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Structural stability of solutions to the Riemann problem for a scalar conservation law ☆

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ABSTRACT

The aim of this paper is to study the structural stability of solutions to the Riemann problem for a scalar conservation law with a linear flux function involving discontinuous coefficients. It is proved that the Riemann solution is possibly instable when one of the Riemann initial data is at the vacuum. Furthermore, we point out that the Riemann solution is also possibly instable even when the Riemann initial data stay far away from vacuum. In order to deal with it, we perturb the Riemann initial data by taking three piecewise constant states and then the global structures and large time asymptotic behaviors of the solutions are obtained constructively. It is also proved that the Riemann solutions are unstable in some certain situations under the local small perturbations of the Riemann initial data by letting the perturbed parameter ε tend to zero. In addition, the interaction of the delta standing wave and the contact vacuum state is considered which appear in the Riemann solutions.

1. Introduction

In this paper, we are concerned with the following scalar conservation law with the linear flux function involving discontinuous coefficients [8]

$$u_t + \big(k(x)u\big)_x = 0. \tag{1.1}$$

The system (1.1) arises in many areas, such as the multi-species chemical reacting model [11], the particle flows in applications [3] and the polymer flooding of an oil reservoir [10], etc. It is more convenient to view (1.1) as a system by adding the trivial term $k_t = 0$ if $k(x)$ is allowed to be discontinuous, namely we can consider the system

$$\begin{cases} k_t = 0, \\ u_t + (ku)_x = 0, \end{cases} \tag{1.2}$$

here k and u may be regarded as the velocity and the density in the discontinuous flow respectively. It is clear to see that the characteristic speeds for (1.2) are $\lambda_0 = 0$ and $\lambda_1 = k$ and both characteristic fields are linear degenerate. Thus the system (1.2) is nonstrictly hyperbolic and the nonlinear resonant situation occurs when $k = 0$.

It is well known that the Riemann problem for the system (1.2) is a special Cauchy problem with initial data

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$$(k,u)(x,0)=\begin{cases}(k_l,u_l), & x<0,\\ (k_r,u_r), & x>0,\end{cases} \tag{1.3}$$

where k_l, k_r, u_l and u_r are all given constants. In particular, the Riemann problem (1.2) and (1.3) here has infinitely many solutions when the Riemann initial data are taken as $k_l<0<k_r$ and $u_l u_r<0$. Hence we confine ourself to only consider the situation $u\geqslant 0$ and it is also reasonable from the physical sense since we can regard u as the flow density.

It is remarkable that the Riemann problem for a nonlinear hyperbolic resonant system is often ill-posed. If $k_l>0>k_r$, then the Riemann problem (1.2) and (1.3) does not possess a weak L^∞-solution. In order to solve the Riemann problem in this nonclassical situation, it is necessary to consider in some measure space. Here the delta standing wave is introduced in the Riemann solution, which is the Dirac delta function supported on the standing wave discontinuity at $x=0$. On the other hand, the vacuum state also appears in the Riemann solution of (1.2) and (1.3) when $k_l<0<k_r$.

Vacuum states are important physical states in gas and fluid dynamics and usually lead to certain singularities in the physical systems, which cause essential analytical difficulties. Thus it is necessary to understand the behavior of vacuum states, namely the behavior of the system in the presence of vacuum. The stability or instability of solutions with respect to perturbations of the initial data is an important problem which is encountered widely in applications. In this paper, we focus on the stability of the Riemann solutions to (1.2) and (1.3) and we can see that the Riemann solutions are possibly instable when one of Riemann initial states is at the vacuum. In order to solve this Riemann problem, we fix the nonvacuum state and then slightly perturb the vacuum state to a nonvacuum state, which was proposed by Liu and Smoller in [12] and also employed by Aw and Rascle in [2].

On the other hand, even when the initial states stay far away from vacuum, we notice that the Riemann solutions are also possibly instable with respect to the small perturbation of the initial states. In order to the explicit calculations as far as possible, we consider the initial data consisting of three piecewise constant states as follows:

$$(k,u)(x,0)=\begin{cases}(k_l,u_l), & x<-\varepsilon,\\ (k_m,u_m), & -\varepsilon<x<\varepsilon,\\ (k_r,u_r), & x>\varepsilon,\end{cases} \tag{1.4}$$

where $\varepsilon>0$ is arbitrarily small and all the $u_l,u_m,u_r>0$. If we pass to the limit as $\varepsilon\to 0$ in (1.4), then we deduce the corresponding Riemann initial data (1.3). So the initial data (1.4) can be viewed as the perturbation of the corresponding Riemann initial data (1.3). Furthermore, the stability of the Riemann solutions of (1.2) and (1.3) can be investigated under this local small perturbation of the Riemann initial data by letting $\varepsilon\to 0$. More precisely, it is proved that the solutions of the perturbed Riemann problem (1.2) and (1.4) do not converge to the solutions of the corresponding Riemann problem (1.2) and (1.3) in some certain situations as $\varepsilon\to 0$, which displays the structural unstable property of the Riemann solutions.

It is remarkable that one-dimensional homogeneous linear transport equation reads

$$u_t+k(x,t)u_x=0, \tag{1.5}$$

by differentiating (1.5) with respect to x, the conservative form can be obtained as

$$\mu_t+\big(k(x,t)\mu\big)_x=0, \tag{1.6}$$

with $\mu=u_x$. An appropriate theoretical framework for (1.6) was established by Bouchut and James [4,5] through studying the solution of (1.6) in the duality sense and also by Poupaud and Rascle [14] through the construction of Filippov characteristics in the multidimensional situation, in which it was shown that μ is a measure in the space variable x. When k is not smooth, the well-posedness of solution for (1.6) is much more delicate [1,3]. It is noticed that (1.1) is the special form of (1.6) with $k(x,t)$ only depending on the variable x. In this note, we take (1.1) as the simplest example of (1.6) to consider the stability of Riemann solutions in a completely explicit construction, which enables us to develop a fairly complete study of the solution concerning both qualitative and quantitative properties.

