+\sqrt{cK(cK-8)}}{2}}$. Hence $\frac{u}{f(u)}$ has a relative minimum at β . Since $\beta \rightarrow \infty$ as $K \rightarrow \infty$ and m_2 is bounded we have $\beta \in [m_2, m_3]$ for large K . Choose $b = \beta$. Next we choose $a \in (m_2, b)$ such that $f(a) = f^*(a) = f(m_1)$. This is possible since $f(m_1)$ is bounded while $f(b) \rightarrow \infty$ as $K \rightarrow \infty$. (See Proposition 3.3 which follows next where it will be established that $\lim_{K \rightarrow \infty} \frac{b}{f(b)} = 1$. But $\lim_{K \rightarrow \infty} b = \infty$. Hence $f(b) \rightarrow \infty$ as $K \rightarrow \infty$.)

Proposition 3.3.

- (i) $b \leq \sqrt{cK}$ and $m_3 > \frac{K}{4}$ for $K \gg 1$.
- (ii) $\lim_{K \rightarrow \infty} \frac{b}{f(b)} = 1$ and $\lim_{K \rightarrow \infty} \frac{m_3}{f(b)} = \infty$.

Proof. (i) We have $b = \sqrt{\frac{cK-2+\sqrt{cK(cK-8)}}{2}} = \sqrt{\frac{cK-2+cK\sqrt{(1-\frac{8}{cK})}}{2}} \leq \sqrt{cK}$. Now $f'(u) = 1 - \frac{2u}{K} - \frac{2cu}{(1+u^2)^2}$. So $f'(\frac{K}{4}) = \frac{1}{2} - \frac{cK}{2(1+\frac{K^2}{16})^2} > 0$ and $f'(\frac{K}{2}) = -\frac{cK}{(1+\frac{K^2}{4})^2} < 0$ for $K \gg 1$. Hence $m_3 > \frac{K}{4}$ for $K \gg 1$.

(ii) We have $\lim_{K \rightarrow \infty} b = \lim_{K \rightarrow \infty} \sqrt{\frac{cK-2+\sqrt{cK(cK-8)}}{2}} = \infty$. From (i) we have $b \leq \sqrt{cK}$. Hence $\lim_{K \rightarrow \infty} \frac{b}{K} \leq \lim_{K \rightarrow \infty} \frac{\sqrt{cK}}{K} = 0$. Thus $\lim_{K \rightarrow \infty} \frac{b}{f(b)} = \lim_{K \rightarrow \infty} \frac{b}{b - \frac{b^2}{K} - c \frac{b^2}{1+b^2}} = \lim_{K \rightarrow \infty} \frac{1}{1 - \frac{b}{K} - c \frac{b}{1+b^2}} = 1$. Finally $\lim_{K \rightarrow \infty} \frac{m_3}{f(b)} \geq \lim_{K \rightarrow \infty} \frac{K}{4(b - \frac{b^2}{K} - c \frac{b^2}{1+b^2})} = \infty$. \square

Now we will discuss the existence of at least three positive solutions for a certain range of λ (see Theorem 1.1) by establishing that $Q(a, b, \Omega) < 1$. In particular we will analyze the following two cases: (A) Ω is a ball in \mathbb{R}^N and (B) Ω is a general bounded domain in \mathbb{R}^N .

(A) When Ω is a ball

We will now prove that when Ω is a ball of radius R (i.e. $\Omega = B_R$) in \mathbb{R}^N with $N < 8$ there exist $K \gg 1$ and c close to 2 such that $Q(a, b, B_R) < 1$.

First for $u \in [0, M_2]$ and $c = 2$ we consider the function $h(u) := \lim_{K \rightarrow \infty} f(u) = u - 2 \frac{u^2}{1+u^2}$. Note that $h'(u) = 1 - \frac{4u}{(1+u^2)^2}$ and solving $h'(u) = 0$ we get $m_1 \approx 0.2956$ and $m_2 = 1$. Solving $f(u) = f(m_1)$, $u \neq m_1$, we get $a \approx 1.5437$ (see Fig. 5). Hence $\frac{a}{f^*(a)} = \frac{a}{f(m_1)} \approx 11.4445$.