The paper is organized in the following way. In Section 2, the Riemann problem for (1.2) and (1.3) is solved in detail. Moreover, we analyze the structural unstable property of the Riemann solution of (1.2) and (1.3) when one of Riemann initial data is at the vacuum. In Section 3, we mainly discuss the interactions of the elementary waves, including the delta standing waves and the contact discontinuities for all the cases when the initial data are three piece constant states (1.4). Furthermore, the perturbed Riemann solutions of (1.2) and (1.4) are constructed globally cases by cases and the stability of the Riemann solutions is analyzed by passing to the limit $\varepsilon\to 0$. It is discovered that the Riemann solution is possibly unstable with respect to the perturbation (1.3) of Riemann initial data even if the Riemann initial data stay far away from vacuum in some situations. Finally, our conclusion is drawn in Section 4.

2. The Riemann problem for (1.2) and (1.3)

In this section, we mainly discuss the Riemann problem for (1.2) and (1.3) which has also been investigated by Hu [10] through the self-similar viscosity vanishing approach proposed by Dafermos [7]. In the following, we adopt a different approach to deal with it which is easier to understand. Furthermore, we draw our attention to the vacuum problem and discuss the stability of the Riemann solutions when one of the initial states contains a vacuum.

2.1. Basic properties of the system (1.2)

The characteristic speeds for (1.2) are $\lambda_0 = 0$ and $\lambda_1 = k$ and the corresponding right characteristic vectors are $\vec{r_0} = (k, -u)^T$ and $\vec{r_1} = (0, 1)^T$ respectively. Obviously, we have $\nabla\lambda_i \cdot \vec{r_i} = 0$ $(i = 0, 1)$, in which ∇ denotes the gradient with respect to (k, u). Thus, (1.2) is a nonstrictly hyperbolic system and both the two characteristic fields are linearly degenerate. Therefore, the associated wave for λ_0 is the standing wave and be denoted by *SW* since its wave speed is always zero, and the associated wave for λ_1 is the contact discontinuity and be denoted by *CD*. It is clear to see that the standing wave has the similar properties with the contact discontinuity.

Since both (1.2) and (1.3) are invariant under the transformation $(x, t) \to (\alpha x, \alpha t)$ (α is constant), the solution of the Riemann problem must be a function of the single variable x/t. Thus, we seek the self-similar solution

$$(k, u)(x, t) = (k, u)(\xi), \quad \xi = x/t. \tag{2.1}$$

Then the Riemann problem of (1.2) and (1.3) is reduced to the boundary value problem of the ordinary differential equations:

$$\begin{cases} -\xi k_\xi = 0, \\ -\xi u_\xi + (ku)_\xi = 0, \end{cases} \tag{2.2}$$

with $(k, u)(-\infty) = (k_l, u_l)$ and $(k, u)(+\infty) = (k_r, u_r)$.

For a bounded discontinuity at $\xi = \sigma$, the Rankine–Hugoniot condition holds:

$$\begin{cases} -\sigma[k] = 0, \\ -\sigma[u] + [ku] = 0, \end{cases} \tag{2.3}$$

here and below, we use the usual notation $[k] = k_+ - k_-$ where k_- and k_+ are the values of function k on the left-hand and right-hand sides of the discontinuity, etc.

Thus, it follows that $\sigma_0 = 0$ and $k_-u_- = k_+u_+$ should be satisfied which exactly corresponds to the standing wave, and then $\sigma_1 = k_- = k_+$ can be reached which exactly corresponds to the contact discontinuity.

2.2. The Riemann solution when $u_l > 0$ and $u_r > 0$

Let us first consider the Riemann problem when the initial state is away from vacuum. If $u_l > 0$ and $u_r > 0$, then we can construct the Riemann solutions of (1.2) and (1.3) in the following four cases.

(i) If $k_l > 0$ and $k_r > 0$, then the Riemann solution consists of a standing wave followed by a forward contact discontinuity and the intermediate state between them is $(k_c, u_c) = (k_r, \frac{k_l u_l}{k_r})$. Let us see Fig. 2.1(a) for detail.

(ii) If $k_l < 0$ and $k_r < 0$, then the Riemann solution consists of a backward contact discontinuity followed by a standing wave and the intermediate state now is $(k_c, u_c) = (k_l, \frac{k_r u_r}{k_l})$ (see Fig. 2.1(b)).

(iii) If $k_l < 0 < k_r$, then the Riemann solution consists of two contact discontinuities with vacuum state between them, which will be named as the contact vacuum states for conciseness. In order to deal with the interaction problem in the sequel, the vacuum state is assumed with different (fake) velocities like as in [12] and the Riemann solution can be expressed as

$$(k, u)(x, t) = \begin{cases} (k_l, u_l), & x < k_l t, \\ (\xi, 0), & k_l t \leqslant x \leqslant k_r t, \\ (k_r, u_r), & x > k_r t, \end{cases} \tag{2.4}$$

where $\xi = \frac{x}{t}$ varies continuously from k_l to k_r, namely a fake vacuum wave like as rarefaction wave is introduced to connect the two states (k_l, u_l) and (k_r, u_r) (see Fig. 2.1(d)).

(iv) If $k_l > 0 > k_r$, then u consists of a jump function plus a weighted Dirac delta function supported on the line $x = 0$, which can be called as the delta standing wave and be denoted by DSW here (see Fig. 2.1(c)). It is noticed that the delta standing wave is similar to the delta contact discontinuity or the delta shock wave which is a weighted Dirac delta function supported on the contact discontinuity [13,17] or on the shock wave [20,15,19]. In order to define the measure solution as above, let us introduce the following definition.