Our aim is to prove that for $c = 2 - \delta$, where $\delta \approx 0$ and $K \gg 1$, $Q(a, b, B_R) < 1$. We have already seen that $\frac{b}{f(b)} \rightarrow 1$ and $\frac{m_3(=M)}{f(b)} \rightarrow \infty$ as $K \rightarrow \infty$. Hence $Q(a, b, B_R) < 1$ if $\max\{\lambda_1(B_R), (\frac{N+1}{N})^{N+1} \frac{N^2}{R^2}\} < \frac{a}{\|e_{B_R}\|_\infty f^*(a)} = \frac{11.4445}{\|e_{B_R}\|_\infty}$, i.e. if $\lambda_1(B_R) \|e_{B_R}\|_\infty < 11.4445$ and $(\frac{N+1}{N})^{N+1} \frac{N^2}{R^2} \|e_{B_R}\|_\infty < 11.4445$.

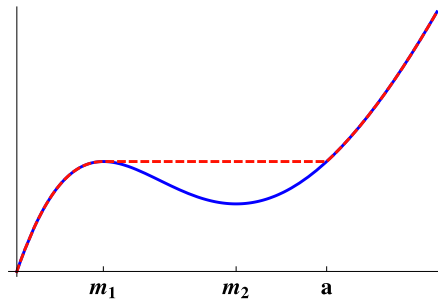


Fig. 5. Graph of $f^*(u)$ (in red). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

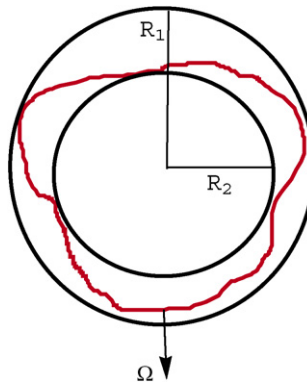


Fig. 6. General domain Ω .

Next we evaluate $\|e_{B_R}\|_\infty$. We have $-\Delta e = 1$ in B_R , $e = 0$ on ∂B_R . Then e is radial, radially decreasing and satisfies

$$\begin{cases} -e''(r) - \frac{N-1}{r}e'(r) = 1, & r \in (0, R), \\ e'(0) = 0 = e(R). \end{cases}$$

Solving this boundary value problem we obtain $e(r) = \frac{1}{2N}(R^2 - r^2)$. From this it follows that $\|e_{B_R}\|_\infty = e(0) = \frac{R^2}{2N}$. Now the principal eigenvalue λ_1 when Ω is a ball of radius R is given by,

$$\lambda_1(B_R) = \begin{cases} \frac{\pi^2}{4R^2} \approx \frac{2.4674}{R^2}, & N = 1, \\ \frac{j_{0,1}^2}{R^2} \approx \frac{5.7832}{R^2}, & N = 2, \\ \frac{j_{\frac{N}{2}-1,1}^2}{R^2}, & N \geq 3, \end{cases}$$

where $j_{n,1}$ is the first zero of the Bessel function of order n (see [5]). From [4] we have $j_{n,1} < (n+1)^{\frac{1}{2}}((n+2)^{\frac{1}{2}}+1)$ for $n > -1$. Hence for $N \geq 3$, we get $\lambda_1(B_R)\|e_{B_R}\|_\infty = \frac{j_{\frac{N}{2}-1,1}^2}{2N} \leq \frac{(\frac{N}{2})^{\frac{1}{2}}((\frac{N}{2}+1)^{\frac{1}{2}}+1)}{2N} = \frac{1}{2\sqrt{2}}(\sqrt{\frac{N+2}{2N}} + \frac{1}{\sqrt{N}}) < 1 < \frac{a}{f^*(a)}$. We also have $\lambda_1(B_R)\|e_{B_R}\|_\infty = \frac{2.4674}{2}$ and $\lambda_1(B_R)\|e_{B_R}\|_\infty = \frac{5.7832}{4}$ when $N = 1$ and $N = 2$ respectively. Thus $\lambda_1(B_R)\|e_{B_R}\|_\infty < \frac{a}{f^*(a)}$ for all N .