Definition 2.1. The two-dimensional weighted δ-measure $w(t)\delta_\Gamma$ supported on the line Γ: $\{x = 0\}$ can be defined by

$$\langle w(t)\delta_\Gamma, \psi(x, t)\rangle = \int_a^b w(t)\psi(0, t)\,dt, \tag{2.5}$$

for any test function $\psi(x, t) \in C_0^\infty(R \times R_+)$, in which $w(t)$ denotes the strength of the delta standing wave at the time t.

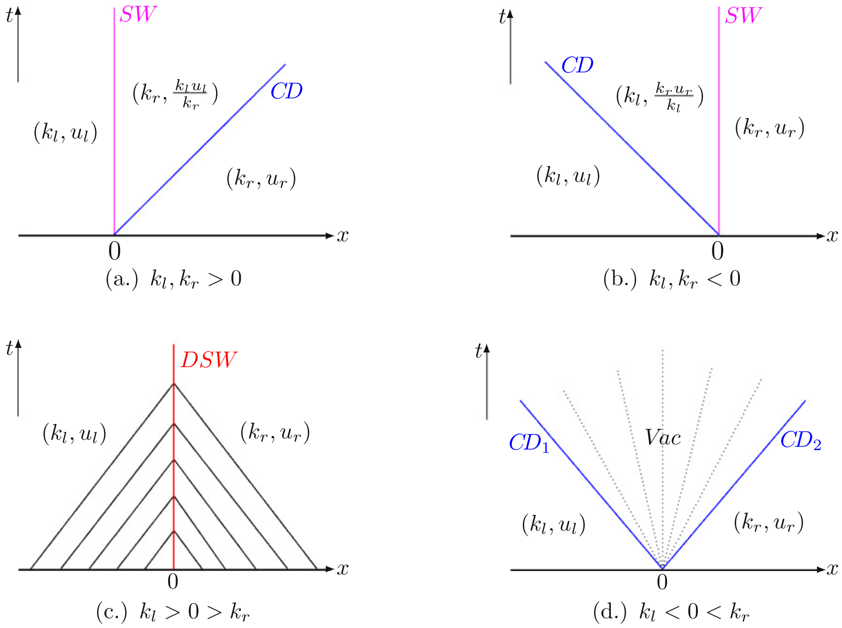


Fig. 2.1. The Riemann solutions when the initial states are away from vacuum.

Then, we have the following theorem to depict the weak solution of (1.2) and (1.3) in the sense of distribution.

Theorem 2.2. *With the above definition, when $k_l > 0 > k_r$, the Riemann solution of* (1.2) *and* (1.3) *is a delta standing wave connecting the two states (k_l, u_l) and (k_r, u_r), which can be expressed as*

$$(k,u)(x,t) = \begin{cases} (k_l, u_l), & x < 0, \\ (0, w(t)\delta(x)), & x = 0, \\ (k_r, u_r), & x > 0, \end{cases} \tag{2.6}$$

where

$$w(t) = (k_l u_l - k_r u_r)t. \tag{2.7}$$

Proof. Let us check that the above constructed δ-measure solution (2.6) with (2.7) should obey Eqs. (1.2) in the sense of distribution, namely the δ-measure solution (k, u) constructed above should satisfy

$$\langle u, \psi_t \rangle + \langle ku, \psi_x \rangle = 0, \tag{2.8}$$

for all test functions $\psi(x,t) \in C_0^\infty(R \times R_+)$.

In fact, by integrating by parts, we have

$$\begin{aligned} \int_0^\infty \int_{-\infty}^\infty (u\psi_t + ku\psi_x)\,dx\,dt &= \int_0^\infty \int_{-\infty}^0 (u_l\psi_t + k_l u_l \psi_x)\,dx\,dt + \int_0^\infty \int_0^\infty (u_r\psi_t + k_r u_r \psi_x)\,dx\,dt + \int_0^\infty w(t)\psi_t(0,t)\,dt \\ &= k_l u_l \int_0^\infty \psi(0,t)\,dt - k_r u_r \int_0^\infty \psi(0,t)\,dt - (k_l u_l - k_r u_r) \int_0^\infty \psi(0,t)\,dt = 0. \end{aligned}$$

In which we use the facts that $\psi(-\infty,t) = \psi(+\infty,t) = \psi(x,0) = 0$ since the function $\psi(x,t)$ is compactly supported in $R \times R_+$.

From above, we can see that (2.6) with (2.7) is indeed the piecewise smooth Riemann solution of (1.2) and (1.3) in the sense of distribution. □

Now, the definition of the elementary waves for (1.2) is given as follows:

Definition 2.3. All of the contact discontinuities *CD*, the standing waves *SW*, and the delta standing waves *DSW*, obtained in solving the Riemann problem, are called the elementary waves for (1.2).

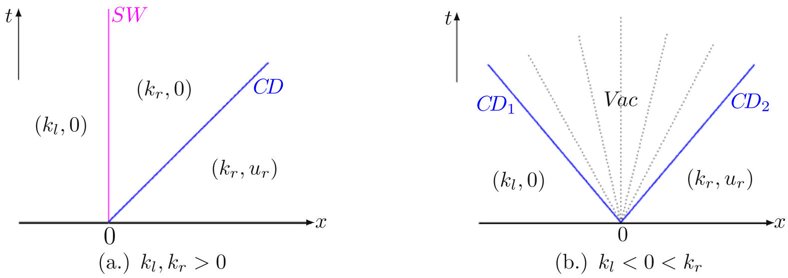


Fig. 2.2. The Riemann solutions for initial data (2.9) satisfying $k_r > 0$.

In the end, let us consider some special situations that $k_l = 0$ or $k_r = 0$.