Next we have $\frac{N^2}{R^2}(\frac{N+1}{N})^{N+1}\|e_{B_R}\|_\infty = \frac{N}{2}(\frac{N+1}{N})^{N+1}$. Further the function $y = \frac{N}{2}(\frac{N+1}{N})^{N+1}$ is increasing for positive N and $\frac{N}{2}(\frac{N+1}{N})^{N+1} < \frac{a}{f^*(a)} (= 11.4445)$ for $N < 8$. Thus for $N < 8$, $Q(a, b, B_R) < 1$ when $c \approx 2$ and $K \gg 1$.

(B) When Ω is a general bounded domain

When Ω is a general bounded region we will establish a sufficient condition on the geometry of the region for our multiplicity result to hold. Let $R_1 > 0$, $R_2 > 0$ be such that $B_{R_2} = B(0, R_2) \subseteq \Omega \subseteq B_{R_1} = B(0, R_1)$ (see Fig. 6). Let $-\Delta e_{B_{R_1}} = 1$ in B_{R_1} , $e_{B_{R_1}} = 0$ on ∂B_{R_1} . Then $e_{B_{R_1}}$ is a supersolution of the problem $-\Delta e_\Omega = 1$ in Ω , $e_\Omega = 0$ on $\partial\Omega$. Hence $\|e_\Omega\|_\infty \leq \|e_{B_{R_1}}\|_\infty$.

As in the case when the domain is a ball, we will try to prove that for $c = 2 - \delta$ (where $\delta > 0$ is very small) and $K \gg 1$, $Q(a, b, \Omega) < 1$. Now $Q(a, b, \Omega) < 1$ if $\max\{\lambda_1(B_{R_2}), (\frac{N+1}{N})^{N+1} \frac{N^2}{R_2^2}\} < \frac{a}{\|e_\Omega\|_\infty f^*(a)} = \frac{11.4445}{\|e_\Omega\|_\infty}$, i.e. if $\lambda_1(B_{R_2})\|e_\Omega\|_\infty < 11.4445$ and $(\frac{N+1}{N})^{N+1} \frac{N^2}{R_2^2} \|e_\Omega\|_\infty < 11.4445$.

For $N \geq 3$, we get $\lambda_1(B_{R_2})\|e_\Omega\|_\infty \leq \lambda_1(B_{R_2})\|e_{B_{R_1}}\|_\infty = \frac{j_{\frac{N}{2}-1,1} R_1^2}{2N R_2^2} \leq \frac{(\frac{N}{2})^{\frac{1}{2}} ((\frac{N}{2}+1)^{\frac{1}{2}}+1) R_1^2}{2N R_2^2} = \frac{1}{2\sqrt{2}} (\sqrt{\frac{N+2}{2N}} + \frac{1}{\sqrt{N}}) \frac{R_1^2}{R_2^2} < \frac{R_1^2}{R_2^2}$. If $N = 1$ we have $\lambda_1(B_{R_2})\|e_\Omega\|_\infty \leq \lambda_1(B_{R_2})\|e_{B_{R_1}}\|_\infty = \frac{2.4674}{2} \frac{R_1^2}{R_2^2} = 1.2337 \frac{R_1^2}{R_2^2}$ and if $N = 2$ we have $\lambda_1(B_{R_2})\|e_\Omega\|_\infty \leq \lambda_1(B_{R_2})\|e_{B_{R_1}}\|_\infty = \frac{5.7832}{4} \frac{R_1^2}{R_2^2} = 1.4458 \frac{R_1^2}{R_2^2}$. Thus $\lambda_1(B_{R_2})\|e_\Omega\|_\infty < \frac{a}{f^*(a)} = 11.4445$ for all N whenever $\frac{R_1^2}{R_2^2} < \frac{11.4445}{1.4458} = 7.91569$.