(1) If $k_l > 0$ and $k_r = 0$, then there is only one delta standing wave connecting the two states (k_l, u_l) and (k_r, u_r), and the singular measure for u at $x = 0$ only comes from the left of *DSW* since all the characteristic lines starting from the ray $(x > 0, t = 0)$ are vertical.
(2) If $k_l = 0$ and $k_r < 0$, then there is also one delta standing wave connecting the two states (k_l, u_l) and (k_r, u_r), but the singular measure for u at $x = 0$ only comes from the right of *DSW* now.
(3) If $k_l < 0$ and $k_r = 0$, then there is a backward contact discontinuity followed by a standing wave and the intermediate state now is the vacuum state $(k_c, u_c) = (k_l, 0)$.
(4) If $k_l = 0$ and $k_r > 0$, then the Riemann solution consists of a standing wave followed by a forward contact discontinuity and the intermediate state between them is also the vacuum state $(k_c, u_c) = (k_r, 0)$.

2.3. The Riemann solution when $u_l = 0$ or $u_r = 0$

Let us now move to the situation that one of the Riemann initial states is at the vacuum. It is well known that the value of k needs not be specified for a vacuum state in any case, since $u = 0$ is always a solution of (1.1) no matter how k is defined. From the physical sense, there is also no meaning to define the velocity k in the vacuum region.

Keeping these concepts in mind, with the similar method in [2,12], we shall show how to solve the Riemann problem when one of the initial states is vacuum. Let us consider the Riemann problem when the left state is vacuum, namely

$$(k, u)(x, 0) = \begin{cases} (k_l, 0), & x < 0, \\ (k_r, u_r), & x > 0, \end{cases} \tag{2.9}$$

where k_l is illusory and $u_r > 0$.

Let us fix the right state (k_r, u_r) and slightly perturb the left state $(k_l, 0)$ to a state (k_l, u_l) with $u_l \ll 1$. If $k_r > 0$, then we are back to Case (i) or Case (iii) based on the value of k_l. More precisely, if $k_l > 0$, then we are back to Case (i) and the Riemann solution is $SW + CD$ which can be represented as

$$(k, u)(x, t) = \begin{cases} (k_l, u_l), & x < 0, \\ (k_r, \frac{k_l u_l}{k_r}), & 0 < x < k_r t, \\ (k_r, u_r), & x > k_r t. \end{cases} \tag{2.10}$$

Now the Riemann solution of (1.2) and (2.9) can be obtained by letting $u_l \to 0$ in (2.10) as (see Fig. 2.2(a)):

$$(k, u)(x, t) = \begin{cases} (k_l, 0), & x < 0, \\ (k_r, 0), & 0 < x < k_r t, \\ (k_r, u_r), & x > k_r t, \end{cases} \tag{2.11}$$

which implies that the right state (k_r, u_r) can be connected with the vacuum state by the contact discontinuity at $x = k_r t$.

Now if $k_l < 0$, then we come back to Case (iii) and the Riemann solution is given by (2.4). Letting $u_l \to 0$ in (2.4), we can see that the right state (k_r, u_r) can also be connected with the vacuum state by the contact discontinuity at $x = k_r t$ (see Fig. 2.2(b)). Compared with (2.11), it is clear that the difference for the solutions only lies in that they have the different virtual velocities in the fan region $k_l t < x < k_r t$.

Thus, we can see if the Riemann initial data are taken in the form

$$(k, u)(x, 0) = \begin{cases} Vacuum, & x < 0, \\ (k_r, u_r), & x > 0, \end{cases} \tag{2.12}$$

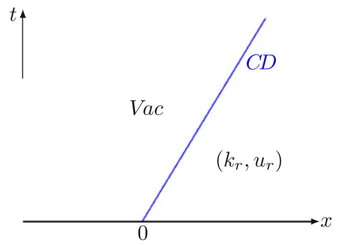


Fig. 2.3. The Riemann solutions for initial data (2.12) satisfying $k_r > 0$.

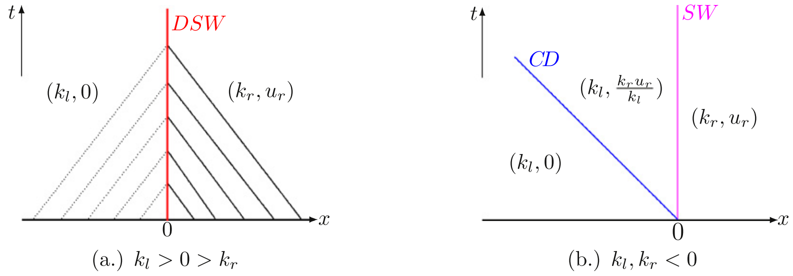


Fig. 2.4. The Riemann solutions for initial data (2.9) satisfying $k_r < 0$.

with $k_r > 0$ and $u_r > 0$, then the Riemann solution is (see Fig. 2.3):

$$(k,u)(x,t) = \begin{cases} Vacuum, & x < k_r t, \\ (k_r, u_r), & x > k_r t. \end{cases} \tag{2.13}$$

From the above results, we can see that the Riemann solution is stable with respect to the perturbation of the Riemann initial data.

On the other hand, if $k_r < 0$, then we are back to Case (ii) or Case (iv) in view of the value of k_l. If $k_l > 0$, then we are back to Case (iv) and the Riemann solution is given by (2.6) and (2.7). Letting $u_l \to 0$ in (2.6) and (2.7), we can see that the right state (k_r, u_r) can be connected with the vacuum state by the delta standing wave at $x = 0$, and the Riemann solution of (1.2) and (2.9) can be (see Fig. 2.4(a)):

$$(k,u)(x,t) = \begin{cases} (k_l, 0), & x < 0, \\ (0, -k_r u_r t\delta(x)), & x = 0, \\ (k_r, u_r), & x > 0. \end{cases} \tag{2.14}$$

If $k_l < 0$, then we are back to Case (ii) and the Riemann solution is $CD + SW$ which can be expressed as

$$(k,u)(x,t) = \begin{cases} (k_l, u_l), & x < k_l t, \\ (k_l, \frac{k_r u_r}{k_l}), & k_l t < x < 0, \\ (k_r, u_r), & x > 0. \end{cases} \tag{2.15}$$

Passing to the limit $u_l \to 0$ in (2.15), we can see that the Riemann solution of (1.2) and (2.9) should be (see Fig. 2.4(b)):

$$(k,u)(x,t) = \begin{cases} (k_l, 0), & x < k_l t, \\ (k_l, \frac{k_r u_r}{k_l}), & k_l t < x < 0, \\ (k_r, u_r), & x > 0, \end{cases} \tag{2.16}$$

which is obviously different from (2.14).

Thus, if we take the Riemann initial data as the form (2.12) with $k_r < 0$ and $u_r > 0$, then the Riemann solution is uncertain. Therefore, a big oscillation occurs and the solution changes dramatically under a small perturbation of the Riemann initial data in this situation.

Now we consider the Riemann problem when the right state is at the vacuum, namely

$$(k,u)(x,0) = \begin{cases} (k_l, u_l), & x < 0, \\ (k_r, 0), & x > 0, \end{cases} \tag{2.17}$$

where k_r is illusory and $u_l > 0$, then the solution of the Riemann problem is similar to the one we have just discussed before. Similarly, when the Riemann initial data are taken as the form

$$(k,u)(x,0) = \begin{cases} (k_l, u_l), & x < 0, \\ Vacuum, & x > 0, \end{cases} \tag{2.18}$$

with $u_l > 0$. If $k_l < 0$, then the Riemann solution of (1.2) and (2.18) is unique and can be represented as

$$(k,u)(x,t) = \begin{cases} (k_l, u_l), & x < k_l t, \\ Vacuum, & x > k_l t. \end{cases} \tag{2.19}$$

Otherwise, if $k_l > 0$, then the Riemann solution of (1.2) and (2.18) is also uncertain and we omit the detailed discussion here.

The above two examples are certainly spectacular examples of discontinuous dependence with respect to the Riemann initial data at the vacuum. Through the above analysis and discussion, we can summarize our results in the following.

Theorem 2.4. *If one of the Riemann initial states is at the vacuum, then the Riemann solution of* (1.2) *is possibly instable with respect to the small perturbation of the vacuum initial state.*

3. Interactions of elementary waves

In this section, we prepare to investigate the stability of the Riemann solutions when the initial states stay far away from vacuum, thus we take the perturbed Riemann initial data (1.4). The propose is to construct the global solutions for the Cauchy problem (1.2) and (1.4) for all the cases and then to determine whether the Riemann solutions of (1.2) and (1.3) are stable or not with respect to this perturbed Riemann initial data (1.4). To accomplish it, we need to study all the possible wave interactions by employing the method of characteristic analysis.

In order to cover all the cases completely, we divide our discussion into the following eight cases according to the different combinations of elementary waves starting from $(-\varepsilon, 0)$ and $(\varepsilon, 0)$ as follows:

(1) $k_l, k_m, k_r > 0$, (2) $k_l, k_m, k_r < 0$, (3) $k_l, k_m < 0$ and $k_r > 0$,

(4) $k_l < 0$ and $k_m, k_r > 0$, (5) $k_l, k_m > 0$ and $k_r < 0$, (6) $k_l > 0$ and $k_m, k_r < 0$,

(7) $k_l, k_r > 0$ and $k_m < 0$, (8) $k_l, k_r < 0$ and $k_m > 0$.

We will deal with this problem case by case along with constructing the solutions.

To study the perturbed Riemann problem (1.2) and (1.4) is to study various possible interactions of elementary waves for (1.2) essentially. Thus we need to encounter the interaction of the delta standing wave with the vacuum state. In this paper, we adopt the idea proposed by Liu and Smoller [12] when they considered the vacuum problem for the isentropic gas dynamic equations, where they make a distinction between two vacuum states with different (fake) velocities. This is equivalent to saying that we deal with the vacuum states like as the rarefaction waves and then approximate them by a large number of small amplitude non-physical shock waves like as the front tracking algorithm proposed by Bressan [6].

For convenience, the upper half plane $(x, t \geqslant 0)$ should be divided into three regions: $\Omega_L = \{(x,t) \mid x < -\varepsilon,\ t \geqslant 0\}$, $\Omega_M = \{(x,t) \mid -\varepsilon < x < \varepsilon,\ t \geqslant 0\}$ and $\Omega_R = \{(x,t) \mid x > \varepsilon,\ t \geqslant 0\}$. Obviously, we can see that $k = k_l$ in Ω_L, $k = k_m$ in Ω_M and $k = k_r$ in Ω_R are true for all the cases when vacuum is not involved.

Case 1. $k_l, k_m, k_r > 0$.

In this case, we consider the situation that a standing wave followed by a forward contact discontinuity emitting from $(-\varepsilon, 0)$ interacts with an another standing wave plus an another forward contact discontinuity $(\varepsilon, 0)$. When t is small enough, the solution of the initial value problem (1.2) and (1.4) can be expressed briefly as (see Fig. 3.1(a)):

$$(k_l, u_l) + SW_1 + (k_1, u_1) + CD_1 + (k_m, u_m) + SW_2 + (k_2, u_2) + CD_2 + (k_r, u_r),$$

where "+" means "followed by", and (k_1, u_1) and (k_2, u_2) are given respectively by

$$(k_1, u_1) = \left(k_m, \frac{k_l u_l}{k_m}\right) \quad \text{and} \quad (k_2, u_2) = \left(k_r, \frac{k_m u_m}{k_r}\right). \tag{3.1}$$

The propagation speed of CD_1 is $k_m > 0$, thus it intersects SW_2 in a finite time and the intersection point can be expressed as $(x_1, t_1) = (\varepsilon, \frac{2\varepsilon}{k_m})$. At the intersection point, a new Riemann problem is formed with the Riemann initial data as

$$(k_-, u_-) = (k_1, u_1) \quad \text{and} \quad (k_+, u_+) = (k_2, u_2). \tag{3.2}$$

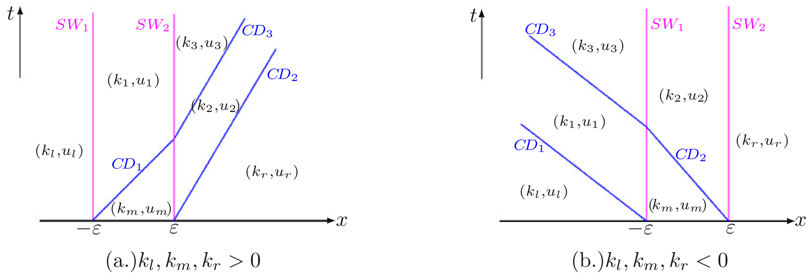


Fig. 3.1. Interactions of $SW+CD$ and $SW+CD$ in (a) and interaction of $CD+SW$ and $CD+SW$ in (b).