Next we have $(\frac{N+1}{N})^{N+1} \frac{N^2}{R_2^2} \|e_\Omega\|_\infty \leq (\frac{N+1}{N})^{N+1} \frac{N^2}{R_2^2} \|e_{B_{R_1}}\|_\infty = \frac{N}{2} (\frac{N+1}{N})^{N+1} \frac{R_1^2}{R_2^2}$. Hence in the general domain case if $\frac{R_1^2}{R_2^2} < \min\{7.91569, \frac{2}{N} (\frac{N}{N+1})^{N+1} 11.4445\}$ then $Q(a, b, \Omega) < 1$ when $c \approx 2$ and $K \gg 1$. Note that $\frac{2}{N} (\frac{N}{N+1})^{N+1} 11.4445 > 1$ when $N < 8$.

Appendix A

Consider the two point boundary value problem

$$\begin{cases} -u'' = \lambda f(u), & x \in (0, 1), \\ u(0) = 0 = u(1), \end{cases} \quad (3)$$

where f satisfies the following hypotheses:

- (G1) $f \in C^2([0, \infty))$ and $f(u) > 0$ for $0 < u < r_0$ for $r_0 > 0$;
- (G2) there exists $k \geq 0$ such that $f(u) - f(v) \geq -k(u - v)$ for all $u, v \in [0, r_0]$ with $u > v$;
- (G3) $r_0 < \infty$ and $f(r_0) = 0$.

Using the quadrature method the solution $u = u(x)$ is defined by

$$\int_0^{u(x)} \frac{dz}{\sqrt{[F(\rho) - F(z)]}} = \sqrt{2\lambda}x, \quad 0 < x < \frac{1}{2},$$

where $F(s) := \int_0^s f(t) dt$, provided

$$\sqrt{\lambda} = \sqrt{2} \int_0^\rho \frac{dz}{\sqrt{[F(\rho) - F(z)]}} := G(\rho).$$

Here $\rho = u(\frac{1}{2}) = \|u\|_\infty$. Since $f(\rho) > 0$ and $F(\rho) > F(z)$ for all $0 \leq z < \rho$, it follows that $G(\rho)$ exists for all $\rho > 0$. Infact $G(\rho)$ is a continuous function. We also have

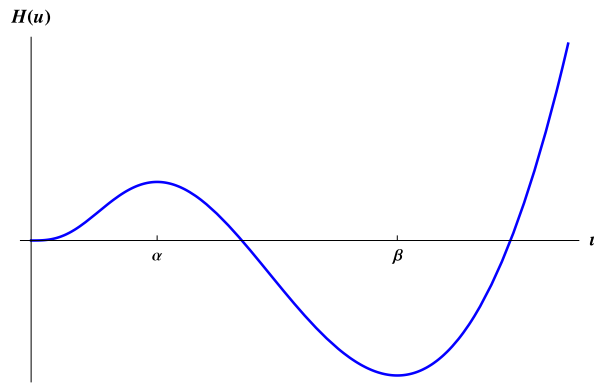
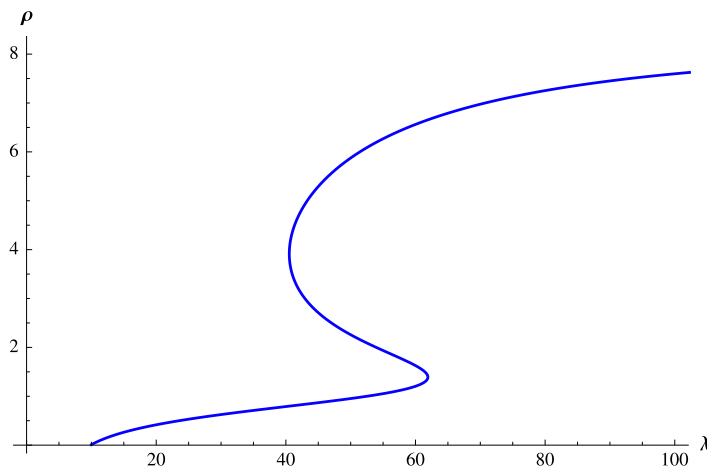
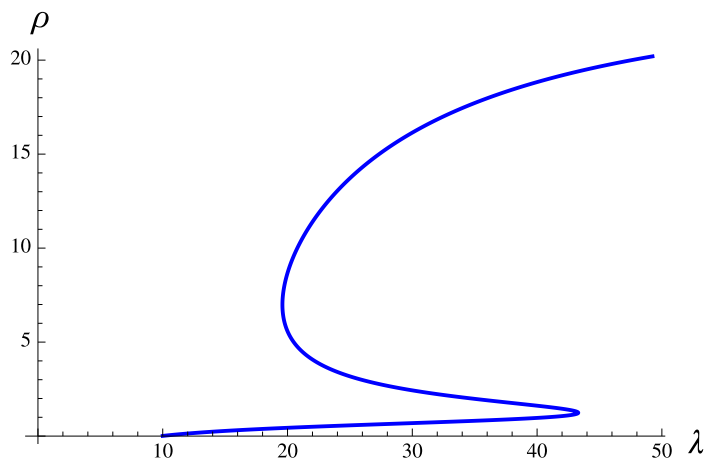
$$G'(\rho) = \sqrt{2} \int_0^1 \frac{H(\rho) - H(\rho s)}{[F(\rho) - F(\rho s)]^{\frac{3}{2}}} ds$$