Thus a new contact discontinuity generates in the region Ω_R and be denoted by CD_3, whose propagation speed is k_r which is the same as that of CD_2. Thus, they are not able to overtake each other and then no interaction happens in the region Ω_R. Now the intermediate state between SW_2 and CD_3 is

$$(k_3,u_3)=\left(k_r,\frac{k_1u_1}{k_r}\right)=\left(k_r,\frac{k_lu_l}{k_r}\right). \tag{3.3}$$

It is easy to see that (x_1,t_1), $(-\varepsilon,0)$ and $(\varepsilon,0)$ tend to $(0,0)$ as $\varepsilon\to 0$. Thus we can see that the region Ω_M disappears and CD_2 and CD_3 coincide with each other in the limit situation, which is exactly the corresponding Riemann solution of (1.2) and (1.3). Thus, the Riemann solution is stable with respect to the perturbation of Riemann initial data in this situation.

Case 2. $k_l,k_m,k_r<0$.

In this case, we consider the situation that a backward contact discontinuity plus a standing wave emits from $(-\varepsilon,0)$ and $(\varepsilon,0)$ respectively. For sufficiently small t, the solution of the initial value problem (1.2) and (1.4) can be described by (see Fig. 3.1(b)):

$$(k_l,u_l)+CD_1+(k_1,u_1)+SW_1+(k_m,u_m)+CD_2+(k_2,u_2)+SW_2+(k_r,u_r),$$

where (k_1,u_1) and (k_2,u_2) are given respectively by

$$(k_1,u_1)=\left(k_l,\frac{k_mu_m}{k_l}\right)\quad\text{and}\quad(k_2,u_2)=\left(k_m,\frac{k_ru_r}{k_m}\right). \tag{3.4}$$

Wave interaction in this case likes as that in Case 1 but now the propagation speed of CD_2 is $k_m<0$, thus it intersects SW_1 in a finite time and the intersection point can be calculated by $(x_1,t_1)=(-\varepsilon,-\frac{2\varepsilon}{k_m})$.

Then a new contact discontinuity generates in the region Ω_L and also be denoted by CD_3, whose propagation speed is k_l which is the same as that of CD_1, thus no interaction occurs anymore in the region Ω_L. Now the intermediate state between CD_3 and SW_1 is

$$(k_3,u_3)=\left(k_l,\frac{k_2u_2}{k_l}\right)=\left(k_l,\frac{k_ru_r}{k_l}\right). \tag{3.5}$$

Similarly to the analysis in Case 1, the limit situation is also a backward contact discontinuity plus a standing wave which is exactly the corresponding Riemann solution.

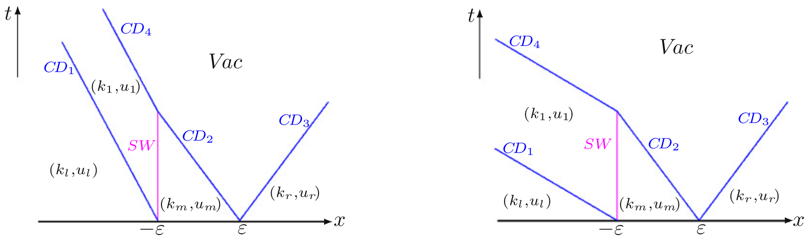
Case 3. $k_l,k_m<0$ and $k_r>0$.

Let us consider the interaction between a backward contact discontinuity followed by a standing wave starting from $(-\varepsilon,0)$ and the contact vacuum states starting from $(\varepsilon,0)$. For sufficiently small t, the solution of the Cauchy problem (1.2) and (1.4) can be constructed by (see Fig. 3.2):

$$(k_l,u_l)+CD_1+(k_1,u_1)+SW+(k_m,u_m)+CD_2+Vac+CD_3+(k_r,u_r),$$

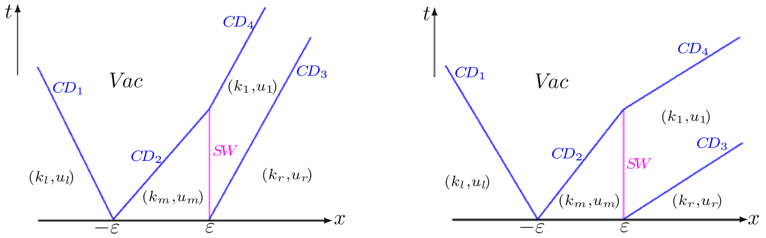
where (k_1,u_1) is given by (3.4).

Similarly to Case 2, the propagation speed of CD_2 is also $k_m<0$ and then it meets SW at a time and the intersection point is also $(x_1,t_1)=(-\varepsilon,-\frac{2\varepsilon}{k_m})$. After the time t_1, it will become the vacuum state gradually from $-\varepsilon$ to $-\infty$ in the region Ω_L along the new contact discontinuity CD_4 whose propagation speed is $k_l<0$. If $k_m<k_l<0$, then the diffusion speed of vacuum state slows down after t_1 (see Fig. 3.2(a)). Otherwise, if $k_l<k_m<0$, then the diffusion speed of vacuum state will speed up after t_1 (see Fig. 3.2(b)).



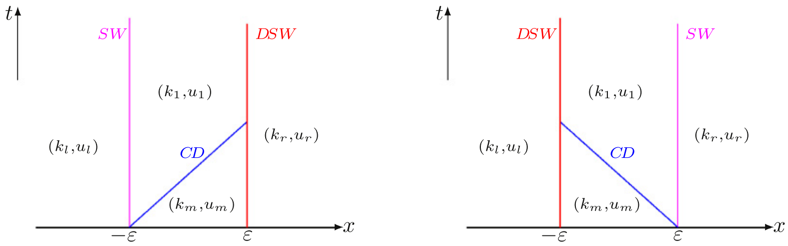
(a.) $k_r > 0 > k_l > k_m$ (b.) $k_r > 0 > k_m > k_l$

Fig. 3.2. Interaction of $CD + SW$ and contact vacuum states.



(a.) $k_l < 0 < k_r < k_m$ (b.) $k_l < 0 < k_m < k_r$

Fig. 3.3. Interaction of contact vacuum states and $SW + CD$.



(a) $k_l, k_m > 0$ and $k_r < 0$ (b) $k_l > 0$ and $k_m, k_r < 0$

Fig. 3.4. Interactions of $SW + CD$ and DSW in (a) and DSW and $CD + SW$ in (b).

The line of CD_4 can be expressed as $x + \varepsilon = k_l(t - t_1)$ and the line of CD_3 can be calculated by $x - \varepsilon = k_r t$. As $\varepsilon \to 0$, CD_1 and CD_4 coincide on the line $x = k_l t$ and CD_3 is on the line $x = k_r t$, and with the vacuum state between them. Thus, the Riemann solution is also stable with respect to the perturbation of Riemann initial data.

Case 4. $k_l < 0$ and $k_m, k_r > 0$.

Let us consider the interaction between the contact vacuum states starting from $(-\varepsilon, 0)$ and a standing wave followed by a forward contact discontinuity starting from $(\varepsilon, 0)$. The wave interaction is similar to that in Case 3 and we omit it in detail. Let us see Fig. 3.3, where $(k_1, u_1) = (k_r, \frac{k_m u_m}{k_r})$.

Case 5. $k_l, k_m > 0$ and $k_r < 0$.

In this case, we pay our attention to the interaction between a standing wave followed by a forward contact discontinuity emitting from $(-\varepsilon, 0)$ and a delta standing wave starting from $(\varepsilon, 0)$. For sufficiently small t, the solution of the Cauchy problem (1.2) and (1.4) is determined by (see Fig. 3.4(a)):

$$(k_l, u_l) + SW + (k_1, u_1) + CD + (k_m, u_m) + DSW + (k_r, u_r),$$

where (k_1, u_1) is given by (3.1).

Like in Case 1, CD meets DSW in a finite time and the intersection point is also $(x_1, t_1) = (\varepsilon, \frac{2\varepsilon}{k_m})$. Before the time t_1, the strength of DSW can be calculated by

$$w(t) = (k_m u_m - k_r u_r)t, \quad \text{for } t \leqslant t_1. \tag{3.6}$$

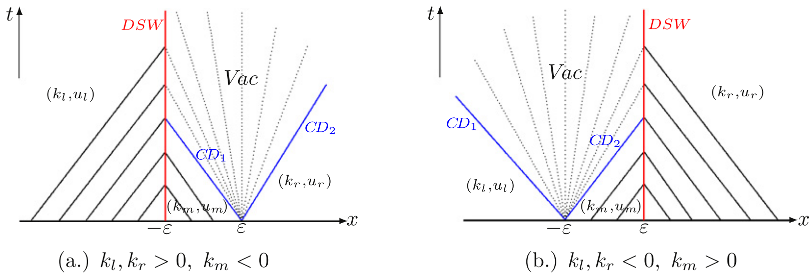


Fig. 3.5. Interaction of *DSW* and contact vacuum states.

At the intersection point, a new Riemann problem is formed whose Riemann initial data can be expressed as

$$(k,u)(x,t_1) = \begin{cases} (k_1,u_1), & x < \varepsilon, \\ (0, w(t_1)\delta(x)), & x = \varepsilon, \\ (k_r,u_r), & x > \varepsilon. \end{cases} \tag{3.7}$$

The Riemann solution is still a delta standing wave connecting (k_1,u_1) and (k_r,u_r), whose strength can be computed by

$$w(t) = (k_l u_l - k_r u_r)(t-t_1) + (k_m u_m - k_r u_r)t_1, \quad \text{for } t > t_1, \tag{3.8}$$

where we have used the fact that $k_1 u_1 = k_l u_l$.

As $\varepsilon \to 0$, the limit situation is still a delta standing wave on the line $x = 0$ with the strength $w(t) = (k_l u_l - k_r u_r)t$, which is also the corresponding Riemann solution.

Case 6. $k_l > 0$ and $k_m, k_r < 0$.

In this case, we deal with the interaction between a delta standing wave starting from $(-\varepsilon, 0)$ and a backward contact discontinuity followed by a standing wave emitting from $(\varepsilon, 0)$. This situation is similar to that in Case 5 and we omit it in detail. Let us see Fig. 3.4(b) in detail, where $(k_1,u_1) = (k_m, \frac{k_r u_r}{k_m})$.

Case 7. $k_l, k_r > 0$ and $k_m < 0$.