where $H(u) = F(u) - \frac{u}{2} f(u)$. Then we have the following lemma from [3].

Lemma 4.1.

- (a) If the bifurcation curve of (3) is S-shaped, then $H'(\rho) < 0$ for some $0 < \rho < r_0$.
- (b) If there exists positive $\rho_0 < r_0$ such that $H(\rho_0) < 0$, then (3) has at least three solutions for a certain range of λ .

Consider the case $f(u) = u - \frac{u^2}{K} - c \frac{u^2}{1+u^2}$. Clearly, given $c < 2$ fixed then for $K \gg 1$ f satisfies (G1)–(G3) (see Proposition 3.2). Hence $G(\rho)$ is defined for all $\rho \in S = (0, r_0)$.

Fig. 7. Graph of $H(u)$.Fig. 8. Bifurcation diagram with $c = 1.5$ and $K = 10$. Here $r_0 = 8.19687$.Fig. 9. Bifurcation diagram with $c = 1.5$ and $K = 25$. Here $r_0 = 23.4004$.

Now we will show that there exists $\rho_0 \in (0, r_0)$ such that $H(\rho_0) < 0$. We have $H(u) = F(u) - \frac{u}{2}f(u) = \frac{u^3}{6K} + c(\frac{u^3}{2(1+u^2)} - u + \tan^{-1}(u))$ and $H'(u) = \frac{1}{2}(f(u) - uf'(u))$. The zeros of $H'(u)$ are the same as the zeros of $(\frac{u}{f(u)})'$ and in Section 4 we have already found that the positive roots of $(\frac{u}{f(u)})'$ are $\alpha = \sqrt{\frac{cK-2-\sqrt{cK(cK-8)}}{2}}$ and $\beta = \sqrt{\frac{cK-2+\sqrt{cK(cK-8)}}{2}}$. Hence $H(u)$ has a maximum at $u = \alpha$ and a minimum at $u = \beta$. From Proposition 3.2 we have $\beta < r_0$ for $K \gg 1$. We will now show that $H(\beta) < 0$ for $K \gg 1$, giving $H(u)$ the shape shown in Fig. 7.

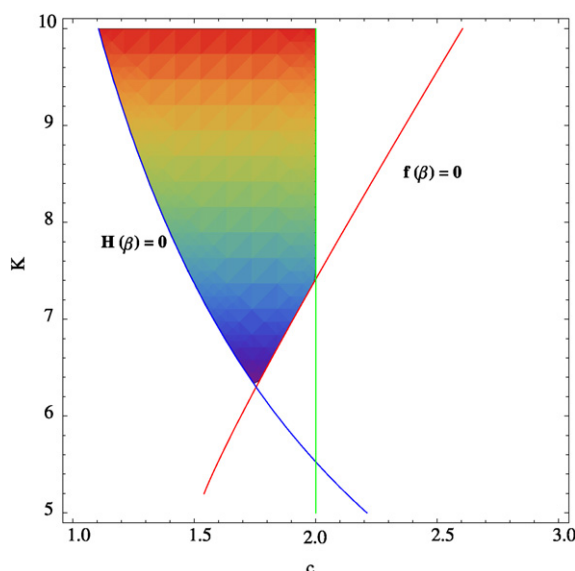


Fig. 10. Feasible region.