In this case, we deal with the situation that a delta standing wave emitting from $(-\varepsilon, 0)$ interacts with the contact vacuum states starting from $(\varepsilon, 0)$. When t is small enough, the solution of the Cauchy problem (1.2) and (1.4) can be denoted by (see Fig. 3.5(a)):

$$(k_l,u_l) + DSW + (k_m,u_m) + CD_1 + Vac + CD_2 + (k_r,u_r).$$

Similarly, *DSW* meets CD_1 at the intersection point $(x_1,t_1) = (-\varepsilon, -\frac{2\varepsilon}{k_m})$. Before the time t_1, the strength of *DSW* can be calculated by

$$w(t) = (k_l u_l - k_m u_m)t, \quad \text{for } t \leqslant t_1. \tag{3.9}$$

After the time t_1, we can see that the right states of *DSW* are the vacuum states. Then a new Cauchy problem is also formed at the intersection point whose initial data can be expressed as

$$(k,u)(x,t_1) = \begin{cases} (k_l,u_l), & x < -\varepsilon, \\ (0, w(t_1)\delta(x)), & x = -\varepsilon, \\ (k(x),0), & x > -\varepsilon, \end{cases} \tag{3.10}$$

where $k(x) = \frac{x-\varepsilon}{t_1}$ is the fake speed in the vacuum region with x varying from $-\varepsilon$ to ε. Here we use the vacuum wave instead of the vacuum state in order to solve this problem in a continuous way with respect to the initial data, namely we make a distinction between two vacuum states with different (fake) velocities [12,2,16,18].

The solution is still a delta standing wave on the line $x = -\varepsilon$, whose strength can now be given by

$$w(t) = k_l u_l(t-t_1) + (k_l u_l - k_m u_m)t_1, \quad \text{for } t > t_1. \tag{3.11}$$

Thus it is clear that the strength of *DSW* will continue to increase but not as quickly as before. This is due to the fact that the right-hand side of the line of *DSW* are the vacuum states and all the added matters come from the left-hand side of the line of *DSW*.

Passing to the limit $\varepsilon \to 0$, we can see that the solution is a delta standing wave followed by a contact discontinuity and with the vacuum state between them, which can be denoted by $(k_l, u_l) + DSW + Vac + CD + (k_r, u_r)$. To be more precise, the solution in the limit situation can be described as

$$(k,u)(x,t) = \begin{cases} (k_l, u_l), & x < 0, \\ (0, k_l u_l t\delta(x)), & x = 0, \\ (\frac{x}{t}, 0), & 0 < x < k_r t, \\ (k_r, u_r), & x > k_r t. \end{cases} \tag{3.12}$$

Clearly, the solution (3.12) in this limit situation is different from the corresponding Riemann solution in Case (i) in Section 2. Thus, we can see that the Riemann solution is unstable under this small perturbation of the Riemann initial data.

Case 8. $k_l, k_r < 0$ and $k_m > 0$.

In the end, let us consider the situation that the contact vacuum states emitting from $(-\varepsilon, 0)$ interacts with a delta standing wave starting from $(\varepsilon, 0)$. For sufficiently small t, the solution of the Cauchy problem (1.2) and (1.4) can be described as (see Fig. 3.5(b)):

$$(k_l, u_l) + CD_1 + Vac + CD_2 + (k_m, u_m) + DSW + (k_r, u_r).$$

This situation likes as that in Case 5 and can be dealt with in the same way. Here CD_2 meets DSW at the intersection point $(x_1, t_1) = (\varepsilon, \frac{2\varepsilon}{k_m})$. Before the time t_1, the strength of DSW is

$$w(t) = (k_m u_m - k_r u_r)t, \quad \text{for } t \leqslant t_1. \tag{3.13}$$

After the time t_1, there is still a delta standing wave on the line $x = \varepsilon$ whose strength now is

$$w(t) = -k_r u_r (t - t_1) + (k_m u_m - k_r u_r)t_1, \quad \text{for } t > t_1. \tag{3.14}$$

It should be noted that all the added matters come from the right-hand side of the line of DSW after the time t_1.

Letting $\varepsilon \to 0$, we can see the solution is a contact discontinuity followed by a delta standing wave and with the vacuum state between them, which can be denoted by $(k_l, u_l) + CD + Vac + DSW + (k_r, u_r)$. To be more precise, the solution in the limit situation can be described as

$$(k,u)(x,t) = \begin{cases} (k_l, u_l), & x < k_l t, \\ (\frac{x}{t}, 0), & k_l t < x < 0, \\ (0, -k_r u_r t\delta(x)), & x = 0, \\ (k_r, u_r), & x > 0. \end{cases} \tag{3.15}$$

Clearly, the solution (3.15) in this limit situation is different from the corresponding Riemann solution in Case (ii) in Section 2. Thus, we can see that the Riemann solution is also unstable under this small perturbation of the Riemann initial data.

So far, the discussion for all kinds of interactions has been finished. The structures of global solutions and the large time asymptotic states for the perturbed Riemann problem (1.2) and (1.4) are given for all the situations. From the above discussion, we can see when $k_l, k_r > 0$ (or $k_l, k_r < 0$) is satisfied in the Riemann initial data (1.3), and if we perturb it in the interval $(-\varepsilon, \varepsilon)$ and the perturbed state (k_m, u_m) obeys $k_m < 0$ (or $k_m > 0$), then the Riemann solution is unstable with respect to the perturbation (1.4) of Riemann initial data (1.3) even if the Riemann initial data (1.3) stay far away from vacuum. Thus the conclusion can be drawn by summarizing the above results as follows:

Theorem 3.1. *Even if the Riemann initial data stay far away from vacuum, then the Riemann solution of* (1.2) *is also possibly instable with respect to the small perturbation* (1.3) *of the Riemann initial state in some certain situations.*

4. Conclusion

It is remarkable that the Riemann solutions of (1.2) and (1.3) contain two extreme situations, namely one is the delta standing wave whose solution is a measure for u and the other is the vacuum state. It is delicate to study the stability of the Riemann solutions when strict hyperbolic fails here, which is also a major difficulty in solving hyperbolic systems of conservation laws. In this paper, we rigorously prove that the uniqueness of solutions to the Riemann problem (1.2) and (1.3) does not hold in some certain situations by taking the detailed examples, which was also checked numerically in [9] that each scheme chooses its own solution.

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