We have

$$\begin{aligned} H(\beta) &= \frac{\beta^3}{6K} + c \left(\frac{\beta^3}{2(1+\beta^2)} - \beta + \tan^{-1}(\beta) \right) \\ &= \frac{\beta^2}{6K} \beta + c \left(\frac{\beta^2}{2(1+\beta^2)} \beta - \beta + \tan^{-1}(\beta) \right) \\ &\leq \frac{c}{6} \beta + c \left(\frac{\beta}{2} - \beta + \frac{\pi}{2} \right) = c \left(-\frac{1}{3} \beta + \frac{\pi}{2} \right). \end{aligned}$$

Clearly $\beta \rightarrow \infty$ as $K \rightarrow \infty$ and hence $\lim_{K \rightarrow \infty} H(\beta) < 0$. Hence by Lemma 4.1 we have the following theorem.

Theorem 4.2. Given $c < 2$ fixed then for $K \gg 1$ the boundary value problem (3) has at least three solutions for a certain range of λ .

We used Mathematica to compute $\sqrt{\lambda} = G(\rho)$ in the case when $f(u) = u - \frac{u^2}{K} - c \frac{u^2}{1+u^2}$ and plotted the bifurcation diagrams. In Figs. 8 and 9 bifurcation diagrams for a certain value of c and K are given. In Fig. 10 the region of (c, K) -plane that satisfies the hypothesis of Theorem 4.2 is given. For all values of c and K , that lies in the region enclosed by these curves, the two point boundary value problem given in (3) will have at least three different solutions for a certain range of λ values.

References

- [1] H. Amann, Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces, *SIAM Rev.* 18 (1976) 620–709.
- [2] R. Aris, On stability criteria of chemical reaction engineering, *Chem. Eng. Sc.* 24 (1969) 149–169.
- [3] K.J. Brown, M.M.A. Ibrahim, R. Shivaji, S-shaped bifurcation curves, *Nonlinear Anal.* 5 (5) (1981) 475–486.
- [4] L.G. Chambers, An upper bound for the first zero of Bessel functions, *Math. Comp.* 38 (158) (1982) 589–591.
- [5] M. Hazewinkel (Ed.), *Encyclopedia of Mathematics*, vol. 3, first ed., 1990, p. 325.
- [6] J.P. Kernevez, G. Joly, M.C. Duban, B. Bunow, D. Thomas, Hysteresis, oscillations and pattern formation in realistic immobilized enzyme systems, *J. Math. Biol.* 7 (1979) 41–56.
- [7] T. Laetsch, The number of solutions of a nonlinear two point boundary value problem, *Indiana Univ. Math. J.* 20 (1) (1970) 1–13.
- [8] R.M. May, Thresholds and breakpoints in ecosystems with a multiplicity of stable states, *Nature* 269 (1977) 471–477.
- [9] I. Noy-Meir, Stability of grazing systems an application of predator-prey graphs, *J. Ecol.* 63 (1975) 459–482.
- [10] J. Jiang, J. Shi, Bistability dynamics in some structured ecological models, in: R.S. Cantrell, C. Cosner, S. Ruan (Eds.), *Spatial Ecology*, in: Chapman & Hall/CRC Math. Comput. Biol. Ser., 2009, pp. 33–62.
- [11] M. Ramaswamy, R. Shivaji, Multiple positive solutions for classes of p -Laplacian equations, *Differential Integral Equations* 17 (11–12) (2004) 1255–1261.
- [12] J.H. Steele, E.W. Henderson, Modelling long term fluctuations in fish stocks, *Science* 224 (1984) 985–987.
- [13] R. Shivaji, A remark on the existence of three solutions via sub-super solutions, in: V. Lakshmikantham (Ed.), *Nonlinear Analysis and Applications*, 1987, pp. 561–566.
- [14] E. H Van Nes, Martin Scheffer, Implications of spatial heterogeneity for catastrophic regime shifts in ecosystems, *Ecology* 86 (7) (2005) 1797–1807.